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Gravitational radiation of systems and the role of their force field

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Contents

Abstract. Gravitational radiation (GR) from compact relativistic systems with a known energy–momentum tensor (EMT) and GR from two masses elliptically orbiting their common center of inertia are considered. In the ultrarelativistic limit, the GR spectrum of a charge rotating in a uniform magnetic field, a Coulomb field, a magnetic moment field, and a combination of the last two fields differs by a factor of $4\pi G m^2 \Gamma^2 / e^2$ (Γ being of the order of the charge Lorentz factor) from its electromagnetic radiation (EMR) spectrum. This factor is independent of the radiation frequency but does depend on the wave vector direction and the way the field behaves outside of the orbit. For a plane wave external field, the proportionality between the gravitational and electromagnetic radiation spectra is exact, whatever the velocity of the charge. Qualitative estimates of Γ are given for a charge moving ultrarelativistically in an arbitrary field, showing that it is of the order of the ratio of the nonlocal and local source contributions to the GR. The localization of external forces near the orbit violates the proportionality of the spectra and reduces GR by about the Lorentz factor squared. The GR spectrum of a rotating relativistic string with masses at the ends is given, and it is shown that the contributions by the masses and string are of the same order of magnitude. In the nonrelativistic limit, the harmonics of GR spectra behave universally for all the rotating systems considered. A trajectory method is developed for calculating the GR spectrum. In this method, the spatial (and hence polarization) components of the conserved EMT are calculated in the long wavelength approximation from the time component of the EMTs of the constituent masses of the system. Using this method, the GR spectrum of two masses moving in elliptic orbits about their common center of inertia is calculated, as are the relativistic corrections to it.

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1. Introduction

The source of gravitational radiation (GR) in the general theory of relativity is the conserved total energy–momentum tensor (EMT) of the system. At the same time, the EMT of the gravitational field is not an unambiguously defined quantity [1, 2], which is one of the reasons why the problem of radiation by gravitational waves is in general so complicated and far from a definitive solution despite the considerable efforts of researchers [1].

For this reason, a detailed study of GR (in the general case, of gravitational fields) from a simple electrodynamic system, such as a charged particle moving in an electromagnetic field, is of great interest. First, such GR shares several common features with the GR of a body moving in a gravitational field. Second, the enormous energy of particles in accelerators being designed opens up the possibility for the experimental verification of the ultrarelativistic effects of the general theory of relativity [3–5]. Equally interesting is to elucidate what information about the dynamic properties of a given system is transmitted by its GR, in particular, to what extent it supersedes the information conveyed by the electromagnetic radiation (EMR) of the system and under which conditions GR acquires the known characteristics of EMR.

As is known from electrodynamics [6, 7], the spectrum of classical EMR of a charge is totally determined by the Fourier components of the conserved current density $j_\alpha(q)$:

$$\mathrm{d}\mathcal{E}_{\mathbf{q}} = \left|j_\alpha(q)\right|^2 \frac{\mathrm{d}^3 q}{16\pi^3}\,, \quad j_\alpha(q) = e\int_{-\infty}^{\infty} \mathrm{d}\tau\, \dot{x}_\alpha(\tau) \exp\left[-\mathrm{i}qx(\tau)\right], \tag{1.1}$$

i.e., by its trajectory $x_\alpha(\tau)$, and is unrelated to the nature of forces driving the charge along this trajectory.

On the other hand, the spectrum of the classical GR of a body with mass m moving along a trajectory $x_\alpha(\tau)$ is determined by Fourier components of the conserved EMT $T_{\alpha\beta}(q)$ of the entire system [7, 8]:

$$\mathrm{d}\mathcal{E}_{\mathbf{q}} = 8\pi G\left[T_{\alpha\beta}(q)T^{\alpha\beta *}(q) - \frac{1}{2}\left|T^\alpha_\alpha(q)\right|^2\right]\frac{\mathrm{d}^3 q}{16\pi^3}\,. \tag{1.2}$$

Because $T_{\alpha\beta}(q)$ is the sum of the EMT of the body in question,

$$t_{\alpha\beta}(q) = m\int_{-\infty}^{\infty}\mathrm{d}\tau\,\dot{x}_\alpha(\tau)\,\dot{x}_\beta(\tau)\exp\left[-\mathrm{i}qx(\tau)\right], \tag{1.3}$$

and the EMT of the force field responsible for moving the body along the given trajectory, the GR spectrum depends essentially on the nature of this field. An exception is the case of a nonrelativistically moving body (or bodies) forming a closed system together with the force field. Then GR becomes quadrupole-like, whatever the nature of the forces acting on the body, and is described by the well-known formulas [6]

$$\begin{aligned} \frac{\mathrm{d}\mathcal{E}_{\mathbf{q}}}{\mathrm{d}t} &= \frac{G}{4\pi}\left[\frac{1}{4}(\dddot{D}_{ij}\,n_i n_j)^2 + \frac{1}{2}\,\dddot{D}^{\,2}_{ij} - \dddot{D}_{ij}\dddot{D}_{ik}n_j n_k\right]\mathrm{d}\Omega\,, \\ \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} &= \frac{G}{5}\,\dddot{D}^{\,2}_{ij}\,, \qquad D_{ij} = \int t_{00}(\mathbf{x}, q^0)\left(x_i x_j - \frac{1}{3}\,r^2\delta_{ij}\right)\mathrm{d}^3 x\,, \end{aligned} \tag{1.4}$$

containing the quadrupole moment D_{ij} of the moving mass distribution; here, $\mathbf{n} = \mathbf{q}/q^0$. On the other hand, in the general relativistic case, the GR spectrum gives an idea of the dynamic properties of its source, and this fact is of considerable interest.

In Section 2, we consider the GR of a body with mass m and charge e driven by electromagnetic forces in a uniform magnetic field, in the Coulomb field of a heavy center, and in a plane-wave field with circular or linear polarizations [9]. Although the GR spectrum in each of these cases has specific features, it coincides with the spectrum $|j_\alpha(q)|^2$ of EMR in the relativistic limit up to the replacement of the squared charge e^2 by the quantity $4\pi G m^2\Gamma^2$, where Γ is proportional to the effective Lorentz factor of the moving body and essentially depends on the character of the external field. In this limit, the radiation wave vector $\mathbf{q}$ is pinned to the plane of the body motion, forming a small angle $\alpha \lesssim \gamma^{-1} \ll 1$ with it, and the radiation frequency is γ^3 times higher than the fundamental frequency ω, $q^0 \equiv |\mathbf{q}| \sim \gamma^3\omega$. Hence, at $\gamma \gg 1$, in the effective range of frequencies and angles of radiation, we have the relation

$$8\pi G\left[T_{\mu\nu}(q)T^{\mu\nu *}(q) - \frac{1}{2}\left|T^\mu_\mu(q)\right|^2\right] \approx \frac{4\pi G m^2\Gamma^2}{e^2}\left|j_\mu(q)\right|^2. \tag{1.5}$$

Because any external electromagnetic field looks like a plane wave in the rest frame of a relativistically moving body, relation (1.5) can be expected to be exact rather than approximate for the GR spectrum of a body with mass m and charge e moving in a plane electromagnetic wave field. Indeed (as is shown in Section 2.3 below), the following strict relation holds for the GR spectrum in the case of such motion:

$$\begin{aligned} &T_{\mu\nu}(q)T^{\mu\nu *}(q) - \frac{1}{2}\left|T^\mu_\mu(q)\right|^2 = \frac{m^2\Gamma^2}{2e^2}\left|j_\mu(q)\right|^2, \\ &\Gamma = \frac{m_* q_\perp}{m q_-} = \gamma_*\cot\frac{\theta}{2}\,, \end{aligned} \tag{1.6}$$

where m_* is the effective mass of the charge equal to its mean kinetic energy in the system of coordinates where it is at rest on average, q_3 and $\mathbf{q}_\perp$ are parallel and perpendicular components of the radiation wave vector $\mathbf{q}$ with respect to the momentum of the wave $\mathbf{k}$, $q_- = q^0 - q_3$, and θ is the angle between $\mathbf{q}$ and $\mathbf{k}$. The effective Lorentz factor γ_* and the velocity v_* are defined by the relation $m_* = m\gamma_* = m(1 - v_*^2)^{-1/2}$.

The value of Γ in the case of ultrarelativistic motion in a circularly polarized wave for the aforementioned high GR frequencies tends to the Lorentz factor γ because, with the plane of the motion orthogonal to the vector $\mathbf{k}$, the angle θ differs from $\pi/2$ by no more than γ^{-1} and the effective mass coincides with the body constant kinetic energy in the system under consideration.

For ultrarelativistic motion in a linearly polarized wave, the quantity Γ is given by formula (1.6) as before; the ultrarelativism only bounds the range of effective angles θ from below: $2\,\mathrm{arccot}\sqrt{2} \leqslant \theta \leqslant \pi$. Γ remains θ-dependent because, as the charge moves along the figure-eight trajectory lying in the plane containing the vector $\mathbf{k}$, the angle between its velocity $\mathbf{v}$ and the vector $\mathbf{k}$ takes all values between $\theta_0 = \arctan(2\sqrt{2}/v_*)$ and π four times (see [6]). In the ultrarelativistic limit, this range of angles widens ($\theta_0 \to 2\,\mathrm{arccot}\sqrt{2}$ at $v_* \to 1$) and restricts the effective radiation angle θ because radiation becomes pinned to the direction of velocity.

We emphasize that the value of $\Gamma(\theta)$ does not coincide with the Lorentz factor of the body at a point where its velocity $\mathbf{v}$ makes the angle θ with the vector $\mathbf{k}$ for the following reason. Although GR is emitted by an ultrarelativistically moving charged body along its velocity vector, it is formed in the region of the order of the trajectory mean curvature radius, whereas EMR is emitted in the ultrarelativistic limit along the velocity vector of the charge and is formed at the segment of the trajectory γ times smaller than the local curvature radius.

The extended region where GR forms is preserved even in the case of ultrarelativistic motion of the body because emission of GR by a local source (the EMT of the body $t_{\mu\nu}$) is accompanied by the emission of GR from an extended source (the EMT $\theta_{\mu\nu}$ of the external and proper electromagnetic fields). The latter mechanism consists of emission of a virtual photon by a local source (the current j_μ), giving rise to a real graviton due to gravitational interaction with a quantum of the external electromagnetic field. In the ultrarelativistic limit, the frequency of the virtual photon is γ^3 times the fundamental frequency ω (defined by the curvature radius r of the trajectory, $\omega \sim c/r$) and its 'mass' is of the order of $\gamma^{3/2}\omega$, i.e., is small compared with the frequency. For this reason, such a photon is emitted almost as a real one, in the direction of the charge velocity vector, and is formed at a small segment ($\sim c/\gamma\omega$) of the charge trajectory, but its gravitational interaction with a quantum of the external field occurs along a length of the order of that quantum wavelength ($\sim c/\omega$). Evidently, the graviton energy and momentum actually coincide with those of the virtual photon, but the probability of the appearance of the graviton depends on the state of the external field over the graviton formation length. The extension of the range of GR formation from $\sim c/\omega\gamma$ to $\sim c/\omega$ results in Γ differing from the Lorentz factor γ at the moment of photon emission and in its dependence on the external field structure. Also important is the fact that the two above mechanisms of GR are coherent. As the charge moves in a plane-wave field, their interference leads to the suppression of GR at the angle $\theta = \pi$, although EMR is not forbidden at this angle.

