LETTERS TO THE EDITORS

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On the relativistic invariance of the Minkowski and Abraham energy-momentum tensors

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In a paper [1] by one of us (VGV) it was pointed out that the Abraham energy–momentum tensor is not relativistically invariant (as opposed to the Minkowski tensor), and it was argued on this basis that it cannot be conceptually considered a tensor. On this last point, though, no supporting mathematics was provided.

It is perhaps for this reason that the above assertions were challenged by the authors of Ref. [2], and accordingly the purpose of this letter is to present calculations which show that the Minkowski tensor in any inertial frame of reference depends in like manner on field components in the same frame, whereas the Abraham tensor does not.

Let us consider two frames, K and K', where K' moves at the velocity v with respect to K along the Ox-axis. Then, for the transition from K to K', the Lorentz transformations of second-rank tensor components can be written out as ([3], p. 357)

$$
\begin{aligned}
T'_{11} &= \gamma^2\big(T_{11} + \mathrm{i}\beta(T_{14} + T_{41}) - \beta^2 T_{44}\big)\,,\\
T'_{12} &= \gamma(T_{12} + \mathrm{i}\beta T_{42})\,,\\
T'_{13} &= \gamma(T_{13} + \mathrm{i}\beta T_{43})\,,\\
T'_{14} &= \gamma^2(T_{14} - \mathrm{i}\beta T_{11} + \beta^2 T_{41} + \mathrm{i}\beta T_{44})\,,\\
T'_{21} &= (T_{21} + \mathrm{i}\beta T_{24})\,,\\
T'_{22} &= T_{22}\,,\\
T'_{23} &= T_{23}\,,\\
T'_{24} &= \gamma(T_{24} - \mathrm{i}\beta T_{21})\,,\\
T'_{31} &= \gamma(T_{31} + \mathrm{i}\beta T_{34})\,,\\
T'_{32} &= T_{32}\,,\\
T'_{33} &= T_{33}\,,\\
T'_{34} &= \gamma(T_{34} - \mathrm{i}\beta T_{31})\,,\\
T'_{41} &= \gamma^2(T_{41} - \mathrm{i}\beta T_{11} + \beta^2 T_{14} + \mathrm{i}\beta T_{44})\,,\\
T'_{42} &= \gamma(T_{42} - \mathrm{i}\beta T_{12})\,,\\
T'_{43} &= \gamma(T_{43} - \mathrm{i}\beta T_{13})\,,\\
T'_{44} &= \gamma^2\big(T_{44} - \mathrm{i}\beta(T_{14} + T_{41}) - \beta^2 T_{11}\big)\,.
\end{aligned}
\tag{1}
$$

Here, we have introduced the notation

$$
\beta = \frac{v}{c}\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,. \tag{2}
$$

The Lorentz transformations for the fields are as follows ([3], p. 375):

$$
\begin{aligned}
E_x &= E'_x\,,\\
E_y &= \gamma(E'_y + \beta B'_z)\,,\\
E_z &= \gamma(E'_z - \beta B'_y)\,,\\
D_x &= D'_x\,,\\
D_y &= \gamma(D'_y + \beta H'_z)\,,\\
D_z &= \gamma(D'_z - \beta H'_y)\,,\\
B_x &= B'_x\,,\\
B_y &= \gamma(B'_y - \beta E'_z)\,,\\
B_z &= \gamma(B'_z + \beta E'_y)\,,\\
H_x &= H'_x\,,\\
H_y &= \gamma(H'_y - \beta D'_z)\,,\\
H_z &= \gamma(H'_z + \beta D'_y)\,.
\end{aligned}
\tag{3}
$$

The energy–momentum tensor can be written down in the form

$$
T_{ik} = \begin{bmatrix} T_{\alpha\beta} & -\mathrm{i}c\mathbf{g} \\ -\dfrac{\mathrm{i}}{c}\,\mathbf{S} & W \end{bmatrix}. \tag{4}
$$

Here, $T_{\alpha\beta}$ are the spatial components of the tensor, so that α, $\beta = x, y, z$, $\mathbf{g}$ is the field momentum density, $\mathbf{S}$ is the Poynting vector (energy flux density), and W is the field energy density.

