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On dual representation in classical electrodynamics

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Contents

Abstract. A discussion is given of the use of the dual representation in solving multipole radiation and electromagnetic wave diffraction problems in classical electrodynamics. In the method discussed, actual electric field sources are replaced by 'magnetic' ones. It is shown that despite the absence of Dirac magnetic monopoles, this formalism allows a physical interpretation of some frequently used methods.

1. Introduction

In various problems of classical electrodynamics, the method that may conventionally called 'dual' is often used. In this method, the real sources of electromagnetic fields — electric charges and currents — are replaced with fictitious magnetic sources. This change, which makes no sense at first glance, has nevertheless turned out to be rather efficient, for instance, in the theory of electromagnetic radiation diffraction from perfectly conducting surfaces. In particular, the well-known method whereby a scattered wave field is represented as the radiation field of a surface current [1, 2] can be related to the method where the source of a diffracted field is the magnetic current that flows over an opening in the screen under consideration (see, e.g., Refs [2–5]). The possibility of such a substitution of the integration surface is the essence of the well-known principle of complementary screens, and the mathematical formulation of dual methods is based on the use of the so-called Kirchhoff–Smythe integral [4–8].

Another example can be adduced. In solving the problem of the Vavilov–Cherenkov radiation generated by a magnetic dipole, in one of his papers, I M Frank considered the magnetic dipole as a Dirac dipole — a pair of magnetic charges of opposite signs [9]. This method was also used in one of the first reviews on Cherenkov radiation [10]. More recently, it turned out that this representation is valid only in the vacuum [11–13].

The equivalence of an ordinary magnetic dipole and the Dirac one in the vacuum also follows from investigations of the radiation of an arbitrarily moving point-like magnetic dipole with a moment $\boldsymbol{\mu}$ [14–16]. Notably, in the studies cited above, it was shown that the solution of this problem can be obtained from the corresponding solution of the problem of the radiation of an electric dipole with a moment $\mathbf{d}$ by the simple substitution of fields $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, with $\mathbf{d} \to \boldsymbol{\mu}$ (in the static case, this property is well known [17]). This substitution corresponds to passing from the ordinary strength tensor $F^{\mu\nu}$ to its dual pseudotensor $\tilde{F}^{\mu\nu} = 1/2\epsilon^{\mu\nu\eta\rho}F_{\eta\rho}$. This is precisely the change involved in passing from electric sources to magnetic ones (see, e.g., Ref. [2]).

The dual method of solution was also used to advantage in problems of the diffraction radiation of a charged particle flying past a perfectly conducting screen [18–20]. Its use allowed a relatively easy derivation of an analytic expression for the density of the surface current induced by the charge field on the screen [20].

Interestingly, while the application of the dual method in the theory of Cherenkov multipole radiation was repeatedly discussed (for instance, by Ginzburg [11] and Frank [12]), this issue went unheeded in the radiation theory of a point-like magnetic moment, despite the 'dual symmetry' of the radiation fields of electric and magnetic dipoles noticed in [14]. On the other hand, the dual representation is ordinarily not given a clear physical interpretation in the diffraction theory (the exception is perhaps provided by the well-known monograph by Vainshtein [2]), which sometimes leads to serious misunderstandings. For instance, in many textbooks on electrodynamics, the issues of diffraction are treated based on the Kirchhoff–Smythe approach [5–7]. However, even in the well-known textbook by Jackson [7], the physical aspect of this method is, in our opinion, discussed inadequately. Recently, Drezet et al. [8] succeeded in producing a more rigorous mathematical proof of the Kirchhoff–Smythe integral; however, its physical interpretation

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remained inadequate as before (see formulas (7)–(9) in Ref. [8]). The aim of this paper is to discuss the use of the dual representation in several problems of classical electrodynamics.

2. Dual representation in the theory of radiation

We consider a homogeneous isotropic medium in which the electromagnetic field is described by the macroscopic displacement tensor $H^{\mu\nu} = (-\mathbf{D}, \mathbf{H})$, i.e., $H^{01,02,03} = \mathbf{D}(\mathbf{r}, t)$ is the electric displacement vector and $H^{23,31,12} = \mathbf{H}(\mathbf{r}, t)$ is the macroscopic magnetic field.[1] The Maxwell equations with a given density of electric sources $j_{\mathrm{e}}^{\mu} = \{c\rho_{\mathrm{e}}, \mathbf{j}_{\mathrm{e}}\}$ have the form

$$\partial_\nu \tilde{F}^{\mu\nu} = 0\,, \quad \partial_\nu H^{\mu\nu} = \frac{4\pi}{c}\, j_{\mathrm{e}}^{\mu}\,, \tag{1}$$

where $\partial_\nu = \partial/\partial r^\nu$ and the first pair of equations, which does not contain charges and currents, is written in terms of the pseudotensor $\tilde{F}^{\mu\nu} = (-\mathbf{B}, -\mathbf{E})$. Here, $\mathbf{E}(\mathbf{r}, t)$ is the electric field and $\mathbf{B}(\mathbf{r}, t)$ is the magnetic induction. As usual, the pseudotensor $\tilde{F}^{\mu\nu}$, which is dual to the true tensor $F^{\mu\nu} = (-\mathbf{E}, \mathbf{B})$, is $\tilde{F}^{\mu\nu} = 1/2\epsilon^{\mu\nu\eta\rho} F_{\eta\rho}$, $\epsilon^{0123} = 1$. For the subsequent discussion, it is expedient to also give the inverse relation: $F^{\mu\nu} = -1/2\epsilon^{\mu\nu\eta\rho}\tilde{F}_{\eta\rho}$.

In the wave zone, in view of the current density continuity equation $\partial_\mu j_{\mathrm{e}}^{\mu} = 0$, the solutions of Maxwell equations (1) for monochromatic fields produced by sources localized in a domain with the size $r_{\mathrm{eff}} \lesssim \lambda$ have the form (for simplicity, we assume that the medium has only the frequency dispersion)

$$\mathbf{E}_{\mathrm{e}}^{\mathrm{R}}(\mathbf{r}_0, \omega) = -\frac{\mathrm{i}(2\pi)^3}{\omega}\,\frac{1}{\varepsilon(\omega)}\,\frac{\exp(\mathrm{i}kr_0)}{r_0}\,\mathbf{k}\times\left[\mathbf{k}\times\mathbf{j}_{\mathrm{e}}(\mathbf{k}, \omega)\right],$$

$$\mathbf{B}_{\mathrm{e}}^{\mathrm{R}}(\mathbf{r}_0, \omega) = \frac{\mathrm{i}(2\pi)^3}{c}\,\mu(\omega)\,\frac{\exp(\mathrm{i}kr_0)}{r_0}\,\mathbf{k}\times\mathbf{j}_{\mathrm{e}}(\mathbf{k}, \omega)\,, \tag{2}$$

