

METHODOLOGICAL NOTES

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On energy and momentum conservation laws for an electromagnetic field in a medium or at diffraction on a conducting plate

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Contents

Abstract. **For a field–matter system, general nonstationary balance equations for energy and momentum densities and their transport velocities are obtained based on a rigorous nonstationary definition of these densities depending on the creation history of the field. We analyze the simplest dispersion law determined by conductivity dissipation; we find the electromagnetic energy density, phase velocity, group velocity, and energy and momentum transport velocities of a plane monochromatic wave. The low-frequency energy density is shown to be given by the electrostatic density in which the dielectric constant is replaced with its real part and the energy transport velocity is equal to the phase velocity. The group velocity can exceed the speed of light. We prove that the Minkowski form of momentum density must be used in the medium, and find the corresponding transport velocity, which in the case under consideration also coincides with the phase velocity. Energy and momentum conservation laws are shown to hold for a plane electromagnetic wave propagating in a medium or diffracted by a conducting plate.**

1. Introduction

A paradoxical situation has reigned in the electrodynamics of continuous media for more than a century now: there is uncertainty in how to choose the correct form of the energy–momentum tensor (EMT) [1–13]. There are two basic forms of the EMT: Minkowski's [1] and Abraham's [2], and many publications argue in favor of Minkowski's definition (e.g., [13]) and against Abraham's definition, or vice versa, in favor of Abraham's definition and against Minkowski's definition (see, e.g., [3–6], recent reviews [11, 12], paper [13] and the references therein). A number of publications (including a physics encyclopedia) hold the opinion that these two definitions are equivalent (see, e.g., [3, 12]) and at the same time advance arguments that Abraham's tensor is preferable or more correct. But Minkowski's tensor can also be used because it is often more convenient and corresponds to a continuous medium [3]. In contrast to this, some other publications claim that Minkowski's tensor is more correct. There are a number of publications on experimental confirmations and refutations of both definitions [12]. Specifically, some papers reported measurements of the Abraham force $\mathbf{f}^{\mathrm{A}}$, adding which to the time derivative 'according to Abraham' $\partial_t \mathbf{g}^{\mathrm{A}}$ yields the derivative $\partial_t \mathbf{g}^{\mathrm{M}}$ of the momentum density 'according to Minkowski' [3]. All these experiments were conducted with quasistationary or transient (pulsed) processes, and Abraham's body force is not equivalent to the sum of Lorentz forces acting on the electric and magnetic polarization currents in matter [5]. In what follows, an electromagnetic (EM) pulse or train is defined (to avoid misunderstanding) as a nonstationary electromagnetic wave (EMW) or spectral wave packet, and the EM field momentum $\mathbf{G}$ is defined as the volume integral of the density of the linear momentum $\mathbf{g}$ of the field–matter system (FMS).

This ambiguity has generated a number of attempts to define the EMT in a different way, e.g., on the basis of microscopic electrodynamics [7, 8] or equations of motion of matter, or by using Noether's theorem [13]. It is believed that the EMT can be defined uniquely for the FMS; this combined EMT yields the rate of transfer of the total momentum $\mathbf{v}_{\mathrm{m}}$, but taken separately, these quantities cannot be defined uniquely. We note that the ambiguity in defining the density of the field momentum leads to an ambiguity in its transfer rate $\mathbf{v}_{\mathrm{m}}^{(\mathrm{EM})}$ and the matter velocity of motion $\mathbf{v}^{(\mathrm{M})}$ and, consequently, to an ambiguity in the force of pressure exerted by the EMW on matter.

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In this article, we obtain time-dependent balance equations for energy and momentum and introduce nonstationary definitions of their densities dependent on the creation history of the field. Using this, we derive general expressions for the energy transport velocity and momentum transfer rate in the FMS, as well as for the field and matter momentum transfer rate. The general results are then applied to a particular case of one-dimensional problems: (a) a plane EMW in a medium with dispersion due to conductivity; (b) diffraction of such a wave on a planar conductive plate. The formulas obtained for monochromatic processes allow constructing the EMT and determining the energy and momentum transfer rate in the FMS; with the dispersion specified above, these rates coincide with the phase velocity.

2. Balance equations for energy and momentum

Whenever paradoxes arise, it must be found out where concepts were swapped or where certain concepts have been improperly applied to phenomena [14]. In the case of the EMT and energy and momentum densities in a medium, the paradox arises due to illegitimate replacement of nonstationary concepts by stationary ones. In particular, the field energy density in matter is defined as $u(\mathbf{r},t) = [\mathbf{D}(\mathbf{r},t)\,\mathbf{E}(\mathbf{r},t) + \mathbf{B}(\mathbf{r},t)\,\mathbf{H}(\mathbf{r},t)]/2$, i.e., the same as this is done in statics, which is wrong [14–16]. Correspondingly, the Abraham momentum density is defined as $\mathbf{g}^{\mathrm{A}} = \mathbf{S}/c^2$, and the Minkowski one as $\mathbf{g}^{\mathrm{M}} = \mathbf{D}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) = n^2\mathbf{S}(\mathbf{r},t)/c^2 = n^2\mathbf{g}^{\mathrm{A}}$, where $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$ is the Poynting vector and $n = \sqrt{\varepsilon\mu}$ is the refractive index (or retardation coefficient). Most papers dealing with the EMT assume that matter equations have the form $\mathbf{D}(\mathbf{r},t) = \varepsilon_0\varepsilon(\mathbf{r})\,\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mu_0\mu(\mathbf{r})\,\mathbf{H}(\mathbf{r},t)$, i.e., it is implicitly assumed that the temporal (frequency) dispersion is absent; this is strictly true only in a static case. These equations correspond to an inhomogeneous medium. It is usually assumed that ε and μ are numerical constants.

We take the matter equations in the generalized Landau–Lifshitz form [17]

$$\begin{aligned}
\mathbf{D}(\mathbf{r},t) &= \varepsilon_0\,\partial_t^{-1}\partial_{\mathbf{r}}^{-1}\big(\hat{\varepsilon}(\mathbf{r},\mathbf{r}',t-t')\,\mathbf{E}(\mathbf{r}',t')\big)\,,\\
\mathbf{B}(\mathbf{r},t) &= \mu_0\,\partial_t^{-1}\partial_{\mathbf{r}}^{-1}\big(\hat{\mu}(\mathbf{r},\mathbf{r}',t-t')\,\mathbf{H}(\mathbf{r}',t')\big)\,,
\end{aligned}\tag{1}$$

although other forms (e.g., Casimir's) are also possible [18]. The following integral operators are used here:

$$\begin{aligned}
\partial_t^{-1}\big(f(t')\big) &= F(t) = \int_0^t f(t')\,\mathrm{d}t'\,,\\
\partial_{\mathbf{r}}^{-1}\big(\varphi(\mathbf{r},\mathbf{r}')\big) &= \Phi(\mathbf{r}) = \int_V \varphi(\mathbf{r},\mathbf{r}')\,\mathrm{d}^3r'\,,
\end{aligned}\tag{2}$$

where the integrand may involve both scalar and vector functions, and $\mathrm{d}^3r' = \mathrm{d}V'$ is the volume element at the source point. The lower limit in the first integral can be set equal to $-\infty$, which yields $f(t)\to 0$ for $t\to-\infty$. The processes are uniform in time, and hence the kernels in (1) are functions of $t-t'$, i.e., the causality principle holds in the form $\hat{\varepsilon}(\mathbf{r},\mathbf{r}',t-t') = \hat{\mu}(\mathbf{r},\mathbf{r}',t-t') = 0$ for $t'>t$. In the second integral, the volume that satisfies the condition $|\mathbf{r}-\mathbf{r}'|/c \leqslant t-t'$ is chosen, i.e., the causality principle holds again. This principle means that only the fields at points located no farther than at the distance $c(t-t')$ contribute to the induction. This implies, in turn, that spatial dispersion is taken into account (here, always $t\geqslant t'$). Typically, spatial dispersion is associated with a much smaller region. In the general case, kernels in (1) are tensors, inhomogeneous in the coordinates, which corresponds to inhomogeneous anisotropic media. We write Maxwell's equations in their general form

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{H}(\mathbf{r},t) &= \partial_t\mathbf{D}(\mathbf{r},t) + \mathbf{J}^{\mathrm{e}}(\mathbf{r},t)\,,\\
-\,\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t) &= \partial_t\mathbf{B}(\mathbf{r},t) + \mathbf{J}^{\mathrm{m}}(\mathbf{r},t)\,,
\end{aligned}\tag{3}$$

where, as usual, $\boldsymbol{\nabla}\equiv\partial_{\mathbf{r}}$ is a vector differential operator. The meaning of Maxwell's equations (3) is simple: they represent the full current balance, and the left-hand sides are the total densities of currents (electric in the first equation and magnetic in the second), and the right-hand sides are the sums of the corresponding displacement currents and external currents. Equations (3) are the most complete, because the entire effect of the medium (including conduction currents) is taken into account in matter equations. The electric conductance in the form $\mathbf{J}_{\sigma}^{\mathrm{e}}(\mathbf{r},t) = \sigma^{\mathrm{e}}(\mathbf{r})\,\mathbf{E}(\mathbf{r},t)$ (i.e., as for dc) can be taken into account by writing the kernel in the form

$$\hat{\varepsilon}(\mathbf{r},\mathbf{r}',t-t') = \delta(t-t')\left[\varepsilon(\mathbf{r},\mathbf{r}') + \frac{\sigma^{\mathrm{e}}(\mathbf{r})}{\varepsilon_0}\,\partial_t^{-1}\right] + \hat{\kappa}^{\mathrm{e}}(\mathbf{r},\mathbf{r}',t-t')\,,$$

where $\hat{\kappa}^{\mathrm{e}}(\mathbf{r},\mathbf{r}',t-t')$ is the kernel of the electric susceptibility and $\partial_t\partial_t^{-1} = I$ is the identity operator.

In the case of frequency dispersion, this method of taking conductivity into account was selected because the spectral function of the dielectric permittivity (DP) acquires a pole at zero frequency [17]. In the case of a frequency-dependent conductivity, we use the Drude formula; this corresponds to the form of the kernel of the integral operator for a plasma. The second equation in (3) incorporates the external magnetic current. Although the magnetic charge (the Dirac monopole) has not been found yet, introducing $\mathbf{J}^{\mathrm{m}}(\mathbf{r},t)$ is very useful for symmetry reasons because external magnetic currents may be equivalent to certain configurations of external electric currents. Next, we assume that the field was zero at times $t<0$. Hence, energy and momentum densities of the field and matter (up to the rest energy of matter) were zero until the moment $t=0$. At the moment $t=t_0=0$, sources are turned on, producing the work needed to create the field and to change the energy and momentum of the field and matter. It is usually assumed that the energy supplied by sources is of a nonelectromagnetic nature, which is convenient from the mathematical standpoint, although physically, this energy is nonetheless electromagnetic but its range of action is outside that of the field. Part of the energy produced dissipates into heat $q(\mathbf{r},t)$. This energy is not of an electromagnetic nature and is ignored in the balance. In the general case, the heating of matter generates a nonequilibrium process and this heated matter begins to radiate in a nonstationary and nonequilibrium manner, which requires solving a kinetic equation. In what follows, we assume the process to be quasi-equilibrium and occurring at a constant temperature, i.e., we treat the intensities of the excited fields as small and the heat capacity of matter as large (infinite).

Following tradition, we obtain the energy balance equations by scalar multiplying each of Eqns (1) by the vector of the other field and adding them, using the identities $\mathbf{a}(\nabla\times\mathbf{b}) - \mathbf{b}(\nabla\times\mathbf{a}) = -\nabla(\mathbf{a}\times\mathbf{b})$. This gives

$$\nabla\mathbf{S}(\mathbf{r},t) + \left[\mathbf{E}(\mathbf{r},t)\,\partial_t\mathbf{D}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\partial_t\mathbf{B}(\mathbf{r},t)\right] = -\left[\mathbf{E}(\mathbf{r},t)\,\mathbf{J}^{\mathrm{e}}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\mathbf{J}^{\mathrm{m}}(\mathbf{r},t)\right]. \quad (4)$$

The right-hand side of (4) includes the density of power exerted by the sources to create the field. This equation has the form characteristic of balance [19]:

$$\nabla\mathbf{S}(\mathbf{r},t) + \partial_t w(\mathbf{r},t) = -\left[\mathbf{E}(\mathbf{r},t)\,\mathbf{J}^{\mathrm{e}}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\mathbf{J}^{\mathrm{m}}(\mathbf{r},t)\right].$$

The first term is the power density of the outflowing field and $\partial_t w(\mathbf{r},t)$ is the density of the accumulated power of the field and matter. To calculate the work expended, that quantity must be integrated:

$$w(\mathbf{r},t) = \partial_t^{-1}\left[\mathbf{E}(\mathbf{r},t')\,\partial_t\mathbf{D}(\mathbf{r},t') + \mathbf{H}(\mathbf{r},t')\,\partial_t\mathbf{B}(\mathbf{r},t')\right]. \quad (5)$$

It is this quantity, not $[\mathbf{E}(\mathbf{r},t)\,\mathbf{D}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\mathbf{B}(\mathbf{r},t)]/2$ (customarily used), that has to be associated with $w(\mathbf{r},t)$ [16]. Energy (5) spent to create the field and change the momentum of matter reflects the entire previous history of the process, which is natural for the electrodynamics of continuous media but unnecessary for a field in the vacuum [20]. By solving (3) simultaneously with (1), we find all the fields in the time interval $(0,t)$. This procedure allows calculating the density of dissipated energy $q(\mathbf{r},t)$ (the heat released by unit volume). Dissipation is connected not only with conductivity but also with delayed polarizations (i.e., the response to applied fields in the form of inductions). Finally, we obtain $e(\mathbf{r},t) = w(\mathbf{r},t) - q(\mathbf{r},t)$ for the field–matter energy density. According to the concept of Umov [21], the energy transfer rate is $\mathbf{v}_{\mathrm{e}}(\mathbf{r},t) = \mathbf{S}(\mathbf{r},t)/e(\mathbf{r},t)$. This rate is defined at each point for each t. Maxwell's equations (3) can now be rewritten as

