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Above the barriers
(I M Khalatnikov's works on the scattering of high-energy particles)

V L Pokrovsky

Relatively recently, in the fall of 1957, I had the good fortune to speak at Landau's seminar on the over-barrier reflection of high-energy particles. I was then working in Novosibirsk, at the Institute of Radiophysics, whose director was one of my teachers Yu B Rumer, and he introduced me to Landau. My coauthors were my fellow students and friends S K Savvinykh and F R Ulinich [1, 2]. The reflection of particles whose energy exceeds the barrier height is a strictly quantum effect: a classical particle just slows down as it approaches the tip of the barrier and then accelerates. We solved the Schrödinger equation in the semiclassical approximation, formally expanding it into a power series in the small parameter λ/a, where λ is the de Broglie wavelength and a is a characteristic size of the potential. The peculiarity of the problem, which was not noticed by other theoreticians, lay in the fact that each consecutive term of the expansion contained a singularity of a higher order than the previous one. As a result, they differed only by universal numerical factors. It turned out to be possible to sum this numerical series using an exactly solvable problem. Landau liked the work, and I was invited to present it at his seminar. Following my talk, I met many celebrities whom I had previously known only through their publications and from legends. Isaak Markovich Khalatnikov showed the most vivid interest. He proposed collaborating, which was flattering for me. He explained his interest by a mission assigned to him by Landau to find a mistake in L Schiff's work on the same topic. This explanation sounded somewhat strange, because we had already found the mistake. Only later did I realize that I became an object of his most sincere and absolutely disinterested affection to any fledgling theorist who came up with an interesting idea. Just this property later made him an ideal director of the Institute of Theoretical Physics and let him gather a unique team, which quickly gained worldwide recognition. I hope, however, that our relationship involved some individual element, the proof of which is our friendship and longstanding research collaboration, which extended to 1992. It would probably have lasted even longer if it had not been interrupted by the turbulent events of that time. The close rapprochement needed for collaborative work became possible due to another of Khalatnikov's rare qualities: his complete lack of both arrogance and servility, as well as his simple and calm way of communicating.

We both realized that the work I presented was just the beginning. Although the method of series summation led to a beautiful and nontrivial result, it was still not physically transparent. It was not clear how to generalize it to similar problems of quantum and classical mechanics. Contemplating this problem, we came to the following idea [3]. Classical and semiclassical particles are reflected at a turning point, where their kinetic energy becomes zero. If the particle energy exceeds the height of the barrier, no turning point exists at a real value of its coordinate. But it appears in the complex coordinate plane if the potential is an analytic function. Going into a complex plane is a rather common operation in quantum mechanics. Going into a complex momentum plane is physically equivalent to tunneling, i.e., penetration into the region of classically forbidden coordinates. Similarly, going into the complex coordinate plane means penetration into the region of classically forbidden momenta. Therefore, we needed to find a suitable path in the complex plane along which a wave travels without reflection to a complex turning point, and then strongly changes in its vicinity. Then the path goes to the real axis, where we can find the reflected wave. In practice, this program was accomplished as shown in Fig. 1. The path begins on the real coordinate axis x at $x \to \infty$. In this region, where the potential can be neglected, only the transmitted wave $\Psi \sim t \exp{(ikx)}$ exists, where t is the transmission amplitude. After that, the path climbs in the upper half of the complex plane until it intersects with the line C_1 going through the turning point x_0 nearest to the real axis, on which the semiclassical action $S(x, x_0) = \int_{x_0}^{x} p(x')\,dx'$,

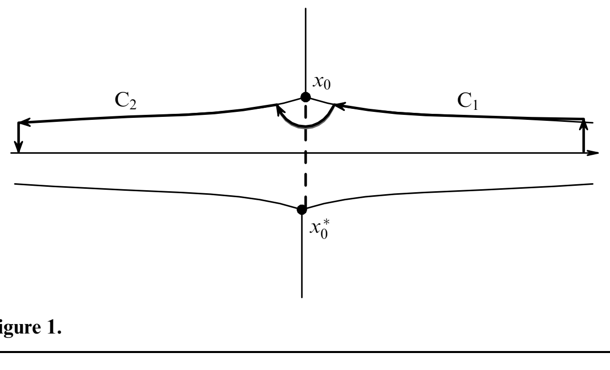

Figure 1.