where $\mathbf{E}_{\mathrm{e}}(\mathbf{r}_0, \omega) = \mathbf{D}_{\mathrm{e}}(\mathbf{r}_0, \omega)/\varepsilon(\omega)$, $\mathbf{B}_{\mathrm{e}}(\mathbf{r}_0, \omega) = \mu(\omega)\,\mathbf{H}_{\mathrm{e}}(\mathbf{r}_0, \omega)$, $k = \omega/c\sqrt{\varepsilon(\omega)\mu(\omega)}$ is the wavenumber, and the wave vector is aligned with the radius vector of the observation point: $\mathbf{k} = k\mathbf{r}_0/r_0 = k\mathbf{e}$. The vector $\mathbf{j}_{\mathrm{e}}(\mathbf{k}, \omega)$ is the Fourier transform of the current density $\mathbf{j}_{\mathrm{e}}(\mathbf{r}, t)$. As can be easily seen, fields (2) have all the properties of radiation fields:

$$\mathbf{E}_{\mathrm{e}}^{\mathrm{R}} = -(\varepsilon\mu)^{-1/2}[\mathbf{e}\times\mathbf{B}_{\mathrm{e}}^{\mathrm{R}}]\,,$$

$$(\mathbf{e}, \mathbf{E}_{\mathrm{e}}^{\mathrm{R}}) = (\mathbf{e}, \mathbf{B}_{\mathrm{e}}^{\mathrm{R}}) = 0\,, \quad E_{\mathrm{e}}^{\mathrm{R}} = (\varepsilon\mu)^{-1/2} B_{\mathrm{e}}^{\mathrm{R}}\,. \tag{3}$$

We now write the equations for magnetic currents in the absence of electric ones (see, e.g., Ref. [2]):

$$\partial_\nu \tilde{F}^{\mu\nu} = \frac{4\pi}{c}\,\tilde{j}_{\mathrm{m}}^{\mu}\,, \quad \partial_\nu H^{\mu\nu} = 0\,, \tag{4}$$

where $\tilde{j}_{\mathrm{m}}^{\mu} = \{c\tilde{\rho}_{\mathrm{m}}, \tilde{\mathbf{j}}_{\mathrm{m}}\}$ is the magnetic current pseudovector. We note that the first equation in (4) is easily obtained from the Lagrangian $L = (1/c)\tilde{j}_\mu \tilde{A}^\mu - (1/16\pi)\tilde{F}_{\alpha\beta}\tilde{F}^{\alpha\beta}$, and the second equation follows from the definition of the dual strength tensor in terms of 'magnetic' potentials: $\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu$ (see Section 3). Therefore, the dual Maxwell equations are obtained from the ordinary ones by the substitution

$$j_{\mathrm{e}}^{\mu} \to j_{\mathrm{m}}^{\mu}\,, \quad H^{\mu\nu} \to \tilde{F}^{\mu\nu}\,, \tag{5}$$

or $\mathbf{E}_{\mathrm{e}} \to \mathbf{H}_{\mathrm{m}}$, $\mathbf{H}_{\mathrm{e}} \to -\mathbf{E}_{\mathrm{m}}$, and $\varepsilon \to \mu$ in terms of fields. Hence, in view of the continuity equation $\partial_\mu \tilde{j}_{\mathrm{m}}^{\mu} = 0$ [which follows from the first of Eqns (4)], we can find the radiation fields of magnetic currents in the wave zone:

$$\mathbf{E}_{\mathrm{m}}^{\mathrm{R}}(\mathbf{r}_0, \omega) = -\frac{\mathrm{i}(2\pi)^3}{c}\,\frac{\exp(\mathrm{i}kr_0)}{r_0}\,\mathbf{k}\times\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{k}, \omega)\,,$$

$$\mathbf{B}_{\mathrm{m}}^{\mathrm{R}}(\mathbf{r}_0, \omega) = -\frac{\mathrm{i}(2\pi)^3}{\omega}\,\frac{\exp(\mathrm{i}kr_0)}{r_0}\,\mathbf{k}\times\left[\mathbf{k}\times\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{k}, \omega)\right]. \tag{6}$$

The equivalence condition for electric and magnetic sources from the standpoint of the equality of their generated fields is easily obtained by comparing formulas (2) and (6). It suffices to impose the requirement, for instance, that the equality $\mathbf{H}_{\mathrm{e}}^{\mathrm{R}} = \mathbf{H}_{\mathrm{m}}^{\mathrm{R}}$ hold. The equality between the electric fields follows automatically due to (3). Therefore, for a given magnetic current, the condition that the magnetic fields be equal is satisfied when the electric current is defined as

$$\mathbf{j}_{\mathrm{e}}(\mathbf{k}, \omega) = -\frac{c}{\omega}\,\frac{1}{\mu(\omega)}\,\mathbf{k}\times\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{k}, \omega)\,, \tag{7}$$

or, in spatial variables,

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega) = \frac{\mathrm{i}c}{\omega}\,\frac{1}{\mu(\omega)}\,\mathrm{rot}\,\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}, \omega)\,. \tag{8}$$

The validity of this relation may be illustrated with the example of the problem of magnetic moment radiation. Let there be Dirac dipoles with a volume dipole-moment density $\mathbf{M}(\mathbf{r}, t)$ and a magnetic current defined by the expression

$$\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}, \omega) = -\mathrm{i}\omega\,\mathbf{M}(\mathbf{r}, \omega)\,, \tag{9}$$

which is similar to the conventional expression for the current $\mathbf{j}_{\mathrm{e}} = -\mathrm{i}\omega\mathbf{P}$ of electric dipoles. According to expression (8), the electric current

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega) = \frac{c}{\mu(\omega)}\,\mathrm{rot}\,\mathbf{M}(\mathbf{r}, \omega) \tag{10}$$

corresponds to this magnetic current. For nonmagnetic media ($\mu(\omega) = 1$), this current coincides with the current produced in the vacuum by the current of an ordinary magnetic moment, for instance, the intrinsic magnetic moment of a particle. Therefore, the problem of the radiation of Dirac dipoles with current (9) is exactly equivalent to the problem of the radiation of electric current (10). Hence, it follows that the solution of the problem of magnetic-dipole radiation in the vacuum can be obtained from the corresponding solution of the problem of electric-dipole radiation by the duality transformation: $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, with $\mathbf{P} \to \mathbf{M}$. This property was confirmed by direct calculations in Refs [14–16].

We emphasize that in the special case of a point-like magnetic moment with the density $\mathbf{M}(\mathbf{r}, t) = \boldsymbol{\mu}(t)\times\delta(\mathbf{r} - \mathbf{r}_\mu(t))$,[2] currents (9) and (10) are defined in its rest frame, because the expression for the current in the laboratory frame also comprises the electric dipole moment density (see below). Indeed, because the volume densities of the electric and magnetic moments $\mathbf{P}$ and $\mathbf{M}$ constitute an

[1] We use the metric $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, in which $r^\mu = \{ct, \mathbf{r}\}$, $r_\mu = \{-ct, \mathbf{r}\}$.