On the other hand, $\Gamma(\theta)$ diverges as θ^{-1} at $\theta \to 0$ and leads to the logarithmic singuliarity $\mathrm{d}\theta/\theta$ in the GR spectrum due to a finite and nonzero current density at the point $\theta = 0$. The appearance of this singularity is related to the fact that as $\theta \to 0$, the leading role in the emission of gravitons is played by the second, nonlocal mechanism. Its amplitude is the sum of two amplitudes proportional to the propagators $(q \pm k)^{-2}$ of the virtual photons present in the source of gravitons, i.e., the EMT of the field of these photons and the plane electromagnetic wave. As $\theta \to 0$, the photon propagators become infinite because $(q \pm k)^{-2} = \mp(2\omega q^0)^{-1}(1 - \cos\theta)^{-1} \approx \mp 1/\omega q^0 \theta^2$, while the remaining factor of the transverse EMT components tends to zero as θ (the electromagnetic field of the photons and the field of the plane electromagnetic wave propagating in the same direction again constitute a plane electromagnetic wave, which, as is well known, cannot be a source of gravitons because its EMT has no transverse components [6]). As a result, the amplitude and therefore Γ diverge like θ^{-1} as $\theta \to 0$.

Because the current density at $\theta = 0$ differs from zero only for the fundamental frequency $q^0 = \omega$, the singularity of the GR spectrum at this point may be due to the subnormal mass ($\sim \omega\theta$) of virtual photons and therefore the very large region ($\sim 1/\omega\theta$) of formation of GR emitted at such small angles.

Such enhancement of the nonlocal mechanism also occurs in the GR process of a charge moving in a constant uniform magnetic field (Section 3) if the field remains uniform over a length l much greater than the radiation wavelength λ, i.e., at $l \gg \lambda$. In this case, the external field is characterized by the wave vector k_α such that $|\mathbf{k}| \sim l^{-1}$ and $k^0 = 0$, and hence the virtual photons emitted by the current j_α have a very small mass $|(q \pm k)^2| \sim q^0 l^{-1}$. If $l = \infty$, $T_{\mu\nu}(q)$ has a pole at $q^2 = 0$. Then the GR spectrum can be represented as an expansion in powers of q^2/ω^2 (ω is the angular frequency in a circular orbit):

$$T_{\mu\nu}(q)T^{\mu\nu *}(q) - \frac{1}{2}\left|T^\mu_\mu(q)\right|^2 = \left(\frac{\omega^2}{q^2}\right)^2 a_{-2} + \frac{\omega^2}{q^2}\,a_{-1} + a_0 + \ldots, \tag{1.7}$$

where the leading term is proportional to the EMR spectrum,

$$a_{-2} = \frac{2m^2\gamma^2 q_\perp^2}{e^2\omega^2}\left|j_\mu(q)\right|^2. \tag{1.8}$$

The proportionality coefficient actually coincides with the corresponding coefficient in (1.6) for the circularly polarized wave. Indeed, the coefficient in (1.6) follows from (1.7) and (1.8) by the replacement $q \to q \pm k$, where k_α is the wave vector with the components $k_3 = k^0 = \omega$, $k_1 = k_2 = 0$:

$$\frac{2m^2\gamma^2\omega^2 q_\perp^2}{e^2 q^4} \to \frac{m^2\gamma^2 q_\perp^2}{2e^2 q_-^2}. \tag{1.9}$$

For finite $l \gg \lambda$, formula (1.7) acquires dependence on the falloff of the magnetic field at distances $\sim l$. For example, for a Gaussian falloff of the field in the motion plane $H(x) = H_0 \exp(-x_\perp^2/l^2)$, q^{-4} in the first term in (1.7) should be replaced by $\pi l^2/16 q_\perp^2$ because the field is characterized in this case by the wave vector $k_{1,2} \sim l^{-1}$, $k_3 = k^0 = 0$. Then the second term vanishes.

Formulas (1.6), (1.7), and (1.8) clearly demonstrate the differences between the GR spectra for different electromagnetic force fields driving a massive charge in the same orbit. Thus, the formation of the GR of a massive charge moving in the external electromagnetic field is greatly promoted by the nonlocal mechanism with the participation of virtual photons. In certain cases (movement in a plane-wave or constant uniform field, ultrarelativistic motion), this mechanism leads to proportionality between the GR and EMR spectra, with the proportionality coefficient carrying information about the nonlocal mechanism and the form of the external field.

Such a relation between the GR and EMR spectra disappears if the external electromagnetic field is replaced by a local force field. We show this with the example of the GR of a body elastically colliding with very massive but small balls arrayed circumferentially at regular intervals such that the resulting motion of the body in the limit of a large number of balls is a uniform circular motion (or motion along a ring-like trough).

In electrodynamics, the current density conservation ensures that its invariant square determining the EMR spectrum can be expressed through transverse current components in the system of coordinates with the 3-axis along the wave vector $\mathbf{q}$, $q^0 = |\mathbf{q}|$:

$$\left|j_\alpha(q)\right|^2 = \left|j_1'(q')\right|^2 + \left|j_2'(q')\right|^2. \tag{1.10}$$

Similarly, the EMT conservation allows expressing invariant (1.2) determining the GR spectrum in terms of the EMT transverse components in the system of coordinates with the 3-axis along $\mathbf{q}$:

$$T_{\alpha\beta}(q)T^{\alpha\beta *}(q) - \frac{1}{2}\left|T^\alpha_\alpha(q)\right|^2 = \frac{1}{2}\left|T_{11}'(q') - T_{22}'(q')\right|^2 + 2\left|T_{12}'(q')\right|^2. \tag{1.11}$$

The use of transverse (polarization) EMT components substantially facilitates the description of a GR system and, specifically, the estimation of contributions by the field and matter components of the EMT. For example, it follows from expression (3.5) for the EMT of the field and formula (3.4) (Section 3.1) that by the order of magnitude, the space components of the field EMT are

$$\theta_{ik}(q) \sim F l j \,, \tag{1.12}$$

where $j = j(q - k_{\rm ef})$, and F is the external field strength in the formation region of the radiated photon. The length l is determined by the photon propagator:

$$\left[(q-k)^2 - {\rm i}\varepsilon\right]^{-1}_{\rm ef} \sim \frac{1}{|\mathbf{q}||\mathbf{k}|_{\rm ef}} \sim \frac{l}{|\mathbf{q}|} \,, \tag{1.13}$$

i.e., it is the photon path length in the field before the photon–graviton conversion. For $\gamma \gg 1$, relation (1.5) is due to the contribution from the field EMT, and the estimated relative amplitude of the conversion (as we refer to Γ hereafter) is

$$\Gamma \sim \frac{eFl}{mc^2} \,. \tag{1.14}$$

We first consider trajectories with a turning angle of the order of 1 or greater (including finite trajectories), for which the curvature radius determined by the motion law $eF \sim mv^2\gamma/r$ does not exceed the field size. If the field varies appreciably at distances much greater than r, then $l \gg r$ and $\Gamma \sim \gamma l/r \gg \gamma$. Such a situation occurs for the GR of a charge moving in a uniform magnetic field extending far beyond the orbit (see Section 2.2). For a field changing considerably at distances of the order of the orbit radius (circular motion in a Coulomb field, see Section 2.4, a magnetic moment field, or a combination of these two fields; see Sections 3.1 and 3.2), $\Gamma \sim \gamma \gg 1$. Finally, for a field extending along the trajectory as far as $l_{\|} \gtrsim r$ but having the small transverse size $l_{\perp} \ll r$ (as in modern cyclic accelerators), $\Gamma \sim \gamma l/r \ll \gamma$, where $l \sim (l_{\perp} r)^{1/2} \ll r$ [see a remark to this effect in Section 3.2 and formula (3.31)]. With this in mind, we cannot agree with the authors of [4] that the GR intensity in existing and projected accelerators can be estimated from the formula for the EMR intensity with the squared charge $e^2/4\pi$ substituted by $Gm^2\gamma^2$; the small factor $l_{\perp}/r$ must be taken into account.

We next consider trajectories with a small rotation angle $\Delta\varphi \lesssim 1$, i.e., infinite trajectories in a field of small extension compared with the curvature radius. Using the motion law $eF \sim mv\gamma\,\Delta\varphi/\Delta t$ and $l \sim v\Delta t$, we obtain $\Gamma \sim \gamma\Delta\varphi \lesssim \gamma$ in accordance with (1.14). This estimate is valid at $\gamma^{-1} \lesssim \Delta\varphi \lesssim 1$. If $\Delta\varphi \lesssim \gamma^{-1}$, the following consideration should be borne in mind. Formula (1.5) relates GR to the EMR produced by the current components transverse to $\mathbf{q}$ at $q^2 = 0$. But a contribution to (1.12) also comes from off-shell current components unrelated to particle acceleration. They are a factor of $1/\Delta\varphi$ larger than the acceleration-related components and have components orthogonal to $\mathbf{q}$ that are $\sim 1/\gamma\Delta\varphi$ times the transverse components responsible for EMR. Consequently, an additional factor $1/\gamma\Delta\varphi$ appears in the above estimate at $\Delta\varphi \ll \gamma^{-1} \ll 1$. Thus, at $\Delta\varphi \lesssim 1$,

$$\Gamma \sim \gamma\Delta\varphi\left(1 + \frac{1}{\gamma\Delta\varphi}\right) = 1 + \gamma\Delta\varphi \,. \tag{1.15}$$

In Section 3, we consider the GR spectrum of a charge rotating in the equatorial plane of a magnetic moment field and show that in the relativistic limit, it coincides with the GR spectrum of a charge rotating in a Coulomb field if the parameter $e\mathfrak{M}/r$ of the charge–field interaction is replaced with $-ee'$. Moreover, the GR spectrum of a charge rotating ultrarelativistically in the combined Coulomb and magnetic moment field is described by the same formula with the quantity in the left-hand side of inequality (3.29) as the interaction parameter. In other words, the GR spectrum is in all cases characterized by the conversion amplitude $\Gamma = \gamma$ [11].

To elucidate the dependence of GR on the field EMT spatial distribution, we considered the GR spectrum of a relativistic string with point masses at the ends and showed that in the relativistic limit, the contributions from the string and the masses are of the same order of magnitude even though the string energy is by the order of magnitude γ times higher than the mass energy.

It has been shown that the GR spectrum of a closed system consisting of a point mass rotating in a central field, in the nonrelativistic approximation when the GR wavelength is much greater than the system size, has a universal character: the leading term of each GR harmonic is independent of the nature of the central field and for all harmonics with $n \geqslant 2$, the contribution of the field EMT is $n-1$ times smaller than that of the mass EMT [11].

The trajectory method for calculating nonrelativistic corrections to GR from a weakly relativistic system is described in Section 4. The method is based on a differential equation relating the spatial components of a conserved EMT to its temporal component. For such systems, the spatial EMT components are small compared with the temporal one. Due to this, the differential equation allows finding them from the temporal component of the matter EMT for a given trajectory, with subsequent adjustment by including the relativistic corrections and taking account of the temporal component of the field EMT. This method was used to find relativistic corrections for all GR harmonics of two masses elliptically orbiting their common center of inertia.

Finally, Section 5 focuses on the GR spectra of an ultrarelativistic charge passing in the equatorial plane of a Coulomb field or a magnetic moment field, with the deflection angle assumed small, $\chi \lesssim \gamma^{-1} \ll 1$. It was shown that the spectra coincide in the effective wave vector region despite somewhat different trajectories and are characterized by a conversion amplitude dependent on the wave vector direction and the deflection angle.