where $p(x) = [2m(E - V(x))]^{1/2}$, is purely real (this line is called an anti-Stokes line). At infinity, line C_1 runs parallel to the real axis. The solution we started with oscillates on this line. Up to a numerical factor, it is given by the typical semiclassical expression $\psi = A/\sqrt{p(x)}\exp[\mathrm{i}S(x,x_0)/\hbar]$. As usual, the semiclassical approximation is invalid in the vicinity of a turning point; but we can bypass this point from below along a large enough arc. In this case, however, the semiclassical exponential increases until reaching the so-called Stokes line, on which $S(x,x_0)$ becomes purely imaginary, and then decreases, and the second exponential with the minus sign in front of $S(x,x_0)$ appears in its background. This change in the asymptotic regime is called the Stokes phenomenon. As a result, on the second anti-Stokes line C_2, which passes through the same turning point x_0 at the angle 120° to C_1 and satisfies the same condition that $S(x,x_0)$ be real, the asymptotic form of the wave function consists of two exponentials:

$$\psi(x)\big|_{C_2} = \frac{A}{\sqrt{p(x)}}\left\{\exp\left[\frac{\mathrm{i}S(x,x_0)}{\hbar}\right] - \mathrm{i}\exp\left[-\frac{\mathrm{i}S(x,x_0)}{\hbar}\right]\right\}. \tag{1}$$

The path continues along C_2 far to the left, where the potential can again be neglected. Along the entire line C_2, asymptotic form (1), corresponding to the two waves propagating in opposite directions with equal absolute values of the amplitude, is valid. On the complex line C_2, the initial wave is completely reflected at the turning point. But when the path goes to the real axis as $x \to -\infty$, one of the exponentials increases, whereas the other decreases. The absolute value of their ratio, which is equal to the reflection amplitude up to a phase factor, can be easily calculated as

$$|r| = \exp\left[-\frac{2}{\hbar}\,\mathrm{Im}\int_0^{x_0} p(x)\,\mathrm{d}x\right] = \exp\left[\frac{\mathrm{i}}{\hbar}\int_{x_0^*}^{x_0} p(x)\,\mathrm{d}x\right]. \tag{2}$$

This result shows that the reflection does not occur in any order in powers of $\hbar$ or of the ratio of the wavelength λ to the characteristic size of the potential a. This effect is exponentially small. This smallness resembles another strictly quantum effect, quantum tunneling. As well as the tunneling amplitude, the over-barrier reflection amplitude contains an imaginary action in the exponent between the two turning points, which, in contrast to tunneling, are in the complex coordinate plane.

In the 1960s, this work was mostly developed by Soviet theorists. Several interesting papers were written by A M Dykhne. In 1961, he considered the motion of a semiclassical particle in a periodic potential [4]. It is well known that the spectrum has a band structure in this case, and the wave functions are modulated Bloch plane waves. The analogue of the over-barrier reflection in this problem is the appearance of band gaps at energies exceeding the maximum of the periodic potential. In this case, the particle reflects from the system of periodically placed turning points x_n, as shown in Fig. 2. All of them are connected by anti-Stokes lines. Dykhne found that the position of the band gaps is given by the 'Bohr' quantization rule $\int_{x_n}^{x_{n+1}} p(x)\,\mathrm{d}x = m\hbar$, while the widths of the band gaps are determined by the above-barrier reflection coefficient: $\Delta = \hbar\omega\exp\left(2\mathrm{i}/\hbar\int_{x_n^*}^{x_n} p(x)\,\mathrm{d}x\right)$. Bands of a finite, exponentially small width appear at energies smaller than the maximum of the potential due to tunneling under the barriers. Dykhne's result shows that in a periodic potential of a general form, the number of bands separated from each other by gaps is infinite. On the other hand, Dubrovin and Novikov [5] showed that for a particular class of potentials, the number of bands is finite. It is still not known how to resolve this controversy. The potentials leading to a finite-band spectrum are elliptic double-periodic functions. This means that the turning points form a regular lattice in the complex plane with the same periods. Presumably, the reflection disappears as a result of interference on this lattice, but this hypothesis has not been proved yet.

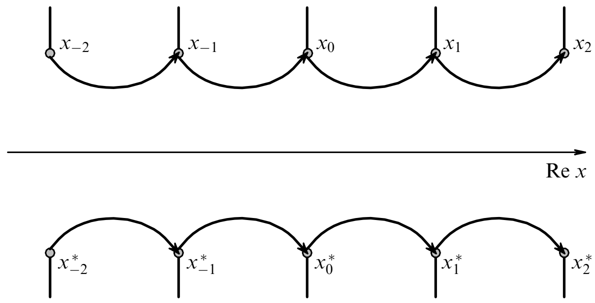

Figure 2.