[2] The time dependence of the orientation of the vector $\boldsymbol{\mu}(t) = \mu\boldsymbol{\zeta}(t)$ is due to a possible precession of the unit vector $\boldsymbol{\zeta}(t)$ of the particle spin (see, e.g., Ref. [16]).

antisymmetric second-rank tensor $M^{\mu\nu} = (\mathbf{P}, \mathbf{M})$, the existence of only the magnetic moment $M'^{\mu\nu} = (\mathbf{0}, \mathbf{M}')$ in the particle rest frame automatically gives rise to an electric moment in the laboratory frame: $\mathbf{P} = \gamma\boldsymbol{\beta} \times \mathbf{M}'$ (here, $\gamma = 1/(1-\beta^2)^{1/2}$ is the Lorentz factor of the particle).

In the derivation of expression (7) for the electric current equivalent to a given magnetic one, we imposed the requirement that the magnetic fields of the radiation of the currents of both types be equal. We now impose the same requirement on the electric fields. The magnetic fields are then also equal due to relations (3). Comparing expressions (2) and (6), we see that

$$\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{k}, \omega) = \frac{c}{\omega} \frac{1}{\varepsilon(\omega)} \mathbf{k} \times \mathbf{j}_{\mathrm{e}}(\mathbf{k}, \omega)\,, \tag{11}$$

or, in terms of spatial variables,

$$\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}, \omega) = -\frac{\mathrm{i}c}{\omega} \frac{1}{\varepsilon(\omega)} \operatorname{rot} \mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega)\,. \tag{12}$$

These formulas permit finding the magnetic current equivalent to a given electric one, i.e., the current producing the same radiation fields. For example, we consider the current produced by electric dipoles with a density $\mathbf{P}$:

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega) = -\mathrm{i}\omega \mathbf{P}(\mathbf{r}, \omega)\,. \tag{13}$$

According to formula (12), the magnetic current

$$\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}, \omega) = -\frac{c}{\varepsilon(\omega)} \operatorname{rot} \mathbf{P}(\mathbf{r}, \omega) \tag{14}$$

corresponds to this current. In the case of a point-like dipole, this implies that for $\varepsilon(\omega) = 1$, the electric dipole in the rest frame is equivalent to the Dirac analog of an ordinary magnetic dipole.

Therefore, it is possible to put every problem of the radiation of an electric current (of an arbitrary multipolarity in general) into correspondence with its 'dual' problem of magnetic currents producing the same radiation fields. An important illustration of this symmetry is the principle of complementary screens in the theory of diffraction, which is considered in Section 3.

3. Dual representation in the theory of diffraction

Let a space (vacuum) be divided by a perfectly conducting planar screen with openings in it through which the field from one half-space can penetrate into the other one. We show that the field of a plane wave diffracted by the openings in the screen can be represented in two equivalent forms: first, as the radiation field of the surface electric current induced on the screen by the incident wave; second, as the radiation field of the magnetic current flowing over the complementary screen [i.e., the opening (see Fig. 1)]. To prove this, we proceed similarly to the method of the vector diffraction theory; but because we are concerned with the symmetry properties under the duality transformation $F^{\mu\nu} \to \tilde{F}^{\mu\nu}$, we use the tensor form whenever possible.

We write the Gauss theorem for a third-rank tensor:

$$\int \partial^{\mu} T_{\alpha\beta\mu} \mathrm{d}\Upsilon = \oint T_{\alpha\beta\mu} n^{\mu} \mathrm{d}\sigma\,. \tag{15}$$

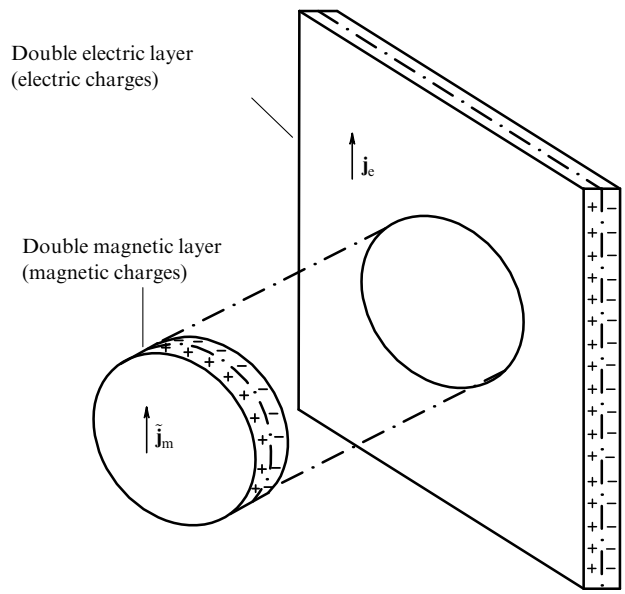

Figure 1. The simplest geometry of diffraction from a planar screen with a round opening (complementary screen).

The integration in the left-hand side is performed over the 4-volume ($\mathrm{d}\Upsilon = \mathrm{d}r^0\,\mathrm{d}r^1\,\mathrm{d}r^2\,\mathrm{d}r^3 = c\,\mathrm{d}t\,\mathrm{d}^3r$) bounded by a closed hypersurface σ with a unit outer normal n^{μ} (a space-like vector). We choose the tensor $T_{\alpha\beta\mu}$ as

$$T_{\alpha\beta\mu} = S_{\alpha\beta}\,\partial_{\mu} G - G\,\partial_{\mu} S_{\alpha\beta}\,. \tag{16}$$

Because external currents are nonexistent in the problem of the diffraction of a plane wave, we take the quantities $S_{\alpha\beta}$ to be the 'free'-field tensor $F_{\alpha\beta} = (\mathbf{E}, \mathbf{H})$, which satisfies the homogeneous Maxwell equations

$$\partial_{\rho} F_{\nu\mu} + \partial_{\nu} F_{\mu\rho} + \partial_{\mu} F_{\rho\nu} = 0, \quad \partial^{\mu} F_{\nu\mu} = 0 \tag{17}$$

inside the closed surface σ and satisfies the boundary conditions for a perfect conductor (see below) on the surface itself. Formula (16) also involves the retarded Green's function [21]:

$$G = \frac{\theta(ct - c\acute{t})}{2\pi}\, \delta\big[(r^{\mu} - \acute{r}^{\mu})^2\big]\,. \tag{18}$$

Here, $\theta(x)$ is the Heaviside function and the prime denotes integration variables in (15). By substituting the tensor $T_{\alpha\beta\mu} = T_{\alpha\beta\mu}(\acute{r}, \acute{t})$ in form (16) in the left-hand side of Gauss theorem (15), we obtain

$$\int \partial^{\mu} T_{\alpha\beta\mu} \mathrm{d}\Upsilon = \int \Big(F_{\alpha\beta}\,\Box G - G\,\Box\, F_{\alpha\beta}\Big)\,\mathrm{d}\Upsilon, \tag{19}$$

where $\Box = \partial^{\mu}\partial_{\mu}$. By virtue of the homogeneity of Eqns (17), the second term vanishes. Because the Green's function satisfies the equation [21]

$$\Box G = -\delta^{(4)}(r^{\mu} - \acute{r}^{\mu})\,, \tag{20}$$

we obtain the value of the strength tensor at the point $r^{\mu} = \{ct, \mathbf{r}\}$ in the left-hand side of (19):

$$\int \partial^{\mu} T_{\alpha\beta\mu}\,\mathrm{d}\Upsilon = -F_{\alpha\beta}(\mathbf{r}, t)\,. \tag{21}$$