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \varepsilon_0\partial_t\mathbf{E}(\mathbf{r},t) + \mathbf{J}^{\mathrm{e}}(\mathbf{r},t) + \mathbf{J}_P^{\mathrm{e}}(\mathbf{r},t),$$

$$-\nabla\times\mathbf{E}(\mathbf{r},t) = \mu_0\partial_t\mathbf{H}(\mathbf{r},t) + \mathbf{J}^{\mathrm{m}}(\mathbf{r},t) + \mathbf{J}_P^{\mathrm{m}}(\mathbf{r},t),$$

and the balance equations as

$$\nabla\mathbf{S}(\mathbf{r},t) + \partial_t w_{\mathrm{EM}}(\mathbf{r},t) = -\left[\mathbf{E}(\mathbf{r},t)\left(\mathbf{J}^{\mathrm{e}}(\mathbf{r},t) + \mathbf{J}_P^{\mathrm{e}}(\mathbf{r},t)\right) + \mathbf{H}(\mathbf{r},t)\left(\mathbf{J}^{\mathrm{m}}(\mathbf{r},t) + \mathbf{J}_P^{\mathrm{m}}(\mathbf{r},t)\right)\right], \quad (6)$$

$$w_{\mathrm{EM}}(\mathbf{r},t) = \partial_t^{-1}\left(\varepsilon_0\mathbf{E}(\mathbf{r},t')\,\partial_t\mathbf{E}(\mathbf{r},t') + \mu_0\mathbf{H}(\mathbf{r},t')\,\partial_t\mathbf{H}(\mathbf{r},t')\right) = \frac{1}{2}\left[\varepsilon_0\mathbf{E}^2(\mathbf{r},t) + \mu_0\mathbf{H}^2(\mathbf{r},t)\right], \quad (7)$$

where $\mathbf{S}$ is the field and matter power flux density. Equation (6) has the same form as the equation for the field excited in the vacuum by sources given by the sum of external currents and the corresponding polarization currents. Expression (7) may therefore be interpreted as the density of the self-energy of the field.

The quantity w_{EM} in (6) is defined up to a constant. Because the total field energy density for $t=0$ is zero, this constant is also zero. The Poynting vector $\mathbf{S}$ is also defined up to an arbitrary solenoidal vector $\mathbf{S}_0$, i.e., a vector satisfying the equation $\nabla\mathbf{S}_0(\mathbf{r},t) = 0$ [19]. The flux of this vector through any closed surface is zero; hence, it does not affect the total energy flux. However, it can be shown that there is no circulation of energy in closed circuits. Because the field was zero at $t=0$, the above equation must be solved under the condition $\mathbf{S}_0(\mathbf{r},0) = 0$, whence $\mathbf{S}_0(\mathbf{r},t) = 0$.

To determine the rate of transfer of the field self-energy, i.e., the energy associated with photons only (quasiphotons [3]), we need to find the density of the flux of matter energy. Once we solve the problem of motion of matter particles in the field, we can calculate the mean velocity $\mathbf{v}(\mathbf{r},t)$ in a physically infinitely small volume. Selecting a volume ΔV bounded by a surface ΔS around a point $\mathbf{r}$, we then determine the flux density in the nonrelativistic limit as

$$\nabla\mathbf{S}_{\mathrm{M}}(\mathbf{r},t) = \lim_{\Delta V\to 0}\frac{1}{2\Delta V}\oint_{\Delta S}\rho(\mathbf{r}',t)\,\mathbf{v}^2(\mathbf{r}',t)\,\mathbf{v}(\mathbf{r}')\,\mathbf{v}(\mathbf{r}',t)\,\mathrm{d}^2r'.$$

The limit is to be understood here as the transition to an infinitely small volume. Hence,

$$\mathbf{S}_{\mathrm{M}}(\mathbf{r},t) = \left[\frac{1}{2}\,\rho(\mathbf{r},t)\,\mathbf{v}^2(\mathbf{r},t)\right]\mathbf{v}(\mathbf{r},t),$$

where $\mathbf{v}(\mathbf{r},t)$ is the velocity of matter and $\rho(\mathbf{r},t)$ is the density. Correspondingly, $\mathbf{S}(\mathbf{r},t) = \mathbf{S}_{\mathrm{M}}(\mathbf{r},t) + \mathbf{S}_{\mathrm{EM}}(\mathbf{r},t)$. In the relativistic case, the known relations must be used [22, 23]. However, this approach is not very constructive because it requires a self-consistent solution of the equation of motion and the equation for the excitation. At the level of a macroscopic description, we need to solve the equation for the excitation by external sources and polarization currents and also equations for the electric and magnetic polarization vectors. Apparently, we cannot divide the balance of power (energy) into components for the field and matter using only the balance relations. To analyze this issue, we rewrite Eqn (6) in the form

$$\nabla\mathbf{S}_{\mathrm{EM}}(\mathbf{r},t) + \partial_t w_{\mathrm{EM}}(\mathbf{r},t) = -\left[\mathbf{E}(\mathbf{r},t)\,\mathbf{J}^{\mathrm{e}}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\mathbf{J}^{\mathrm{m}}(\mathbf{r},t)\right], \quad (8)$$

$$\nabla\mathbf{S}_{\mathrm{EM}}(\mathbf{r},t) = \nabla\mathbf{S}(\mathbf{r},t) + \mathbf{E}(\mathbf{r},t)\,\mathbf{J}_P^{\mathrm{e}}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\mathbf{J}_P^{\mathrm{m}}(\mathbf{r},t). \quad (9)$$

Equation (8) is the same as in the vacuum. We now ask whether it is possible to unambiguously determine the vector $\mathbf{S}_{\mathrm{EM}}$ from (9). According to the Helmholtz theorem, the vector field $\mathbf{S}_{\mathrm{M}}$ can be represented as the sum of its potential and solenoidal parts:

$$\mathbf{S}_{\mathrm{M}}(\mathbf{r},t) = \mathbf{S}(\mathbf{r},t) - \mathbf{S}_{\mathrm{EM}}(\mathbf{r},t) = \nabla\Phi(\mathbf{r},t) + \nabla\times\mathbf{C}(\mathbf{r},t). \quad (10)$$

This yields the Poisson equation for the quantity in (10):

$$\nabla^2\Phi(\mathbf{r},t) + \mathbf{E}(\mathbf{r},t)\,\mathbf{J}_P^{\mathrm{e}}(\mathbf{r},t) + \mathbf{H}(\mathbf{r},t)\,\mathbf{J}_P^{\mathrm{m}}(\mathbf{r},t) = 0. \quad (11)$$

We note that this power flux density is defined up to the curl of the vector $\mathbf{C}(\mathbf{r},t)$. Even though this vector is solenoidal and generates no flux, we nevertheless have $\mathbf{C}(\mathbf{r},t)\neq 0$ in general. Moreover, it is possible that $\mathbf{C}(\mathbf{r},0)\neq 0$, i.e., fluxes can circulate at the time of creation of the field in matter. Hence, vector (10) is not uniquely defined.

To solve Eqn (11), we must assume that matter exists in some limited volume V (which is a natural requirement in the nonstationary case). In this case, if we know $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, the solution can be obtained, for example, by the method of Green's function $\Gamma(\mathbf{r},\mathbf{r}') = (4\pi|\mathbf{r}-\mathbf{r}'|)^{-1}$ for the Poisson equation. Then the self-energy of the field is transferred at the rate $\mathbf{v}_{\rm e}^{\rm EM}(\mathbf{r},t) = \mathbf{S}_{\rm EM}(\mathbf{r},t)/w_{\rm EM}(\mathbf{r},t)$. Apparently, this approach cannot be extended to an infinite medium. Moreover, the rate $\mathbf{v}_{\rm e}^{\rm EM}(\mathbf{r},t)$, unlike $\mathbf{v}_{\rm e}(\mathbf{r},t)$, has no definite physical meaning.

As an example, we consider a monochromatic plane wave in an infinite medium. The amplitudes E_0 of the electric and H_0 of the magnetic field in this medium are related by $H_0 = Z_0\rho_0 E_0$, where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ and ρ_0 is the real normalized impedance. In the general case of losses in this wave, the field shifts in phase by an angle φ. For a monochromatic wave, we approximately have $\langle \mathbf{E}(\mathbf{r},t)\,\mathbf{J}_P^{\rm e}(\mathbf{r},t)\rangle \approx 0$ and $\langle \mathbf{H}(\mathbf{r},t)\,\mathbf{J}_P^{\rm m}(\mathbf{r},t)\rangle \approx 0$, where the Dirac brackets $\langle\ldots\rangle$ indicate averaging over the period; on average, therefore, there is no exchange of energy between the field and matter. These conditions hold strictly in a medium with zero dispersion, i.e., for the ideal matter equations $\mathbf{D} = \varepsilon_0\varepsilon\mathbf{E}$, $\mathbf{B} = \mu_0\mu\mathbf{H}$. Then quantity (10) can be neglected on average and we calculate $\langle\mathbf{S}\rangle$ and $\langle w_{\rm EM}\rangle$. We assume that the wave propagates along the unit vector $\mathbf{z}_0$ of the z axis. Then we would have to obtain the propagation velocity of the pure EM energy

$$v_{\rm e}^{\rm EM} \approx \frac{2c\cos\varphi}{\rho_0 + 1/\rho_0} \leqslant c\,. \qquad (12)$$

In the case of these matter equations, we have $\varphi = 0$, $\rho_0 = \sqrt{\mu/\varepsilon}$, $\mathbf{J}_P^{\rm e} = \varepsilon_0(\varepsilon-1)\,\partial_t\mathbf{E}$, and $\mathbf{J}_P^{\rm m} = \mu_0(\mu-1)\,\partial_t\mathbf{H}$. If $\rho_0 = 1$, i.e., $\varepsilon = \mu$, then Eqn (12) yields the speed of light, whereas $v_{\rm e} = c/n = c/\varepsilon$. If we take balance equation (8), we see that the second term in the right-hand side of (9) is naturally added to $w_{\rm EM}$, which yields the total energy density $w = u = nw_{\rm EM}$, whereas $\mathbf{S}_{\rm EM} = \mathbf{S}$ and $S_{\rm M} = 0$.

We consider the momentum balance. We left-multiply the first equation in (3) vectorially by $\mathbf{B}(\mathbf{r},t)$ and the second by $\mathbf{D}(\mathbf{r},t)$ and subtract it from the first:

$$\begin{aligned}&\big[\mathbf{B}(\mathbf{r},t)\times\boldsymbol{\nabla}\times\mathbf{H}(\mathbf{r},t) + \mathbf{D}(\mathbf{r},t)\times\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t)\big]\\ &\quad + \partial_t\big(\mathbf{D}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\big)\\ &\quad = -\big(\mathbf{J}^{\rm e}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) + \mathbf{D}(\mathbf{r},t)\times\mathbf{J}^{\rm m}(\mathbf{r},t)\big) = -\mathbf{f}^{\rm L}(\mathbf{r},t)\,.\end{aligned} \qquad (13)$$

The right-hand side of (13) involves the Lorentz force $\mathbf{f}^{\rm L}(\mathbf{r},t)$ with reversed sign acting on external currents, i.e., the power applied by external sources to generate the momentum of the field and matter. The second term in the left-hand side of (13) is the time derivative of the density of momentum in the FMS. Accordingly, the momentum density, up to a constant vector $\mathbf{g}_0^{\rm M}(\mathbf{r})$, becomes $\mathbf{g}^{\rm M}(\mathbf{r},t) = \mathbf{D}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)$, i.e., it should be taken in the Minkowski form. The first term in (13) can be rewritten as

$$\begin{aligned}&\mathbf{B}(\mathbf{r},t)\times\boldsymbol{\nabla}\times\mathbf{H}(\mathbf{r},t) + \mathbf{D}(\mathbf{r},t)\times\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t)\\ &\quad = \boldsymbol{\nabla}\hat{\Sigma}(\mathbf{r},t) = \partial_\nu\hat{\Sigma}_{\nu'}^{\nu}(\mathbf{r},t)\,,\end{aligned} \qquad (14)$$

where $\nu = x,y,z$ and $\hat{\Sigma}(\mathbf{r},t)$ is a second-rank tensor in three-dimensional space. Therefore, its divergence (the contraction over one index) yields the vector found in the left-hand side of (14). The above tensor is also defined up to an arbitrary tensor for which $\boldsymbol{\nabla}\hat{\Sigma}_0(\mathbf{r},t) = 0$. Because the field was zero at $t = t_0 = 0$, the initial conditions $\mathbf{g}_0^{\rm M}(\mathbf{r}) = 0$, $\hat{\Sigma}_0(\mathbf{r},0) = 0$ must be imposed. Then at any instant of time, $\mathbf{g}^{\rm M}(\mathbf{r},t)$ and $\hat{\Sigma}(\mathbf{r},t)$ are obtained unambiguously by solving the excitation problem, i.e., in terms of the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ at each previous instant. We note that to determine $\mathbf{g}^{\rm M}(\mathbf{r},t)$, we have to calculate integrals (1), and to obtain $\hat{\Sigma}(\mathbf{r},t)$, it is additionally required to solve differential equation (14). To solve (14), we can also use the Helmholtz theorem and thus transform (14) into a Poisson equation. Consequently, the balance equation is written as

$$\partial_\nu\hat{\Sigma}_{\nu'}^{\nu}(\mathbf{r},t) + \partial_t g_{\nu'}^{\rm M} = -f_{\nu'}^{\rm L} = -\partial_t\partial_t^{-1}f_{\nu'}^{\rm L}\,,\quad \nu' = x,y,z\,. \qquad (15)$$

