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Honoring the 90th birthday of Academician I M Khalatnikov
(Scientific session of the Physical Sciences Division of the Russian Academy of Sciences, 21 October 2009)

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21 October 2009, in the conference hall of the Lebedev Physical Institute, Russian Academy of Sciences, a scientific session of the Physical Sciences Division was held honoring the 90th birthday of Academician I M Khalatnikov. The following talks were given at the session:

(1) **Andreev A F** (Kapitza Institute of Physical Problems, Russian Academy of Sciences, Moscow) "Momentum deficit in quantum glasses";

(2) **Kamenshchik A Yu** (Dipartimento di Fisica and Istituto Nazionale di Fisica Nucleare, Bologna, Italy; Landau Institute for Theoretical Physics RAS, Moscow) "The problem of singularities and chaos in cosmology";

(3) **Pokrovsky V L** (Landau Institute for Theoretical Physics, RAS, Moscow; Department of Physics, Texas A&M University, USA) "I M Khalatnikov's works on scattering of high-energy particles";

(4) **Khriplovich I B** (Budker Institute of Nuclear Physics, Novosibirsk) "Screening and antiscreening of charge in gauge theories."

Brief versions of talks 2 – 4 are given below.

Isaak Markovich Khalatnikov

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The problem of singularities and chaos in cosmology

A Yu Kamenshchik

1. Introduction

We consider different aspects of the problem of cosmological singularities, such as the Belinsky–Khalatnikov–Lifshitz (BKL) oscillatory approach to a singularity, the new features of cosmological dynamics in the neighborhood of a singularity in multidimensional and superstring cosmological models, and their connections with modern branches of mathematics such as infinite-dimensional Lie algebras. The chaoticity of the oscillatory approach to the cosmological singularity is also discussed. The conclusions contain some thoughts about the past and the future of the Universe in light of the oscillatory approach to the Big Bang and the Big Crunch cosmological singularities.

Many years ago, in conversations with his students, Lev Davidovich Landau used to say that three problems were the most important for theoretical physics: the problem of the cosmological singularity, the problem of phase transitions, and the problem of superconductivity [1]. We now know that the great breakthrough was achieved in the explanation of the phenomena of superconductivity [2] and phase transitions [3]. The cosmological singularity problem has been extensively studied during the last 50 years and many important results have been obtained, but it still preserves some intriguing aspects. Moreover, some quite unexpected facets of the problem of the cosmological singularity were discovered. Isaak Markovich Khalatnikov, who was one of the students of Landau, made a significant contribution to the discovery and elaboration of different aspects of the problem of the cosmological singularity and the chaos

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arising in the process of the asymptotic approach to this singularity.

In our review [4] published 10 years ago in an issue of this journal dedicated to the 90th anniversary of Landau's birth, we discussed some issues connected with the problem of singularity in cosmology. In a paper dedicated to the 100th birthday of Landau [5], we dwelled on relations between the well-known old results of these studies and new developments in this area.

In the present paper, dedicated to the 90th birthday of I M Khalatnikov, I give a brief review of some old and new ideas connected with the development of the theory of the asymptotic approach to the cosmological singularity, and try to argue why this could be interesting not only for physicists and mathematicians but also for a wider audience.

To begin with, we recall that Penrose and Hawking [6 – 8] proved the impossibility of indefinite continuation of geodesics under certain conditions. This was interpreted as pointing to the existence of a sigularity in the general solution of the Einstein equations. These theorems, however, did not allow finding the particular analytic structure of the singularity. The analytic behavior of the general solutions of the Einstein equations in the neighborhood of a singularity was investigated by Lifshitz and Khalatnikov [9 – 12] and Belinsky, Lifshitz, and Khalatnikov [13 – 15]. These papers revealed the enigmatic phenomenon of an oscillatory approach to the singularity, which has become known also as the *Mixmaster Universe* [16]. The model of a closed homogeneous but anisotropic universe with three degrees of freedom (the Bianchi type-IX cosmological model) was used to demonstrate that the universe approaches the singularity in such a way that its contraction along two axes is accompanied by an expansion along the third axis, and the axes change their roles according to a rather complicated law that reveals a chaotic behavior [14 – 18].

The study of the dynamics of the universe in the vicinity of a cosmological singularity has exploded as a developing field of modern theoretical and mathematical physics. We first note a generalization of the oscillatory approach to the cosmological singularity in multidimensional cosmological models. It was noted in [19 – 21] that the approach to the cosmological singularity in multidimensional (Kaluza–Klein type) cosmological models has a chaotic character in spacetimes whose dimension is not higher than ten, while in spacetimes of higher dimensions, the universe enters a monotonic Kasner-type contracting regime after undergoing a finite number of oscillations.

The development of cosmological studies based on superstring models has revealed some new aspects of the dynamics in the vicinity of the singularity [23 – 25]. First, it was shown that mechanisms for changing Kasner epochs exist in these models, and they are due not to gravitational interactions but to the influence of other fields present in these theories. Second, it was proved that cosmological models based on the six main superstring models plus the $D = 11$ supergravity model exhibit a chaotic oscillatory approach toward the singularity. Third, the connection between cosmological models manifesting an oscillatory approach toward a singularity and a special subclass of infinite-dimensional Lie algebras [26], so-called hyperbolic Kac–Moody algebras, was discovered (a comprehensive review of the corresponding mathematical tools with their application to BKL studies was given in [27]). The study of the algebraic structures underlying the chaotic approach to the cosmological singularity opens some new (although still very weakly elaborated) prospects for the development of a consistent quantum gravity theory [28].

In speaking about the new aspects of the oscillatory approach to the cosmological singularity in multidimensional and superstring theories, we must not forget that the 'classical' BKL behavior for the $3+1$ dimensional general relativity has not yet been totally understood, and requires further study. In addition, we try to attract attention to some philosophical aspects of this phenomenon, which have so far been underestimated.