The components of the Minkowski energy–momentum tensor are given by ([3], p. 377)

$$
T_{\alpha\beta} = \frac{1}{4\pi}\,(E_\alpha D_\beta + H_\alpha B_\beta) - \frac{1}{8\pi}\,\delta_{\alpha\beta}(\mathbf{ED} + \mathbf{HB})\,, \tag{5}
$$

$$
\mathbf{S} = \frac{c}{4\pi}\,[\mathbf{EH}]\,, \quad \mathbf{g} = \frac{1}{4\pi c}\,[\mathbf{DB}]\,, \quad W = \frac{1}{8\pi}\,(\mathbf{ED} + \mathbf{HB}), \tag{6}
$$

and those of its Abraham counterpart are defined as ([3], p. 357)

$$
\begin{aligned}
T_{\alpha\beta} = {} & \frac{1}{8\pi}\,(E_\alpha D_\beta + E_\beta D_\alpha + H_\alpha B_\beta + H_\beta B_\alpha)\\
& - \frac{1}{8\pi}\,\delta_{\alpha\beta}(\mathbf{ED} + \mathbf{HB})\,,
\end{aligned}
\tag{7}
$$

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$$\mathbf{S}=\frac{c}{4\pi}\,[\mathbf{EH}]\,,\;\;\mathbf{g}=\frac{1}{4\pi c}\,[\mathbf{EH}]\,,\;\;W=\frac{1}{8\pi}\,(\mathbf{ED}+\mathbf{HB})\,. \quad (8)$$

Let us demonstrate that the Minkowski energy–momentum tensor retains its form at transition from one frame of reference to another. As an example, we shall consider the component T_{xx} which, according to Eqn (5), takes the form

$$T_{xx}=\frac{1}{4\pi}\,(E_xD_x+H_xB_x)-\frac{1}{8\pi}\,\delta_{xx}(\mathbf{ED}+\mathbf{HB})\,. \quad (9)$$

From Eqn (4) and the first formula of set (1) one finds

$$T'_{xx}=\gamma^2\left(T_{xx}+\mathrm{i}\beta\left(-\frac{\mathrm{i}}{c}\,S_x-\mathrm{i}cg_x\right)-\beta^2W\right). \quad (10)$$

Substituting the expressions for T_{xx}, S_x, g_x, and W from Eqns (6) and (9) into Eqn (10) yields

$$T'_{xx}=\gamma^2\left\{\frac{1}{4\pi}\,(E_xD_x+H_xB_x)+\frac{\beta}{4\pi}\,\big([\mathbf{EH}]_x+[\mathbf{DB}]_x\big) -\frac{1}{8\pi}\,(1+\beta^2)(\mathbf{ED}+\mathbf{HB})\right\}. \quad (11)$$

Now, writing the scalar and vector products as

$$\mathbf{ED}+\mathbf{HB}=E_xD_x+E_yD_y+E_zD_z+H_xB_x +H_yB_y+H_zB_z\,, \quad (12)$$

$$[\mathbf{EH}]_x+[\mathbf{DB}]_x=E_yH_z-E_zH_y+D_yB_z-D_zB_y\,, \quad (13)$$

and substituting the Lorentz-transformed fields from Eqn (3) in Eqns (12), (13), we obtain

$$\mathbf{ED}+\mathbf{HB}=E'_xD'_x+H'_xB'_x+\gamma^2(1+\beta^2) \times(E'_yD'_y+E'_zD'_z+H'_yB'_y+H'_zB'_z) +2\gamma^2\beta\big([\mathbf{E}'\mathbf{H}']_x+[\mathbf{D}'\mathbf{B}']_x\big)\,, \quad (14)$$

$$[\mathbf{EH}]_x+[\mathbf{DB}]_x=\gamma^2\big\{(1+\beta^2)\big([\mathbf{E}'\mathbf{H}']_x+[\mathbf{D}'\mathbf{B}']_x\big) +2\beta(E'_yD'_y+E'_zD'_z+H'_yB'_y+H'_zB'_z)\big\}\,. \quad (15)$$