Here, the first term is the flux of the ν' component of the total momentum. This equation implies that the transfer rate of the ν' component of momentum in the FMS is [19]

$$v_{{\rm m}\nu'} = \frac{\partial_\nu\hat{\Sigma}_{\nu'}^{\nu}(\mathbf{r},t)}{g_\nu^{\rm M}}\,, \qquad (16)$$

with $g_{\nu'}^{\rm M}$ being the total generated momentum of the field and matter. If we take a volume bounded by a surface with the radius $r = ct$, then it maintains a constant total momentum of the field, matter, and the source $G_{\nu'} = \partial_{\mathbf{r}}^{-1}g_{\nu'} = \partial_{\mathbf{r}}^{-1}(g_{\nu'}^{\rm M} + \partial_t^{-1}f_{\nu'}^{\rm L}) = 0$, because the flux through the surface is zero. We consider the momentum transferred to matter. Obviously, this transfer occurs via the polarization currents

$$\mathbf{J}_P^{\rm e} = \partial_t(\mathbf{D} - \varepsilon_0\mathbf{E}) = \sigma\mathbf{E} + N_p^{\rm e}\partial_t\mathbf{p}^{\rm e}\,,$$
$$\mathbf{J}_P^{\rm m} = \partial_t(\mathbf{B} - \mu_0\mathbf{H}) = N_p^{\rm m}\,\partial_t\mathbf{p}^{\rm m}\,.$$

As has already been mentioned, the conduction current $\sigma\mathbf{E} = e\mathbf{v}N$ (where N is the number of charge carriers per unit volume) is taken into account in the polarization current. Here, $N_p^{\rm e}$ and $N_p^{\rm m}$ are the numbers of electric and magnetic dipoles with moments $\mathbf{p}^{\rm e}$ and $\mathbf{p}^{\rm m}$. Specific momentum transferred to matter is expressed as

$$\begin{aligned}&\mathbf{G}_{\rm M} = \partial_t^{-1}\mathbf{g}_{\rm M}\,,\\ &\mathbf{g}_{\rm M}(\mathbf{r},t) = \mathbf{J}_P^{\rm e}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) + \mathbf{D}(\mathbf{r},t)\times\mathbf{J}_P^{\rm m}(\mathbf{r},t)\,.\end{aligned} \qquad (17)$$

It remains to find the flux density of matter. For this, we rewrite Eqn (3) as

$$\begin{aligned}&\boldsymbol{\nabla}\times\mathbf{H}(\mathbf{r},t) = \varepsilon_0\,\partial_t\mathbf{E}(\mathbf{r},t) + \mathbf{J}_P^{\rm e}(\mathbf{r},t) + \mathbf{J}^{\rm e}(\mathbf{r},t)\,,\\ &-\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t) = \mu_0\,\partial_t\mathbf{H}(\mathbf{r},t) + \mathbf{J}_P^{\rm m}(\mathbf{r},t) + \mathbf{J}^{\rm m}(\mathbf{r},t)\end{aligned} \qquad (18)$$

and with this form of Maxwell's equations, write the balance of momentum by left-multiplying the first equation in (18) vectorially by $\mu_0\mathbf{H}(\mathbf{r},t)$ and the second by $\varepsilon_0\mathbf{E}(\mathbf{r},t)$ and subtracting one from the other:

$$\boldsymbol{\nabla}\hat{\Sigma}_{\rm EM}(\mathbf{r},t) + \partial_t\mathbf{g}^{\rm A}(\mathbf{r},t) = -\mathbf{f}^{\rm L}(\mathbf{r},t)\,. \qquad (19)$$

The Lorentz force

$$\mathbf{f}_{\rm M}^{\rm L}(\mathbf{r},t) = \mathbf{J}_P^{\rm e}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) + \mathbf{D}(\mathbf{r},t)\times\mathbf{J}_P^{\rm m}(\mathbf{r},t)\,,$$

acting on matter is moved to the left-hand side of (19) and taken into account in a certain flux density $\hat{\Sigma}_{\rm EM}$ that satisfies the differential equation

$$\begin{aligned}\boldsymbol{\nabla}\hat{\Sigma}_{\rm EM}(\mathbf{r},t) &= \big[\mu_0\mathbf{H}(\mathbf{r},t)\times\boldsymbol{\nabla}\times\mathbf{H}(\mathbf{r},t)\\ &\quad + \varepsilon_0\mathbf{E}(\mathbf{r},t)\times\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t)\big] + \mathbf{f}_{\rm M}^{\rm L}(\mathbf{r},t)\\ &= \mu_0\left[\frac{\boldsymbol{\nabla}\mathbf{H}^2(\mathbf{r},t)}{2} - \big(\mathbf{H}(\mathbf{r},t)\boldsymbol{\nabla}\big)\,\mathbf{H}(\mathbf{r},t)\right]\\ &\quad + \varepsilon_0\left[\frac{\boldsymbol{\nabla}\mathbf{E}^2(\mathbf{r},t)}{2} - \big(\mathbf{E}(\mathbf{r},t)\boldsymbol{\nabla}\big)\,\mathbf{E}(\mathbf{r},t)\right] + \mathbf{f}_{\rm M}^{\rm L}(\mathbf{r},t)\,.\end{aligned} \qquad (20)$$

We discuss the essence of balance equation (19). Abraham's density $\mathbf{g}^{\mathrm{A}}(\mathbf{r},t)$ is the density proper of the EM momentum of the field created by the primary (external) sources and secondary sources, i.e., the matter polarization currents that determine $\hat{\Sigma}_{\mathrm{EM}}(\mathbf{r},t)$. In the absence of sources ($\mathbf{f}^{\mathrm{L}}=0$), Eqn (19) is a typical conservation law. Consequently, the tensor quantity $\hat{\Sigma}_{\mathrm{EM}}(\mathbf{r},t)$ defines flux densities of the components of the field momentum proper. This quantity is also defined up to a tensor $\hat{\Sigma}^{0}_{\mathrm{EM}}(\mathbf{r},t)$ whose divergence is zero; under zero initial conditions, we have $\hat{\Sigma}^{0}_{\mathrm{EM}}(\mathbf{r},0)=0$. The matter flux momentum is given by the tensor $\hat{\Sigma}_{\mathrm{M}}(\mathbf{r},t)=\hat{\Sigma}(\mathbf{r},t)-\hat{\Sigma}_{\mathrm{EM}}(\mathbf{r},t)$. Now we can find the transfer rate of the field and matter momentum proper:

$$v^{\mathrm{EM}}_{\mathrm{m}\nu'}=\frac{\partial_\nu \hat{\Sigma}_{\mathrm{EM}}{}^{\nu}_{\nu'}(\mathbf{r},t)}{g^{\mathrm{A}}_{\nu'}(\mathbf{r},t)}\,,\qquad v^{\mathrm{M}}_{\mathrm{m}\nu'}=\frac{\partial_\nu \hat{\Sigma}_{\mathrm{M}}{}^{\nu}_{\nu'}(\mathbf{r},t)}{g_{\mathrm{M}\nu'}(\mathbf{r},t)}\,. \tag{21}$$

To transform (20), we use the vector identity $\boldsymbol{\nabla}(\mathbf{ab})=(\mathbf{a}\boldsymbol{\nabla})\mathbf{b}+(\mathbf{b}\boldsymbol{\nabla})\mathbf{a}+\mathbf{a}\times\boldsymbol{\nabla}\times\mathbf{b}+\mathbf{b}\times\boldsymbol{\nabla}\times\mathbf{a}$, which takes the form $\boldsymbol{\nabla}\mathbf{a}^2=2(\mathbf{a}\boldsymbol{\nabla})\mathbf{a}+2\mathbf{a}\times\boldsymbol{\nabla}\times\mathbf{a}$ if $\mathbf{a}=\mathbf{b}$. Likewise, to transform the tensors introduced, for example, the tensor $\hat{\Sigma}(\mathbf{r},t)$, we can use the vector–tensor identity

$$\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b})+\mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}) = -\boldsymbol{\nabla}\big[\hat{I}(\mathbf{ab})-\mathbf{a}\otimes\mathbf{b}-\mathbf{b}\otimes\mathbf{a}\big]+\mathbf{a}(\boldsymbol{\nabla}\mathbf{b})+\mathbf{b}(\boldsymbol{\nabla}\mathbf{a})\,,$$

which for identical vectors becomes $2\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{a})=\boldsymbol{\nabla}[\hat{I}\mathbf{a}^2-2\mathbf{a}\otimes\mathbf{a}]+2\mathbf{a}(\boldsymbol{\nabla}\mathbf{a})$. For the vacuum, the tensor $\hat{\Sigma}(\mathbf{r},t)$ is equal to the Maxwell stress tensor $\hat{\sigma}^{\nu}_{\nu'}$ taken with the opposite sign. It can also be transformed, given that in view of (3),

$$\boldsymbol{\nabla}\mathbf{D}(\mathbf{r},t)=-\partial_t^{-1}\big(\boldsymbol{\nabla}\mathbf{J}^{\mathrm{e}}(\mathbf{r},t')\big)=\partial_t^{-1}\big(\partial_{t'}\rho^{\mathrm{e}}(\mathbf{r},t')\big)=\rho^{\mathrm{e}}(\mathbf{r},t)\,,$$

$$\boldsymbol{\nabla}\mathbf{B}(\mathbf{r},t)=-\partial_t^{-1}\big(\boldsymbol{\nabla}\mathbf{J}^{\mathrm{m}}(\mathbf{r},t')\big)=\partial_t^{-1}\big(\partial_{t'}\rho^{\mathrm{m}}(\mathbf{r},t')\big)=\rho^{\mathrm{m}}(\mathbf{r},t)\,,$$

because external sources satisfy the continuity equations $\boldsymbol{\nabla}\mathbf{J}^{\mathrm{e}}(\mathbf{r},t)+\partial_t\rho^{\mathrm{e}}(\mathbf{r},t)=0$ and $\boldsymbol{\nabla}\mathbf{J}^{\mathrm{m}}(\mathbf{r},t)+\partial_t\rho^{\mathrm{m}}(\mathbf{r},t)=0$. In the absence of external magnetic charges, i.e., for $\rho^{\mathrm{m}}(\mathbf{r},t)=0$, the density of the external magnetic current is solenoidal and can be represented as a curl of the density of some electric current.

Hence, in a nondispersive medium, the Minkowski momentum density becomes the density of the matter–field complex, but its rate of transfer is equal to the phase velocity. A generalization of this velocity for dispersive media leads to transfer rate (16), which is the rate of transfer of the total field–matter momentum. In this case, to completely determine all quantities, we need to obtain a rigorous solution of the nonstationary problem of excitation, with all the above quantities depending on the previous history of the process. The local (differential) balance relations thus obtained can be written as integral relations for a certain volume V. Then integrals of u and $\mathbf{g}^{\mathrm{M}}$ over the specified volume are the total energy U and the total momentum $\mathbf{G}^{\mathrm{M}}$ of this volume, which are stored there in the sense of a global conservation law. Two cases are possible.

(1) The volume contains field sources. Then the total energy and momentum balances are in fact inhomogeneous balance relations; in their right-hand sides, we find quantities corresponding to energy and momentum production in the volume. Negative energy production means dissipation.

(2) There were no sources in the volume until an instant t. In this case, sources are located outside the volume and the instant when the field enters the volume can be taken as the time t_0. Then the energy and momentum in the volume are conserved in the sense that the quantity $\partial_t(U+Q)$ is equal at every instant to the power leaving the volume, while the change in the total momentum of the volume is equal to the momentum emerging from it.

Specific forms of the quantities mentioned above can be obtained in the case of a simple, stationary (monochromatic) field (wave) for a number of simple dispersion laws. In this case, densities averaged over one period 'forget' their initial values in the limit transition from a quasistationary to a stationary excitation, i.e., cease to depend on them, and the EM process reaches stationary level values. All the values of the EMT, the energy density, and momentum for the vacuum coincide with those written in Abraham's form. We note that in our case, all quantities are uniquely defined. For example, the solution of the differential equations $\boldsymbol{\nabla}\hat{\Sigma}_0(\mathbf{r},t)=0$ with zero initial conditions gives zero components of the tensor $\hat{\Sigma}_0(\mathbf{r},t)$.