Dykhne applied the same method to solve the problem of transitions when two levels cross in the complex time plane [6, 7]. The same problem when the levels cross in real time is known as the Landau–Zener problem (or theory) [8, 9]. This is one of the most important results of nonstationary quantum mechanics. Landau, and independently from him Zener, considered a nonstationary two-level system that can be described by the Hamiltonian

$$H_{\mathrm{LZ}} = \begin{pmatrix} E_1(t) & \Delta \\ \Delta^* & E_2(t) \end{pmatrix}. \tag{3}$$

The diagonal elements of Hamiltonian (3) are called diabatic levels, while the quantities $E_\pm = (E_1 + E_2)/2 \pm \left\{[(E_1 + E_2)/2]^2 + \Delta^2\right\}^{1/2}$, which are obtained by formal diagonalization of this Hamiltonian, are called adiabatic levels. It is assumed that the process occurs adiabatically, with the exception of a short time interval close to the instant of intersection of the diabatic levels. Without a loss of generality, we can assume that this instant occurs at $t = 0$. After that, we can assume the dependence of the diabatic levels on time to be linear. Finally, we assume that $E_1 = -E_2 = \hbar\dot{\Omega}t/2$. The amplitude of survival on one of the diabatic levels, found by Landau and Zener, is given by

$$A_{\mathrm{LZ}} = \exp\left(-\frac{2\pi\Delta^2}{\hbar^2\dot{\Omega}}\right). \tag{4}$$

What happens if the levels do not cross on the real time axis? Following what was said, it is obvious that the crossing point must be found in the complex time plane and the problem must be solved near that point. I suggested this formulation of the problem to Dykhne as the initiation of his PhD dissertation, but I did not participate in solving this problem. The solution was found by Dykhne and simultaneously by Landau, who discovered a mistake in the original version of Dykhne's solution. Landau's solution was published in the third and subsequent editions of *Quantum*

Mechanics [10]. Landau reduced this problem to one of over-barrier reflection. It is not surprising that these results look similar. The transition amplitude from one level to another, found by Dykhne and Landau, is

$$A_{\mathrm{DL}} = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0^*}^{t_0}\left[E_2(t) - E_1(t)\right]\mathrm{d}t\right), \tag{5}$$

where t_0 is the crossing point of the levels in the complex time plane. We note that in this case, the crossing amplitude is exponentially small. Equation (5) is known in the literature as the Dykhne formula or the Landau–Dykhne formula. Landau–Zener formula (4) directly follows from it. Indeed, according to the assumptions of this theory, $E_+(t) - E_-(t) = \left[(\hbar\Omega t)^2 + 4\Delta^2\right]^{1/2}$ and $t_0 = \mathrm{i}2\Delta/\hbar\dot{\Omega}$. The integration in the exponential in Eqn (5) is done along the imaginary axis and immediately leads to Eqn (4).

This problem is related to the question of the change in an adiabatic invariant in classical mechanics. It is known that under a slow variation of the Hamiltonian, the classical action per period is approximately conserved. This action is an adiabatic invariant. What is the accuracy of this approximate conservation law? The answer depends on time intervals within which the perturbation acts and the observation is performed. In the simplest case, when the perturbation tends to zero sufficiently fast as $t \to \pm\infty$, and the observation is made when the perturbation can be neglected, the change in the adiabatic invariant first found by Dykhne can be rather easily linked to the Dykhne–Landau problem, at least in the case of one-dimensional motion. It is known that in the semiclassical approximation, the action within a period is quantized with the period $2\pi\hbar$. Up to this factor, the action coincides with the level number n. In the language of quantum mechanics, the change in the adiabatic invariant means a transition from one level to another. The value of this change is $\Delta I = 2\pi\hbar\sum_{n'}(n' - n)\, w_{n,n'}$, where $w_{n,n'}$ denotes the probability of transition from level n to level n'. In the adiabatic regime, the transitions between the nearest levels $n' - n = \pm 1$ are the most probable, while other transitions are much less probable. Because of a weak dependence of $w_{n,n+1}$ on n, we obtain [6, 7, 11]