The structure of the paper is as follows. In Section 2, we briefly discuss the Landau theorem on the singularity, which was not published in a separate paper and was reported in book [29] and review [9]; in Section 3, we recall the main features of the oscillatory approach to the singularity in relativistic cosmology, including its chaoticity; Section 4 is devoted to the modern development of BKL ideas and methods, including dynamics in the presence of a massless scalar field, multidimensional cosmology, superstring cosmology, and the correspondence between chaotic cosmological dynamics and hyperbolic Kac–Moody algebras; in the concluding Section 5, we express some thoughts about the past and the future of the Universe in light of the BKL phenomenon.

2. Landau theorem on the singularity

We consider a synchronous reference frame with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\,, \tag{1}$$

where $\gamma_{\alpha\beta}$ is the spatial metric. Landau pointed out that the determinant g of the metric tensor in a synchronous reference frame must tend to zero at some finite time if some simple conditions on the equation of state are satisfied. To prove this statement, it is convenient to write the $0-0$ component of the Ricci tensor as

$$R_0^0 = -\frac{1}{2}\,\frac{\partial K_\alpha^\alpha}{\partial t} - \frac{1}{4}\,K_\alpha^\beta K_\beta^\alpha\,, \tag{2}$$

where $K_{\alpha\beta}$ is the extrinsic curvature tensor defined as

$$K_{\alpha\beta} = \frac{\partial\gamma_{\alpha\beta}}{\partial t}\,, \tag{3}$$

and the spatial indices are raised and lowered by the spatial metric $\gamma_{\alpha\beta}$. The Einstein equation for R_0^0 is

$$R_0^0 = T_0^0 - \frac{1}{2}\,T\,, \tag{4}$$

where the energy–momentum tensor is

$$T_i^j = (\rho + p)\,u_i u^j - \delta_i^j p\,, \tag{5}$$

where ρ, p, and u_i are the energy density, the pressure, and the four-velocity. The quantity in the right-hand side of Eqn (4),

$$T_0^0 - \frac{1}{2}\,T = \frac{1}{2}\,(\rho + 3p) + (\rho + p)\,u_\alpha u^\alpha\,, \tag{6}$$

is positive whenever

$$\rho + 3p > 0\,. \tag{7}$$

Hence, it follows from Eqn (4) that

$$\frac{1}{2}\frac{\partial K^\alpha_\alpha}{\partial t} + \frac{1}{4} K^\beta_\alpha K^\alpha_\beta \leqslant 0\,. \tag{8}$$

Because of the algebraic inequality

$$K^\beta_\alpha K^\alpha_\beta \geqslant \frac{1}{3}\,(K^\alpha_\alpha)^2\,, \tag{9}$$

we have

$$\frac{\partial K^\alpha_\alpha}{\partial t} + \frac{1}{6}\,(K^\alpha_\alpha)^2 \leqslant 0 \tag{10}$$

or

$$\frac{\partial}{\partial t}\frac{1}{K^\alpha_\alpha} \geqslant \frac{1}{6}\,. \tag{11}$$

If $K^\alpha_\alpha > 0$ at some instant of time, then as t decreases, the quantity t decreases to zero within a finite time. Hence, K^α_α tends to $+\infty$, and because of the identity

$$K^\alpha_\alpha = \gamma^{\alpha\beta}\,\frac{\partial g_{\alpha\beta}}{\partial t} = \frac{\partial}{\partial t}\ln g\,, \tag{12}$$

this means that the determinant g tends to zero [no faster than g according to inequality (11)]. If $K^\alpha_\alpha < 0$ at the initial instant, then the same result follows for the increasing time. A similar result was obtained in [30, 31] for dust-like matter and in [32].

This result does not prove that a true physical singularity inevitably exists in spacetime itself, irrespective of the chosen reference system. However, it played an important role in stimulating discussion about the existence and generality of singularities in cosmology. We note that the energodominance condition in (7) used in the proof of the Landau theorem also appears in the proof of the Penrose and Hawking singularity theorem [6–8]. Moreover, the breakdown of this condition is necessary for an explanation of the phenomenon of cosmic acceleration.

The Landau theorem is deeply connected with the appearance of caustics studied by Lifshitz, Khalatnikov, and Sudakov [33, 34] and discussed between them and Landau in 1961. In trying to geometrically construct a synchronous reference frame, one starts from the three-dimensional Cauchy surface and designs a family of geodesics orthogonal to this surface. The length along these geodesics serves as the time measure. It is known that these geodesics intersect on some two-dimensional caustic surface. This geometry constructed for empty space is also valid in the presence of dust-like matter ($p = 0$). Such matter moving along the geodesics concentrates on caustics, but the increase in density cannot be unbounded because the arising pressure destroys the caustics.[1] This question was studied by Grishchuk [35]. Later, Arnold, Shandarin, and Zeldovich [36] used caustics for the explanation of the initial clustering of dust, which, while not creating physical singularities, is nevertheless responsible for the creation of so-called pancakes. These pancakes represent the initial stage of the development of the large-scale structure of the universe.

[1] In an empty space, the caustic is a mathematical, but not a physical, singularity. This follows simply from the fact that we can always shift its location by changing the initial Cauchy surface.

3. Oscillatory approach to the singularity in relativistic cosmology

One of the first exact solutions found in the framework of general relativity was the Kasner solution [22] for the Bianchi type-I cosmological model representing a gravitational field in an empty space, with the Euclidean metric depending on time according to the formula

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^{2p_1}\,\mathrm{d}x^2 - t^{2p_2}\,\mathrm{d}y^2 - t^{2p_3}\,\mathrm{d}z^2\,, \tag{13}$$

where the exponents p_1, p_2, and p_3 satisfy the relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1\,. \tag{14}$$

Remarkably, this solution was the first nonstationary cosmological solution, found before the isotropic Friedmann solution. Perhaps because of its 'exoticity,' it was for many years ignored by working cosmologists and became appreciated only in the 1950s.