Because $n^\mu \partial_\mu = (\mathbf{n}, \nabla)$ in the right-hand side of (15) in the rest frame of the screen, we finally obtain

$$F_{\mu\nu}(\mathbf{r}, t) = \oint \big(F_{\mu\nu}(\mathbf{n}, \nabla)\, G - G(\mathbf{n}, \nabla)\, F_{\mu\nu}\big) \mathrm{d}\sigma\,, \tag{22}$$

where the normal $\mathbf{n}$ is now directed toward the interior of the domain bounded by the surface σ, whose integration element is written as $\mathrm{d}\sigma = c\,\mathrm{d}t'\,\mathrm{d}S$, where $\mathrm{d}S$ is the integration element of an ordinary two-dimensional surface. Because the tensor $F^{\mu\nu}$ is made up of electric and magnetic field components, expression (22) is an ordinary Kirchhoff integral written for time-dependent quantities. It is easily seen that after the integration over time and the Fourier transformation $t \to \omega$, formula (22) exactly coincides with the generally accepted formulation of the Kirchhoff integral for monochromatic fields (see, e.g., Ref. [7]):

$$F_{\mu\nu}(\mathbf{r}, \omega) = \oint \big(F_{\mu\nu}(\mathbf{r}', \omega)(\mathbf{n}, \nabla)\, G_\omega - G_\omega(\mathbf{n}, \nabla)\, F_{\mu\nu}(\mathbf{r}', \omega)\big)\, \mathrm{d}S, \tag{23}$$

where $G_\omega = 1/(4\pi) \exp\{\mathrm{i}\omega|\,\mathbf{r}' - \mathbf{r}\,|/c\}/|\,\mathbf{r}' - \mathbf{r}\,|$ is the Fourier component of the retarded Green's function.

Quite frequently, the closed surface S is chosen as the surface of the screen, with an infinite-radius half-sphere bearing on the screen; on the half-sphere surface, the fields satisfy the radiation conditions. But this choice requires invoking additional assumptions about the field behavior at the boundary between the screen and the openings. In the general case, a line integral along the edge of the opening should be added to surface integral (23) in order to obtain the exact solution of the problem (see Refs [2, 6]). We note that a similar situation also occurs when the Kirchhoff integral is used to calculate the diffraction radiation of a charged particle flying through an opening in a perfectly conducting screen [22]. However, another way of choosing the integration surface S in formula (23) exists, which leads to physically more lucid results. Namely, the surface can be chosen in the form of a 'double layer' and an infinite-radius sphere, on whose surface the radiation conditions for the fields $F^{\mu\nu}$ are also satisfied (see, e.g., Refs [7, 19]). In this case, after several simple but rather cumbersome transformations, formula (23) reduces to the form (to be compared with the three-dimensional form in Refs [6, 7, 19])

$$F_{\mu\nu}(\mathbf{r}, \omega) = \partial_\mu \int 2F_{\nu\eta}\, n^\eta G_\omega\, \mathrm{d}S - \partial_\nu \int 2F_{\mu\eta}\, n^\eta G_\omega\, \mathrm{d}S\,, \tag{24}$$

where integration is performed only over the screen surface and where the property $\partial_\mu G_\omega = \partial G_\omega/\partial \acute{r}^\mu = -\partial G_\omega/\partial r^\mu$ is used, which permits eliminating the 4-gradient from the integrand. The 4-vectors $F_{\nu\eta}\, n^\eta$ appearing here allow a simple physical interpretation: they are surface currents and charges induced by the incident wave on the screen.

We integrate Maxwell equations (1), by analogy with the three-dimensional case, over the 4-volume bounded by the surface of a hypercylinder whose bases, parallel to the surface boundary, are assumed to be so small that the fields may be considered constant on their surfaces. According to the four-dimensional Gauss theorem, as the cylinder height δ tends to zero, we have

$$\int_{\Upsilon \to 0} \partial_\nu H^{\mu\nu}\, \mathrm{d}\Upsilon = \oint_{\delta \to 0} \big(H_2^{\mu\nu} - H_1^{\mu\nu}\big)\, \mathrm{d}\sigma_\nu\,, \tag{25}$$

where $\mathrm{d}\sigma_\nu = n_\nu\, \mathrm{d}\sigma$ is an element of the boundary; the minus sign appears because the normals to the opposite cylinder bases are directed oppositely. The right-hand side of the second pair of Eqns (1), which contains currents and charges, does not vanish upon integration over the infinitely small hypercylinder volume only when there is a surface current on the boundary. In other words, the equality $j_{\mathrm{e}}^\mu = j_{\mathrm{s}}^\mu \delta(r^{\mathrm{i}} - r_{\mathrm{s}}^{\mathrm{i}})$ must be satisfied, whence the boundary condition

$$(H_2^{\mu\nu} - H_1^{\mu\nu})\, n_\nu = \frac{4\pi}{c}\, j_{\mathrm{s}}^\mu \tag{26}$$

follows. Written in three-dimensional notation, this condition takes the well-known form

$$(\mathbf{n}, \mathbf{D}_2 - \mathbf{D}_1) = 4\pi\rho_{\mathrm{s}}, \quad [\mathbf{n} \times \mathbf{H}_2 - \mathbf{H}_1] = \frac{4\pi}{c}\, \mathbf{j}_{\mathrm{s}}\,. \tag{27}$$

Similarly, by performing the integration of the first pair of Eqns (1) with the use of the Gauss theorem, we obtain the second boundary condition,

$$(\tilde{F}_2^{\mu\nu} - \tilde{F}_1^{\mu\nu})\, n_\nu = 0\,, \tag{28}$$

or, in the three-dimensional form,

$$(\mathbf{n}, \mathbf{B}_2 - \mathbf{B}_1) = 0\,, \quad [\mathbf{E}_2 - \mathbf{E}_1 \times \mathbf{n}] = 0\,. \tag{29}$$

If we introduce magnetic currents in lieu of electric ones, boundary conditions (26) become homogeneous and the conditions derived from the first pair of the Maxwell equations take the form

$$(\tilde{F}_2^{\mu\nu} - \tilde{F}_1^{\mu\nu})\, n_\nu = \frac{4\pi}{c}\, \tilde{j}_{\mathrm{s}}^\mu\,, \tag{30}$$

with the pseudovector of the surface magnetic current

$$\tilde{j}_{\mathrm{s}}^\mu = \frac{c}{4\pi} \big\{(\mathbf{n}, \mathbf{B}_2 - \mathbf{B}_1), [\mathbf{E}_2 - \mathbf{E}_1 \times \mathbf{n}]\big\}\,. \tag{31}$$

Because the field inside a perfect conductor is nonexistent in the problem of diffraction from a perfectly conducting screen, the boundary conditions take the simple form

$$F^{\mu\nu} n_\nu = \frac{4\pi}{c}\, j_{\mathrm{s}}^\mu\,, \quad \tilde{F}^{\mu\nu} n_\nu = \frac{4\pi}{c}\, \tilde{j}_{\mathrm{s}}^\mu\,, \tag{32}$$

where $F^{\mu\nu}$ is the field strength tensor in the vacuum.