$$\Delta I = 2\pi h\,\frac{\mathrm{d}w_{n,n+1}}{\mathrm{d}n} = \mathrm{i}2\pi h^2\int_{t_0^*}^{t_0}\frac{\partial\omega}{\partial I}\,\mathrm{d}t\,\exp\left[2\mathrm{i}\int_{t_0^*}^{t_0}\omega(t)\,\mathrm{d}t\right], \tag{6}$$

where ω is the frequency of classical motion, which slowly depends on time. The change in the adiabatic invariant turns out to be exponentially small. But if the measurement is done within a finite and not exponentially large period of time, then the change becomes much larger; it oscillates in time and simultaneously decays as $1/t$, just like the transition probabilities. This phenomenon, unknown at that time, leads to a disagreement with the experimental results.

A more general situation with several periodic motions was investigated by A A Slutskin in the framework of classical mechanics. The description of Slutskin's work can be found in the last editions of *Mechanics* by Landau and Lifshitz [12] in Pitaevskii's treatment.

A three-dimensional generalization of the issue of over-barrier reflection was achieved in a sequence of papers [13–16] by Patashinskii, Pokrovsky, and Khalatnikov, published in 1962–1964. This work was started during Landau's scientific life and was discussed with him repeatedly. In the course of this work, Khalatnikov and I invented the poles of the scattering amplitude in the complex momentum plane: back then, these poles were not yet called Regge poles. This nut, however, was so hard to crack that we were able to finish this work only several years later, with the participation of Sasha Patashinskii. The formulation of the problem was as follows. Classical mechanics allows scattering in a definite cone of angles. Quantum mechanics does not have this limitation. What is the amplitude of semiclassical scattering at a classically forbidden angle? In classical mechanics, each allowed scattering angle θ corresponds to a definite value of the impact parameter ρ. Following the same line of reasoning as in the case of over-barrier reflection, it can be conjectured that the scattering at a classically forbidden angle should correspond to a complex impact parameter. Usually, the semiclassical approximation in scattering theory is obtained by means of the Watson transform of the Faxen–Holtzmark formula for the scattering amplitude:

$$f(\theta) = \frac{1}{2\mathrm{i}k}\sum_{l=0}^{\infty}(2l+1)\exp(2\mathrm{i}\delta_l)\,P_l(\cos\theta)$$
$$= -\frac{1}{2\mathrm{i}k}\int_\Gamma \nu S(\nu)\,P_{\nu-1/2}(-\cos\theta)\,\frac{\mathrm{d}\nu}{\cos\nu\pi}\,. \tag{7}$$

The integration contour Γ is shown in Fig. 3. It has to be deformed if possible in order to pass through the saddle point in the direction of the steepest descent. The value ν at the saddle point is the impact parameter up to some factor ($\rho = \nu/k$), which corresponds to the scattering angle θ. Free contour deformation is obstructed by poles of the function $S(\nu)$. Therefore, in a certain region of parameters, the contribution of the poles dominates in the scattering amplitude and the use of a complex impact parameter depending on the scattering angle becomes invalid. A detailed description of the result is inappropriate in this short note; but it is possible to show how the poles of the reflection amplitude appear. The function $S(\nu)$ is defined by the asymptotic form of the radial wave function,

$$R_{\nu-1/2}(r) \sim \frac{1}{r}\left\{A(\nu)\exp\left[\mathrm{i}\left(kr - \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}\right)\right]\right.$$
$$\left. - B(\nu)\exp\left[-\mathrm{i}\left(kr - \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}\right)\right]\right\},$$

as $S(\nu) = A(\nu)/B(\nu)$. The pole appears when $B(\nu) = 0$. We consider how the radial wave function behaves in the complex r plane. Typical anti-Stokes lines passing through the turning point r_1 nearest to the real axis are shown in Fig. 4a. The radial wave function decays near the coordinate origin, i.e., has only one exponential $R = \exp(\mathrm{i}S(r, r_1)/\hbar)$ in the sector left of the turning point. In passing to the right anti-Stokes line, the radial wave function acquires the second exponential, $R = \exp(\mathrm{i}S(r, r_1)/\hbar) - \mathrm{i}\exp(-\mathrm{i}S(r, r_1)/\hbar)$, whose coeffi-

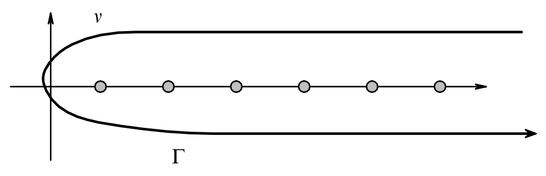

Figure 3.