3. Monochromatic plane wave in a conducting magnetodielectric medium

We consider a monochromatic plane wave incident on a layer of conductive magnetodielectric material of thickness d. Let this layer in the region $0\leqslant z\leqslant d$ have constant real spectral permittivities ε' and μ, such that $\varepsilon',\,\mu\geqslant 1$. This means zero frequency dispersion, which is true in a certain frequency range $0<\omega\leqslant\omega_{\min}$, where $\omega_{\min}$ is a certain minimum frequency in a set including the resonance eigenfrequencies of the material, the frequency at which the normal skin effect (if it is nonzero) is violated, and the plasma frequency of free carriers ω_{p}. If the plasma frequency is nonzero, it defines the conducting medium with a complex spectral dielectric permittivity [17]

$$\varepsilon(\omega)=\varepsilon'-\mathrm{i}\varepsilon''=\varepsilon'-\frac{\mathrm{i}\sigma}{\varepsilon_0\omega} \tag{22}$$

and the magnetic permeability $\mu=\mathrm{const}$. For example, for water, the frequency $\omega_{\min}\sim\omega_{\mathrm{c}}=1/\tau$ is in the 100 GHz range, ω_{c} is the frequency of collisions, and τ denotes the relaxation time in Debye's formula. For metals, $\omega_{\min}$ may lie in the range from the infrared to ultraviolet spectral regions [24, 25]. We first consider the quasimonochromatic plane wave (long train), excited at infinity $t=-\infty$ and arriving at the layer boundary at the instant $t_0=0$ from the left, and also its diffraction on it. In formal mathematical terms, this plane wave can be excited in the direction of both sides by an external current sheet with the density $\mathbf{J}^{\mathrm{e}}_{\mathrm{inc}}(x,y,z,t)=\mathbf{x}_0\chi(t+\tau)\,I_x(t)\,\delta(z+l)$ located at $z=-l$, which was initiated at the moment $t=-\tau$ [26]. Here, χ is the Heaviside function, and for a quasi-monochromatic wave, it is convenient to take $I_x(t)=\sin(\omega t)$ and introduce the current density

$$\mathbf{J}^{\mathrm{e}}_{\mathrm{inc}}(x,y,z,t)=\mathbf{x}_0\chi(t+\tau)\,I_x\delta(z+l)\big[1-\exp(-\delta t)\big]\sin(\omega t)\,. \tag{23}$$

We analyze the field at large values of time $t\gg\tau+l/c$. As a special case, we also evaluate the excitation with density (23) in an infinite homogeneous magnetodielectric conducting medium. The selected current density creates a plane wave with components $E_x = E$ and $H_y = H$ (in what follows, we often omit the indices x and y). This wave satisfies Maxwell's equations in the form

$$\partial_z H = -\varepsilon_0 \varepsilon' \partial_t E - \sigma E, \\ \partial_z E = -\mu_0 \mu \partial_t H. \tag{24}$$

In the vacuum, $\varepsilon' = \mu = 1$ and $\sigma = 0$. We know that for a monochromatic plane wave with the complex dependence $\exp[\mathrm{i}(\omega t - \beta(\omega)z) - \alpha(\omega)z]$, propagating along z in a dispersive medium, we can introduce two speeds determining the transfer of the inherent physical substances of the wave or its characteristics: the velocity of energy motion v_e and the velocity of the EM momentum transfer v_m [27–29] (because of one-dimensionality, the vector notation is omitted). For a dispersion law $\beta(\omega)$, we can introduce two more quantities describing the motion of mathematical (kinematic) characteristics of the wave: the phase velocity $v_p(\omega) = \omega/\beta(\omega)$ and the group velocity $v_g(\omega) = (\partial\beta(\omega)/\partial\omega)^{-1}$. The former characterizes the rate of advance of the phase, and the latter, the velocity of phase perturbations or interference pattern (beats) of two waves with infinitely close frequencies with the same amplitude (as introduced by Stokes). We note that a positive value of the derivative corresponds to a positive dispersion, or a direct wave, and a negative value corresponds to a negative dispersion, or a backward wave. The coefficient $n'(\omega) = c/v_p(\omega)$ determines the retardation and refraction of waves at the interface, and the coefficient $n''(\omega)$ determines losses. Normal dispersion corresponds to the values $\partial n'(\omega)/\partial\omega > 0$ and $\partial v_p(\omega)/\partial\omega < 0$, and anomalous dispersion corresponds to $\partial n'(\omega)/\partial\omega < 0$ and $\partial v_p(\omega)/\partial\omega > 0$ [30]. In the general case, the last two velocities do not correspond to motions of any physical substance and are just convenient mathematical concepts for wave description [14–16, 31–41], although $v_g(\omega)$ is often identified with $v_e(\omega)$, which is wrong for dissipative media [14–16, 33–35, 39, 40]. Because dissipation occurs in all real environments, in one way or another, the equality $v_g(\omega) = v_e(\omega)$ holds only for some ideal model cases, for example, that of ideal collisionless plasma.

The case of dispersion in perfectly conducting waveguides, including periodic ones, is quite different: if the wave is harmonic, there is no special frequency spectral group of waves and the dispersion arises owing to the spatial spectral groups of partial waves propagating at an angle θ to the axis of the waveguide with a phase velocity equal to the speed of light. This group depends, generally, on two angles, one of which determines the energy transfer rate along the axis, while the other may have a continuous spectrum of values [16]. We note that $v_p = c/\cos\theta$, $v_g = c\cos\theta$, and $v_p v_g = c^2$. This case is trivial and corresponds to electrodynamic structures, not to media. It is maintained in a number of papers (e.g., in [41]) that $v_g = v_e$ always; hence, $v_g \leqslant c$, even though this is not true [14–16, 39, 40]. The same paper [41] analyzes the possibility of $v_g \to \infty$ for waves with negative energy. Strictly speaking, in the problem of propagation of a pulse (spectral wave packet), an infinite number of variables having the dimension of velocity can be introduced, for example, $v_n = (\omega^{n-1}\partial^n\beta(\omega)/\partial\omega^n)^{-1}$. In the mathematical description, the corresponding complex velocities are used with β replaced by a complex constant of propagation $\gamma(\omega) = \beta(\omega) - \mathrm{i}\alpha(\omega)$ [15, 37]. Accordingly, a complex refractive index is introduced as $n(\omega) = n'(\omega) - \mathrm{i}n''(\omega)$ [17]. A complex v_p for the signal can have (and has) a mathematical meaning, unlike a complex v_g, which can be introduced only for complex signals.

Paper [42] introduces the group velocity 4-vector, which is justified in the case of a self-adjoint Hamiltonian, when the Lagrangian is a quadratic function in generalized coordinates and momenta. For conservative (nondissipative) systems, the well-known Leontovich–Lighthill theorem holds [27–29, 31, 43–45], which states that with the Hamiltonian given above, the relation $v_g = v_e$ holds. In our situation (and generally with a nonzero dissipation), this is not the case. Traditionally, v_g is introduced by expanding the phase constant $\beta(\omega)$ entering the spectral integral in a Taylor series in the neighborhood of a certain frequency (e.g., the carrier frequency) and keeping zeroth and first-order terms (the first approximation of dispersion theory) [36–38]. Sometimes, the inverse expansion $\omega(\beta)$ is also used [35]. Taking higher-order terms into account leads just to the appearance of the velocities mentioned above, and in the first approximation, the second derivative $\partial^2\beta/\partial\omega^2$ then characterizes the rate of spreading of the momentum as a whole [33–37]. Similar complex velocities, also depending on the damping constant $\alpha(\omega)$, can be defined in dissipative media. All such expansions are asymptotic [38, 46], i.e., not necessarily convergent. In the simplest one-dimensional case of a simple monochromatic wave, no frequency group of waves is formed, and hence there is no reason for introducing the group velocity (although it can formally be introduced for the dispersion $\beta(\omega)$, exactly as we do here).

Processes in conducting media are often relatively slow, and we can therefore neglect the dispersion not related to conduction. The corresponding frequency intervals may form at ultralow frequencies (the Heaviside layer in the ionosphere), at radio frequencies (sea water, metals), and in the ultrahigh-frequency (microwave) and high-frequency ranges (metals, semiconductors). When dispersion is determined only by the frequency-independent conduction σ, the above law has the form [35]

$$\beta(\omega) = \frac{\omega}{c}\left\{\frac{\varepsilon'\mu}{2}\left[1 + \sqrt{1 + \frac{\sigma^2}{\varepsilon_0^2\varepsilon'^2\omega^2}}\right]\right\}^{1/2} = \frac{\omega n'(\omega)}{c}, \tag{25}$$

$$\alpha(\omega) = \frac{\omega}{c}\left\{\frac{\varepsilon'\mu}{2}\left[-1 + \sqrt{1 + \frac{\sigma^2}{\varepsilon_0^2\varepsilon'^2\omega^2}}\right]\right\}^{1/2} = \frac{\omega n''(\omega)}{c}. \tag{26}$$

Direct application of formula (22) immediately demonstrates that the group velocity may exceed $c/\tilde{n}$, where $\tilde{n} = \sqrt{\varepsilon'\mu}$ is the retardation or refraction coefficient in a medium with zero conductivity. Moreover, the group velocity can exceed the speed of light in the vacuum, c. Let $\mu = 1$. Setting $\tilde{\omega} = \sigma/(\varepsilon_0\varepsilon)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\beta(\omega) = \frac{\beta(\omega)}{\omega} \times \left\{1 - \frac{(\tilde{\omega}/\omega)^2}{2\left(1 + \sqrt{1 + (\tilde{\omega}/\omega)^2}\right)\sqrt{1 + (\tilde{\omega}/\omega)^2}}\right\}_{\omega=\tilde{\omega}} = \frac{0.8535\ldots}{v_p(\tilde{\omega})} = \frac{\tilde{n}/c}{1.0663\ldots}. \tag{27}$$

The speed exceeds the speed of light after $\tilde{n} < 1.0663\ldots$; this may occur, for instance, in weakly ionized air under the conditions $\omega_c \gg \omega_p$ and $\omega_c \gg \tilde{\omega}$. The introduction of dispersion of the dielectric permittivity ε' allows making this effect even more impressive [16]. We cite two publications [39, 40], which were among the first papers emphasizing the fact that the group velocity of light increases above the speed of light in the region of anomalous dispersion (see also [15]) and simultaneously showing that the signal then propagates at a speed $v < c$. Therefore, in conducting media, $v_{\rm g}$ cannot characterize the velocity of energy propagation. On the contrary, in this medium with an anomalous positive dispersion, the phase velocity is always below the speed of light: $v_{\rm p}(\omega) = c/n'(\omega) < c$.

An additional purpose of this paper is to prove the relations $v_{\rm e} = v_{\rm p}$ and $v_{\rm m} = v_{\rm p}$ in the case formulated above. We note that the relation $v_{\rm e} = v_{\rm p}$ for media with anomalous positive dispersion and with the dielectric permittivity described by the Debye formula (i.e., for polar dielectrics with rigid dipoles) has been proved in [16] by two independent methods.

4. The density of the electromagnetic energy of a monochromatic wave

We assume that a linearly polarized monochromatic plane wave with the electric field component E_x propagates in a conducting medium. This wave creates a conduction current density $J_{x\sigma} = \sigma E_x$, which leads to dissipation of the wave energy. The distribution of dissipation along the z axis is exponential, given by $\exp(-2\alpha(\omega)z)$, and it heats up infinite space nonuniformly along the z axis. Nonuniform heating, in turn, generates thermal radiation in the directions $\pm z$; this radiation has all possible spectral components. The process is therefore nonequilibrium from the start. We assume it to be equilibrium and single-frequency, assuming the amplitude of the waves on the interval of interest to be small or the heat capacity of the medium to be infinite. In dissipative media, an undamped harmonic wave can propagate only at the expense of the energy of distributed external sources that compensate the loss of the wave energy to dissipation of heat Q [17]. We assume that such sources are outside the zone in which the wave is considered (typically, at infinity). For plane waves, the source energy should be infinite even with zero damping, which characterizes the wave as a convenient mathematical abstraction (solution of homogeneous Maxwell equations). Formally, the mathematical plane wave is initiated in both directions by an external current sheet with current density (23), applied for an indefinitely long period. If the sources are at infinity, then the plane wave is a limit case of a spherical wave. Dielectric permittivity (22) is obtained by direct substitution of exponential expressions for fields in the Maxwell equations taking the conduction current into account. Another derivation of this quantity can be given. Namely, it is necessary to calculate polarization per unit volume averaged over one period and use the relation

$$D_x(\omega,t,z) = D(\omega,t,z) = \varepsilon_0\varepsilon(\omega)\,E(\omega,t,z) = \varepsilon_0 E(\omega,t,z) + P_x(\omega,t,z)\,, \qquad (28)$$

where $E(\omega,t,z) = E_x(\omega,0,0)\exp(\mathrm{i}\omega t - \mathrm{i}\gamma(\omega)z)$. We set $E_0 = E_x(\omega,0,0)$. Expression (28) must be averaged over one period of oscillations. In our case, $\langle P_x(\omega,t,z)\rangle = (\kappa_1 + \kappa_2(\omega))\langle E(\omega,t,z)\rangle$. For the susceptibilities introduced, we write $\kappa_1 = \varepsilon' - 1$ and $\kappa_2(\omega) = -\mathrm{i}\sigma/(\varepsilon_0\omega)$. Indeed, polarization per unit volume in a medium is created by the polarization of the medium proper and by the motion of free charges that are scattered by atoms and molecules of matter and by each other. The first of these polarizations is instantaneous by virtue of our assumption that the characteristic frequencies of matter are very high. The motion of charges is described by the equation $\dot{x}(t) = \sigma(Ne)^{-1}E_x(\omega,t,z)$, and it occurs such that the potential energy of the charges is zero, and their average kinetic energy is

$$\langle U_{\rm K}\rangle = \frac{m\sigma^2E_0^2}{4Ne^2}\exp\left(-\alpha(\omega)z\right),$$

where $\sigma = Ne^2/(m\omega_{\rm c})$. This can be obtained from the dielectric permittivity of the plasma

$$\kappa_{\rm p}(\omega) = -\frac{\omega_{\rm p}^2}{\omega(\omega - \mathrm{i}\omega_{\rm c})}\,, \qquad (29)$$

assuming that $\omega \ll \omega_{\rm p}$ and $\omega \ll \omega_{\rm c}$, i.e., that the plasma frequency and collision frequency are very high; in addition, $\sigma = \varepsilon_0\omega_{\rm p}^2/\omega_{\rm c}$ is the conductivity at zero (infinitely low) frequency. In other words, the properties of the conducting medium are the same as those of plasma at ultralow frequencies. The mean density of the electric part of the EM energy and the dielectric permittivity of the gas of oscillators with an eigenfrequency ω_0 were obtained in [15]:

$$\left\langle U_E(t,z)\right\rangle = \frac{\varepsilon_0E_0^2}{4}\left\{1 + \frac{\omega_{\rm p}^2(\omega^2+\omega_0^2)}{(\omega^2-\omega_0^2)^2+\omega_{\rm c}^2\omega^2}\right\}\exp\left(-2\alpha(\omega)z\right), \qquad (30)$$

$$\varepsilon(\omega) = 1 - \frac{\omega_{\rm p}^2(\omega^2-\omega_0^2+\mathrm{i}\omega\omega_{\rm c})}{(\omega^2-\omega_0^2)+\omega_{\rm c}^2\omega^2}\,. \qquad (31)$$