In view of these equalities, as is easily seen, the integrals entering formula (24) are the potentials of the radiation field generated by the doubled surface current:

$$A_{\mathrm{rad}}^\mu(\mathbf{r}, \omega) = \frac{4\pi}{c} \int 2j_{\mathrm{s}}^\mu G_\omega\, \mathrm{d}S\,, \tag{33}$$

or, in an explicitly covariant form,

$$A_{\mathrm{rad}}^\mu(\mathbf{r}, t) = \frac{4\pi}{c} \int j^\mu G\, \mathrm{d}\Upsilon, \tag{34}$$

where $j^\mu = 2j_{\mathrm{s}}^\mu \delta(r^{\mathrm{i}} - r_{\mathrm{s}}^{\mathrm{i}})$. Therefore, the 'free'-field tensor in the presence of boundaries is expressed in terms of surface currents flowing along these boundaries, which is sometimes treated as the electrodynamic formulation of the Huygens principle [2]. In this notation, formula (24) is rewritten as

$$F^{\mu\nu} = \partial^\mu A_{\mathrm{rad}}^\nu - \partial^\nu A_{\mathrm{rad}}^\mu\,. \tag{35}$$

The mechanism of the emergence of the 'extra' factor 2 in the right-hand side of expressions (24) and (33) was explained, for instance, in Refs [2, 8, 23]. Physically, the selection of the integration surface in the form of a double (rather than simple) layer signifies that the surface current flowing over the screen is formed by electric dipoles and that the surface itself is a double electric layer [13]. This formalism also has the effect that the integral over the opening surface is identically zero due to the absence of a surface current. This obviates the use of a line integral and leads to the same results [2, 6].

We now consider the formulation of the dual representation in the diffraction problem. Because the choice of the tensor in formula (16) is arbitrary, we can choose the dual pseudotensor $\tilde{F}^{\mu\nu}$ instead of the ordinary strength tensor (or simply multiply Eqn (22) by a totally antisymmetric expression). It is evident that the equality following therefrom,

$$\tilde{F}_{\mu\nu}(\mathbf{r},\omega) = \oint \big(\tilde{F}_{\mu\nu}(\mathbf{r}',\omega)(\mathbf{n},\nabla)\, G_\omega - G_\omega(\mathbf{n},\nabla)\, \tilde{F}_{\mu\nu}(\mathbf{r}',\omega)\big)\, \mathrm{d}S\,, \tag{36}$$

is exactly analogous to expression (23). But with the corresponding choice of the integration surface in the form of a double layer, expression (36) is transformed into a formula that differs from (24) (to be compared with the three-dimensional form in Refs [6, 7, 19]),

$$\tilde{F}_{\mu\nu}(\mathbf{r},\omega) = \partial_\mu \int 2\tilde{F}_{\nu\eta}\, n^\eta G_\omega\, \mathrm{d}S - \partial_\nu \int 2\tilde{F}_{\mu\eta}\, n^\eta G_\omega\, \mathrm{d}S\,. \tag{37}$$

Formula (37) differs from expression (24) in that the wave field in (37) is represented as the radiation field of surface magnetic current (32), which generates 'magnetic' potentials $\tilde{A}^\mu$:

$$\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu_{\mathrm{rad}} - \partial^\nu \tilde{A}^\mu_{\mathrm{rad}}\,. \tag{38}$$

Because the tangential components of the electric field vanish on the surface of a perfect conductor, the integral in formula (37) reduces to the integral over only the opening, i.e., over the screen complementary to the one under consideration (see Fig. 1). This signifies that a replacement of ordinary surface currents with magnetic ones corresponds to the substitution of the complementary screen for the integration surface, i.e., a transition to the dual representation: $F^{\mu\nu} \to \tilde{F}^{\mu\nu}$. In this case, the surface of the opening, which carries the doubled magnetic current, is equivalent to a doubled magnetic layer [13] because the Dirac dipoles that make up this surface current are equivalent to ordinary magnetic dipoles (to an elementary current loop [2]). We emphasize that only this property permits providing a physical interpretation to the mathematical formalism widely used in diffraction theory (e.g., in Refs [3–8, 18]).

4. Dual representation with Hertz potentials

The dual symmetry in diffraction theory indicated above becomes even more evident when Kirchhoff integrals (23) and (36) are written in terms of the antisymmetric Hertz tensor (see, e.g., Ref. [16]) $Z^{\mu\nu} = (\mathbf{Z}^{\mathrm{e}}, \mathbf{Z}^{\mathrm{m}})$. Here, the electric Hertz vector (potential) $\mathbf{Z}^{\mathrm{e}}$ is related to the volume density of the dipole moment $\mathbf{P} = \mathbf{d}\delta(\mathbf{r} - \mathbf{r}_{\mathrm{d}})$ (for one dipole), and the magnetic one $\mathbf{Z}^{\mathrm{m}}$ is related to the volume density of the magnetic moment $\mathbf{M} = \boldsymbol{\mu}\,\delta(\mathbf{r} - \mathbf{r}_\mu)$ as

$$\Box Z^{\mu\nu} = -4\pi M^{\mu\nu}\,. \tag{39}$$

The antisymmetric tensor $M^{\mu\nu} = (\mathbf{P}, \mathbf{M})$, which enters the right-hand side of Eqn (39), is sometimes referred to as the polarization tensor. It is noteworthy that Eqn (39) is easily derived from the Lagrangian $L = M^{\mu\nu} Z_{\mu\nu} - (1/8\pi)\, \partial^\eta Z^{\mu\nu} \partial_\eta Z_{\mu\nu}$ [16], where the role of generalized coordinates is played by the Hertz tensor $Z^{\mu\nu}$ and the role of generalized velocities is played by its derivatives $\partial^\eta Z^{\mu\nu}$. A twofold increase in the number of the degrees of freedom in comparison with the ordinary approach (six independent tensor components in lieu of the three independent components of a 4-potential A^μ that satisfies the Lorentz condition $\partial^\mu A_\mu = 0$) is attributable to the fact that the electric and magnetic dipole densities are defined independently of each other.