The requirement of ultrarelativism may be relaxed to relativism to obtain a more general expression for the electromagnetic current squared; it is not segregated from the GR spectrum, however, but may serve to evaluate it to the order of magnitude.

Notation. Greek letters $\alpha, \beta, \mu, \nu, \ldots$ take values 1, 2, 3, 0 and Latin letters $i, j, k, l, \ldots$ take values 1, 2, 3. We use the metric $g_{\alpha\beta} = {\rm diag}\,(1,1,1,-1)$, Heaviside units for the charge and electromagnetic field and the speed of light $c = 1$, except when relativism has to be emphasized and in Section 4.

2. Gravitational radiation from simple electromagnetic systems

2.1 Energy–momentum tensor conservation and the gravitational radiation spectrum of a body moving in a ring-like trough

We consider the trough as a system of very massive small balls with a mass M much bigger than the mass m of the elastic body colliding with them and EMTs in the form (1.3) with m

replaced with M. It is easy to see that the spatial components of these tensors are smaller by a factor of the order of $M/m\gamma$ than the corresponding EMT components of the moving body. Therefore, they can be neglected in the limit $(M/m\gamma) \to \infty$ and the spatial components of tensor (1.3) can be used as the EMT spatial components of the whole system, i.e., $T_{ij}(q) = t_{ij}(q)$. The remaining four components of the tensor $T_{\mu\nu}$ can be found from the four conservation laws:

$$q^{\mu} T_{\mu\nu}(q) = 0\,, \quad \nu = 1, 2, 3, 0\,. \tag{2.1}$$

Then

$$T_{0j}(q) = -\frac{q^i}{q^0}\, t_{ij}(q)\,, \qquad T_{00}(q) = \frac{q^i q^j}{q^{0\,2}}\, t_{ij}(q)\,. \tag{2.2}$$

For uniform motion with the speed $v = \omega r$ on a circle of radius r in the 1, 2 plane, the following spatial components are nonzero:

$$\begin{aligned} t_{11,22}(q) &= \frac{mv^2\gamma}{2} \sum_n 2\pi\delta(q^0 - n\omega) \\ &\times \left[J_n \pm \frac{1}{2}\, J_{n+2} \exp(-\mathrm{i}2\varphi) \pm \frac{1}{2}\, J_{n-2} \exp(\mathrm{i}2\varphi) \right] \exp(-\mathrm{i}n\varphi)\,, \\ t_{12}(q) &= \frac{mv^2\gamma}{4\mathrm{i}} \sum_n 2\pi\delta(q^0 - n\omega) \\ &\times \left[J_{n-2} \exp(\mathrm{i}2\varphi) - J_{n+2} \exp(-\mathrm{i}2\varphi) \right] \exp(-\mathrm{i}n\varphi)\,, \end{aligned} \tag{2.3}$$

where $J_n = J_n(z)$ is the Bessel function, $z = q_{\perp} r = |n| v \sin\theta$, θ, φ are polar and azimuthal angles of the vector $\mathbf{q}$, and the sum is taken over integer $n \geqslant 0$. Hence and from formula (2.2), the expression for the GR spectrum is

$$\begin{aligned} &T_{\mu\nu}(q) T^{\mu\nu *}(q) - \frac{1}{2} \left| T^{\mu}_{\mu}(q) \right|^2 \\ &= t \sum_n 2\pi\delta(q^0 - n\omega)\, \frac{m^2 v^4 \gamma^2}{4} \left[J^2_{n+2} + J^2_{n-2} \right. \\ &- \sin^2\theta (J^2_{n+2} + J^2_{n-2} + J_n J_{n+2} + J_n J_{n-2}) \\ &\left. + \frac{1}{2} \sin^4\theta \left(J_n + \frac{1}{2}\, J_{n+2} + \frac{1}{2}\, J_{n-2} \right)^2 \right]. \end{aligned} \tag{2.4}$$

It is essentially different from the EMR spectrum of a charge moving circumferentially:

$$\left| j_{\mu}(q) \right|^2 = t \sum_n 2\pi\delta(q^0 - n\omega)\, e^2 \left[\cot^2\theta J_n^2 + v^2 J_n'^2 \right]. \tag{2.5}$$

The difference persists in the ultrarelativistic limit $\gamma \gg 1$, where $z \approx n \sim \gamma^3$ and $\alpha \equiv (\theta - \pi/2) \sim \gamma^{-1}$ are effective and the square brackets in Eqns (2.4) and (2.5) become

$$\begin{aligned} [\ldots]_{\mathrm{GR}} &\approx 8 \left[\left(\alpha^2 + \frac{1}{2\gamma^2} \right)^2 J_n^2 + \alpha^2 J_n'^2 \right], \\ [\ldots]_{\mathrm{EMR}} &\approx \alpha^2 J_n^2 + J_n'^2\,, \end{aligned}$$

and instead of J_n and J_n', their asymptotic representations in terms of the Airy function $\Phi(y)$ should be used:

$$\begin{aligned} J_n(z) &\approx \frac{1}{\pi} \left(\frac{2}{n} \right)^{1/3} \Phi(y)\,, \qquad J_n'(z) \approx -\frac{1}{\pi} \left(\frac{2}{n} \right)^{2/3} \Phi'(y)\,, \\ y &= \left(\frac{n}{2} \right)^{2/3} \left(1 - \frac{z^2}{n^2} \right). \end{aligned} \tag{2.6}$$

It is easy to see that relation (1.5) is not satisfied; in viewing it as an order-of-magnitude relation, we should take $\Gamma \sim 1$ and not $\Gamma \sim \gamma$, as it would be for a system with a nonlocal EMT.

In the nonrelativistic limit, the sum in (2.4) contains only the quadrupole terms $n = \pm 2$. According to (1.2), the $n = 2$ term gives rise to the GR intensity

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{Gm^2 v^4 \omega^2}{8\pi} \int \mathrm{d}\Omega\, (1 + 6\cos^2\theta + \cos^4\theta) = \frac{8}{5}\, Gm^2 v^4 \omega^2\,, \tag{2.7}$$

accounting for 1/4 the GR intensity of the same body orbiting in a force field with the extended EMT (see [6], par. 110).

This means that for a nonrelativistic system with the force field having an extended EMT, the contributions of local and nonlocal channels to the GR amplitude coincide and hence the total amplitude is twice that of the local channel and the corresponding intensities differ by a factor of 4. We demonstrate this in Section 2.4 by direct calculation using the example of the GR of a charge held on a circular orbit by the Coulomb center.

Unlike (1.6), the differential distribution in (2.4) and (2.7) does not vanish at $\theta = \pi$.

It is appropriate to use formulas (2.2) and write the conserved EMT $T^{\alpha\beta}$ in the coordinate representation as the sum $T^{\alpha\beta}(x) = t^{\alpha\beta}(x) + \tau^{\alpha\beta}(x)$ of the point-mass EMT

$$t^{\alpha\beta}(x) = m \int \mathrm{d}\tau\, \dot{x}^{\alpha}(\tau)\, \dot{x}^{\beta}(\tau)\, \delta\big(x - x(\tau)\big)\,, \tag{2.8}$$

and the tensor $\tau^{\alpha\beta}(x)$ that has zero spatial and nonzero mixed and temporal components:

$$\begin{aligned} &\tau^{ij}(x) = 0\,, \qquad i, j = 1, 2, 3\,, \\ &\tau^{0\beta}(x) = -\frac{1}{2}\, m \int \mathrm{d}\tau\, \ddot{x}^{\beta}(\tau)\, \mathrm{sgn}\, \big(x^0 - x^0(\tau)\big)\, \delta\big(\mathbf{x} - \mathbf{x}(\tau)\big) \\ &\qquad + \delta_0^{\ \beta}\, \frac{1}{2}\, m \int \mathrm{d}\tau \left| x^0 - x^0(\tau) \right| \ddot{x}^i(\tau)\, \partial_i \delta\big(\mathbf{x} - \mathbf{x}(\tau)\big)\,, \\ &\qquad \beta = 1, 2, 3, 0\,. \end{aligned} \tag{2.9}$$

We note that the tensor $\tau^{\alpha\beta}(x)$, unlike (2.8), contains only the spatial δ-function in the integrand and lacks the temporal one.

The tensor $\tau^{\alpha\beta}(x)$ differs from zero only at points $\mathbf{x} = \mathbf{x}(\tau)$ of the mass trajectory where the mass undergoes acceleration $\ddot{\mathbf{x}}(\tau) \neq 0$. In this sense, the tensor $\tau^{\alpha\beta}$ is local, but its values at these points depend on the time x^0. However, the divergence of this tensor

$$\frac{\partial}{\partial x^{\alpha}}\, \tau^{\alpha\beta}(x) = -m \int \mathrm{d}\tau\, \ddot{x}^{\beta}(\tau)\, \delta\big(x - x(\tau)\big) \tag{2.10}$$

is entirely local. It has the same value (but the opposite sign) as density of the force accelerating the mass, i.e.,

$$\frac{\partial}{\partial x^{\alpha}}\, t^{\alpha\beta}(x) = m \int \mathrm{d}\tau\, \ddot{x}^{\beta}(\tau)\, \delta\big(x - x(\tau)\big)\,. \tag{2.11}$$

These two divergences compensate each other, demonstrating the conservation of $T^{\alpha\beta}(x)$ in the coordinate representation.

It would be possible to give the tensor $\tau^{\alpha\beta}$ a physical interpretation and to call it the trough EMT if force (2.11) acting on the mass m from the trough could be related, e.g., to the trough elastic properties and the change in its energy in a certain effective volume near the contact with the mass. The variation of the energy in a volume moving along the trough together the with the mass would be an additional extended source of GR.

In the present review, much attention is given to the force field holding the mass on the orbit and to the corresponding EMT.

2.2 Gravitational radiation by a charge moving on a circle in a constant uniform magnetic field

The source of the GR of a charge moving in an electromagnetic field is the conserved tensor $T_{\mu\nu} = t_{\mu\nu} + \theta_{\mu\nu}$ consisting of the EMT $t_{\mu\nu}$ of point-like charge (1.3) and the EMT $\theta_{\mu\nu}$ of the external ($\varphi_{\alpha\beta}$) and proper ($f_{\alpha\beta}$) electromagnetic fields [6, 8]:

$$\begin{aligned} \theta_{\mu\nu} &= -F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{4}\,g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\,, \\ F_{\alpha\beta} &= \varphi_{\alpha\beta} + f_{\alpha\beta}\,. \end{aligned} \tag{2.12}$$

The terms of the tensor $f_{\alpha\beta}$, quadratic in $\theta_{\mu\nu}$, can be omitted because we disregard the action of the proper field of the charge on itself. At the quantum level, this corresponds to neglecting radiative corrections. In this approximation, the tensor $T_{\mu\nu}$ is strictly conserved and the expression for the Lorentz force contains only the external (with respect to the charge) field $\varphi_{\alpha\beta}$, rather than $\varphi_{\alpha\beta} + f_{\alpha\beta}$.