Choosing the order of the exponents as

$$p_1 < p_2 < p_3\,, \tag{15}$$

we can parameterize them as [9, 10]

$$p_1 = \frac{-u}{1+u+u^2}\,,\quad p_2 = \frac{1+u}{1+u+u^2}\,,\quad p_3 = \frac{u(1+u)}{1+u+u^2}\,. \tag{16}$$

As the parameter u varies in the range $u \geqslant 1$, p_1 and p_2 take all their permissible values:

$$-\frac{1}{3} \leqslant p_1 \leqslant 0,\quad 0 \leqslant p_2 \leqslant \frac{2}{3}\,,\quad \frac{2}{3} \leqslant p_3 \leqslant 1\,. \tag{17}$$

The values $u < 1$ lead to the same range of values of p_1, p_2, and p_3 because

$$p_1\left(\frac{1}{u}\right) = p_1(u),\; p_2\left(\frac{1}{u}\right) = p_3(u),\; p_3\left(\frac{1}{u}\right) = p_2(u)\,. \tag{18}$$

The parameter u, introduced in the early 1960s, is very useful, and its properties have attracted the attention of researchers in various contexts. For example, in recent paper [37], a connection was established between the Lifshitz–Khalatnikov parameter u and the invariants arising in the context of Petrov's classification of Einstein spaces [38].

In the case of Bianchi type-VIII or Bianchi type-IX cosmological models, the Kasner regime in (13) and (14) is no longer an exact solution of the Einstein equations; however, generalized Kasner solutions can be constructed [11–15]. It is possible to construct some kind of perturbation theory where the exact Kasner solution in (13) and (14) plays the role of the zeroth-order approximation, while the role of perturbations is played by those terms in the Einstein equations that depend on spatial curvature tensors (apparently, such terms are absent in the Bianchi type-I cosmology). This perturbation theory is effective in the vicinity of a singularity or, in other terms, at $t \to 0$. The remarkable feature of these perturbations is that they imply a transition from the Kasner regime with one set of parameters to the Kasner regime with another set.

The metric of the generalized Kasner solution in a synchronous reference system can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\,, \tag{19}$$

where

$$a = t^{p_l}, \quad b = t^{p_m}, \quad c = t^{p_n}. \tag{20}$$

The three-dimensional vectors **l**, **m**, and **n** define the directions along which the spatial distances vary with time according to power laws (20). We set $p_l = p_1$, $p_m = p_2$, and $p_n = p_3$, such that

$$a \sim t^{p_1}, \quad b \sim t^{p_2}, \quad c \sim t^{p_3}, \tag{21}$$

i.e., the Universe is contracting in directions given by the vectors **m** and **n** and is expanding along **l**. It was shown in [14] that the perturbations caused by spatial curvature terms make the variables a, b, and c undergo a transition to another Kasner regime characterized by the formulas

$$a \sim t^{p'_l}, \quad b \sim t^{p'_m}, \quad c \sim t^{p'_n}, \tag{22}$$

where

$$p'_l = \frac{|p_1|}{1 - 2|p_1|}, \quad p'_m = -\frac{2|p_1| - p_2}{1 - 2|p_1|}, \quad p'_n = -\frac{p_3 - 2|p_1|}{1 - 2|p_1|}. \tag{23}$$

The effect of the perturbation is therefore to replace one 'Kasner epoch' by another such that the negative power of t is transformed from the **l** to the **m** direction. During the transition, the function $a(t)$ reaches a maximum and $b(t)$ a minimum. Hence, the previously decreasing quantity b now increases, the quantity a decreases, and $c(t)$ remains a decreasing function. The previously increasing perturbation that caused the transition from regime (21) to (22) is damped and eventually vanishes. Then another perturbation begins to grow, which leads to a new replacement of one Kasner epoch by another, and so on.

We emphasize that just the fact that the perturbation implies a change in dynamics that suppresses this perturbation allows using the perturbation theory so successfully. We note that the effect of changing the Kasner regime exists already in cosmological models that are simpler than those of Bianchi type IX and Bianchi type VIII. As a matter of fact, in a Bianchi type-II universe, only one type of perturbations exists, connected with spatial curvature, and this perturbation leads to a change in the Kasner regime (one bounce). This fact was known to Lifshitz and Khalatnikov at the beginning of the 1960s, and they discussed this topic with Landau (just before his tragic accident), who appreciated it highly. The results describing the dynamics of the Bianchi type-IX model were reported by Khalatnikov in his talk given in January 1968 at the Henri Poincaré Seminar in Paris. J A Wheeler, who was present there, pointed out that the dynamics of the Bianchi type-IX universe represent a nontrivial example of a chaotic dynamical system. Later, K Thorn distributed a preprint with the text of this talk.

Returning to the rules governing the bouncing of a negative power of time from one direction to another, we emphasize that the very complicated system of nonlinear partial differential equations is reduced in the vicinity of a singularity to a rather simple system of ordinary differential equations. To extract information about rules (23), it was enough to analyze them qualitatively. This analysis may be compared with a description of the motion of a ball climbing up a hill: after reaching the highest possible point, it stops and begins rolling down. At the foot of the hill, its velocity is equal to its initial velocity, but with the opposite sign. Moreover, some kind of a conservation law for the sum of velocities corresponding to the expansion (contraction) of different space directions was used in [14, 29].