Substituting the expressions for E_x, D_x, H_x, and B_x from Eqn (3) and also expressions from Eqns (14) and (15) into formula (11), and collecting similar terms, Eqn (11) becomes

$$T'_{xx}=\gamma^2\left\{\frac{1}{8\pi}\,(1-\beta^2)(E'_xD'_x+H'_xB'_x) -\frac{1}{8\pi}\,(1-\beta^2)(E'_yD'_y+E'_zD'_z+H'_yB'_y+H'_zB'_z)\right\}, \quad (16)$$

or equivalently

$$T'_{xx}=\frac{1}{4\pi}\,(E'_xD'_x+H'_xB'_x)-\frac{1}{8\pi}\,\delta_{xx}(\mathbf{E}'\mathbf{D}'+\mathbf{H}'\mathbf{B}')\,. \quad (17)$$

The implication of this transformation is that the form of the tensor component at hand remains unchanged at the transition to K'.

We now repeat the above for the component T_{xx} in the Abraham form of the energy–momentum tensor. From Eqn (7) it follows that

$$T_{xx}=\frac{1}{4\pi}\,(E_xD_x+H_xB_x)-\frac{1}{8\pi}\,\delta_{xx}(\mathbf{ED}+\mathbf{HB})\,. \quad (18)$$

From the first formula of set (1) and using Eqn (4), one finds

$$T'_{xx}=\gamma^2\left(T_{xx}+\mathrm{i}\beta\left(-\frac{\mathrm{i}}{c}\,S_x-\mathrm{i}cg_x\right)-\beta^2W\right). \quad (19)$$

Substituting the expressions for T_{xx}, S_x, g_x, and W from Eqns (8) and (18) in formula (19) gives

$$T'_{xx}=\gamma^2\left\{\frac{1}{4\pi}\,(E_xD_x+H_xB_x)+\frac{\beta}{4\pi}\,\big(2[\mathbf{EH}]_x\big) -\frac{1}{8\pi}\,(1+\beta^2)(\mathbf{ED}+\mathbf{HB})\right\}. \quad (20)$$

With $\mathbf{ED}+\mathbf{HB}$ already calculated above, we only need obtaining expressions for $[\mathbf{EH}]_x$:

$$[\mathbf{EH}]_x=E_yH_z-E_zH_y=\gamma^2\big\{[\mathbf{E}'\mathbf{H}']_x +\beta(E'_yD'_y+E'_zD'_z+H'_yB'_y+H'_zB'_z)+\beta^2[\mathbf{D}'\mathbf{B}']_x\big\}\,. \quad (21)$$

Here, we have again taken advantage of the Lorentz field transformation (3).

Substituting the expressions for E_x, D_x, H_x, and B_x from Eqn (3) and also expressions from Eqns (12) and (21) into formula (20), and collecting similar terms, we reduce Eqn (20) to the form

$$T'_{xx}=\gamma^2\left\{\frac{1}{8\pi}\,(1-\beta^2)(E'_xD'_x+H'_xB'_x) -\frac{1}{8\pi}\,(1-\beta^2)(E'_yD'_y+E'_zD'_z+H'_yB'_y+H'_zB'_z) +\frac{\beta}{4\pi}\,[\mathbf{E}'\mathbf{H}']_x-\frac{\beta}{4\pi}\,[\mathbf{D}'\mathbf{B}']_x\right\}. \quad (22)$$

We obtain finally

$$T'_{xx}=\frac{1}{4\pi}\,(E'_xD'_x+H'_xB'_x)-\frac{1}{8\pi}\,\delta_{xx}(\mathbf{E}'\mathbf{D}'+\mathbf{H}'\mathbf{B}') +\frac{\gamma^2\beta}{4\pi}\,\big([\mathbf{E}'\mathbf{H}']_x-[\mathbf{D}'\mathbf{B}']_x\big)\,. \quad (23)$$

Thus, we see that upon transformation the components of the Abraham energy–momentum tensor acquire an additional term which depends on the velocity of motion of the reference system K' relative to K. It is precisely this fact that underlies the assertion in Ref. [1] that the Abraham tensor is not relativistically invariant.

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