For the external fields considered below, the terms of $\theta_{\mu\nu}$ quadratic in $\varphi_{\alpha\beta}$ are not a source of GR and can be omitted. We therefore use the following equation for the Fourier transform $\theta_{\mu\nu}(q)$:

$$\begin{aligned} \theta_{\mu\nu}(q) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4} \bigg[& \varphi_{\mu\alpha}(k)f^{\alpha}{}_{\nu}(q-k) + \varphi_{\nu\alpha}(k)\,f^{\alpha}{}_{\mu}(q-k) \\ & + \frac{1}{2}\,g_{\mu\nu}\varphi_{\alpha\beta}(k)\,f^{\alpha\beta}(q-k)\bigg]\,, \end{aligned} \tag{2.13}$$

where the proper field can be expressed, in accordance with the Maxwell equation, in terms of the current density:

$$f_{\alpha\beta}(q) = \frac{\mathrm{i}}{q^2}\left[q_\alpha\, j_\beta(q) - q_\beta\, j_\alpha(q)\right]. \tag{2.14}$$

For a constant uniform magnetic field **H** directed along the 3-axis, only the following components are nonzero:

$$\varphi_{12}(k) = -\varphi_{21}(k) = H(2\pi)^4\delta(k)\,. \tag{2.15}$$

For a charge moving along a circular trajectory in such a field,

$$\begin{aligned} x_1(\tau) &= r\sin\Omega\tau\,, \qquad & x_2(\tau) &= r\cos\Omega\tau\,, \\ x_3 &= 0\,, & x^0(\tau) &= \gamma\tau\,, \end{aligned} \tag{2.16}$$

with the eigenfrequency $\Omega = \omega\gamma$ fixed by the field ($\Omega = eH/m$) and the spatial components of $t_{\mu\nu}$ given in (2.3). The nonzero spatial components of the tensor $\theta_{\mu\nu}$ are defined by the formulas

$$\begin{aligned} &\theta_{11}(q) = \theta_{22}(q) = -\theta_{33}(q) = \frac{\mathrm{i}H}{q^2}\left[q_1\, j_2(q) - q_2\, j_1(q)\right], \\ &\theta_{13}(q) = \frac{\mathrm{i}H}{q^2}\, q_3\, j_2(q)\,, \qquad \theta_{23}(q) = -\frac{\mathrm{i}H}{q^2}\, q_3\, j_1(q)\,, \end{aligned} \tag{2.17}$$

in terms of the spatial current density components

$$j_1(q) \pm \mathrm{i}\, j_2(q) = ev\sum_n 2\pi\delta(q^0 - n\omega)\, J_{n\mp1}(z)\exp\left[-\mathrm{i}(n\mp1)\varphi\right]. \tag{2.18}$$

The remaining mixed and temporal components of $t_{\mu\nu}$ and $\theta_{\mu\nu}$ can be found from the same formulas (1.3) and (2.13)–(2.15); on the other hand, the corresponding components of the conserved tensor $T_{\mu\nu}$ can be reconstructed from its spatial components by formulas (2.2). Both ways lead to the same result (1.7) for the GR spectrum.

Conversion of the photon propagator to infinity implies a cascade process: first, the current emits a real photon, which then turns into a graviton as it moves in a constant field [12–14]. Interestingly, the constant field transfers the zero 4-momentum to the graviton.

Because a real magnetic field sooner or later loses uniformity, it may be assumed that its sources do not contribute to the Fourier components of the EMT being considered. If such a field decreases at distances $\sim l \gg \lambda = 2\pi\omega^{-1}$, e.g., like $H(x) = H\exp(-x_\perp^2/l^2)$ or $H(x) = H\exp(-|x_3|/l)$, then the factor $1/q^2$ in formulas (2.17) for θ_{ik} is replaced by $\mathrm{i}\sqrt{\pi}\, l/4q_\perp$ or $\mathrm{i}l/2|q_3|$, which leads to the corresponding substitution in expression (1.7) for the spectrum.

2.3 Gravitational radiation by a charge moving in the field of a plane electromagnetic wave

The GR of a charge in an external plane-wave field originates from the same sources $t_{\mu\nu}$ and $\theta_{\mu\nu}$, with the contribution of the latter mediated through a virtual photon of the proper field of the charge in order that the graviton be real.

We first consider the GR of a charge in the field of a circularly polarized wave

$$\begin{aligned} &\varphi_{\alpha\beta}(x) = -\varphi^{(1)}_{\alpha\beta}\sin(kx) + \varphi^{(2)}_{\alpha\beta}\cos(kx)\,, \\ &\varphi^{(i)}_{\alpha\beta} = k_\alpha a^i_\beta - k_\beta a^i_\alpha\,, \quad a^i_\alpha a^{j\alpha} = a^2\delta_{ij}\,, \quad k_\alpha a^{i\alpha} = k^2 = 0\,. \end{aligned} \tag{2.19}$$

We choose a system of coordinates where the charge is on the average at rest and the wave propagates along the 3-axis with the wave vector $k_1 = k_2 = 0$, $k_3 = k^0 = \omega$ and with the potential amplitudes $a^i_\alpha = a\delta^i_\alpha$. Then the trajectory of the charge is a circle in the plane $x_3 = \mathrm{const}$, along which it moves with the speed $v = \xi$, $\xi = ea/m_*$, $m_* = (m^2 + e^2a^2)^{1/2}$, and a phase as in (2.16) if $x_3 = \pi/\omega$ is chosen instead of $x_3 = 0$. Therefore, the components of $t_{\mu\nu}$ are the same as in (2.3) but with the phase factor $p \equiv \exp(-\mathrm{i}q_3\pi/\omega)$. To these, we add the mixed and temporal components

$$\begin{aligned} t^0_1 \pm \mathrm{i}t^0_2 &= mv\gamma\sum_n 2\pi\delta(q^0 - n\omega)\, J_{n\mp1}\exp\left[-\mathrm{i}(n\mp1)\varphi\right], \\ t_{00} &= m\gamma\sum_n 2\pi\delta(q^0 - n\omega)\, J_n\exp(-\mathrm{i}n\varphi)\,, \end{aligned} \tag{2.20}$$

which must be accompanied by the same factor p.

Because the Fourier components of the field reduce to two δ-functions,

$$\begin{aligned} &\varphi_{\alpha\beta}(q) = (2\pi)^4\,\frac{\mathrm{i}}{2}\left[\Phi_{\alpha\beta}\,\delta(q-k) - \Phi^*_{\alpha\beta}\,\delta(q+k)\right], \\ &\Phi_{\alpha\beta} = \varphi^{(1)}_{\alpha\beta} - \mathrm{i}\varphi^{(2)}_{\alpha\beta}\,, \end{aligned} \tag{2.21}$$

the tensor $\theta_{\mu\nu}(q)$ can easily be found in terms of the current components $j_\alpha(q \mp k)$, which differ from the components of the current for the motion in (2.16) by the factor $\exp[-\mathrm{i}(q_3 \mp \omega)\pi/\omega] = -p$. The omission of the phase factor p common for tensors $t_{\mu\nu}$ and $\theta_{\mu\nu}$ results in

$$
\begin{aligned}
\theta_{11} = -\theta_{22} &= \frac{mv^2}{4}\,\gamma \sum_n 2\pi\delta(q^0 - n\omega) \\
&\times \left\{ \left(\frac{q_\perp}{vq_-} J_{n-1} - J_{n-2} \right) \exp\left[-\mathrm{i}(n-2)\varphi\right] \right. \\
&+ \left. \left(\frac{q_\perp}{vq_-} J_{n+1} - J_{n+2} \right) \exp\left[-\mathrm{i}(n+2)\varphi\right] \right\}, \\
\theta_{12} &= \frac{mv^2\gamma}{4\mathrm{i}} \sum_n 2\pi\delta(q^0 - n\omega) \\
&\times \left\{ \left(\frac{q_\perp}{vq_-} J_{n-1} - J_{n-2} \right) \exp\left[-\mathrm{i}(n-2)\varphi\right] \right. \\
&- \left. \left(\frac{q_\perp}{vq_-} J_{n+1} - J_{n+2} \right) \exp\left[-\mathrm{i}(n+2)\varphi\right] \right\}, \\
\theta_{13} = \theta^0_1 &= \frac{mv^2\gamma}{8} \sum_n 2\pi\delta(q^0 - n\omega) \\
&\times \left\{ \left[2\,\frac{q_3 - \omega}{vq_-} J_{n-1} - \frac{q_\perp}{q_-}(J_{n-2} - J_n) \right] \exp\left[-\mathrm{i}(n-1)\varphi\right] \right. \\
&+ \left. \left[2\,\frac{q_3 + \omega}{vq_-} J_{n+1} - \frac{q_\perp}{q_-}(J_{n+2} - J_n) \right] \exp\left[-\mathrm{i}(n+1)\varphi\right] \right\}, \\
\theta_{23} = \theta^0_2 &= \frac{mv^2\gamma}{8\mathrm{i}} \sum_n 2\pi\delta(q^0 - n\omega) \times \\
&\times \left\{ \left[2\,\frac{q_3 - \omega}{vq_-} J_{n-1} - \frac{q_\perp}{q_-}(J_{n-2} - J_n) \right] \exp\left[-\mathrm{i}(n-1)\varphi\right] \right. \\
&- \left. \left[2\,\frac{q_3 + \omega}{vq_-} J_{n+1} - \frac{q_\perp}{q_-}(J_{n+2} - J_n) \right] \exp\left[-\mathrm{i}(n+1)\varphi\right] \right\}, \\
\theta_{33} = \theta^0_3 = \theta_{00} &= \frac{mv^2\gamma}{2} \sum_n 2\pi\delta(q^0 - n\omega) \\
&\times \frac{q^0 + q_3 - q^0 v^{-2}}{q_-} J_n \exp(-\mathrm{i}n\varphi)\,.
\end{aligned}
\tag{2.22}
$$

Here, $q_- = q^0 - q_3$; the Bessel functions depend on $z = q_\perp r$.

Calculation of the GR spectrum from formulas (2.3), (2.19), and (2.22) gives

$$
\begin{aligned}
T_{\mu\nu} T^{\mu\nu *} &- \frac{1}{2}\,|T^\mu_\mu|^2 \\
&= t \sum_n 2\pi\delta(q^0 - n\omega)\,\frac{m^2\gamma^2 q_\perp^2}{2q_-^2} \left[\frac{q_3^2}{q_\perp^2} J_n^2 + v^2 J_n'^2 \right].
\end{aligned}
\tag{2.23}
$$

A comparison with the EMR spectrum (2.5) of a charge moving in a circular orbit shows that the GR and EMR spectra are related by simple equation (1.6) with the proportionality coefficient independent of the radiation frequency q^0 or the harmonic number (see also [10]).

We now turn to the GR of a charge in the field of a linearly polarized wave:

$$
\begin{aligned}
&\varphi_{\alpha\beta}(x) = -\Phi_{\alpha\beta} \sin(kx)\,, \qquad \Phi_{\alpha\beta} = k_\alpha a_\beta - k_\beta a_\alpha\,, \\
&k^2 = ak = 0\,.
\end{aligned}
\tag{2.24}
$$

In a coordinate system where the charge is on the average at rest and the wave vector is directed along the 3-axis, $k_1 = k_2 = 0$, $k_3 = k^0 = \omega$, the charge has a figure-eight trajectory lying in the plane of the wave vector $\mathbf{k}$ and the amplitude of the electric field $\mathbf{E} = \omega\mathbf{a}$. The trajectory is described by equations

$$
\begin{aligned}
&x_1(\tau) = -\frac{\xi}{\omega} \sin\Omega_*\tau\,, \qquad x_2(\tau) = 0\,, \\
&x_3(\tau) = \frac{\xi^2}{8\omega} \sin 2\Omega_*\tau\,, \qquad x^0(\tau) = \gamma_*\tau + \frac{\xi^2}{8\omega} \sin 2\Omega_*\tau\,,
\end{aligned}
\tag{2.25}
$$

where

$$
\Omega_* = \omega\gamma_*\,, \qquad \gamma_* = \frac{m_*}{m}\,, \qquad \xi = \frac{ea}{m_*}\,;
$$

$m_* = (m^2 + (1/2)e^2a^2)^{1/2}$ is the effective mass of the charge equal to its mean kinetic energy in the system of interest.