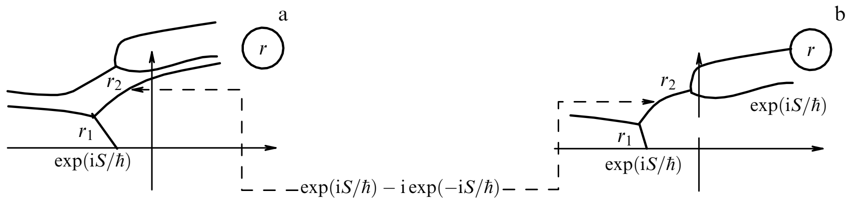

Figure 4.

cient remains the same as $r \to +\infty$. This means that $B(v) \neq 0$ and $S(v)$ does not have a pole. The pole appears at the value of v defined by two conditions: 1) a second turning point r_2 appears on the same anti-Stokes line (Fig. 4b); 2) the action between the two turning points obeys Bohr's rule $S(r, r_1) = n\pi\hbar$. Under this condition, the second exponential disappears after passing the second turning point.

In the following rather long time period, activity in this area almost disappeared and the above-mentioned papers were seldom cited. Interest in them was suddenly resumed in the late 1980s — early 1990s because of the development of new areas in physics and mathematics. In physics, it was the pattern formation theory, in particular, fractal crystal growth and the theory of motion of the interface between viscous and ideal fluids trapped between two parallel plates (Hele–Shaw flow). The new mathematical science is called 'asymptotics beyond-all-orders' (in the sense of perturbation theory). Among the scientists who significantly contributed to this new discipline are M Kruskal, M Berry, J Boyd, J Langer, H Segur, H Levine, H Muller-Krumbhaar, S Tanveer, B Shraiman, D Bensimon, M Mineev, V Mel'nikov, E Brenner, and P Wiegmann. The first step was taken by Kruskal and Segur in their work devoted to dendritic crystal growth [17], in which our method was first generalized to a nonlinear problem. This research is active even nowadays. In addition to the original papers, many collections of papers, reviews, and monographs have been published. I refer to two of them. The first is a collection of articles [18] named *Asymptotics beyond all orders*, published in 1991. It contains several important reviews of the above-mentioned problems. The second is a book by J Boyd, *Weekly nonlocal solitary waves and Beyond-All-Orders Asymptotics* [19], published in 1999. Even though the title looks more specialized, this book contains a detailed and clear description of general methods and related areas, and it can therefore be recommended as a primer on the subject. With the permission of the author, I reproduce some excerpts from this book related to our work of 1969.

Boyd calls our method "Matched asymptotics in the complex plane" and characterizes it as rather general and applicable to a large number of different problems. Here is what he writes in the introduction to the corresponding chapter:

"The earliest use of matched asymptotics in the complex plane was by Pokrovsky and Khalatnikov (1961), who generalized the semiclassical theory to calculate exponentially small reflection of waves from a potential barrier whose height is everywhere less than the energy of the waves. Kruskal and Segur (1985, 1991) applied their ideas to a nonlinear phenomenon: Dendritic fingering of a solid-liquid interface. Later, Segur and Kruskal (1987) and Pomeau, Ramani, and Grammaticos (1988) applied the method to solitary waves. Since then, there have been many applications; Akylas and Grimshow (1992) study of nonlocal higher mode of internal gravity solitons is particularly readable. Grimshow and Joshi (1995) have extended Pomeau et al. (1988) to the higher order with corrections."

While describing our work of 1961, Boyd intentionally uses rather vague and extremely general terminology. To characterize it, we consider Fig. 5, by which he substitutes our more precise Fig. 1. All details are omitted; what is left is the general idea of motion with a known solution from one infinity to a complex turning point, and then from this point with the other known solution to the other infinity. In an even more abstract form, the method of matched asymptotics is illustrated in Fig. 6. It shows an external region in which the asymptotic form of the solution must be found, two adjacent regions where the asymptotic forms are known up to several unknown constants, separated by the line of the change in the asymptotic regime (Stokes line), and the internal region in the complex plane where the asymptotics are invalid. It is required to solve the problem in the internal region. Usually, it is possible to use the proximity of this region to a certain point at which the short-wavelength approximation is strictly

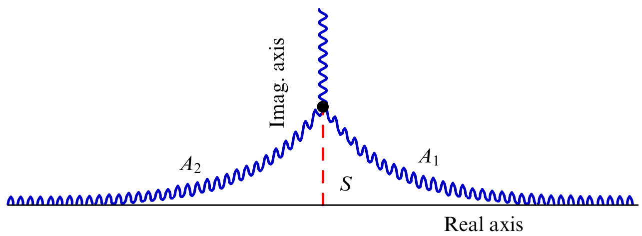

Figure 5.