For a nonconducting medium without dispersion, we assume that $\omega \ll \omega_0$, which gives

$$\varepsilon(\omega) = 1 + \frac{\omega_{\rm p1}^2}{\omega_0^2} = 1 + \kappa_1 = \varepsilon' = \mathrm{const}\,, \qquad (32)$$

$$\left\langle U_E(t,z)\right\rangle = \frac{1}{4}\,\varepsilon_0E_0^2\left(1+\frac{\omega_{\rm p1}^2}{\omega_0^2}\right)\exp\left(-2\alpha(\omega)z\right) = \frac{1}{4}\,\varepsilon_0\varepsilon E_0^2\exp\left(-2\alpha(\omega)z\right). \qquad (33)$$

Here, the subscript 1 marks the plasma frequency associated with the concentration of dipoles in matter. In this environment, the energy at frequencies below $\omega_{\min}$ propagates at the phase velocity. For a conducting medium with a plasma of charge carriers, we need to add terms from formulas (30) and (31) with $\omega_0 = 0$. In this case, $\omega_{\rm p2} \gg \omega$, and hence

$$\left\langle U_E(t,z)\right\rangle = \frac{\varepsilon_0E_0^2}{4}\left(1+\frac{\omega_{\rm p1}^2}{\omega_0^2}+\frac{\omega_{\rm p2}^2}{\omega_{\rm c}^2}\right)\exp\left(-2\alpha z\right), \qquad (34)$$

$$\varepsilon(\omega) = 1 + \frac{\omega_{\rm p1}^2}{\omega_0^2} - \frac{\mathrm{i}\omega_{\rm p2}^2}{\omega_{\rm c}\omega} = 1 + \kappa_1 + \kappa_2\,. \qquad (35)$$

5. Energy propagation velocity

In each wave process, the transfer rate of some substance is determined, according to [21], by its density and by the density vector of its flux per unit time. In our case of energy transfer, this vector is the Poynting vector $\mathbf{S}(z,t) = \mathbf{z}_0S(z,t)$, and hence

$$\mathbf{v}_{\rm e}(z,t) = \mathbf{z}_0v_{\rm e}(z,t) = \frac{\mathbf{S}(z,t)}{u(z,t)} = \frac{\mathbf{z}_0E(z,t)\,H(z,t)}{u(z,t)}\,, \qquad (36)$$

which follows from conservation law (4) under general assumptions. Equations starting with Eqn (36) involve real physical fields. In the case of harmonic fields in homogeneous media, averaging (36) over time and taking (30) and the magnetic energy into account, we find that

$$v_{\mathrm{e}}(\omega) = c\,\frac{\sqrt{2}}{\sqrt{\varepsilon'\mu}}\,\frac{\sqrt{1+\delta}}{1+\zeta+\delta} < v_{\mathrm{p}}(\omega)\,, \tag{37}$$

where $\delta = \sqrt{1+\zeta^2}$ and $\zeta = \sigma/(\varepsilon_0\varepsilon'\omega)$. If conditions $\omega \ll \omega_{\mathrm{c}}$ and $\sigma \gg \varepsilon_0\varepsilon'\omega_{\mathrm{c}}$ hold, then the second term in the denominator of (37) can be neglected, and we obtain

$$v_{\mathrm{e}}(\omega) \approx c\,\frac{\sqrt{2}}{\sqrt{\varepsilon'\mu}\sqrt{1+\delta}} = v_{\mathrm{p}}(\omega)\,. \tag{38}$$

With these approximations, the unity under the root can also be neglected, and hence

$$v_{\mathrm{e}}(\omega) = v_{\mathrm{p}}(\omega) = \frac{c}{\sqrt{\varepsilon'\mu}}\sqrt{\frac{2\varepsilon_0\varepsilon'\omega}{\sigma}} = \sqrt{\frac{2\omega}{\mu_0\mu\sigma}} = \frac{v_{\mathrm{g}}(\omega)}{2}\,. \tag{39}$$

If $\mu = 1$ in dispersion (39) and if retardation is not large, then the group velocity can somewhat exceed the speed of light in the vacuum. Namely, the restriction $v_{\mathrm{g}} > c$ leads to the condition $\sigma < 8\omega\varepsilon_0$. Using the condition under which formula (39) has been derived, we find $\omega\varepsilon_0\varepsilon \ll \sigma < 8\omega\varepsilon_0$. For media with $\varepsilon \sim 1$, this last inequality can be assumed satisfied as σ approaches to $8\omega\varepsilon_0$ from below. However, it becomes impossible to neglect the displacement current completely in comparison with the conduction current at frequencies of the order of $\sigma/(8\varepsilon_0)$, and therefore the rigorous relation (25) must be used.

Result (38) can also be obtained differently, by passing from a quasistationary to a stationary process [16]. Let the source of field-generating surface current (23) emerge at $z = 0$ at the instant $t = 0$. There was no field at $t < 0$. In the range $|z| > 0$, this source generates a plane wave. The density of work (energy) w that the source spends to create field (5) can then be written, for the sake of clarity, as

$$\begin{aligned} w(z,t) &= \int_0^t \Big\{E(z,t')\,\partial_{t'}D(z,t') + H(z,t')\,\partial_{t'}B(z,t')\Big\}\,\mathrm{d}t' \\ &= \frac{\varepsilon_0 E^2(z,t) + \mu_0 H^2(z,t)}{2} \\ &+ \int_0^t \Big\{\varepsilon_0\hat{\kappa}^{\mathrm{e}}(z,0)\,E^2(z,t') + \mu_0\hat{\kappa}^{\mathrm{m}}(0,z)\,H^2(z,t') \\ &+ \int_0^{t'} \big[\varepsilon_0\mathbf{E}(z,t')\,\partial_t\hat{\kappa}^{\mathrm{e}}(t'-t'',z)\,\mathbf{E}(z,t'') \\ &+ \mu_0\mathbf{H}(z,t')\,\partial_t\hat{\kappa}^{\mathrm{m}}(t'-t'',z)\,\mathbf{H}(z,t'')\big]\,\mathrm{d}t''\Big\}\,\mathrm{d}t'\,, \end{aligned} \tag{40}$$

where we have introduced kernels of the integral operators of the dielectric permittivity and magnetic permeability $\hat{\kappa}^{\mathrm{e}}$ and $\hat{\kappa}^{\mathrm{m}}$. In view of the homogeneity, we ignore their dependence on z, although the fields are exponential functions of z. In (40), we can choose $-\infty$ as the lower limit. For this particular dispersion law, $\hat{\kappa}^{\mathrm{m}}(t) = (\mu-1)\,\delta(t)$, and the dielectric permittivity operator takes the form

$$\hat{\kappa}^{\mathrm{e}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\kappa^{\mathrm{e}}(\omega)\exp(\mathrm{i}\omega t)\,\mathrm{d}\omega = (\varepsilon'-1)\,\delta(t) + \frac{\chi(t)\sigma}{\varepsilon_0}\,,$$

where $\chi(t)$ is the Heaviside function. To determine the energy density of field (40), we need to subtract the density of dissipated energy q (heat) [15–17]. In determining these quantities for large times $t \gg 1/\delta$ and $t \gg 1/\omega$, we obtain the mentioned parameters for a quasi-monochromatic process. Averaging over the period of oscillations $2\pi/\omega$ and passing to the limit $t \to \infty$, we find the energy of the monochromatic process. Substituting the expressions for each quantity in terms of spectral integrals in the integrals of type (40), isolating the delta functions, integrating over time, and then calculating the spectral integral using the theory of residues, we arrive at formula (30). We note that the result $\langle U_E\rangle = \varepsilon_0\varepsilon'|\mathbf{E}|^2/4$ for a conducting medium ($\omega_0 = 0$) for low frequencies is immediately implied by the Umov–Poynting theorem in complex form [34, 35]. The same result follows from the explicit representation of the fields in Eqns (24) [35]:

$$\begin{aligned} E_x &= E = E_0\cos(\omega t - \beta z)\exp(-\alpha z)\,, \\ H_y &= H = H_0\cos(\omega t - \beta z - \varphi)\exp(-\alpha z)\,. \end{aligned} \tag{41}$$

Here, $\varphi = \arctan(\alpha/\beta)$ is the phase shift given by formula (6.32) in [35], and the ratios of the amplitudes in (18) give the real impedance

$$Z = \frac{E_0}{H_0} = \frac{\omega\mu_0\mu}{\sqrt{\alpha^2+\beta^2}}\,.$$

Also,

$$Z = Z_0\left[\frac{\mu}{\sqrt{\varepsilon'^2 + \sigma^2/(\omega^2\varepsilon_0^2)}}\right]^{1/2}$$

(formula (6.31) in [35]), where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. This result implies that

$$\begin{aligned} v_{\mathrm{e}} &= \frac{\langle S\rangle}{\langle U\rangle} = 2\,\frac{E_0^2\cos(\varphi)/Z}{\varepsilon_0\varepsilon' E_0^2 + \mu_0\mu E_0^2/Z^2} \\ &= \frac{2}{\omega\mu_0\mu}\,\frac{\beta}{\varepsilon_0\varepsilon' + \varepsilon_0\varepsilon'\sqrt{1+\zeta^2}} \\ &= \frac{2}{\omega^2}\,\frac{\beta\omega}{\varepsilon_0\varepsilon'\mu_0\mu(1+\delta)} = \frac{\omega}{\beta} = v_{\mathrm{p}}\,. \end{aligned}$$

6. Momentum transfer rate

We address the question of the transfer rate of the EM momentum of the field. The momentum density in the vacuum is given unambiguously by the Abraham vector $\mathbf{g}^{\mathrm{A}}(t,z) = \mathbf{z}_0 S(t,z)/c^2$. Traditionally, two forms are used for a continuous medium: Abraham's $\mathbf{g}^{\mathrm{A}}$ and Minkowski's $\mathbf{g}^{\mathrm{M}}(t,z) = \mathbf{z}_0\tilde{n}^2 S(t,z)/c^2 = \tilde{n}^2\mathbf{g}^{\mathrm{A}}(t,z)$ and, as we have mentioned, the dilemma of which of them is preferable remains unresolved [3–13, 41]. In the case of zero dispersion, both definitions yield equivalent conservation laws for the energy–momentum tensors of the field in the Minkowski and Abraham forms [3]. We have already mentioned the claim that choosing the form of the total momentum density is impossible without solving the equations of motion of matter in the field and without determining the EMT of the medium. However, without finding the momentum density and its flux (of at least one of these quantities), it is also impossible to determine the momentum transfer rate. This ambiguity is, in fact, typical of the electrodynamics of continuous media. For example, the introduction of electrodynamic potentials in a medium is also ambiguous [47, 48], and gauge transformations are not the only cause of it. For a plane wave, $\mathbf{g}^{\mathrm{M}}(t,z)$ and $\mathbf{g}^{\mathrm{A}}(t,z)$ have the form

$$\mathbf{g}^{\mathrm{A}}(t,z)=\mathbf{z}_0 g^{\mathrm{A}}(t,z)=\frac{\mathbf{z}_0 E_0^2}{c^2 Z}\cos\left(\omega t-\beta(\omega)z\right)\times\cos\left(\omega t-\beta(\omega)z-\varphi\right)\exp\left(-2\alpha(\omega)z\right), \tag{42}$$

$$\mathbf{g}^{\mathrm{M}}(t,z)=\mathbf{z}_0 g^{\mathrm{M}}(t,z)=\frac{\mathbf{z}_0 \tilde{n}^2 E_0^2}{c^2 Z}\cos\left(\omega t-\beta(\omega)z\right)\times\cos\left(\omega t-\beta(\omega)z-\varphi\right)\exp\left(-2\alpha(\omega)z\right). \tag{43}$$