Then the nonzero components of the EMT of the charge are

$$
\begin{aligned}
t_{11}(q) &= m_*\xi^2 \sum_s 2\pi\delta(q^0 - s\omega) A_2(s\alpha\beta)\,, \\
t_{13}(q) &= -\frac{m_*\xi^3}{4} \sum_s 2\pi\delta(q^0 - s\omega)(2A_3 - A_1)\,, \\
t_{33}(q) &= \frac{m_*\xi^4}{16} \sum_s 2\pi\delta(q^0 - s\omega)(4A_4 - 4A_2 + A_0)\,, \\
t^0_1(q) &= -m_*\xi \sum_s 2\pi\delta(q^0 - s\omega) \left[\left(1 - \frac{\xi^2}{4}\right) A_1 + \frac{\xi^2}{2} A_3 \right], \\
t^0_3(q) &= \frac{m_*\xi^2}{4} \sum_s 2\pi\delta(q^0 - s\omega) \\
&\quad\times \left[\xi^2 A_4 + (2 - \xi^2) A_2 - \left(1 - \frac{\xi^2}{4}\right) A_0 \right], \\
t_{00}(q) &= m_* \sum_s 2\pi\delta(q^0 - s\omega) \\
&\quad\times \left[\frac{\xi^4}{4} A_4 + \xi^2 \left(1 - \frac{\xi^2}{4}\right) A_2 + \left(1 - \frac{\xi^2}{4}\right)^2 A_0 \right].
\end{aligned}
\tag{2.26}
$$

Here,

$$
A_n(s\alpha\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\varphi\, \cos^n\varphi \exp\left[\mathrm{i}(\alpha\sin\varphi - \beta\sin 2\varphi - s\varphi)\right]
\tag{2.27}
$$

are the functions introduced in [15], with arguments $\alpha = -\xi q_1/\omega$ and $\beta = \xi^2 q_-/8\omega$.

With the Fourier components of the field now equal to

$$
\varphi_{\alpha\beta}(q) = (2\pi)^4\,\frac{\mathrm{i}}{2}\,\Phi_{\alpha\beta}\left[\delta(q-k) - \delta(q+k)\right],
\tag{2.28}
$$

formulas (2.13) and (2.14) give the field EMT

$$
\begin{aligned}
\theta_{11}(q) = -\theta_{22}(q) &= \frac{a}{4}\left[j_1 + j_1 - \frac{q_1}{q_-}(j_- + j_-) \right], \\
\theta_{12}(q) &= -\frac{aq_2}{4q_-}(j_- + j_-)\,, \\
\theta_{13}(q) = \theta^0_1(q) &= \frac{a}{4q_-}\left[(q^0 - k^0)\, j_3 - (q_3 - k_3)\, j^0 \right. \\
&\quad \left. + (q^0 + k^0)\, j_3 - (q_3 + k_3)\, j^0 \right], \\
\theta_{23}(q) = \theta^0_2(q) &= -\frac{aq_2}{4q_-}(j_1 + j_1)\,, \\
\theta_{33}(q) = \theta^0_3(q) = \theta_{00}(q) &= \frac{a}{4q_-}\left[q_1(j^0 + j_3 + j^0 + j_3) \right. \\
&\quad \left. - (q^0 - k^0 + q_3 - k_3)\, j_1 - (q^0 + k^0 + q_3 + k_3)\, j_1 \right].
\end{aligned}
\tag{2.29}
$$

Here, j_α are the Fourier components of the current density; the first of the two identical symbols in the brackets depends on $q-k$ and the second on $q+k$. The expressions for these components differ from those for $j_\alpha(q)$ by the replacement of the functions $A_n(s)$ with $A_n(s\mp 1)$:

$$j_1(q\mp k) = -e\xi \sum_s 2\pi\delta(q^0 - s\omega)\, A_1(s\mp 1, \alpha\beta)\,,$$
$$j_3(q\mp k) = \frac{e\xi^2}{4}\sum_s 2\pi\delta(q^0 - s\omega) \times \left[2A_2(s\mp 1,\alpha\beta) - A_0(s\mp 1,\alpha\beta)\right], \quad (2.30)$$
$$j^0(q\mp k) = e\sum_s 2\pi\delta(q^0 - s\omega) \times \left[\left(1-\frac{\xi^2}{4}\right)A_0(s\mp 1,\alpha\beta) + \frac{\xi^2}{2}\,A_2(s\mp 1,\alpha\beta)\right].$$

Forming the conserved EMT $T_{\mu\nu}(q)$, we obtain an expression for the GR spectrum in the form

$$T_{\mu\nu}(q)T^{\mu\nu *}(q) - \frac{1}{2}\left|T^\mu_\mu(q)\right|^2 = t\sum_s 2\pi\delta(q^0 - s\omega)\,\frac{m^2 q_\perp^2}{2q_-^2}\left[-A_0^2 + x^2(A_1^2 - A_0A_2)\right], \quad (2.31)$$

where $x = ea/m$ and $A_n \equiv A_n(s\alpha\beta)$. The following relations for the functions A_n were used in this derivation:

$$(s-2\beta)A_0 - \alpha A_1 + 4\beta A_2 = 0\,, \quad (2.32)$$
$$A_n(s-1) - A_n(s+1) = \frac{2}{n+1}\left[4\beta A_{n+3}(s) - \alpha A_{n+2}(s) + (s-2\beta)A_{n+1}(s)\right], \quad (2.33)$$
$$A_n(s-1) + A_n(s+1) = 2A_{n+1}(s)\,. \quad (2.34)$$

Spectrum (2.31) is related to the EMR spectrum of a charge in a linearly polarized wave

$$\left|j_\mu(q)\right|^2 = t\sum_s 2\pi\delta(q^0 - s\omega)\,\frac{e^2}{\gamma_*^2}\left[-A_0^2 + x^2(A_1^2 - A_0A_2)\right] \quad (2.35)$$

by expression (1.6).

Therefore, the GR and EMR spectra of a charge in a plane-wave field differ only in the radiation frequency-independent coefficient

$$\frac{4\pi G m^2}{e^2}\,\gamma_*^2\cot^2\frac{\theta}{2} \quad (2.36)$$

[here, the factor $8\pi G$ omitted in (2.23) and (2.31) is taken into account; see (1.2)]. Even though relation (1.6) was already discussed in the Introduction, it should be remembered that *it arises from the joint action of the local and nonlocal GR mechanisms*. As a result, *the final answer depends on $j_\alpha(q)$, whereas $\theta_{\mu\nu}(q)$ depended on $j_\alpha(q\pm k)$*. In the ultrarelativistic limit, the nonlocal mechanism becomes dominant. This can be illustrated by the example of formula (2.31). If only the field source $\theta_{\mu\nu}$ is taken instead of $T_{\mu\nu}$ in the left-hand side, then $(\theta^\mu_\mu = 0)$

$$\theta_{\mu\nu}(q)\theta^{\mu\nu *}(q) = t\sum_s 2\pi\delta(q^0 - s\omega) \times m^2\left[-\frac{x^2}{2+x^2}\,A_0A_2 + \frac{q_\perp^2}{2q_-^2}\,x^2(A_1^2 - A_0A_2)\right], \quad (2.37)$$

i.e., expressions (2.31 and (2.37) differ essentially in the first terms. But in the ultrarelativistic case, when $x \gg 1$, the main contribution to the integral for A_n in (2.27) comes from the saddle point $\varphi = \psi$, where

$$\cos\psi\Big|_{x\gg 1} \approx -\frac{q_1}{\sqrt{2}\,q_-}$$

(see [15]). With the azimuthal angle pinned to 0 or π, it follows that $q_1 \approx \pm q_\perp$. Therefore, $A_2 \approx \cos^2\psi A_0 \approx (q_\perp^2/2q_-^2)A_0$, and Eqn (2.37) becomes (2.31), where the relevant asymptotic expressions should naturally be used for the functions A_0^2 and $A_1^2 - A_0A_2$ (see [15, 16]).

We note that formulas (2.23) and (2.31) allow passing to the limit of an infinitely heavy charge mass; in this case, they describe the angular distribution of GR produced when a plane electromagnetic wave is incident on a fixed Coulomb center:

$$T_{\mu\nu}T^{\mu\nu *} - \frac{1}{2}|T^\mu_\mu|^2\Big|_{m\to\infty} = t\sum_{n=\pm 1} 2\pi\delta(q^0 - n\omega)\,\frac{e^2a^2}{8}\cot^2\frac{\theta}{2}\begin{cases}1+\cos^2\theta\,,\\ 1-\sin^2\theta\cos^2\varphi\,,\end{cases} \quad (2.38)$$

where the top and bottom lines respectively refer to circular and linear polarizations. This result is consistent with [17].

2.4 Gravitational radiation by a charge rotating in the field of a Coulomb center

We consider the motion of a charge e on a circle (2.16) in the Coulomb gravitational field of a fixed charge e'. This field, unlike the magnetic and plane-wave fields, has the continuous wave vector spectrum

$$\varphi_{\alpha\beta}(k) = -\mathrm{i}\,\frac{e'}{k^2}(k_\alpha\delta^0_\beta - k_\beta\delta^0_\alpha)2\pi\delta(k^0)\,. \quad (2.39)$$

Using the causal proper time representations for the propagators k^{-2} and $(q-k)^{-2}$ in integral (2.13), it is possible to perform the Gaussian integration over $\mathbf{k}$ and over one of the proper times. The tensor $\theta_{\mu\nu}$ is then represented as an integral over the dimensionless variable u (the ratio of one of the proper times to their sum):

$$\theta_{\mu\nu}(q) = -\frac{\mathrm{i}ee'}{8\pi|\mathbf{q}|}\int_0^1 \mathrm{d}u\int_{-\infty}^{\infty}\mathrm{d}\tau\,\exp(\mathrm{i}f)a_{\mu\nu}\,,$$
$$f = -q_\alpha x^\alpha(\tau) + u\big(\mathbf{q}\mathbf{x}(\tau) + |\mathbf{q}|r\big)\,. \quad (2.40)$$

The following expressions can be obtained for the components $a_{\mu\nu}$:

$$a_{ij} = \bigg[(1-u)(\mathbf{q}^2\delta_{ij} - 2q_iq_j) + (1-2u)\frac{|\mathbf{q}|}{r}(\delta_{ij}\,\mathbf{q}\mathbf{x} - q_ix_j - q_jx_i) + u\,\frac{\mathbf{q}^2}{r^2}(2x_ix_j - \delta_{ij}r^2) + \mathrm{i}\,\frac{2|\mathbf{q}|}{r^3}\,x_ix_j\bigg]\dot{x}^0 + q^0\left[q_i\dot{x}_j + q_j\dot{x}_i - \delta_{ij}\mathbf{q}\dot{\mathbf{x}} + \frac{|\mathbf{q}|}{r}(x_i\dot{x}_j + x_j\dot{x}_i)\right], \quad (2.41)$$

$$a_i^0 = \left[(1-2u)\left(\mathbf{q}^2 + \frac{|\mathbf{q}|}{r}\,\mathbf{qx}\right) + \mathrm{i}\,\frac{|\mathbf{q}|}{r}\right]\dot{x}_i + u\,\frac{|\mathbf{q}|}{r}\,\mathbf{q}\dot{\mathbf{x}}x_i - (1-u)\mathbf{q}\dot{\mathbf{x}}q_i\,, \tag{2.42}$$

$$a_{00} = \left[(1-2u)\left(\mathbf{q}^2 + \frac{|\mathbf{q}|}{r}\,\mathbf{qx}\right) + 2\mathrm{i}\,\frac{|\mathbf{q}|}{r}\right]\dot{x}^0 - q^0\mathbf{q}\dot{\mathbf{x}}\,. \tag{2.43}$$

We note that $a_{ii} = a_{00}$ by virtue of the tracelessness of the EMT of the electromagnetic field. The circular motion is described by coordinates (2.16).