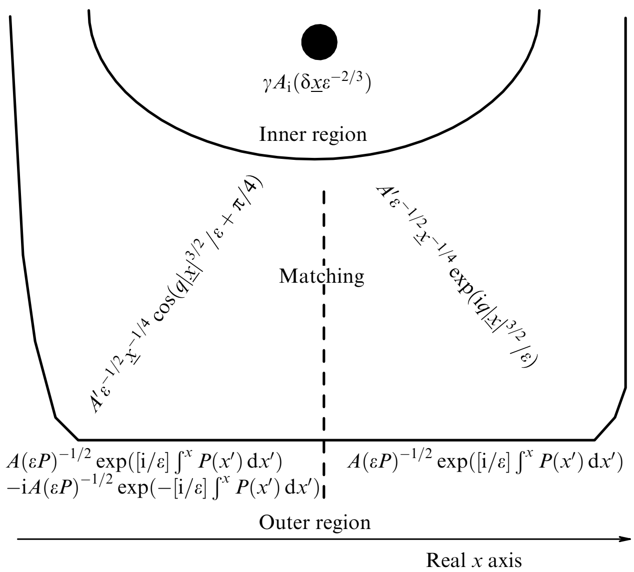

Figure 6.

invalid (the analog of the turning point in classical mechanics), solve the internal problem, if not analytically then numerically, and match it with the two different asymptotic forms in the adjacent regions. In this form, the method is valid even for nonlinear problems, for which different types of wave solutions are known, for example, solitons, automodel solutions, and shock waves. Apart from considering the previously discussed problems of over-barrier reflection, Boyd illustrates the general method by the original solution of the problem of a pendulum driven by a force slowly depending on time. The corresponding equation of motion is

$$u_{tt} + u = f(\varepsilon t)\,, \tag{8}$$

where ε is a small parameter. Let the solution at $t < 0$ be close to $f(\varepsilon t)$. For $t > 0$, it differs by a solution of the homogeneous equation: $u(t \to +\infty) = f(\varepsilon t) + c \sin t$. To find the constant c, we need to solve the problem in the vicinity of the pole of $f(x)$ and match the two asymptotic forms of u with the solution in the internal region. The problem was solved in the case where the pole of $f(x)$ located at a point x_s is of the second order. The function $f(x)$ in the internal region is substituted by the function $(x - x_s)^{-2}$, and the solution of the standard equation obtained from (8) for $U = \varepsilon^2 u$, namely, $U_{tt} + U = (t - x_s/\varepsilon)^{-2}$, is the so-called Borel logarithm $U(\tau) = \mathrm{Bo}(\tau) = \int_0^\infty \exp(-s) \ln(1 + s^2/\tau^2)\, \mathrm{d}s$, where $\tau = t - x_s/\varepsilon$. When τ changes its sign after circulating around the origin, the Borel logarithm acquires the additional term $2\pi \mathrm{i} \exp(-\mathrm{i}\tau)$. Matching this solution with the asymptotic expression in the region $1 < \tau < 1/\varepsilon$ and then going down to the real axis t, we find $c = (2\pi/\varepsilon^2) \exp(\mathrm{i} x_s/\varepsilon)$. As expected, this is an exponentially small quantity. Surprisingly, the solution of this linear problem appeared to be a key to the solution of the much more complicated nonlinear problem on the variation of a soliton in the framework of the Korteweg–de Vries equation with the added fifth derivative when the soliton slowly propagates from one end of the line to another [20]. The solution is too cumbersome, and it is difficult to describe it briefly, but the very formulation of the problem gives an idea of what class of problems can be solved by the method of matched asymptotics.