Let the half-space bounded by the screen surface contain no 'external' dipoles and the tensor $Z^{\mu\nu}$ satisfy homogeneous equation (39). Then, by selecting the tensor $S_{\alpha\beta}$ in formula (16) in the form $Z_{\alpha\beta}$, we obtain the Kirchhoff integral written in terms of the Hertz potentials:

$$Z_{\mu\nu}(\mathbf{r},\omega) = \oint \big(Z_{\mu\nu}(\mathbf{r}',\omega)(\mathbf{n},\nabla)\, G_\omega - G_\omega(\mathbf{n},\nabla)\, Z_{\mu\nu}(\mathbf{r}',\omega)\big)\, \mathrm{d}S\,. \tag{40}$$

Evidently, this relation can also be written in terms of the dual tensor $\tilde{Z}^{\mu\nu}$. It was suggested recently that integral (40) written in terms of the electric potential $\mathbf{Z}^{\mathrm{e}}$ can be used in solving diffraction problems [24, 25]. On the face of it, when the integration surface is taken in the form of a screen and an infinite-radius half-sphere, the resultant relation contains the same drawback: the necessity of fixing approximate values of the Hertz potentials on the opening surface. But because the initial sources of the potentials $Z^{\mu\nu} = (\mathbf{Z}^{\mathrm{e}}, \mathbf{Z}^{\mathrm{m}})$ are dipoles, there is no need to select the integration surface in the form of a double layer. Indeed, the potentials entering the right-hand side of Eqn (40) are taken from the screen surface and therefore satisfy Eqn (39) with the right-hand side $M^{\mu\nu} = M_{\mathrm{s}}^{\mu\nu}\delta(r^{\mathrm{i}} - r_{\mathrm{s}}^{\mathrm{i}})$, where $M_{\mathrm{s}}^{\mu\nu} = (\mathbf{P}_{\mathrm{s}}, \mathbf{M}_{\mathrm{s}})$ is the surface density of the dipole moments induced on the screen by the incident wave. The solution of this equation can be written similarly to the solution of the ordinary wave equation for A^μ:

$$Z^{\mu\nu}(\mathbf{r}',\omega) = 4\pi \int M_{\mathrm{s}}^{\mu\nu} G_\omega\, \mathrm{d}S\,. \tag{41}$$

It follows from the foregoing that we should set $\mathbf{M}_{\mathrm{s}} = 0$ on the screen surface and $\mathbf{P}_{\mathrm{s}} = 0$ on the opening surface. Then the screen is a double electric layer, the surface of distributed dipole moment, and its complementary screen (the opening) is a double magnetic layer. Therefore, the advantage of integral (40) over the ordinary Kirchhoff integral (23) is that the electric vector $\mathbf{Z}^{\mathrm{e}}$ in the right-hand side of (40) is nonzero only on the screen surface and the magnetic potential $\mathbf{Z}^{\mathrm{m}}$ is nonzero only on the opening. This obviates the necessity of invoking additional assumptions in the solution of the problem. This circumstance is not obvious; however, it clearly follows from the foregoing analysis and may be regarded as the principle of complementary screens in the formalism of Hertz potentials.

We note that Eqn (39) has been used to advantage to describe the symmetry properties of Maxwell equations with

Dirac poles [26]. This stems from the fact that Eqn (39), unlike conventional equations (1), is completely dual-symmetric in form. We use this property here to once again demonstrate the equivalence of certain electric and magnetic sources. Passing from the Hertz potentials to the ordinary 4-potential A^μ is effected as follows. We multiply both sides of (39) by ∂_ν. By introducing the notation $A^\mu = \partial_\nu Z^{\mu\nu}$, we obtain the 4-current components $j_{\mathrm{e}}^\mu = c\,\partial_\nu M^{\mu\nu}$ as

$$j_{\mathrm{e}}^\mu = \left\{ -c\ \mathrm{div}\,\mathbf{P},\ \frac{\partial \mathbf{P}}{\partial t} + c\ \mathrm{rot}\,\mathbf{M} \right\}, \tag{42}$$

which is the standard expression for the current density written in terms of polarizations (see Ref. [10]). Expression (39) then implies the standard formula

$$\Box A^\mu = -\frac{4\pi}{c}\, j_{\mathrm{e}}^\mu\,. \tag{43}$$

By multiplying both sides of relation (39) by $1/2\epsilon_{\rho\eta\mu\nu}$, we obtain a similar equation for dual quantities. Next, multiplying this equation by ∂^η, we find

$$\Box \partial^\eta \tilde{Z}_{\rho\eta} = -4\pi \partial^\eta \tilde{M}_{\rho\eta}\,. \tag{44}$$

The pseudovector $-\partial^\eta \tilde{Z}_{\rho\eta}$ is naturally denoted as $\tilde{A}_\rho$ and termed the 'magnetic' 4-potential, and the quantity $\tilde{j}_\rho = -c\ \partial^\eta \tilde{M}_{\rho\eta}$ can be referred to as the magnetic current equivalent to the given electric one. Its components are

$$\tilde{j}_{\mathrm{m}}^\mu = \left\{ -c\ \mathrm{div}\,\mathbf{M},\ \frac{\partial \mathbf{M}}{\partial t} - c\ \mathrm{rot}\,\mathbf{P} \right\}. \tag{45}$$

Hence, it is evident that the magnetic current density $\tilde{\mathbf{j}}_{\mathrm{m}} = \partial \mathbf{M}/\partial t - c\ \mathrm{rot}\,\mathbf{P}$ is in a sense equivalent to the given electric current density $\mathbf{j}_{\mathrm{e}} = \partial \mathbf{P}/\partial t + c\,\mathrm{rot}\,\mathbf{M}$. In the special cases of only electric or only magnetic dipoles, this reduces to the previously obtained results, formulas (10) and (14), where we must set $\varepsilon(\omega) = 1$, $\mu(\omega) = 1$. We emphasize that we are dealing with the equivalence of electric and magnetic sources from the standpoint of the equality of the fields they produce. In this case, defining the field strength tensor in terms of the 4-potential $A^\mu = \partial_\nu Z^{\mu\nu}$ by the formula $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ automatically leads to the equation $\partial_\nu \tilde{F}^{\mu\nu} = 0$, i.e., to the zero density of magnetic sources. In this case, the fields and the Hertz potentials are related by the formulas

$$\mathbf{E}_{\mathrm{e}}(\mathbf{r}, t) = \mathrm{rot}\left(\mathrm{rot}\,\mathbf{Z}^{\mathrm{e}} - \frac{1}{c}\frac{\partial \mathbf{Z}^{\mathrm{m}}}{\partial t} \right) - 4\pi \mathbf{P}\,,$$

$$\mathbf{H}_{\mathrm{e}}(\mathbf{r}, t) = \mathrm{rot}\left(\mathrm{rot}\,\mathbf{Z}^{\mathrm{m}} + \frac{1}{c}\frac{\partial \mathbf{Z}^{\mathrm{e}}}{\partial t} \right). \tag{46}$$

Defining the dual tensor $\tilde{F}^{\mu\nu}$ in terms of the magnetic potential $\tilde{A}^\mu = -\partial_\nu \tilde{Z}^{\mu\nu}$ by the formula $\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu$ automatically leads to the equation $\partial_\nu F^{\mu\nu} = 0$, i.e., to the zero density of electric sources. In this case, the fields and the Hertz potentials are related as

$$\mathbf{E}_{\mathrm{m}}(\mathbf{r}, t) = \mathrm{rot}\left(\mathrm{rot}\,\mathbf{Z}^{\mathrm{e}} - \frac{1}{c}\frac{\partial \mathbf{Z}^{\mathrm{m}}}{\partial t} \right),$$

$$\mathbf{H}_{\mathrm{m}}(\mathbf{r}, t) = \mathrm{rot}\left(\mathrm{rot}\,\mathbf{Z}^{\mathrm{m}} + \frac{1}{c}\frac{\partial \mathbf{Z}^{\mathrm{e}}}{\partial t} \right) - 4\pi \mathbf{M}\,, \tag{47}$$

whence it is evident that the quantity $\mathrm{div}\,\mathbf{H}_{\mathrm{m}} = -4\pi\ \mathrm{div}\,\mathbf{M}$ plays the role of a magnetic charge density in accordance with expression (45). A comparison of formulas (46) and (47) suggests that the fields of electric and magnetic sources are equal only at those points in space that are free from the dipoles: $\mathbf{P} \equiv \mathbf{M} \equiv 0$. It is valid to say that sources of both types are equivalent only for distances that exceed the size of the system producing the fields. And this is precisely the condition with which we started in deriving the formulas for the radiation fields in Section 2.