On the other hand, it was shown that bouncing rules (23) can be conveniently expressed by means of parameterization (16):

$$p_l = p_1(u), \quad p_m = p_2(u), \quad p_n = p_3(u), \tag{24}$$

and then

$$p'_l = p_2(u-1), \quad p'_m = p_1(u-1), \quad p'_n = p_3(u-1). \tag{25}$$

The greater of the two positive exponents remains positive.

Successive changes (25), accompanied by a bouncing of the negative power between the directions **l** and **m**, continue as long as the integral part of u is not exhausted, i.e., until u becomes less that unity. Then, according to Eqn (18), the value $u < 1$ transforms into $u > 1$, and at this moment either the exponent p_l or p_m is negative and p_n becomes the smaller of the two positive numbers ($p_n = p_2$). The next sequence of changes bounces the negative power between the directions **n** and **l** or **n** and **m**. We emphasize that the Lifshitz–Khalatnikov parameter u is useful because it allows encoding rather complicated laws of transitions between different Kasner regimes (23) in such simple rules as $u \to u - 1$ and $u \to 1/u$.

Consequently, the evolution of our model toward a singular point consists of successive periods (called eras) in which distances along two axes oscillate, while the distance along the third axis decreases monotonically, and the volume decreases $\sim t$. In the transition from one era to another, the axes along which the distances decrease monotonically are interchanged. The order in which the pairs of axes are interchanged and the order in which eras of different lengths follow each other acquire a stochastic character.

To every (sth) era, there corresponds a decreasing sequence of values of the parameter u. This sequence has the form $u_{\max}^{(s)}, u_{\max}^{(s)} - 1, \ldots, u_{\min}^{(s)}$, where $u_{\min}^{(s)} < 1$. We introduce the notation

$$u_{\min}^{(s)} = x^{(s)}, \quad u_{\max}^{(s)} = k^{(s)} + x^{(s)}, \tag{26}$$

i.e., $k^{(s)} = [u_{\max}^{(s)}]$ (the square brackets denote the greatest integer $\leqslant u_{\max}^{(s)}$). The number $k^{(s)}$ defines the era length. For the next era, we obtain

$$u_{\max}^{(s+1)} = \frac{1}{x^{(s)}}, \quad k^{(s+1)} = \left[\frac{1}{x^{(s)}}\right]. \tag{27}$$

The ordering with respect to the lengths $k^{(s)}$ of successive eras (measured by the number of Kasner epochs contained in them) asymptotically acquires a stochastic character. The random nature of this process arises because of rules (26) and (27), which define the transitions from one era to another in an infinite sequence of values of u. If this infinite sequence begins with some initial value $u_{\max}^{(0)} = k^{(0)} + x^{(0)}$, then the lengths $k^{(0)}$, $k^{(1)}, \ldots$ are the numbers appearing in an expansion into a continuous fraction:

$$k^{(0)} + x^{(0)} = k^{(0)} + \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \ldots}}. \tag{28}$$

We can describe this sequence of eras statistically if, instead of a given initial value $u_{\max}^{(0)} = k^{(0)} + x^{(0)}$, we consider a distribution of $x^{(0)}$ over the interval $(0, 1)$ governed by some probability law [17]. Then we also obtain some distributions of the values of $x^{(s)}$ that terminate every sth series of numbers. It can be shown that as s increases, these distributions tend to a stationary (independent of s) probability distribution $w(x)$ in which the initial value $x^{(s)}$ is completely 'forgotten':

$$w(x) = \frac{1}{(1+x)\ln 2}\,. \tag{29}$$

It follows from Eqn (29) that the probability distribution of the lengths k is given by

$$W(k) = \frac{1}{\ln 2}\ln\frac{(k+1)^2}{k(k+2)}\,. \tag{30}$$

The source of stochasticity arising at the oscillatory approach to the cosmological singularity can be described as follows: the transition from one Kasner era to another is described by the transformation

$$Tx = \left\{\frac{1}{x}\right\}, \quad \text{i.e.,} \quad x_{s+1} = \left\{\frac{1}{x_s}\right\} \tag{31}$$

of the interval $[0, 1]$ into itself; the curly brackets denote the fractional part. This transformation expands and has the property of exponential instability. It is not a one-to-one transformation; its inverse is not unique. In other words, fixing the value of the parameter u, we can predict the evolution toward the singularity, but we cannot describe the past.

We can pass from a one-sided infinite sequence

$$(x_0, x_1, x_2, \ldots) \tag{32}$$

to a doubly infinite sequence [18]

$$X = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)\,. \tag{33}$$

The sequence X is equivalent to the sequence of integers

$$K = (\ldots, k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots) \tag{34}$$

such that

$$k_s = \left[\frac{1}{x_{s-1}}\right]. \tag{35}$$

Conversely,

$$x_s = \cfrac{1}{k_{s+1} + \cfrac{1}{k_{s+2} + \ldots}} \equiv x_{s+1}^{+}\,, \tag{36}$$

$$x_s^{+} = [k_s, k_{s+1}, \ldots]\,, \tag{37}$$

$$x_s^{-} \equiv [k_{s-1}, k_{s-2}, \ldots]\,. \tag{38}$$

The shift of the entire sequence X to the right means a joint transformation

$$x_{s+1}^{+} = \left\{\frac{1}{x_s^{+}}\right\}, \quad x_{s+1}^{-} = \frac{1}{\left(\left[\dfrac{1}{x_s^{+}}\right] + x_s^{-}\right)}\,. \tag{39}$$

This is a one-to-one map in the unit square, which permits exactly calculating the probability distributions for other parameters describing successive eras, such as the parameter δ giving the relation between the amplitudes of the logarithms of the functions a, b, and c and the logarithmic time [18]. Thus, we see from the results of statistical analysis of evolution in the neighborhood of a singularity [17] that the stochasticity and probability distributions of parameters occur already in classical general relativity.