We introduce another definition, which is closer to Minkowski's: $\tilde{\mathbf{g}}=\mathbf{z}_0\tilde{g}^{\mathrm{M}}$, where $\tilde{g}^{\mathrm{M}}=S/v_{\mathrm{p}}^2=n'^2(\omega)g^{\mathrm{A}}$. For $\sigma=0$, we have $g^{\mathrm{M}}=\tilde{g}^{\mathrm{M}}$, $\alpha=0$, and $\beta=\omega\sqrt{\varepsilon'\mu}/c$; we note that the momentum is not transferred to matter on average (except in the cases of reflections from interfaces) because the Lorentz force and the electric polarization current $\mathbf{J}_P^{\mathrm{e}}=\varepsilon_0(\varepsilon'-1)\,\partial\mathbf{E}/\partial t$ are phase-shifted by $\pi/2$ and the momentum propagates at the phase velocity $v_{\mathrm{m}}=v_{\mathrm{p}}=c/\tilde{n}$ [29]. The Lorentz force $\varepsilon_0\varepsilon'\mathbf{E}\times\mathbf{J}_P^{\mathrm{m}}$ is also shifted by the same amount for the magnetic polarization current $\mathbf{J}_P^{\mathrm{m}}=\mu_0(\mu-1)\,\partial\mathbf{H}/\partial t$. In dissipative media, momentum is transferred to matter via photon absorption, the momentum flux is directed along z, and we can therefore write the balance equation for the momentum density vector as [19, 29]

$$-\partial_t\left[g(t,z)+g_\sigma(t,z)\right]=\partial_z\left[v_{\mathrm{m}}(t,z)\,g(t,z)\right]=v_{\mathrm{m}}(t,z)\,\partial_z g(t,z), \tag{44}$$

where we introduce the momentum $g_\sigma(t,z)$ transferred to matter. It is assumed here that owing to the homogeneity of the medium, v_{m} is independent of z. For any of the three forms, we have the functional dependence $g(t,z)=G(\omega t-\beta z)\exp(-2\alpha z)$. The rate of change of the momentum transferred to unit area of the layer of thickness dz is equal to the difference between the fluxes g through the sections z and $z+\mathrm{d}z$, and hence

$$\partial_t g_\sigma(t,z)=2\alpha v_{\mathrm{m}}(t,z)\,g(t,z). \tag{45}$$

The value given by (45) is equal to the pressure exerted by the field on the layer of unit thickness. To calculate the pressure on a finite layer, expression (45) should be integrated over coordinates inside the layer. In an infinite layer, the entire momentum of the field is transferred to matter. Substituting (45) in (44), we obtain that the momentum transfer rate is constant and (for $\alpha\ll\beta$) is equal to the phase velocity:

$$v_{\mathrm{m}}(t,z)=-\frac{\partial_t g(t,z)}{\partial_z g(t,z)+2\alpha(\omega)\,g(t,z)}=\frac{\omega}{\beta(\omega)}=v_{\mathrm{p}}(\omega). \tag{46}$$

If $\alpha\sim\beta$, we obtain $v_{\mathrm{p}}/2$. Here, the form in which the momentum is written is not specified. The same relation can be obtained using the following line of reasoning. Maxwell's equations are written in form (24), and hence the balance equation is

$$\partial_z\left(\frac{\varepsilon_0\varepsilon' E^2}{2}+\frac{\mu_0\mu H^2}{2}\right)=-\partial_t g^{\mathrm{M}}-\mu_0\mu\sigma EH. \tag{47}$$

Because $\partial S/\partial t=-v_{\mathrm{p}}(\partial S/\partial z+2\alpha S)$, the only form of the quantity $g=\eta S/c^2$ that cancels the Lorentz force $\mu_0\mu\sigma S$ acting on charges is obtained using the form $\tilde{g}^{\mathrm{M}}$. Indeed, we impose the requirement that the balance equation $2\alpha\eta S v_{\mathrm{p}}/c^2=\mu_0\mu\sigma S$ hold. The balance is written for charges in the medium, whence $\eta=c^2\omega\mu_0\mu\sigma\beta/(2\alpha\omega^2)=c^2/v_{\mathrm{p}}^2$. In view of this, we use the substitution $\partial g^{\mathrm{M}}/\partial t=\partial\tilde{g}^{\mathrm{M}}/\partial t+\tilde{f}$, where $\tilde{f}=c^{-2}(\tilde{n}^2-n'^2)\,\partial S/\partial t$, in the right-hand side of (47). The average energy density in (47) has the form $\langle u\rangle=\langle S\rangle/v_{\mathrm{p}}=E_0^2\cos(\varphi)/(2Zv_{\mathrm{p}})=E_0^2/(2v_{\mathrm{p}}^2\mu_0\mu)$. Likewise, we obtain the third form of momentum as $\langle\tilde{g}^{\mathrm{M}}\rangle=n'^2\langle S\rangle/c^2=n'^2E_0^2/(2v_{\mathrm{p}}\mu_0\mu)$. Its rate of transfer is therefore given by

$$\tilde{v}_{\mathrm{m}}^{\mathrm{M}}=\frac{\langle u\rangle}{\langle\tilde{g}^{\mathrm{M}}\rangle}=\frac{c^2}{n'^2 v_{\mathrm{p}}}=v_{\mathrm{p}}(\omega). \tag{48}$$

In this case, the balance equation acquires an additional term similar to the Abraham force $\tilde{f}$ acting on the medium. But the emergence of this force (in complete analogy to the Abraham force) should not generate any objections concerning violation of the balance equations because $\langle\partial S/\partial t\rangle=0$ for any phase shift φ. Consequently, neither the above force nor the Abraham force transfer any momentum to matter on average. In a nonconducting medium, $\tilde{f}=0$ and $\tilde{g}^{\mathrm{M}}=g^{\mathrm{M}}$, while the energy and momentum propagate at the frequency-independent phase velocity $v_{\mathrm{p}}=\omega/\beta=c/\tilde{n}$. Moving the divergence part $\tilde{g}^{\mathrm{M}}$ to the left-hand side of (47), we obtain another balance equation: $\langle u-S/v_{\mathrm{p}}\rangle=0$, which is an identity. The remaining zero term $\langle\tilde{f}\rangle$ in the right-hand side is the averaged force acting on the medium, but it does not transfer any momentum per period on average. We note that using the traditional Minkowski vector momentum density leads to no physically clear expression for the momentum transfer rate in a conducting medium. But if there is a localized external electric current, and $\sigma=0$, then the introduction of $\mathbf{g}^{\mathrm{M}}$ is more convenient and leads to the total momentum transfer rate $v_{\mathrm{m}}=v_{\mathrm{p}}=v_{\mathrm{g}}=v_{\mathrm{e}}=c/\tilde{n}$, which is trivial because there is no dispersion in this case. In fact, the EMT is traditionally defined in media in this manner [3–5].

The 'correct' Abraham momentum density is defined in many papers as $g^{\mathrm{A}}(t,z)=S(t,z)/c^2$ [3–5]. This produces the Abraham bulk force $f^{\mathrm{A}}=c^{-2}(\varepsilon'\mu-1)\,\partial S/\partial t$. This definition is considered to be more strict, although Minkowski's definition is more convenient in the electrodynamics of continuous media [3, 5], because it complies with experiments on pressure by light. We rewrite balance equation (47) in the form

$$\partial_z\left(\frac{\varepsilon_0\varepsilon' E^2}{2}+\frac{\mu_0\mu H^2}{2}\right)=-\partial_t g^{\mathrm{A}}-f^{\mathrm{A}}-\mu_0\mu\sigma S. \tag{49}$$

The change in the momentum density per second, affecting matter, is now equal to $(2\alpha\omega\varepsilon'\mu/\beta)S/c^2$. This is the contribution of the first and second terms in the right-hand side of (49). In fact, this contribution does not compensate the last term because $\tilde{n}\neq n'$. Obviously, no such compensation is achieved if we use the first or second term in the right side of (49) individually. Nor does g^{A} used as a momentum density lead to working out some sort of definition of the speed of momentum density propagation, which would not exceed c in nonzero-dispersion media. The only exception is the case of a nonconducting nondispersive medium with $\tilde{n}=n'$. Because

$$c^{-2}(\varepsilon'\mu-1)\,\partial_t S=-v_{\mathrm{p}}c^{-2}(\varepsilon'\mu-1)\,\partial_z S,$$

we can add the divergence component f^{A} to the flux density and find

$$\begin{aligned} v_{\mathrm{m}}^{\mathrm{A}} &= \frac{\langle u - v_{\mathrm{p}}(\varepsilon'\mu - 1)S/c^2\rangle}{\langle g^{\mathrm{A}}\rangle} \\ &= \frac{c^2\big(\langle S\rangle/v_{\mathrm{p}} - v_{\mathrm{p}}(\varepsilon'\mu - 1)\langle S\rangle/c^2\big)}{\langle S\rangle} \\ &= c\tilde{n} - v_{\mathrm{p}}(\tilde{n}^2 - 1) = v_{\mathrm{p}} = \frac{c}{\tilde{n}}\,. \end{aligned}$$

In this case, therefore, the field momentum is transferred at the phase velocity, which is less than c and is the same as the velocity of energy transfer in nondispersive media.

We calculate the momentum transferred to the medium per unit time. For this, we rewrite the balance equation in the form

$$\begin{aligned} \partial_z\left(\frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2}\right) &= -\partial_t g^{\mathrm{A}} - \mu_0\mu\sigma S \\ &+ \mu_0(\mu - 1)\sigma S - c^2(\varepsilon' - 1)H\,\partial_t E - c^2(\mu - 1)E\,\partial_t H\,. \quad (50) \end{aligned}$$

Equation (50) was also obtained from Maxwell's equations when the medium was taken into account in the form of polarization currents, electric $J_{Px}^{\mathrm{e}} = \varepsilon_0(\varepsilon' - 1)\,\partial_t E$ and magnetic $J_{Py}^{\mathrm{m}} = \mu_0(\mu - 1)\,\partial_t H$. This approach is rather typical of microscopic electrodynamics [7, 8], but the description becomes complicated. The Abraham force does not give an exhaustive description of the impact on the medium, while the last three terms in (50) correctly reflect this impact. Namely, $f_\sigma^{\mathrm{L}} = \mu_0\mu\sigma S = \sigma EB$ is the Lorentz force acting on the conduction current, $f_{P\mathrm{e}}^{\mathrm{L}} = c^{-2}(\varepsilon' - 1)H\,\partial_t E = J_{Px}^{\mathrm{e}} B$ is the Lorentz force acting on the current of electric polarization of the medium, and $f_{P\mathrm{m}}^{\mathrm{L}} = c^{-2}(\mu - 1)E\,\partial_t H = D_x J_{Py}^{\mathrm{m}}$ is the Lorentz force acting on the magnetic polarization current of the medium (here, $B = \mu_0\mu H$). For the first of these forces, we obtain the average value

$$\langle f_\sigma^{\mathrm{L}}\rangle = \mu_0\mu\sigma\langle S\rangle = \frac{\sigma E_0^2\beta}{2\omega} = \frac{n'\sigma E_0^2}{2c}\,.$$

For the second and third forces, we have $\langle f_{P\mathrm{e}}^{\mathrm{L}}\rangle = -(\varepsilon' - 1)E_0^2\sigma/(4n'c)$ and $\langle f_{P\mathrm{m}}^{\mathrm{L}}\rangle = (\mu - 1)E_0^2\sigma/(4n'c)$. Typically, $n' \gg 1$, and hence the first term plays a critical role in the transfer of momentum to mobile charges. These charges are scattered by molecules and atoms of matter and thereby transfer momentum to matter. The energy transferred per unit surface area per unit time is $S = v_{\mathrm{p}} u$. The power lost per unit volume, averaged over one period, is $\sigma E_0^2/2$. On the other hand, the same quantity can be redefined as $\langle \partial u_\sigma/\partial t\rangle = v_{\mathrm{p}}^{-1}\langle \partial S_\sigma/\partial t\rangle$, which is equivalent to $2\alpha\langle S\rangle = \alpha E_0^2\cos(\varphi)/Z = \sigma E_0^2/2$. The lost energy corresponds to the transferred momentum $\langle \partial g_\sigma/\partial t\rangle$. For the three definitions introduced above, we thus have

$$\begin{aligned} \langle \partial_t g_\sigma^{\mathrm{M}}\rangle &= 2\alpha\tilde{n}^2 v_{\mathrm{p}}\,\frac{E_0^2\cos(\varphi)}{c^2 Z} = \sigma\tilde{n}^2\,\frac{E_0^2 v_{\mathrm{p}}}{2c^2}\,, \\ \langle \partial_t g_\sigma^{\mathrm{A}}\rangle &= 2\alpha v_{\mathrm{p}}\,\frac{E_0^2\cos(\varphi)}{c^2 Z} = \sigma\,\frac{E_0^2 v_{\mathrm{p}}}{2c^2}\,, \\ \langle \partial_t \tilde{g}_\sigma^{\mathrm{M}}\rangle &= 2\alpha n'^2 v_{\mathrm{p}}\,\frac{E_0^2\cos(\varphi)}{c^2 Z} = \sigma\,\frac{E_0^2}{2v_{\mathrm{p}}}\,. \end{aligned}$$

Because free charges satisfy the equation of motion

$$Ne\dot{x}(t) = \sigma E_0\cos(\omega t - \beta(\omega)z)\exp(-\alpha(\omega)z)\,,$$

it may seem that matter has an oscillating x-component of momentum. But this is not true. Because we assume the medium to be electrically neutral, matter always contains charges moving in the opposite direction, and the oppositely directed momentum is transferred to them (in metals, for instance, this is the crystal lattice). A question may arise: what created the momentum of the field generated by a dipole or a system of dipoles (in this case, a plane of dipoles) if the sources had none? The answer is trivial: the source excited two waves directed along the positive and negative directions of z and having equal and opposite momenta.

Elementary field momentum quanta are transferred in matter by photons at the speed of light c between the acts of scattering by particles of matter. Momentum transfer at the phase velocity is a collective result of the above elementary events, with the corresponding phase delays and interference taken into account. Formally, the effect of matter can be described by introducing polarization currents. In the case of nondissipative media (i.e., with zero dispersion ε' and μ), the shift between them and the fields is $\pi/2$; we note that these currents do not transfer momentum to the medium: only the conduction current does. Balance equation (50) (and a similar equation for the balance of power), which involves the density $u_0 = (\varepsilon_0 E^2 + \mu_0 H^2)/2$, is also inconvenient for determining the effects of transfer by a monochromatic wave in the medium.