It follows that $a_{\mu\nu}$ are polynomials of degree no higher than second in the coordinates $x_{1,2}(\tau)$ with coefficients that are quadratic in the components q_α and depend linearly on u and the velocities $\dot{x}_{1,2}(\tau)$, $\dot{x}^0(\tau)$, where τ is the proper time of the charge e. The dependence on the velocities $\dot{x}_{1,2}$, $\dot{x}^0$ arises from the use of formula (2.14) for the proper field and expression (1.1) for the current. The integral over τ in (2.40) can be represented as a series over Bessel functions with the argument $z_1 = (1-u)z$, $z = q_\perp r$. The product of charges is eliminated by using the equation of motion $mv\gamma\omega = -ee'/4\pi r^2$.

Besides the EMT $\theta_{\mu\nu}(q)$ of the external and proper fields, the conserved EMT of the entire system contains the EMT $t_{\mu\nu}(q)$ of the charge e [see (2.3), (2.20)] and the EMT $\tau_{\mu\nu}(q)$ of the Coulomb center with vanishing spatial and nonvanishing mixed and temporal components of the $n = \pm 1$ harmonics:

$$\tau_1^0(q) \pm \mathrm{i}\tau_2^0(q) = -mv\gamma(1 - v^2 - \mathrm{i}v)\exp(\mathrm{i}v)\,2\pi\delta(q^0 \mp \omega)\,,$$
$$\tau_{00}(q) = -\frac{mv\gamma q_\perp}{2\omega}(1 - v^2 - \mathrm{i}v)\exp(\mathrm{i}v)\times \left[\exp(-\mathrm{i}\varphi)\,2\pi\delta(q^0 - \omega) - \exp(\mathrm{i}\varphi)\,2\pi\delta(q^0 + \omega)\right]. \tag{2.44}$$

The divergence of this tensor $\mathrm{i}q_\alpha\tau^{\alpha\beta}(q)$ coincides with the force density exerted on the fixed charge e' by the charge e rotating around it. Its nontrivial components are

$$f^1(q) \pm \mathrm{i}f^2(q) = \pm\mathrm{i}mv\gamma\omega(1 - v^2 - \mathrm{i}v)\exp(\mathrm{i}v)\,2\pi\delta(q^0 \mp \omega)\,,$$
$$f^0(q) = 0\,. \tag{2.45}$$

At the same time, the divergence $\mathrm{i}q^\alpha t_\alpha^\beta(q)$ of the EMT $t_{\alpha\beta}(q)$ of the charge e moving in a circle coincides with the density of the force $g^\beta(q)$ acting on this charge. Its nontrivial components are

$$g^1(q) \pm \mathrm{i}g^2(q) = \mp\mathrm{i}mv\gamma\omega\sum_n 2\pi\delta(q^0 - n\omega) \times J_{n\mp1}(z)\exp\left[-\mathrm{i}(n\mp1)\varphi\right],$$
$$g^0(q) = 0\,. \tag{2.46}$$

We recall that $z = q_\perp r = |n|v\sin\theta$, and θ and φ are the polar and azimuthal angles of the vector $\mathbf{q}$.

We postpone the discussion of the relation between the forces $f^\beta(q)$, $g^\beta(q)$ and the divergence $\mathrm{i}q^\alpha\theta_\alpha^\beta(q)$ of the EMT of the Coulomb and proper fields until the next section.

In this section, unlike in the previous ones where the GR spectrum was calculated from the invariant product of EMT components [see (1.2)], we find the spectrum as the sum of squares of two independent polarization amplitudes denoted as $T_+(q)$ and $T_\times(q)$. Indeed, writing the invariant expression for the spectrum in a reference frame where the wave vector $\mathbf{q}$ is directed along the 3-axis, labeling the tensor components in this system with a prime, and using the conservation law $q^\alpha T_{\alpha\beta}(q) = q'^\alpha T'_{\alpha\beta}(q') = 0$ and the equality $q^2 = q'^2 = 0$, we obtain

$$T_{\alpha\beta}(q)T^{\alpha\beta*}(q) - \frac{1}{2}\left|T_\alpha^\alpha(q)\right|^2 = \frac{1}{2}\left|T'_{11}(q') - T'_{22}(q')\right|^2 + 2\left|T'_{12}(q')\right|^2. \tag{2.47}$$

The expressions

$$T_+(q) = T'_{11}(q') - T'_{22}(q')\,, \qquad T_\times(q) = T'_{12}(q')\,, \tag{2.48}$$

where the three spatial EMT components in the right-hand sides can again be expressed in terms of $T_{ij}(q)$, are the two transverse components describing the GR of a system with independent polarizations.

Almost all systems considered in this review have an axial symmetry, meaning that the angular GR distribution in a spherical system with the polar axis coincident with the symmetry axis must be independent of the azimuthal angle φ of the vector $\mathbf{q}$. By choosing the vector $\mathbf{q}$ in the 1, 3 plane and denoting its polar angle by θ, we have the following relations in such systems:

$$T'_{11} - T'_{22} = T_{11}\cos^2\theta - 2T_{13}\sin\theta\cos\theta + T_{33}\sin^2\theta - T_{22}\,, \tag{2.49}$$

$$T'_{12} = T_{12}\cos\theta - T_{32}\sin\theta\,. \tag{2.50}$$

Transverse components of the total EMT are represented by the sums $T_A = \theta_A + t_A$, $A = +, \times$, of transverse components of the field and material body EMTs. The EMT of the Coulomb center makes no contribution to the transverse components of the total EMT because it lacks space components.

Using formula (2.40) for the transverse components $\theta_{ij}(q)$ of the field EMT and formulas (2.49) and (2.50) relating the transverse components of the tensor to its spatial components, we obtain the following expressions for the two transverse components $\theta_A(q)$, $A = +, \times$ (see also [18]):

$$\theta_A(q) = \sum_n 2\pi\delta(q^0 - n\omega) \times m\gamma v^2\int_0^1 \mathrm{d}u\,\exp(\mathrm{i}\zeta u)\left[\alpha_A J_n(z_1) + \beta_A J'_n(z_1)\right], \tag{2.51}$$

$$\alpha_+ = \frac{\zeta^2}{z_1^2}(1 + \cos^2\theta) - (\mathrm{i}\zeta u - 1)\left[\frac{n^2}{z_1^2}(1 + \cos^2\theta) - \cos^2\theta\right], \tag{2.52}$$

$$\beta_+ = \frac{1}{z_1}(1 + \cos^2\theta)(\mathrm{i}\zeta u - \zeta^2 - 1)\,, \tag{2.53}$$

$$\alpha_\times = \mathrm{i}n\left[-\frac{1}{2}\,v^2 + \frac{1}{z_1^2}(\zeta^2 + 1 - \mathrm{i}\zeta u)\right]\cos\theta\,, \tag{2.54}$$

$$\beta_\times = \frac{\mathrm{i}n}{z_1}(\mathrm{i}\zeta u - 1 - v^2)\cos\theta\,. \tag{2.55}$$

Here, $z_1 = (1-u)z$, $z = |n|v\sin\theta$, and $\zeta = |\mathbf{q}|r = |n|v$. Peters's remark cited under (5.25) in Section 5.1 can be useful in verifying expressions (2.52)–(2.55).

The transverse components of the EMT of mass m are

$$t_{+}(q)=\sum_{n}2\pi\delta(q^{0}-n\omega)m\gamma v^{2}\left\{\left[\frac{n^{2}}{z^{2}}-1+\frac{n^{2}}{z^{2}}\cos^{2}\theta\right]J_{n}(z)-\frac{1}{z}(1+\cos^{2}\theta)\,J_{n}'(z)\right\},\tag{2.56}$$

$$t_{\times}(q)=\sum_{n}2\pi\delta(q^{0}-n\omega)\,m\gamma v^{2}\,\frac{\mathrm{i}n}{z}\left[\frac{1}{z}\,J_{n}(z)-J_{n}'(z)\right]\cos\theta\,.\tag{2.57}$$

It follows from the above expressions for θ_A and t_A that the GR spectrum determined by the invariant $1/2\,|T_+|^2+2|T_\times|^2$, $T_A=\theta_A+t_A$ is very complicated. Therefore, we consider it here only in the nonrelativistic and ultrarelativistic limits.

We define the amplitude $T_{An}(q)$ of the nth harmonic by the relation

$$T_{A}(q)=\sum_{n}2\pi\delta(q^{0}-n\omega)T_{An}(q)\,,\qquad T_{An}(q)=\theta_{An}(q)+t_{An}(q)\,.\tag{2.58}$$

It is then possible to show that in the nonrelativistic approximation, when $z\ll 1$,

$$\theta_{+n}(q)\approx m\gamma v^{2}\begin{cases}\dfrac{z}{16}(1-3\cos^{2}\theta)\,, & n=1\,,\\[2mm] \dfrac{z^{n-2}(1+\cos^{2}\theta)}{2^{n}(n-1)!}\,, & n\geqslant 2\,,\end{cases}\tag{2.59}$$

$$\theta_{\times n}(q)\approx m\gamma v^{2}\begin{cases}\dfrac{\mathrm{i}z}{16}\cos\theta\,, & n=1\,,\\[2mm] -\dfrac{\mathrm{i}z^{n-2}\cos\theta}{2^{n}(n-1)!}\,, & n\geqslant 2\,.\end{cases}\tag{2.60}$$

For the analogous amplitudes of a material body,

$$t_{+n}(q)\approx m\gamma v^{2}\begin{cases}-\dfrac{z}{8}(3-\cos^{2}\theta)\,, & n=1\,,\\[2mm] \dfrac{z^{n-2}(1+\cos^{2}\theta)}{2^{n}(n-2)!}\,, & n\geqslant 2\,,\end{cases}\tag{2.61}$$

$$t_{\times n}(q)\approx m\gamma v^{2}\begin{cases}\dfrac{\mathrm{i}z}{8}\cos\theta\,, & n=1\,,\\[2mm] -\dfrac{\mathrm{i}z^{n-2}\cos\theta}{2^{n}(n-2)!}\,, & n\geqslant 2\,.\end{cases}\tag{2.62}$$

The coefficient $m\gamma v^2$, which is here equal to mv^2, is kept in the relativistic form only for the convenience of comparing the approximate formulas and relativistic ones, (2.51), (2.56), and (2.57). Evidently, for the quadrupole and higher harmonics, the contribution of the field EMT is $n-1$ times smaller than that of the mass EMT. We emphasize that for the quadrupole harmonic, which is the leading one in the nonrelativistic approximation, the field and matter contributions are identical.