I reproduce here two tables extracted from the same book by Boyd, collecting the information about the class of problems, excluding the solitons, in which exponentially small effects localized in the complex plane appear (Table 1, 2). The method of matched asymptotics can be

Table 1. Nonsoliton, nonquantum exponential smallness.

Phenomenon	Field	References
Dendritic crystal growth	Condensed matter	Kessler, Koplik & Levine (1988)
Viscous fingering (Saffman–Taylor problem)	Fluid dynamics	Shraiman (1986), Hong & Langer (1986), Combescot et al. (1986), Tanveer (1990, 1991)
Diffusive front merger. Exponentially flow	Reaction-diffusion systems	Carr (1992), Hale (1992), Carr & Pego (1989), Fusco & Hale (1989), Laforgue & O'Malley (1994, 1995), Reyna & Ward (1994, 1995), Ward & Reyna (1995)
Stokes' phenomenon in asymptotic expansions	Applied mathematics	Dingle (1973), Berry (1989, 1995), Berry & Howls (1990, 1991, 1993, 1994), Olver (1974, 1991, 1993), Olde Daalhuis (1992), Paris & Wood (1992), Paris (1992), Howls (1997), Jones (1997)
Rapidly-forced pendulum	Classical physics	Chang (1991), Scheurle et al. (1991)
Resonant sloshing in a tank	Fluid mechanics	Byatt-Smith & Davie (1991)
Laminar flow in porous pipes	Fluid mechanics. Space plasmas	Berman (1953), Robinson (1976), Terril (1965, 1973), Terril & Thomas (1969), Grundy & Allen (1994)
Jeffrey–Hamel flow stagnation points	Higher-order boundary layer	Bulakh (1964)
Shocks in a nozzle	Fluid mechanics	Adamson & Richey (1973)
Slow viscous flow past a circle, sphere	Fluid mechanics (log & power series)	Proudman & Pearson (1957), Chester & Breach (1969), Skinner (1975), Kropinski & Ward & Keller (1995)
Equatorial Kelvin wave instability	Meteorology, oceanography	Boyd & Christidis (1982, 1983), Boyd & Natarov (1998)
Error: midpoint rule	Numerical analysis	Hildebrand (1974)
Radiation leakage from fiber optics waveguide	Nonlinear optics	Kath & Kriegsmann (1988), Paris & Wood (1989)
Particle channeling in crystals	Condensed matter physics	Dumas (1991)

Table 1 (continued)

Phenomenon	Field	References
Island-trapped waves	Oceanography	Lozano & Meyer (1976), Meyer (1980)
Rising bubbles	Fluids	Vanden-Broeck (1984, 1986, 1988, 1992)
Chaos onset	Physics	Holmes, Marsden & Scheurle (1988)
Separatrix separation	Applied mathematics	Hakim & Mallick (1993)
Slow manifold in geophysical fluids	Meteorology, oceanography	Lorenz & Krishnamurthy (1987), Boyd (1994), Camassa (1995)

Table 2. Selected examples of exponentially small quantum phenomena.

Phenomenon	Field	References
energy of a quantum double well (H_2^+, etc.)	Atomic physics, quantum chemistry	Fröman (1966), Cizek et al. (1986)
Imaginary part of eigenvalue of a metastable quantum species: stark effect (external electric field)	Atomic physics, quantum chemistry	Oppenheimer (1928), Reinhardt (1982), Hinton & Shaw (1985), Benassi et al. (1979)
(Im E): cubic anharmonicity	Quantum chemistry	Alvarez (1988)
(Im E): quadratic Zeeman effect (external magnetic field)	Atomic physics, quantum chemistry	Cizek and Vrscay (1982)
Transition probability, two-state quantum system (exponentially small in speed of variations)	Quantum mechanics	Berry & Lim (1993)
Superoscillations in Fourier integrals, quantum billiards, etc.	Applied Mathematics. Quantum Mechanics electromagnetics	Berry (1994)
Width of stability bands for Hill's equation	Quantum physics, astronomy	Weinstein and Keller (1985, 1987)
Above-the-barrier scattering	Quantum physics	Pokrovskii & Khalatnikov (1961)

applied to many of these problems. The variety of phenomena united by a similar mathematical structure is striking. Among others, these are abstract mathematics, hydrodynamics, meteorology, solid state physics, and quantum mechanics. I believe that many things have yet to be discovered.

I hope that this brief review will renew I M Khalatnikov's interest in this circle of questions. According to my observations, his interest in science and his research activity have not weakened.

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