At the end of this section, a historical remark is in order. Continuous fraction (28) was shown in 1968 to I M Lifshitz (Landau had already passed away), and he immediately noticed that formula (29) for a stationary distribution of the values of x can be derived. Later, it became known that this formula was derived in the nineteenth century by Gauss, who had not published it but described it in a letter to a colleague.

4. Oscillatory approach to the singularity: modern development

The oscillatory approach to the cosmological singularity described in the preceding section was developed for empty spacetime. It is not difficult to understand that in a universe filled with a perfect fluid with the equation of state $p = w\rho$, where p is the pressure, ρ is the energy density, and $w < 1$, the presence of this matter cannot change the dynamics in the vicinity of the singularity. Indeed, using the energy conservation equation, it can be shown that

$$\rho = \frac{\rho_0}{(abc)^{w+1}} = \frac{\rho_0}{t^{w+1}}\,, \tag{40}$$

where ρ_0 is a positive constant. Therefore, the term representing matter in the Einstein equations behaves as $1/t^{1+w}$ and at $t \to 0$ is weaker than the terms of geometric origin coming from the time derivatives of the metric, which behave as $1/t^2$, to say nothing of perturbations due to the spatial curvature, which are responsible for changes to the Kasner regime and behave as $1/t^{2+4|p_1|}$. But the situation changes drastically if the parameter w is equal to unity, i.e., the pressure is equal to the energy density. Such kind of matter is called 'stiff matter' and can be represented by a massless scalar field. In this case, $\rho \sim 1/t^2$ and the contribution of matter is of the same order as the leading term of geometric origin. Hence, it is necessary to find a Kasner-type solution taking the presence of terms connected with stiff matter (a massless scalar field) into account. This was studied in [39]. It was shown that the scale factors a, b, and c can again be respectively represented as t^{2p_1}, t^{2p_2}, and t^{2p_3}, where the Kasner indices satisfy the relations

$$p_1 + p_2 + p_3 = 1\,, \quad p_1^2 + p_2^2 + p_3^2 = 1 - q^2\,, \tag{41}$$

where the number q^2 reflects the presence of stiff matter and is bounded by

$$q^2 \leqslant \frac{2}{3}\,. \tag{42}$$

It follows that if $q^2 > 0$, then combinations of positive Kasner indices satisfying relations (41) exist. Moreover, if $q^2 \geqslant 1/2$, only sets of three positive Kasner indices can satisfy relations (41). If a universe finds itself in a Kasner regime with three positive indices, the perturbative terms existing due to spatial curvature are too week to change this Kasner regime, and it therefore becomes stable. This means that in the presence of stiff matter, after a finite number of changes in

Kasner regimes, the universe finds itself in a stable regime and oscillations stop. Thus, the massless scalar field plays an 'antichaotizing' role in the process of cosmological evolution [39]. The Lifshitz–Khalatnikov parameter can also be used in this case. The Kasner indices satisfying relations (41) are conveniently represented as [39]

$$\begin{aligned} p_1 &= \frac{-u}{1+u+u^2}\,, \\ p_2 &= \frac{1+u}{1+u+u^2}\left[u - \frac{u-1}{2}\left(1-(1-\beta^2)^{1/2}\right)\right], \\ p_3 &= \frac{1+u}{1+u+u^2}\left[1 + \frac{u-1}{2}\left(1-(1-\beta^2)^{1/2}\right)\right], \\ \beta^2 &= \frac{2(1+u+u^2)^2}{(u^2-1)^2}\,. \end{aligned} \tag{43}$$

The range of u is now $-1 \leqslant u \leqslant 1$, while the admissible values of the parameter q at a given u are

$$q^2 \leqslant \frac{(u^2-1)^2}{2(1+u+u^2)^2}\,. \tag{44}$$

It can be easily shown that after one bounce, the value of q^2 changes according to the rule

$$q^2 \to q'^2 = q^2\,\frac{1}{(1+2p_1)^2} > q^2\,. \tag{45}$$

Hence, the value of q^2 increases and the probability of finding all three Kasner indices to be positive therefore increases. This again confirms the statement that after a finite number of bounces, in the presence of a massless scalar field, the universe finds itself in the Kasner regime with three positive indices and the oscillations stop.

In the second half of the 1980s, a series of papers was published [19–21] where solutions of the Einstein equations were studied in the vicinity of the singularity for $(d+1)$-dimensional spacetimes. A multidimensional analog of a Bianchi type-I universe was considered, where the metric is a generalized Kasner metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{i=1}^{d} t^{2p_i}\,\mathrm{d}x^{i\,2}\,, \tag{46}$$

where the Kasner indices p_i satisfy the conditions

$$\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} p_i^2 = 1\,. \tag{47}$$

In the presence of spatial curvature terms, a transition from one Kasner epoch to another occurs and is described by the following rule: the new Kasner exponents are equal to

$$p'_1, p'_2, \ldots, p'_d = \text{ordering of} \quad q_1, q_2, \ldots, q_d, \tag{48}$$

$$\begin{aligned} q_1 &= \frac{-p_1 - P}{1+2p_1+P}\,, \quad q_2 = \frac{p_2}{1+2p_1+P}\,, \ldots, \\ q_{d-2} &= \frac{p_{d-2}}{1+2p_1+P}\,, \quad q_{d-1} = \frac{2p_1+P+p_{d-1}}{1+2p_1+P}\,, \\ q_d &= \frac{2p_1+P+p_d}{1+2p_1+P}\,, \end{aligned} \tag{49}$$

where

$$P = \sum_{i=2}^{d-2} p_i\,. \tag{50}$$

However, such a transition from one Kasner epoch to another occurs if at least one of the numbers