Balance relations involving polarization currents and external currents that create the field are very efficient in the case of nonstationary excitation [16]. In this case, the energy and momentum accumulated in a certain volume depend on the preceding history of the process of creating the field. For example, in plasma, the energy and momentum of the FMS should be taken into account. For plasma, this means taking the kinetic energy of oscillations of charged particles and the momentum transferred to particles into account [15, 16]. Solving the equations of motion of the FMS allows in principle finding the field and matter EMT at any instant of time, and hence instantaneous values $\mathbf{v}_{\mathrm{e}}(\mathbf{r}, t)$ and $\mathbf{v}_{\mathrm{m}}(\mathbf{r}, t)$. In the macroscopic electrodynamics of a continuous medium, averaging over a physically infinitesimal volume (homogenization) leads to matter equations that are analogues of the equations of motion. For example, we write one-dimensional nonstationary equations (24) in a homogeneous medium taking only the temporal (frequency) dispersion into account:

$$-\frac{\partial H}{\partial z} = \frac{\partial D}{\partial t} + J^{\mathrm{e}}\,, \qquad -\frac{\partial E}{\partial z} = \frac{\partial B}{\partial t} + J^{\mathrm{m}}\,. \quad (51)$$

All quantities in (51) are functions of time and a single coordinate z, which simplifies the analysis. We take the matter equations that determine the dispersion in the form

$$\begin{aligned} D(z, t) &= \varepsilon_0\int_0^t \varepsilon(z, t - t')\,E(z, t')\,\mathrm{d}t'\,, \\ B(z, t) &= \mu_0\int_0^t \mu(z, t - t')\,H(z, t')\,\mathrm{d}t'\,. \end{aligned} \quad (52)$$

Without loss of generality, we assume that the field is zero at $t = 0$. If sources are in the plane $z = 0$, then at $t > 0$, the field is located within $|z| \leqslant ct$. We rewrite the dielectric permittivity as

$$\varepsilon(z,\omega)=\int_{-\infty}^{\infty}\varepsilon(z,t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t=\int_0^t\varepsilon(z,t')\exp(-\mathrm{i}\omega t')\,\mathrm{d}t'\,.$$

This takes into account that $\varepsilon(z,t)=0$ for $t<0$. It is most convenient to use the plasma model. For waves in a plasma, we can introduce the Landau damping and exclude the pole at $\omega=0$. Writing the balance equation for momentum, we obtain

$$\partial_z u_0(z,t)=-\big[\mu_0 H\,\partial_t D(z,t)+\varepsilon_0 E\,\partial_t B(z,t)\big]-\mu_0 HJ^{\mathrm{e}}-\varepsilon_0 EJ^{\mathrm{m}}\,. \tag{53}$$

The last two terms are responsible for creating the field and matter momenta. However, they correspond to the Lorentz force in the vacuum, not in the medium. Assuming that a quantity of the type of $\partial_t g(z,t)$ should be in the square brackets in the right side of (53), after integration we obtain

$$\begin{aligned}g(z,t)&=\partial_{t'}^{-1}\big(\varepsilon_0 E(z,t')\,\partial_{t'}B(z,t')+\mu_0 H(z,t')\,\partial_{t'}D(z,t')\big)\\&=\varepsilon_0\varepsilon(z,0)\,E(z,t)+\mu_0\mu(z,0)\,H(z,0)\\&+\partial_{t'}^{-1}\big(\varepsilon_0\,\partial_t\varepsilon(z,t-t')\,E(z,t')+\mu_0\,\partial_t\mu(z,t-t')\,H(z,t')\big)\\&=\mu_0 H(z,t)\,D(z,t)+\varepsilon_0 E(z,t)\,B(z,t)\\&-\partial_{t'}^{-1}\big(\mu_0\,\partial_{t'}H(z,t')\,D(z,t')+\varepsilon_0\,\partial_{t'}E(z,t')\,B_y(z,t')\big)\,,\end{aligned} \tag{54}$$

which implies that this z-component at an instant t depends not only on the values taken by the fields at this instant but also on all their previous values. For a wave in a homogeneous plasma, we have $\mu(z,t)=\delta(t)$, $\varepsilon(z,t)=\delta(t)+(\omega_{\mathrm{p}}^2/\omega_{\mathrm{c}})[1-\exp(-\omega_{\mathrm{c}}t)]\exp(-\omega_{\mathrm{L}}t)$, where the last exponential determines the Landau damping, and we can take $\omega_{\mathrm{L}}\to 0$ in the final results. Then, $\varepsilon(\omega)=1+\omega_{\mathrm{p}}^2/(\omega^2-\mathrm{i}\omega\omega_{\mathrm{c}})$. Taking the electric current density in the form $J^{\mathrm{e}}=I[1-\exp(-t/\tau)]\sin(\omega t)$, we can solve Eqns (51) and find density (54) for large values of t when the process becomes quasistationary. But expression (54) does not correspond to momentum in a continuous medium. For the vacuum, it becomes trivial: $g=g^{\mathrm{A}}=S/c^2$. Continuous media agree better with the density $g=g^{\mathrm{M}}=DB$ and the momentum balance equation, which for (51) has the form

$$\partial_z\Sigma+\partial_t g^{\mathrm{M}}=-B_y J_x^{\mathrm{e}}-D_x J_y^{\mathrm{m}}\,. \tag{55}$$

Here, $\partial_z\Sigma=B\,\partial_z H+D\,\partial_z E$, $g^{\mathrm{M}}=DB$, and Σ is the momentum flux density in the direction z, while the right-hand side of (55) includes real Lorentz forces describing the collective effect of motion of all charges in the continuous medium. It appears that relation (55) works better for continuous media, because with zero dispersion, i.e., with the matter relations $D=\varepsilon_0\varepsilon E$ and $B=\mu_0\mu H$, we have $g^{\mathrm{M}}=\varepsilon\mu S/c^2$, $\Sigma=u=(\varepsilon_0\varepsilon E^2+\mu_0\mu H^2)/2$, and $v_{\mathrm{m}}^{\mathrm{M}}=c/\sqrt{\varepsilon\mu}=v_{\mathrm{p}}$, and therefore we have obtained a generalization of the momentum density in Minkowski's form. The difficulty in the nonstationary case lies in the fact that the calculation of $g^{\mathrm{M}}=DB$ and Σ requires the knowledge of the entire previous history of the EM process; in addition, Σ has to be found by solving a differential equation. In the one-dimensional case of diffraction on a plate, integration over z is sufficient. In contrast to this, the introduction of the Abraham force in (55) as an addition to the Abraham momentum density (in order to arrive at the Minkowski momentum) only complicates the analysis.

We also give a similar balance equation for the power $p=\partial_t w=\partial_t(u+q)$: $\partial_t w+\partial_z S=-EJ_x^{\mathrm{e}}-HJ_y^{\mathrm{m}}$, which corresponds to (51). Here, $q(z,t)$ is the heat or the dissipated work of field sources, which can always be calculated. In the simplest case of a conducting medium, we have $q(z,t)=\sigma\,\partial_{t'}^{-1}(E^2(z,t'))$, whence the expression for the field and matter energy density is

$$u(z,t)=\int_0^t\big(E(z,t')\,\partial_{t'}D(z,t')+H(z,t')\,\partial_{t'}B(z,t')\big)\,\mathrm{d}t'-q(z,t)\,. \tag{56}$$

7. Conclusion

Representations for the energy and momentum densities in dispersive media were obtained in general and in the special case of a monochromatic plane wave with dispersion determined by the conduction current only. It was shown that in this case, the energy and momentum are transferred at the phase velocity, which is lower than the speed of light c in the vacuum, but the group velocity may exceed the speed of light. We considered several forms of representation of the momentum density and showed that Minkowski's form is preferable in this case to Abraham's form. The result for the energy transfer rate was obtained in several ways, for instance, by using the dispersion law for a gas of oscillators. The results were generalized to the case of multiple resonance frequencies of oscillators and to an internal molecular field. It is essential here that the calculation of polarizability uses the equation of motion of particles, which is a first-order equation. In this case, the potential energy in matter does not accumulate, natural oscillations are not excited, and the kinetic energy is zero.

The results we report above can be generalized to the case of conducting polar dielectrics, for example, water with an admixture of conduction ions. Retardation in ideally distilled water varies from 9 to about 1 (we neglect the effect of resonances in the infrared and ultraviolet parts of the spectrum), while the energy transfer velocity equals the phase velocity [16]. Retardation and loss coefficients in ion-containing conducting water tend to infinity in the range of ultralow frequencies. In seawater ($\sigma=4$ Sm m^{-1}), the displacement current equals the conduction current at a frequency about 900 MHz, and behaves like a metal at much lower frequencies. The knowledge of the energy and accordingly the momentum transfer velocity is important for sending messages, for example, in communicating with submarines at ultralow frequencies. The signals are transmitted in this case at a velocity approximately equal to the phase velocity, while the group velocity is twice as large. For example, a pulse with the carrier frequency 1 kHz and length 2 ms (two periods) reaches the submarine at a depth of 100 m from the surface in 2 ms (retardation 6000) and is detected at the maximum of the envelope (we ignore the response time of the detector), while the greatly smeared front of the pulse becomes several times broader, arrives in 1 ms, and is undetectable [38].

The problem of the form of the EMT in electrodynamics of continuous media and of the corresponding densities is known as the Abraham–Minkowski controversy (for more details, see [49–54]). Various model examples that are often given as arguments either in favor of some forms of the EMT or against it, often contain inaccuracies; we have no space to analyze these arguments in this paper (see also [12]). The vast majority of discussions of this issue known to the author use static matter equations $\mathbf{D} = \varepsilon_0\varepsilon\mathbf{E}$ and $\mathbf{B} = \mu_0\mu\mathbf{H}$. The main objection against Minkowski's EMT is that it is nonsymmetric and allegedly fails to agree with the momentum conservation law [4]. In both forms of the EMT, the component T_{44} is defined as in statics, i.e., as $T_{44} = u = (\varepsilon_0\varepsilon E^2 + \mu_0\mu H^2)/2$, whereas in dynamics, the field interacts with matter, and vice versa, and hence expressions like (56) should be used [15, 16]. It is not clear why the energy density in a medium should depend on the parameters of the medium, whereas the momentum density $\mathbf{g}^{\mathrm{A}}$ should be independent of them and should be the same as in the vacuum. Clearly, the interaction between the field and matter can lead to an asymmetry in the EMT of the FMS [13]. As we have shown above, the momentum density depends on the temporal process and therefore on its previous history. Apparently, a specific algebraic form of the EMT can only be discussed for harmonic processes in media with simple dispersion laws, when these processes were obtained by a limit transition from the corresponding quasi-monochromatic processes of field creation in which the components $\hat{T}$ averaged over one period reach the stationary level. The total EMT for the FMS in the case of an arbitrary (temporal and spatial) dispersion has not been constructed yet, and hence the issue of demanding that it be symmetric remains unanswered. But if the field and matter tensors are independently symmetric, then there is no interaction between matter and the field in the sense of energy and momentum exchange.

As an example, we can show the implementation of the momentum conservation law for Minkowski's form. Let a planar extended quasistationary (quasimonochromatic) train or wave packet of length l with a rectangular envelope, a carrier frequency ω, and constant ε and μ approach the boundary of a layer of thickness d at the instant $t_0 = 0$. The quasistationary nature means that $l \gg \lambda_0 = 2\pi/k_0 = 2\pi c/\omega$. This property is required for using the quasimonochromatic values of the quantities. The wavelength in the layer is $\lambda = \lambda_0/n$, where $n = \sqrt{\varepsilon\mu}$. We assume for simplicity that the thickness d is adjusted such that by the instant t_1, the entire wave train has gone into the layer and filled it up. Because the velocity of motion is $v_{\mathrm{p}} = c/n$, we have $t_1 = d/v_{\mathrm{p}} = nd/c$ for this instant of time. For this, the following should hold: $l = nd$. For simplicity, we adjust the large length l such that it equals an integer number M of wavelengths: $l = M\lambda$, $M \gg 1$, and in addition $d/\lambda = M$. A reflected pulse is also formed. At the instant t_1, it is located in the region $-l \leqslant z \leqslant 0$, just as the incident momentum is at the instant t_0. The coefficient of reflection from the interface for the electric field of the normally incident monochromatic plane wave is expressed as $R = (\rho_0 - 1)/(\rho_0 + 1)$, where $\rho_0 = \sqrt{\mu/\varepsilon}$ is the impedance normalized to Z_0; for the coefficient of penetration into the bulk, we have $T = 2\rho_0/(\rho_0 + 1)$. The incident wave has form (41) for $\alpha = 0$ and $\varphi = 0$.

Next, we consider the balance for unit area and average values of momentum densities. The field momentum at the instant t_0 was $G_0^{\mathrm{M}} = g^{\mathrm{M}} l = \langle S_0\rangle l/c^2$ in the direction of the z axis, while the plate momentum was zero. First, we consider the ideal case of a matched (stealth) plate $\varepsilon = \mu$. In this case, there is no reflection and no momentum is transferred to the plate. When the momentum density is integrated over z or after averaging over one period, the factor $1/2$ arises at the amplitudes of the variables. Correspondingly,

$$g^{\mathrm{M}} = \frac{Z_0 n^2 E_0^2}{2c^2} = \frac{\langle u\rangle n}{c}\,,$$

where $\langle u\rangle = \varepsilon_0\varepsilon E_0^2/2$. At the time $t_1 = l/c$ of complete filling of the plate, the field momentum is

$$G_1^{\mathrm{M}} = g^{\mathrm{M}} d = \frac{dn^2\langle S_1\rangle}{c^2} = \frac{dn\langle u_1\rangle}{c} = nG_0^{\mathrm{M}}\,.$$

Once the wave train has completely left the plate, i.e., when $t_2 \geqslant 2t_1$, we again have $G_1^{\mathrm{M}} = G_0^{\mathrm{M}}$. The Minkowski momentum is 'sort of' nonconserved, because the plate has not moved. This 'momentum nonconservation' in Minkowski's form was formulated as the main argument against it [4] and in favor of Abraham's momentum. We consider what happens to Abraham's momentum. At t_0, we have: $G_0^{\mathrm{A}} = G_0^{\mathrm{M}} = \langle S_0\rangle l/c^2 = \langle u_0\rangle l/c$, and at t_1, correspondingly, $G_1^{\mathrm{A}} = \langle S_1\rangle d/c^2 = \langle u_0\rangle d/c = G_1^{\mathrm{M}}/n^2$.