5. Example: the dual method in the problem of transition radiation of a charged particle

As an illustration, we consider the solution of the problem of transition radiation of a charged particle with the use of the dual representation. In the simplest case, this radiation emerges when a charge that executes a uniform rectilinear motion in the vacuum traverses a perfectly conducting screen. To solve this problem (as well as its kindred problem of the diffraction radiation emerging in the particle motion near the screen), the representation of the radiation field as the field of a surface electric current induced in the screen by the field of the charge was used in several papers (by analogy with the representation in the electromagnetic radiation diffraction theory, see, e.g., Refs [27–29]). The dual representation was proposed relatively recently for the solution of problems of this kind, i.e., the use of a surface magnetic current as the radiation source [18]. In this case, the advantage of the dual method is that it enabled relatively easily reproducing the results known from the literature, which were obtained by other, quite often cumbersome methods. On the other hand, it turned out that the solution of the transition radiation problem with the aid of the 'ordinary' method of surface currents does not lead to the correct result (see below), even in the simplest case of normal particle incidence on the interface [19, 20].

We consider a charged particle moving at a constant velocity $\beta = v/c$ along the z axis (Fig. 2). As the particle crosses an infinite perfectly conducting screen located in the $z = 0$ plane, a burst of transition radiation emerges that may be treated as the radiation of the surface current induced on the screen by the field of the charge. To determine the surface current density in the theory of plane wave diffraction, the integral Fock equation is used [1, 29],

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}', \omega) = \frac{c}{2\pi}\,\mathbf{n} \times \mathbf{H}^{\mathrm{R}} - \frac{1}{2\pi}\,\mathbf{n} \times \int_S \mathbf{j}_{\mathrm{e}}(\mathbf{r}'', \omega) \times \mathrm{grad}\,\frac{\exp\left(\mathrm{i}\omega|\,\mathbf{r}' - \mathbf{r}''|/c\right)}{|\mathbf{r}' - \mathbf{r}''|}\,\mathrm{d}S''\,, \tag{48}$$

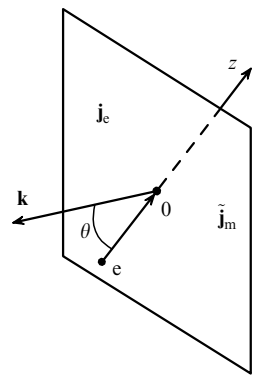

Figure 2. Schematic of the transition radiation generation by a charged particle.

where $\mathbf{n}$ is the normal to the screen surface. The derivation of Eqn (48) involved conventional boundary conditions (27) and the fact that the field $\mathbf{H}^{\mathrm{R}}$ satisfies the homogeneous Maxwell equations (in this case, the mechanism of the emergence of the 'extra' factor 2 is the same as the mechanism discussed in Section 3). When the particle is ultrarelativistic and its field $\mathbf{E}^0$ is virtually transverse, the charge field $\mathbf{H}^0$ may be taken as $\mathbf{H}^{\mathrm{R}}$. This approach to the transition radiation problem was used, e.g., in Ref. [29]. In the general case, however, only the radiation field given by the difference of the total field and the intrinsic charge field, $\mathbf{H}^{\mathrm{R}} = \mathbf{H} - \mathbf{H}^0$, satisfies the homogeneous equations in the problem with an external source. In this case, Eqn (48) must be modified as [20]

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}', \omega) = \frac{c}{2\pi}\,\mathbf{n}\times\mathbf{H}^0 - \frac{1}{2\pi}\,\mathbf{n}\times\int \mathbf{j}_{\mathrm{e}}(\mathbf{r}'', \omega) \times \operatorname{grad}\frac{\exp\left(\mathrm{i}\omega|\,\mathbf{r}'-\mathbf{r}''|/c\right)}{|\,\mathbf{r}'-\mathbf{r}''|}\,\mathrm{d}S'' + \frac{c}{(2\pi)^2}\,\mathbf{n} \times\int[\mathbf{n}\times\mathbf{H}^0]\times\operatorname{grad}\frac{\exp\left(\mathrm{i}\omega|\,\mathbf{r}'-\mathbf{r}''|/c\right)}{|\,\mathbf{r}'-\mathbf{r}''|}\,\mathrm{d}S''. \quad (49)$$

Using boundary conditions (31), which contain surface magnetic currents, it is possible to derive a similar equation for magnetic currents [20]:

$$\tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}', \omega) = -\frac{c}{2\pi}\,\mathbf{n}\times\mathbf{E}^0 - \frac{1}{2\pi}\,\mathbf{n}\times\int \tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}'', \omega) \times \operatorname{grad}\frac{\exp\left(\mathrm{i}\omega|\,\mathbf{r}'-\mathbf{r}''|/c\right)}{|\,\mathbf{r}'-\mathbf{r}''|}\,\mathrm{d}S'' - \frac{c}{(2\pi)^2}\,\mathbf{n} \times\int[\mathbf{n}\times\mathbf{E}^0]\times\operatorname{grad}\frac{\exp\left(\mathrm{i}\omega|\,\mathbf{r}'-\mathbf{r}''|/c\right)}{|\,\mathbf{r}'-\mathbf{r}''|}\,\mathrm{d}S''. \quad (50)$$

In the classical theory of electromagnetic radiation diffraction, Eqn (48) is solved by the method of successive approximations [1]. The problem under consideration is substantially different in that Eqns (49) and (50) can be solved exactly. These solutions have the form

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega) = \frac{c}{2\pi}\,\mathbf{n}\times\mathbf{H}^0,\ \tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}, \omega) = -\frac{c}{2\pi}\,\mathbf{n}\times\mathbf{E}^0\,. \quad (51)$$

By analogy with the diffraction theory, the solution of the ordinary and dual Maxwell equations with currents (51) in the right-hand side would be expected to yield exact expressions for the transition radiation field. In reality, this is the case only for the magnetic current.