$$\alpha_{ijk} \equiv 2p_i + \sum_{l \neq j,k,i} p_l \quad (i \neq j,\ i \neq k,\ j \neq k)\,. \tag{51}$$

is negative. For spacetimes with $d < 10$, one of the α is always negative, and hence one change in the Kasner regime is followed by another, implying the oscillatory behavior of the universe in the neighborhood of the cosmological singularity. But for spacetimes with $d \geqslant 10$, there exist such combinations of Kasner indices that satisfy Eqn (47) and for which all the α_{ijk} are positive. If a universe enters the Kasner regime with such indices (the so-called "Kasner stability region"), its chaotic behavior disappears and this Kasner regime preserves itself. The hypothesis was put forward that in spacetimes with $d \geqslant 10$, after a finite number of oscillations, the universe under consideration finds itself in the Kasner stability region and the oscillating regime is replaced by a monotonic Kasner behavior.

The discovery that the chaotic character of the approach to the cosmological singularity disappears in spacetimes with $d \geqslant 10$ was unexpected and looked like an accidental result of an interplay between real numbers satisfying generalized Kasner relations (49). It later became clear that a deep mathematical structure, the hyperbolic Kac–Moody algebras, are underlying this fact. Indeed, in the series of works by Damour, Henneaux, Nicolai, and others (see, e.g., Ref. [16]) on cosmological dynamics in models based on superstring theories and living in 10-dimensional spacetime and on the $d+1 = 11$-dimensional supergravity model, it was shown that these models reveal a BKL-type oscillating behavior in the vicinity of the singularity. The important new feature of the dynamics in these models is the role played by nongravitational bosonic fields (p-forms), which are also responsible for transitions from one Kasner regime to another. For a description of these transitions, the Hamiltonian formalism [16] is very convenient.

In the framework of this formalism, the configuration space of the Kasner parameters describing the dynamics of the universe can be treated as billiards, while the curvature terms in the Einstein theory and the p-form potentials in superstring theories play the role of cushions on these billiard tables. The transition from one Kasner epoch to another is the rebound off one of the cushions. There is a correspondence between the rather complicated dynamics of a universe in the vicinity of the cosmological singularity and the motion of an imaginary ball on a billiard table.

However, a more striking and unexpected correspondence exists between the chaotic behavior of the universe in the vicinity of the singularity and such an abstract mathematical object as the hyperbolic Kac–Moody algebras [23–25]. We briefly explain what this means. Every Lie algebra is defined by its generators h_i, e_i, f_i, $i = 1, \ldots, r$, where r is the rank of the Lie algebra, i.e., the maximal number of its generators h_i that commute with each other (these generators constitute the Cartan subalgebra). The commutation relations between generators are

$$\begin{aligned} [e_i, f_j] &= \delta_{ij}\,h_i\,, \\ [h_i, e_j] &= A_{ij}\,e_j\,, \\ [h_i, f_j] &= -A_{ij} f_j\,, \\ [h_i, h_j] &= 0\,. \end{aligned} \tag{52}$$

The coefficients A_{ij} constitute the $r \times r$ generalized Cartan matrix, such that $A_{ii} = 2$, its off-diagonal elements are nonpositive integers, and $A_{ij} = 0$ for $i \neq j$ implies $A_{ji} = 0$. The e_i may be called raising operators, similar to the well-known operator $L_+ = L_x + \mathrm{i}L_y$ in the theory of angular momentum, while the f_i are lowering operators like $L_- = L_x - \mathrm{i}L_y$. The generators h_i of the Cartan subalgebra can be compared with the operator L_z. The generators must also satisfy the Serre relations

$$(\mathrm{ad}\ e_i)^{1-A_{ij}}\, e_j = 0\,,$$
$$(\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0\,, \tag{53}$$

where $(\mathrm{ad}A)B \equiv [A, B]$.

The Lie algebras $\mathcal{G}(A)$ built on a symmetrizable Cartan matrix A have been classified according to the properties of their eigenvalues:

if A is positive definite, $\mathcal{G}(A)$ is a finite-dimensional Lie algebra;

if A admits one zero eigenvalue and the others are all strictly positive, $\mathcal{G}(A)$ is an affine Kac–Moody algebra;

if A admits one negative eigenvalue and all the others are strictly positive, $\mathcal{G}(A)$ is a Lorentz KM algebra.

A correspondence exists between the structure of a Lie algebra and a certain system of vectors in an r-dimensional Euclidean space, which essentially simplifies the task of classification of the Lie algebras. These vectors, called roots, represent the raising and lowering operators of the Lie algebra. The vectors corresponding to the generators e_i and f_i are called simple roots. The system of positive simple roots (i.e., those roots corresponding to the raising generators e_i) can be represented by the nodes of Dynkin diagrams, while the edges connecting (or not connecting) the nodes give information about the angles between simple positive root vectors.

An important subclass of Lorentz KM algebras can be defined as follows: a KM algebra such that the deletion of one node from its Dynkin diagram gives a sum of finite or affine algebras is called a hyperbolic KM algebra. These algebras are all known. In particular, no hyperbolic algebras exist with a rank higher than 10.

We recall some more definitions from the theory of Lie algebras. Reflections with respect to hyperplanes orthogonal to simple roots leave the systems of roots invariant. The corresponding finite-dimensional group is called the Weyl group. Finally, the hyperplanes mentioned above divide the r-dimensional Euclidean space into regions called Weyl chambers. The Weyl group transforms one Weyl chamber into another.