We have arrived at a paradox: neither Minkowski's momentum nor Abraham's momentum is conserved! We note that for the former, the additional momentum first appears but then disappears, while for the latter, it starts with part of the momentum disappearing but then reappearing! What is going on here? Whenever we produce a paradox, we need to find out where there was a covert substitution or improper use of concepts. Similar reasoning is given in [4] for a train and a plate with $\mu = 1$ and $n = \sqrt{\varepsilon}$, and it was assumed that the plate was perfectly matched using an antireflective layer. Although a plate cannot be perfectly matched, it is still possible to obtain a sufficiently low reflection coefficient for a monochromatic process by preparing a multi-layer coating or a nonuniform antireflecting layer. Without a doubt, they would have to be taken into account. However, we consider a train with rectangular fronts containing all frequencies, and a nonstationary process. The main spectral intensity definitely concentrates near the carrier frequency; nevertheless, the plate would gain some small momentum. If $\varepsilon = \mu$, then nothing is reflected, but this condition can only be met in dispersive artificial media in a fairly narrow band, and spectral permittivities would always be complex. This means that a real plate made of a stealth material would also gain a certain momentum. In what follows, we disregard all this. We note that formula (1.13) in [4], which was interpreted as demonstrating the conservation of Abraham's momentum, in reality demonstrates its violation: $G = e/nc$. Here, e is the total energy of the train, which remains the same regardless of its position. In the vacuum, $e = \langle u_0\rangle l$. In a medium, $e = \langle u\rangle d$, and, because the train is compressed by a factor of n, $\langle u\rangle = n\langle u_0\rangle$, which is immediately apparent from (41). Also, the total momentum is $G = e/c$ in both cases.

How can we resolve the paradox? Clearly, the local balance equation in our case of no Lorentz force has the form

$$\frac{\partial u}{\partial z} = -\frac{\partial g^{\mathrm{M}}}{\partial t}\,, \qquad (57)$$

which is implied by (41). If the local balance holds, then the global (integral) balance also holds, and it must be thoroughly calculated. We remark in this connection that $u(z,t)$ displays steps $u_0(0,t)(n-1)$ at $0 \leqslant t \leqslant t_1$ and $u_0(d,t)(1-n)$ at $t_1 \leqslant t \leqslant t_2$. In these intervals, terms with delta functions emerge in (57): $u_0(0,t)(n-1)\delta(z)$

and $U_0(d,t)(1-n)\delta(z-d)$. The momentum at the time instant t_1 should be calculated as

$$G_1^{\mathrm{M}} = -\int_0^{t_1}\left[\left[\int_0^d u(z,t)\,\mathrm{d}z + u_0(0,t)(n-1)\right]\mathrm{d}t\right.$$
$$= nG_0^{\mathrm{M}} - (n-1)G_0^{\mathrm{M}} = G_0^{\mathrm{M}}\,.$$

This conclusion concerning the nonconservation of G^{M} in [4] is based on the constancy of the center-of-mass velocity of the system 'fixed plate–moving EM-train' (the photon). The plate has zero velocity, while the wave train (or photon) moving in one direction has zero mass [40, 41]! We note, however, that a 'photon mass' is introduced in [4] using the formula $m = e/c^2$ and its momentum is $G = mv_{\mathrm{p}} = e/(cn)$! Speaking of a train or a photon, we should bear in mind that the energy density increases by a factor of n, and therefore the number of photons increases as well, while the rate of energy transfer decreases. This means that the plate contains photons propagating in both directions at the speed of light c between acts of elementary interactions with particles of matter, while a photon with the momentum $n\hbar\omega/c$ is essentially a quasiparticle [3]. The complete-interference result for a macroscopic wave with a sufficiently high energy containing many photons is implied by the cancellation theorem, which states that a wave in a medium propagates in the forward direction at the phase velocity. In the general case of reflection from a nonabsorbing plate at $t > t_2$, the energy balance becomes $1 = |R|^2 + |T|^2$, where we introduce the total reflection coefficient

$$R = \frac{(\rho_0^2 - 1)\tan\theta}{(\rho_0^2 + 1)\tan\theta - 2\mathrm{i}\rho_0}$$

and the transmission coefficient

$$T = \left[\cos\theta + \frac{\mathrm{i}\sin\theta(Z^2+1)}{2Z}\right]^{-1},$$

with $\theta = \gamma l = 2\pi l/\lambda_0$, and hence the reflected momentum is $G_{\mathrm{r}}^{\mathrm{M}} = e|R|^2/c$ and the transmitted momentum is $G_{\mathrm{t}}^{\mathrm{M}} = e|T|^2/c$, i.e., the momentum transferred to the plate is $e(1 + |R|^2 - |T|^2)/c$. If $R = 0$, then $T = 1$ and the plate is stationary. But if $\sigma \neq 0$, then γ is a complex quantity and the energy balance is $|R|^2 + |T|^2 < 1$, i.e., the layer receives part of the momentum contained in it. If $|R|$ is very small or very large, then the entire traveling momentum is transferred to the plate. If $\sigma \to \infty$, we have $R \to -1$ and $T \to 0$, and the perfectly reflecting plate receives twice the momentum. Here, we again applied stationary formulas to quasistationary processes instead of solving a more complex nonstationary problem, which is justified for a long train. In the stationary case, it makes sense to speak only about the pressure $\langle u\rangle(1 + |R|^2 - |T|^2)/c$ or about the momentum transferred per second, and these formulas are already accurate.

As regards the momentum G^{A}, it is conserved in the case under consideration if the momentum due to the Abraham force is added to it. This force arises in a nondispersive medium only as the train enters the plate or exits from it [5]! It was the need to add a certain momentum to G^{A} to keep the validity of the conservation law that provided the main argument against Abraham's momentum. Another argument against Abraham's momentum in a medium is that the Abraham force is not precisely equal to the Lorentz forces acting on polarization currents.

As an example, we consider a system of two atoms with masses m_1 and m_2, stationary in the laboratory reference frame at $t < 0$. The energy of the system is $e = e_1 + e_2$, where $e_1 = m_1c^2$ and $e_2 = m_2c^2$. The first atom, in an excited state with the excitation energy $\hbar\omega$, is located at the origin, and the second at a point $(0,0,z)$, $z > 0$. The excited atom is stationary (as a quantum particle) in the sense that its wave function is time independent [or rather, has a factor $\exp(-\mathrm{i}e_1t/\hbar)$]. We note that the probability of detection of an atom at the origin is maximal and that the uncertainty relation $\Delta z\Delta p_z \geqslant \hbar/2$ holds. The same is true for the second atom at z. Assuming that there is no interaction, we write the total wave function as a product of atomic wave functions, which is independent of time until the instant $t_0 = 0$. At $t_0 = 0$, the first atom emits a photon with the energy $\hbar\omega$ and momentum $\mathbf{p} = \mathbf{z}_0 p = \mathbf{z}_0\hbar\omega/c$. The atom receives the momentum $-\mathbf{p}$ and moves to the left, which means that its wave function begins to depend on time. This wave function takes the form of a wave packet, an eigenfunction of the momentum operator. This wave packet shifts to the left at a rate that satisfies the relation

$$\frac{v_1/c}{\sqrt{1-(v_1/c)^2}} = \frac{p}{m_1'c}\,,$$

where

$$m_1' = \sqrt{\frac{(e_1 - \hbar\omega)^2}{c^4} - \frac{p^2}{c^2}} = m_1\sqrt{1 - \frac{2p}{m_1c}}$$

is the change in the mass as a result of the interaction. The excitation energy $\hbar\omega$ of the atom plays the role of the internal nonelectromagnetic energy creating the field.

Let a photon at a point $z = ct_1$ be absorbed by another atom at an instant t_1. Having gained this momentum $\mathbf{p}$, the atom lifts to an excited state with the energy

$$e_2 + \hbar\omega = \frac{m_2'c^2}{\sqrt{1-(v_2/c)^2}}\,,$$

and its mass changes to $m_2' = m_2\sqrt{1 + 2p/(m_2c)}$. The speed of the atom also follows from the relation

$$\frac{v_2/c}{\sqrt{1-(v_2/c)^2}} = \frac{p}{m_2'c}\,.$$

Until the instant t_0, the system of two noninteracting atoms had zero momentum, the mass $m_1 + m_2$, and the energy $(m_1 + m_2)c^2$. In the time interval $t_0 < t < t_1$, the mass of the atoms becomes equal to $m_1' + m_2$ (lower than their initial mass); the photon mass is zero, but the mass of the entire FMS has not changed because its total momentum is zero. In addition, energy and momentum are conserved, and the position of the photon is undefined. The photon localizes (and the field vanishes) at the time of absorption (interaction) at the point z. The mass of the whole system, $m_1 + m_2$, remains unchanged and continues to stay unchanged, and the total momentum continues to be zero. The total energy is also conserved,

$$e = \frac{m_1'c^2}{\sqrt{1-(v_1/c)^2}} + \frac{m_2'c^2}{\sqrt{1-(v_2/c)^2}}\,,$$

the total mass of the atoms being $m_1' + m_2' < m_1 + m_2$. The mass defect arises because, as a result of the interaction, the atoms gained oppositely directed momenta. Nevertheless, the energy, mass, and momentum of the entire closed FMS remains unchanged. If $2p/(m_i c) \ll 1$, $i = 1, 2$, we can use expansion in a small parameter. We immediately see that in the first order, the mass defect is zero.

This simple qualitative example based on the representation of a nonstationary interaction in terms of stationary processes with two point interactions is included to show the need to integrate the external field sources in the balance, including the case where it has a nonzero momentum. This example is also useful in connection with the example in [4]. Of course, rigorous analysis must be based here on nonstationary approaches of quantum electrodynamics, e.g., on the method of multiparticle spacetime Green's functions [55]. This analysis if very cumbersome, unfortunately. Wave propagation in a continuous medium at the microscopic level is based on many such interaction events and the propagation velocity depends on the lifetimes of atoms in excited states, i.e., on the nature of scattering: whether it is elastic or not, and to what extent. In such a wave, there is always a predominant photon flux in the direction of the energy flow, although oppositely moving photons are also possible. Absolutely elastic scattering of photons in the plasma of free charge carriers occurs in the limit of infinite conductivity, i.e., as $\omega_p^2/\omega_c \to \infty$, or $\omega \to 0$, which corresponds to the reflection from a perfect electric wall. As follows from the above, photon exchange processes are always at least quasistationary. Therefore, the monochromatic wave is a convenient abstraction, which cannot be reproduced in experiments in principle. In a macroscopic description, the wave creates polarization currents, which in fact sustain its propagation.

In conclusion, we remark that the real dispersion of specific substances is very complicated. We would need to consider the internal molecular or crystalline structure, several (or several dozen) natural resonance frequencies, and spatial dispersion, and also to apply a nonstationary approach. This means that the field–matter EMT is not determined exclusively by the values of fields at the current instant of time; it is determined by the previous history of creation of the field by the sources and correspondingly the previous history of the effects of sources and field on matter. If there are no sources in the volume, then this prehistory must be taken into account from the moment of entering the field created by external sources into the specified volume. Therefore, we should take the momentum density in the form $\mathbf{g}^{\mathrm{M}} = \mathbf{D} \times \mathbf{B}$ and take (1) into account, and define expressions like (56) for the density [15, 16]. We note that the total EMT for the FMS is not symmetric. If dispersion is absent, the total EMT transforms into Minkowski's EMT. For plane waves, the quantity $\partial_z u$ defined by (56) does not provide the momentum flux density. In the general case, using matter equations only allows separating the momenta of the field and matter and the fluxes corresponding to them. Apparently, such separation carried out without solving the equations of motion cannot be successful for the energy flux density. The reason is that the power density of polarization currents in the right-hand side of (9), containing time derivatives, rather more refers to the total power density than it represents the divergence of a vector. Furthermore, the above vector is not uniquely defined. Moreover, in the general case, the energy of the interacting FMS cannot be separated into the energies of field and matter without taking their interaction energy into account. The main objection to Minkowski's EMT stems from it being nonsymmetric. The symmetry restriction arises from the need for uniqueness in the definition of the EMT, plus the condition that the momentum tensor must be determined in terms of the EMT using the standard formulas of mechanics, resorting to the relation between the components of the momentum and those of the EMT, the same as for the field in the vacuum [56, p. 107]. Obviously, this EMT represents a tensor field in the vacuum and is identical to Abraham's tensor. In our case of nonstationary excitation in the medium, the EMT is uniquely defined and an additional condition is not required. In contrast to this, the angular momentum tensor of the FMS must be determined separately. We must also take into account whether matter had an angular momentum before the creation of the field, as well as include the moment transferred from the field to external sources in the process of field generation.

Expressions for the EMT given above depend on the form of matter equations. If matter equations were given, for example, in Casimir's form [18], different relations would result. Recently, the problem gained importance regarding the EMT and the corresponding densities (and velocities) in artificial media with spatial dispersion, including bi-anisotropic left-handed media (see, e.g., an unsuccessful attempt at such an analysis in [57]), whose models are extremely complicated. In a number of cases, however, dispersion laws in certain frequency ranges can be described by simpler relations (models), and we did use this in the present article.

A recent publication [58] showed the relativistic covariance of Minkowski's EMT, which is more evidence in its favor. The EMT for the FMS is nonsymmetric because the system is not closed and becomes closed only if balance relations of external sources creating the field are taken into account. In the case of a monochromatic plane wave, they are at infinity and cannot be taken into account.

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