The fields of surface currents in the wave zone are given by the analogs of expressions (2) and (6):

$$\mathbf{E}^{\mathrm{R}}_{\mathrm{e}}(\mathbf{r}_0, \omega) = -\frac{\mathrm{i}}{\omega}\,\frac{\exp\left(\mathrm{i}kr_0\right)}{r_0}\,\mathbf{k}\times\mathbf{k}\times\int \mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega)\,\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r}\right)\mathrm{d}S,$$
$$\mathbf{E}^{\mathrm{R}}_{\mathrm{m}}(\mathbf{r}_0, \omega) = -\frac{\mathrm{i}}{c}\,\frac{\exp\left(\mathrm{i}kr_0\right)}{r_0}\,\mathbf{k}\times\int \tilde{\mathbf{j}}_{\mathrm{m}}(\mathbf{r}, \omega)\,\exp(-\mathrm{i}\mathbf{k}\mathbf{r})\,\mathrm{d}S, \quad (52)$$

where the integrations are performed over the screen surface. We have the following expressions for the particle field strengths (see, e.g., Ref. [7]):

$$\mathbf{E}^0(\mathbf{r}, \omega) = \frac{e\omega}{\pi v^2\gamma}\left(\frac{\boldsymbol{\rho}}{\rho}\,K_1\left[\frac{\omega\rho}{v\gamma}\right] - \frac{\mathrm{i}}{\gamma}\,\frac{\mathbf{v}}{v}\,K_0\left[\frac{\omega\rho}{v\gamma}\right]\right) \times\exp\left(\mathrm{i}\,\frac{\omega}{v}\,z\right),\ \mathbf{H}^0 = \boldsymbol{\beta}\times\mathbf{E}^0, \quad (53)$$

where e is the particle charge, $\boldsymbol{\rho} = (x, y)$, γ is the Lorentz factor, and K_0 and K_1 are the Macdonald functions. Integration in expressions (52) is performed in the polar coordinate system using the well-known relations

$$\int_0^{2\pi}\cos\phi\,\exp\left(-\mathrm{i}a_1\rho\cos\phi\right)\mathrm{d}\phi = -2\mathrm{i}\pi J_1(a_1\rho),$$
$$\int_0^{2\pi}\exp\left(-\mathrm{i}a_1\rho\cos\phi\right)\mathrm{d}\phi = 2\pi J_0(a_1\rho),$$
$$\int_0^{\infty}\rho J_1(a_1\rho)\,K_1(a_2\rho)\,\mathrm{d}\rho = \frac{a_1}{a_2}\,\frac{1}{a_1^2+a_2^2}\,, \quad (54)$$

where J_0 and J_1 are the Bessel functions, and a_1 and a_2 are real positive constants. The energy radiated in a unit frequency interval into a unit solid angle is given by the square of the field modulus:

$$\left.\frac{\mathrm{d}^2W}{\mathrm{d}\omega\,\mathrm{d}\Omega}\right|_{\mathrm{e}} = cr_0^2|\mathbf{E}^{\mathrm{R}}_{\mathrm{e}}|^2 = \frac{e^2}{\pi^2 c}\,\frac{\beta^4\sin^2\theta\cos^2\theta}{(1-\beta^2\cos^2\theta)^2},$$
$$\left.\frac{\mathrm{d}^2W}{\mathrm{d}\omega\,\mathrm{d}\Omega}\right|_{\mathrm{m}} = cr_0^2|\mathbf{E}^{\mathrm{R}}_{\mathrm{m}}|^2 = \frac{e^2}{\pi^2 c}\,\frac{\beta^2\sin^2\theta}{(1-\beta^2\cos^2\theta)^2}\,, \quad (55)$$

where θ is the polar radiation angle measured from the negative direction of the particle velocity (backward radiation [see Fig. 2]). We see that only the last expression obtained using the dual method coincides with the well-known Ginzburg–Frank formula [11], while the formula obtained on the basis of the 'ordinary' approach has an additional factor $\beta^2\cos^2\theta$, which permits its use only in the ultrarelativistic case, when the radiation is concentrated in the angular interval $\theta \sim \gamma^{-1} \ll 1$. It can be shown that in the more general case of oblique charged-particle incidence on a screen surface, the dual method also leads to the well-known results of Korkhmazyan and Pafomov, while the method of electric currents introduces an error, which increases with the angle of incidence (i.e., with the angle between the particle velocity vector and the normal to the screen surface) [19, 20].

Therefore, it turns out that just the dual method allows obtaining reliable results in the problem of transition radiation (as well as in the more general case of diffraction radiation), while the application of the approach with electric currents is limited to ultrarelativistic energies. The cause of the apparent contradiction lies with the initially incorrect physical formulation of the problem. In the theory of plane wave diffraction from a perfectly conducting and infinitely thin screen, it is assumed that the surface current has two tangential components, because $\mathbf{j}_{\mathrm{e}} \propto \mathbf{n}\times\mathbf{H}$ by virtue of the boundary conditions. In reality, all quantities in macroscopic electrodynamics are averaged over a physically infinitely small volume, and therefore even a macroscopically infinitely thin screen nevertheless has a finite thickness that exceeds the mean free path of conduction electrons, as well as the dimensions of the averaging domain. Hence, it follows that in the general case, there are no grounds to believe that the electric current induced by the field of a charge has only two components.

With the knowledge of the exact expression for magnetic surface current (51), it is possible to find its corresponding expression for the electric current by using the results in Section 2. Indeed, from formula (7) for the electric current equivalent to a given magnetic current, we find

$$\mathbf{j}_{\mathrm{e}}(\mathbf{r}, \omega) = \frac{c}{2\pi}\,\mathbf{e}\times[\mathbf{n}\times\mathbf{E}^0]\,, \quad (56)$$

where $\mathbf{e} = \mathbf{k}/k$. By substituting expression (56) in expression (52), we readily check that it leads exactly to the Ginzburg–Frank formula, as well as to the corresponding results in the case of oblique incidence (see also Ref. [20]). A characteristic property of this expression is the existence of a current density component perpendicular to the screen surface, and the contribution of this component is significant only for large radiation angles because $j_z \propto \sin\theta$. In the relativistic case, the radiation is concentrated in the domain of small θ and the normal current component may be neglected, which determines the applicability conditions for the methods of the theory of plane wave diffraction in the theory of transition and diffraction radiation (see, e.g., Refs [27–29]).

In conclusion, we emphasize once again: not only is the use of the dual method helpful in solving several particular problems in the theory of transition (or diffraction) radiation, but it also permits relatively easily finding the exact expression for the density of the surface electric current induced by the particle field on a screen.

6. Summary

The dual representation in classical electrodynamics, i.e., the use of formulas of the form $\mathbf{E} \propto \mathrm{rot}\tilde{\mathbf{A}}$, is most frequently encountered in the vector theory of diffraction; however, the physical interpretation of this method is not given in the majority of papers. As shown in this paper, the use of this formalism is possible due to the equivalence of certain electric and magnetic sources. In this case, the use of the formalism of Hertz potentials is especially lucid, which permits solving the problems of the radiation of multipoles and the diffraction of electromagnetic waves in a completely dual-symmetric form. This method also proves to be highly beneficial for solving the problems of the transition and diffraction radiation of charged particles.

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