Now, we can briefly formulate the results of the approach in [40] following papers [23–25]: the links between the billiards describing the evolution of the universe in the neighborhood of a singularity and their corresponding Kac–Moody algebra can be described as follows:

the Kasner indices describing the 'free' motion of the universe between rebounds from the cushions correspond to elements of the Cartan subalgebra of the KM algebra;

the dominant cushions, i.e., the terms in the equations of motion responsible for the transition from one Kasner epoch to another, correspond to simple roots of the KM algebra;

the group of reflections on the cosmological billiard table is the Weyl group of the KM algebra;

the billiard table can be identified with the Weyl chamber of the KM algebra.

Two types of billiard tables can be imagined: infinite ones where linear motion without collisions with the cushions is possible (nonchaotic regime), and those where rebounds from the cushions are inevitable and the regime can only be chaotic. Remarkably, Weyl chambers of hyperbolic KM algebras are designed such that infinitely repeating collisions with the cushions occur. It has been shown that all the theories with the oscillating approach to the singularity such as the Einstein theory in dimensions $d < 10$ and superstring cosmological models correspond to hyperbolic KM algebras.

The existence of links between the BKL approach to the singularity and the structure of some infinite-dimensional Lie algebras has inspired some authors to declare a new program of development of quantum gravity and cosmology [28]. They propose "to take seriously the idea that near the singularity (i.e. when the curvature gets larger than the Planck scale) the description of a spatial continuum and space-time based (quantum) field theory breaks down, and should be replaced by a much more abstract Lie algebraic description. Thereby the information previously encoded in the spatial variation of the geometry and of the matter fields gets transferred to an infinite tower of Lie-algebraic variables depending only on 'time'. In other words we are led to the conclusion that space–and thus, upon quantization also space-time — actually *disappears* (or 'de-emerges') as the singularity is approached."

5. Conclusion: some thoughts about the past and future of the Universe

In the preceding section, we outlined the newest developments in the theory of the BKL approach to the cosmological singularity connected with superstring-inspired cosmological models and infinite-dimensional Lie algebras. But already in the 'standard' $(3 + 1)$-dimensional general relativity, the effect of the oscillatory approach to the singularity and the chaoticity implied by it is of great interest. Indeed, the discovery of nonstatic time-dependent cosmological solutions in general relativity, first and foremost the Friedmann solutions, has given birth to animated discussions on such questions as:

Did the Universe have a beginning ?

Will the Universe have an end?

Can the Universe exist during a *finite* interval of time?

What was *before* the beginning and what will be *after* the end?

These questions look quite reasonable because we know that in all three Friedmann models — flat, open, and closed — the Universe has a beginning and this beginning is nothing but the Big Bang singularity. In the closed Friedmann model, the Universe also has the end — the Big Crunch singularity — and exists during a finite period of time. Moreover, according to the so-called Standard Cosmological Model, based on a rather large set of observational data, something like the Big Bang took place approximately 13.7 billion years ago (measured in terms of cosmic, i.e., synchronous, time). The more or less accepted existence of the beginning of the evolution of the Universe and the possible existence of the end of the Universe can be a source of joy for those who believe in the creation of the Universe and for whom its possible end can also confirm their philosophical or theological beliefs. It is curious that the Pontifical Academy of Sciences organized a special conference at the Vatican in October–November of 2008 with the title "Scientific insights into the evolution of the universe and of life." On the other hand, the possibility of a finite-time existence of the Universe can provoke some kind of psychological discomfort in those for whom this finite duration seems to be senseless. For some, the fact that their own existence takes place in the Universe that exists only for a finite period of time can appear depressing.

In analyzing these aspects of the problem of the evolution of the Universe, we should ask ourself which time parametrization we should use in speaking about the time of the existence of the Universe. As we know since the creation of special relativity, time is relative. In the framework of general relativity, time becomes even more relative and can run with different rates at different spatial points. Making conformal transformations (for example, in constructing the Penrose conformal diagrams [6 – 8]), we can turn an infinite time interval into a finite time interval. Why should we then use cosmic time? The answer to this question is simple: cosmic time for a particle staying at rest in a Friedmann homogeneous and isotropic Universe coincides with the proper time introduced in special relativity. Hence, when we are considering the present-day Universe, it is quite reasonable to discuss it in terms of cosmic time and to say that the Universe was created 13.7 billion years ago. But when we consider the vicinity of the Big Bang cosmological singularity in the past, or when we admit the possibility of the existence of a Big Crunch singularity in the distant future, the situation changes drastically. The Universe is extremely anisotropic in the neighborhood of such singularities and is described by a chaotic succession of Kasner epochs and eras, as was discussed above. (We can be precise here: by choosing very special isotropic initial conditions, we can avoid the chaoticity in the neighborhood of the Big Bang singularity, which can have the Friedmannian form in principle; it is impossible not to have a chaotic regime in the vicinity of the Big Crunch singularity, because the inhomogeneities developed during the evolution of the Universe make its contracting stage highly anistropic [41]).

Therefore, while the evolution from an arbitrary instant of cosmic time to the instant corresponding to the initial Big Bang or final Big Crunch singularity occupies a finite interval of cosmic time, an infinite number of events occurs during this finite period. The infinite chaotic succession of Kasner epochs and eras renders cosmic time as a measure of cosmological evolution senseless. Indeed, we have an infinite history that separates us from the birth of the Universe at the Big Bang. If the contraction of the Universe culminating in the encounter with the Big Crunch singularity awaits us in the future [42, 43], we still have an infinite number of events in front of us. Thus, the BKL oscillatory regime of approaching the cosmological singularity screens us from the Big Bang and the Big Crunch.

From the mathematical standpoint, this means that the natural time parameter in the vicinity of a singularity is not cosmic time but logarithmic time. As cosmic time runs from the zero instant corresponding to the singularity to some finite instant t_1, logarithmic time runs from $-\infty$ to $\ln t_1$, spanning an infinite interval of time.