We give the differential and total GR intensities for the first and second harmonics in the nonrelativistic limit:

$$\left(T_{\mu\nu}T^{\mu\nu*}-\frac{1}{2}\,|T_{\mu}^{\mu}|^{2}\right)_{n=1,\,v\to 0}\approx t\,2\pi\delta(q^{0}-\omega)\,\frac{9}{64}\,m^{2}v^{6}\sin^{2}\theta\left(1-\frac{2}{3}\sin^{2}\theta+\frac{1}{72}\sin^{4}\theta\right),\tag{2.63}$$

$$\left.\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right|_{n=1,\,v\to 0}\approx\frac{5}{28}\,Gm^{2}\omega^{2}v^{6}\,,\tag{2.64}$$

$$\left(T_{\mu\nu}T^{\mu\nu*}-\frac{1}{2}\,|T_{\mu}^{\mu}|^{2}\right)_{n=2,\,v\to 0}\approx t\,2\pi\delta(q^{0}-2\omega)m^{2}v^{4}\left(1-\sin^{2}\theta+\frac{1}{8}\sin^{4}\theta\right),\tag{2.65}$$

$$\left.\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right|_{n=2,\,v\to 0}\approx\frac{32}{5}\,Gm^{2}\omega^{2}v^{4}\,.\tag{2.66}$$

As expected, the last two formulas for quadrupole radiation define the leading contribution and coincide with the known results given by the Einstein formula (see [6], par. 110). They are 4 times larger than the differential and total GR intensities of a body with the localized EMT [see (2.7)].

In the ultrarelativistic limit, the leading contribution comes from the pure field source $\theta_{\mu\nu}$, and hence

$$\left.\frac{1}{2}|T_{+}|^{2}+2|T_{\times}|^{2}\right|_{v\to 1}\approx\left.\frac{1}{2}|\theta_{+}|^{2}+2|\theta_{\times}|^{2}\right|_{v\to 1}\approx t\sum_{n}2\pi\delta(q^{0}-n\omega)\,\frac{m^{2}}{2\pi^{2}}\left(\frac{2}{n}\right)^{2/3}\times\left[-\Phi^{2}+\gamma^{2}\left(\frac{2}{n}\right)^{2/3}(y\Phi^{2}+\Phi'^{2})\right].\tag{2.67}$$

Here, we used asymptotic expressions (2.6) for the Bessel functions with $z_1\approx n\sim\gamma^3\gg 1$ and the effective values of the integration variable $u\sim n^{-1}\sim\gamma^{-3}$. We note that the contribution by the local tensor $t_{\mu\nu}$ and the interference contribution are γ^2 and γ times smaller than the field contribution (2.67). Because the expression for $|\,j_\alpha(q)|^2$ in the same limit differs from (2.67) by the replacement $m^2\to 2e^2/\gamma^2$, the relation between the GR and EMR spectra is defined by formula (1.5) with $\Gamma=\gamma$. This result is consistent with the one in [18].

2.5 On the conservation of the EMT

The well-known EMT of mass m in formula (1.3) has components (2.3) and (2.20) in the case of uniform circular motion. Less known is the EMT of a Coulomb center at rest, whose components enter (2.44). Therefore, we here propose its derivation, which is conveniently performed by finding a force that acts on the charge e' in the center from a circularly moving charge e.

Evidently, the Fourier transform of this force density can be represented by the integral

$$\mathbf{f}(q)=\int\mathrm{d}^{4}x\,\exp(-\mathrm{i}qx)e'\mathbf{E}(x)\delta(\mathbf{x})\,,\tag{2.68}$$

where e' is the charge of a heavy Coulomb center located at $\mathbf{x}=0$, and $\mathbf{E}(x)$ is the electric field strength created by the moving charge e at the point $x^\alpha=(\mathbf{x},t)$. Representing the field $\mathbf{E}(x)$ by the Fourier integral with components

$$f_{i0}(k)=\frac{\mathrm{i}}{k^{2}}\left[k_{i}j_{0}(k)-k_{0}j_{i}(k)\right],\qquad i=1,2\,,\tag{2.69}$$

[cf. (2.14)], we express the nonzero components of the vector $\mathbf{f}(q)$ as

$$f^{i}(q)=\mathrm{i}e'\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}}\,\frac{k_{i}j_{0}-q_{0}j_{i}}{\mathbf{k}^{2}-q_{0}^{2}}\,,\qquad i=1,2\,,\tag{2.70}$$

where after the integration over the x-space, the zeroth component of the 4-vector k_α becomes equal to q_0. Using representation (1.1) for the current $j_\alpha(k)$ and the proper time representation for the propagator $(\mathbf{k}^2 - q_0^2)^{-1}$ allows integrating over both $\mathbf{k}$ and the proper time of the virtual photon. As a result, $\mathbf{f}(q)$ turns out to be an integral over only the proper time of the charge e:

$$\mathbf{f}(q) = \frac{\mathrm{i}ee'}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau \exp\left(\mathrm{i}q^0\gamma\tau + \mathrm{i}|\mathbf{q}|r\right) \left[\frac{q^0\dot{\mathbf{x}}}{r} + \gamma\left(\frac{|\mathbf{q}|}{r^2} + \frac{\mathrm{i}}{r^3}\right)\mathbf{x}\right]. \tag{2.71}$$

The use of motion law (2.16) leads to representation (2.45) for $\mathbf{f}$.

Because $\mathbf{f}(q)$ must be the divergence of the corresponding EMT $\tau_{\alpha\beta}(q)$, it follows that

$$f^{\beta}(q) = \mathrm{i}q^{\alpha}\tau_{\alpha}^{\beta}(q). \tag{2.72}$$

For $\beta = 1, 2$, the right-hand side of (2.72) reduces to a single term $\mathrm{i}q^0\tau_0^{\beta}(q)$, which allows finding the mixed components $\tau_0^{\beta}(q) = -\tau_{\beta}^{0}(q)$ [see (2.44)]. For $\beta = 0$, the left-hand side of (2.72) is zero, while the right-hand side contains three terms, two of which are already known. Hence, there is a possibility of finding the third term $\mathrm{i}q^0\tau_0^0(q)$, i.e., the time component $\tau_0^0(q) = -\tau_{00}(q)$ [see (2.44)].

By calculating the divergence of the field tensor $\theta_{\alpha\beta}$ in (2.40)–(2.43), it can be shown that

$$\mathrm{i}q^{\alpha}\theta_{\alpha}^{\beta}(q) = -f^{\beta}(q) - g^{\beta}(q). \tag{2.73}$$

This means that the EMT $T_{\alpha\beta} = \theta_{\alpha\beta} + \tau_{\alpha\beta} + t_{\alpha\beta}$ of the whole system is conserved.

3. Gravitational radiation from systems with a more complicated force field

3.1 Gravitational radiation by a charge rotating in a magnetic moment field

We consider the circular motion of a charge in the equatorial plane of the field

$$\mathbf{H} = \frac{3\mathbf{r}(\mathfrak{M}\mathbf{r}) - \mathfrak{M}r^2}{4\pi r^5} = \boldsymbol{\nabla}(\mathfrak{M}\boldsymbol{\nabla})\,\frac{1}{4\pi r} \tag{3.1}$$

produced by a magnetic moment $\mathfrak{M}$. According to the equation of motion

$$mv^2\gamma = \frac{e\omega\mathfrak{M}_\omega}{4\pi c r}, \tag{3.2}$$

the projection $\mathfrak{M}_\omega$ of the magnetic moment on the direction of the charge angular velocity vector must have the same sign as the charge: $e\mathfrak{M}_\omega = |e\mathfrak{M}| > 0$.

Using the Fourier components

$$\varphi_{\alpha\beta}(k) = -\frac{\mathfrak{M}\mathbf{k}}{k^2}\,\varepsilon_{\alpha\beta\gamma 0}k^{\gamma}2\pi\delta(k^0) \tag{3.3}$$

of field (3.1) and the Fourier components of the charge proper field

$$f_{\alpha\beta}(q) = \frac{\mathrm{i}}{q^2}\left[q_\alpha j_\beta(q) - q_\beta j_\alpha(q)\right], \tag{3.4}$$

it is easy to construct the components of the field EMT

$$\theta_{\mu\nu}(q) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\varphi_{\mu\alpha}(k)f^{\alpha}{}_{\nu}(q-k) + \varphi_{\nu\alpha}(k)f^{\alpha}{}_{\mu}(q-k) + \frac{1}{2}\,g_{\mu\nu}\varphi_{\alpha\beta}(k)\,f^{\alpha\beta}(q-k)\right] \tag{3.5}$$

by omitting quadratic combinations $\varphi\varphi$ and ff on the grounds outlined in Section 2. It is convenient to perform the integration over wave vectors $\mathbf{k}$ of the external field using causal proper time representations for the propagators k^{-2} and $(q-k)^{-2}$ and the representation

$$j_\alpha(q) = e\int_{-\infty}^{\infty} \mathrm{d}\tau\,\dot{x}_\alpha(\tau)\exp\Big(-\mathrm{i}qx(\tau)\Big) \tag{3.6}$$

for the current density $j_\alpha(q-k)$. Then $\theta_{\mu\nu}(q)$ is given by

$$\theta_{\mu\nu}(q) = \frac{e\mathfrak{M}_\omega}{8\pi|\mathbf{q}|}\int_0^1 \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}\tau \exp\left(\mathrm{i}f\right)a_{\mu\nu}, \tag{3.7}$$

where $f = -q_\alpha x^\alpha(\tau) + u[\mathbf{qx}(\tau) + |\mathbf{q}|r]$, $u = t(s+t)^{-1}$, with s and t being the proper times of the quanta of the external and proper fields, and $a_{\mu\nu}$ are second or lower-order polynomials in coordinates $x_{1,2}(\tau)$ or velocities $\dot{x}_{1,2}(\tau)$.

Using formulas (2.49), (2.50), and (3.7), we obtain the two transverse components of the field EMT as

$$\theta_A(q) = \sum_n 2\pi\delta(q^0 - n\omega)m\gamma v^2 \times \int_0^1 \mathrm{d}u \exp\left(\mathrm{i}\zeta u\right)\left[\alpha_A J_n(z_1) + \beta_A J_n'(z_1)\right]. \tag{3.8}$$

Here, the subscript A is either $+$ or $\times$, and the coefficients α_A and β_A are

$$\alpha_+ = \frac{3}{2}(1 - \mathrm{i}\zeta u)\sin^2\theta - \left(\frac{n^2}{z_1^2} - \frac{1}{2}\right)\left[2\zeta^2u(1-u)\cos^2\theta + (\mathrm{i}\zeta u - 1)(1+\cos^2\theta)\right], \tag{3.9}$$

$$\beta_+ = \frac{1}{z_1}\left[2\zeta^2u(1-u)\cos^2\theta + (\mathrm{i}\zeta u - 1)(1+\cos^2\theta)\right] + \mathrm{i}\sin\theta\left[\mathrm{i}\zeta(1-u) - \zeta u(\mathrm{i}+\zeta u)\cos^2\theta\right], \tag{3.10}$$

$$\alpha_\times = \frac{n}{2z_1}(-\zeta^2u^2 + \mathrm{i}\zeta - 2\mathrm{i}\zeta u)\sin\theta\cos\theta - \frac{\mathrm{i}n}{z_1^2}\left[\frac{1}{2}\,\zeta^2u(1-u)(1+\cos^2\theta) + \mathrm{i}\zeta u - 1\right]\cos\theta, \tag{3.11}$$

$$\beta_\times = \frac{\mathrm{i}n}{z_1}\left[\frac{1}{2}\,\zeta^2u(1-u)(1+\cos^2\theta) + \mathrm{i}\zeta u - 1\right]\cos\theta, \tag{3.12}$$

$$z_1 = (1-u)z, \qquad z = |n|v\sin\theta, \qquad \zeta = |\mathbf{q}|r = |n|v. \tag{3.13}$$

To obtain the transverse components T_+ and $T_\times$ of the total EMT, it is necessary to add the transverse components of the material body EMT to (3.8):

$$t_+(q) = \sum_n 2\pi\delta(q^0 - n\omega)m\gamma v^2\left\{\left[\frac{n^2}{z^2} - 1 + \frac{n^2}{z^2}\cos^2\theta\right]J_n(z) - \frac{1}{z}(1+\cos^2\theta)\,J_n'(z)\right\}, \tag{3.14}$$

$$t_\times(q) = \sum_n 2\pi\delta(q^0 - n\omega)m\gamma v^2\,\frac{\mathrm{i}n}{z}\left[\frac{1}{z}\,J_n(z) - J_n'(z)\right]\cos\theta. \tag{3.15}$$

These expressions follow from the formulas (2.3) and (2.48)–(2.50) in Section 2.