Remarkably, a comment concerning the importance of logarithmic time can already be found in the penultimate paragraph of the Landau and Lifshitz monograph [29]: "The successive series of oscillations crowd together as we approach the singularity. An infinite number of oscillations are contained between any finite world time t and the moment $t = 0$. The natural variable for describing the time behavior of this evolution is not the time t, but its logarithm, $\ln t$, in terms of which the whole process of approach to the singular point is spread out to $-\infty$."

A similar idea is also expressed in paper [28] cited above: "There is no 'quantum bounce' bridging the gap between an incoming collapsing and an outgoing expanding quasiclassical universe. Instead 'life continues' at the singularity for an *infinite affine time*, but with the understanding that (i) dynamics no longer "takes place" in space, and (ii) the infinite affine time [measured, say, by the Zeno-like time coordinate t] corresponds to a sub-Planckian interval $0 < T < T_{\mathrm{Planck}}$ of geometrical proper time." Curiously, the analog of the object that the authors of [28] call Zeno-like time is the so-called spatial tortoise coordinate in the Schwarzschild geometry [44]. Both these names have their origin in Zeno's paradox about Achilles and the tortoise, which is, perhaps, the first example of transforming a finite time interval into an infinite one (see, e.g., section I of the third part of the third volume of *War and Peace* by Leo Tolstoy [45]).

Concluding, I would like to say semiseriously that the discovery of the oscillatory approach to the cosmological singularity has a practical meaning: it liberates us from the fear of the end of the world.

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Above the barriers

(I M Khalatnikov's works on the scattering of high-energy particles)

V L Pokrovsky

Relatively recently, in the fall of 1957, I had the good fortune to speak at Landau's seminar on the over-barrier reflection of high-energy particles. I was then working in Novosibirsk, at the Institute of Radiophysics, whose director was one of my teachers Yu B Rumer, and he introduced me to Landau. My coauthors were my fellow students and friends S K Savvinykh and F R Ulinich [1, 2]. The reflection of particles whose energy exceeds the barrier height is a strictly quantum effect: a classical particle just slows down as it approaches the tip of the barrier and then accelerates. We solved the Schrödinger equation in the semiclassical approximation, formally expanding it into a power series in the small parameter λ/a, where λ is the de Broglie wavelength and a is a characteristic size of the potential. The peculiarity of the problem, which was not noticed by other theoreticians, lay in the fact that each consecutive term of the expansion contained a singularity of a higher order than the previous one. As a result, they differed only by universal numerical factors. It turned out to be possible to sum this numerical series using an exactly solvable problem. Landau liked the work, and I was invited to present it at his seminar. Following my talk, I met many celebrities whom I had previously known only through their publications and from legends. Isaak Markovich Khalatnikov showed the most vivid interest. He proposed collaborating, which was flattering for me. He explained his interest by a mission assigned to him by Landau to find a mistake in L Schiff's work on the same topic. This explanation sounded somewhat strange, because we had already found the mistake. Only later did I realize that I became an object of his most sincere and absolutely disinterested affection to any fledgling theorist who came up with an interesting idea. Just this property later made him an ideal director of the Institute of Theoretical Physics and let him gather a unique team, which quickly gained worldwide recognition. I hope, however, that our relationship involved some individual element, the proof of which is our friendship and longstanding research collaboration, which extended to 1992. It would probably have lasted even longer if it had not been interrupted by the turbulent events of that time. The close rapprochement needed for collaborative work became possible due to another of Khalatnikov's rare qualities: his complete lack of both arrogance and servility, as well as his simple and calm way of communicating.

We both realized that the work I presented was just the beginning. Although the method of series summation led to a beautiful and nontrivial result, it was still not physically transparent. It was not clear how to generalize it to similar problems of quantum and classical mechanics. Contemplating this problem, we came to the following idea [3]. Classical and semiclassical particles are reflected at a turning point, where their kinetic energy becomes zero. If the particle energy exceeds the height of the barrier, no turning point exists at a real value of its coordinate. But it appears in the complex coordinate plane if the potential is an analytic function. Going into a complex plane is a rather common operation in quantum mechanics. Going into a complex momentum plane is physically equivalent to tunneling, i.e., penetration into the region of classically forbidden coordinates. Similarly, going into the complex coordinate plane means penetration into the region of classically forbidden momenta. Therefore, we needed to find a suitable path in the complex plane along which a wave travels without reflection to a complex turning point, and then strongly changes in its vicinity. Then the path goes to the real axis, where we can find the reflected wave. In practice, this program was accomplished as shown in Fig. 1. The path begins on the real coordinate axis x at $x \to \infty$. In this region, where the potential can be neglected, only the transmitted wave $\Psi \sim t \exp(ikx)$ exists, where t is the transmission amplitude. After that, the path climbs in the upper half of the complex plane until it intersects with the line C_1 going through the turning point x_0 nearest to the real axis, on which the semiclassical action $S(x, x_0) = \int_{x_0}^{x} p(x')\,dx'$,

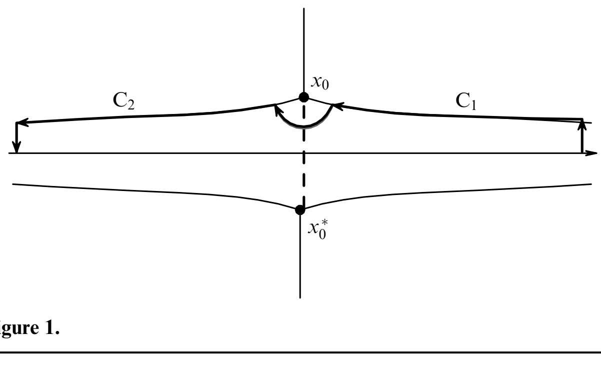

Figure 1.