In the nonrelativistic limit, when $|n|v \ll 1$, the arguments of the Bessel functions and the parameter ζ are small: $z_1 \sim z \sim \zeta \ll 1$. Physically, this condition implies the smallness of orbit dimensions compared with the radiation wavelength. Expansion of the Bessel functions allows expressing the integrals defining the field EMT as

$$\int_0^1 \mathrm{d}u\, \exp(\mathrm{i}\zeta u)(\alpha_+ J_n + \beta_+ J_n') \approx \begin{cases} \dfrac{z}{16}(1 - 3\cos^2\theta)\,, & n = 1\,, \\ \dfrac{z^{n-2}(1 + \cos^2\theta)}{2^n (n-1)!}\,, & n \geqslant 2\,, \end{cases} \tag{3.16}$$

$$\int_0^1 \mathrm{d}u\, \exp(\mathrm{i}\zeta u)(\alpha_\times J_n + \beta_\times J_n') \approx \begin{cases} \dfrac{\mathrm{i}z}{16}\cos\theta\,, & n = 1\,, \\ -\dfrac{\mathrm{i}z^{n-2}\cos\theta}{2^n (n-1)!}\,, & n \geqslant 2\,. \end{cases} \tag{3.17}$$

For analogous quantities (3.14) and (3.15) in the EMT of the material body, we have

$$\left(\frac{n^2}{z^2} - 1 + \frac{n^2}{z^2}\cos^2\theta\right) J_n - \frac{1}{z}(1 + \cos^2\theta) J_n' \approx \begin{cases} -\dfrac{z}{8}(3 - \cos^2\theta)\,, & n = 1\,, \\ \dfrac{z^{n-2}(1 + \cos^2\theta)}{2^n (n-2)!}\,, & n \geqslant 2\,, \end{cases} \tag{3.18}$$

$$\frac{\mathrm{i}n}{z}\cos\theta\left(\frac{1}{z}\, J_n - J_n'\right) \approx \begin{cases} \dfrac{\mathrm{i}z}{8}\cos\theta\,, & n = 1\,, \\ -\dfrac{\mathrm{i}z^{n-2}\cos\theta}{2^n (n-2)!}\,, & n \geqslant 2\,. \end{cases} \tag{3.19}$$

We note that for harmonics with $n \geqslant 2$, the contribution from the field EMT is $n - 1$ times smaller than from the body EMT.

Finally, for the GR spectrum in the nonrelativistic approximation, we have

$$\frac{1}{2}|T_+|^2 + 2|T_\times|^2\Big|_{n=1,\, v \ll 1} = t 2\pi\delta(q^0 - \omega)\,\frac{9}{64}\, m^2 v^6 \sin^2\theta \left(1 - \frac{2}{3}\sin^2\theta + \frac{1}{72}\sin^4\theta\right), \tag{3.20}$$

$$\frac{1}{2}|T_+|^2 + 2|T_\times|^2\Big|_{n \geqslant 2,\, nv \ll 1} = t 2\pi\delta(q^0 - n\omega) m^2 v^4 \left(\frac{n z^{n-2}}{2^{n-1}(n-1)!}\right)^2 \times \left(1 - \sin^2\theta + \frac{1}{8}\sin^4\theta\right). \tag{3.21}$$

As expected, the largest contribution comes from the second harmonic; it coincides with the value given by the Einstein formula.

In the ultrarelativistic limit, when $\gamma \gg 1$, the most important harmonics and angles in the GR spectrum are those for which $n \approx z_1 \approx z \approx \zeta \sim \gamma^3$, $\alpha = \theta - \pi/2 \sim \gamma^{-1}$ and the effective values $u \sim \gamma^{-3}$. Then integrals (3.16) and (3.17) become

$$\int_0^1 \mathrm{d}u\, \exp(\mathrm{i}\zeta u)(\alpha_+ J_n + \beta_+ J_n') \approx -\mathrm{i}J_n'(z)\,, \tag{3.22}$$

$$\int_0^1 \mathrm{d}u\, \exp(\mathrm{i}\zeta u)(\alpha_\times J_n + \beta_\times J_n') \approx -\frac{1}{2}\, J_n(z)\cos\theta\,, \tag{3.23}$$

where the asymptotic representations in terms of the Airy function should be used for $J_n(z)$ and $J_n'(z)$:

$$J_n(z) \approx \frac{1}{\pi}\left(\frac{2}{n}\right)^{1/3} \Phi(y)\,, \qquad J_n'(z) \approx -\frac{1}{\pi}\left(\frac{2}{n}\right)^{2/3} \Phi'(y)\,,$$
$$\Phi(y) = \int_0^\infty \mathrm{d}t\, \cos\left(yt + \frac{t^3}{3}\right), \qquad y = \left(\frac{n}{2}\right)^{2/3}\left(1 - \frac{z^2}{n^2}\right). \tag{3.24}$$

Expressions (3.18) and (3.19) for the EMT of a material body in the limit under consideration are

$$\left(\frac{n^2}{z^2} - 1 + \frac{n^2}{z^2}\cos^2\theta\right) J_n - \frac{1}{z}(1 + \cos^2\theta) J_n' \approx 2\left(\cos^2\theta + \frac{1}{2\gamma^2}\right) J_n(z)\,, \tag{3.25}$$

$$\frac{\mathrm{i}n}{z}\cos\theta\left(\frac{1}{z}\, J_n - J_n'\right) \approx -\mathrm{i}\cos\theta\, J_n'(z)\,. \tag{3.26}$$

Evidently, the transverse components of the body EMT are a factor of γ smaller than those of the field EMT and can be neglected. The GR spectrum in the ultrarelatvistic limit is then given by

$$\frac{1}{2}|T_+|^2 + 2|T_\times|^2\Big|_{\gamma \gg 1} \approx t\sum_n 2\pi\delta(q^0 - n\omega)\,\frac{1}{2}\, m^2\gamma^2(\alpha^2 J_n^2 + J_n'^2) = \frac{m^2\gamma^2}{2e^2}\,\left|j_\mu(q)\right|^2. \tag{3.27}$$

Therefore, the GR and EMR spectra for the system of interest are related in the ultrarelativistic limit by Eqn (1.5) with $\Gamma = \gamma$.

3.2 Gravitational radiation by a charge rotating in the field of a charged center with a magnetic moment

The equation of motion for a charge e in a circle orbit of radius r, speed v, and angular frequency ω in the equatorial plane of the center carrying a charge e' and a magnetic moment $\mathfrak{M}$ has the form

$$m\gamma v^2 = \frac{1}{4\pi r}\left(-ee' + \frac{e\mathfrak{M}_\omega\omega}{c}\right). \tag{3.28}$$

For such an orbit to exist, it is necessary that

$$-ee' + \frac{e\mathfrak{M}_\omega\omega}{c} > 0\,. \tag{3.29}$$

The conserved EMT of the whole system consists of the EMT $t_{\alpha\beta}$ of the material body, the field EMT $\theta^{\mathrm{M}}_{\alpha\beta}$ proportional to the magnetic moment, and the field EMT $\theta^{\mathrm{C}}_{\alpha\beta}$ proportional to the Coulomb field. The transverse components of the first two tensors are given by formulas (3.14), (3.15), and (3.8). The transverse components of EMT $\theta^{\mathrm{C}}_{+,\times}$ were presented in Section 2 for the purely Coulomb problem.

In the present problem with the two external fields acting simultaneously, the coefficients $m\gamma v^2$ in components (3.8) and (2.51) must be respectively replaced with the coefficients

$$\frac{e\mathfrak{M}_\omega \omega}{4\pi c r} \equiv k^{\mathrm{M}} m\gamma v^2\,, \qquad -\frac{ee'}{4\pi r} \equiv k^{\mathrm{C}} m\gamma v^2\,;$$

the field EMT is given by the sum $k^{\mathrm{M}}\theta_A^{\mathrm{M}} + k^{\mathrm{C}}\theta_A^{\mathrm{C}}$.

It can be shown that in the nonrelativistic limit, the integrals

$$\int_0^1 \mathrm{d}u\, \exp\left(\mathrm{i}\zeta u\right)\left[\alpha_A J_n + \beta_A J_n'\right]$$

for any n coincide with (3.16) and (3.17); in other words, they are given by the right-hand sides of these formulas. For the effective values of n and θ in the ultrarelativistic limit, they coincide with (3.22) and (3.23).

Hence, the components $\theta_A^{\mathrm{M}}(q)$ and $\theta_A^{\mathrm{C}}(q)$ are identical in both nonrelativistic and ultrarelativistic limits. Then

$$k^{\mathrm{M}}\theta_A^{\mathrm{M}}(q) + k^{\mathrm{C}}\theta_A^{\mathrm{C}}(q) \approx \theta_A^{\mathrm{M}}(q) \approx \theta_A^{\mathrm{C}}(q)$$

because $k^{\mathrm{M}} + k^{\mathrm{C}} = 1$ by virtue of equation of motion (3.28). This means that the GR spectrum of a charge rotating in this composite field is given by formulas (3.20) and (3.21) in the nonrelativistic region and by (3.27) in the ultrarelativistic region. We note that in the intermediate domain where the velocity of the charge is neither too low nor too close to 1, the transverse components $\theta_A^{\mathrm{M}}(q)$ and $\theta_A^{\mathrm{C}}(q)$ are quite different and the GR spectrum is sensitive to the character of the field in which the charge moves.

One of the main reasons for undertaking the present study was a desire to elucidate those properties of the field that are most important in determining the conversion amplitude Γ. Because it defines the proportionality factor between two invariants [see (1.5)], Γ itself must be an integral invariant of the system. In all four electromagnetic systems considered in Sections 2 and 3 with the same charge orbits but different fields (circular motion in the field of a circularly polarized wave, a magnetic moment field, a Coulomb field, and a combination of the last two fields), the conversion amplitude Γ is the same and equal to γ.

It appears that for a circular trajectory, Γ is given by such a simple formula because the field has no inherent scale length. Indeed, if we consider the GR spectrum of a charge in a circular orbit in a screened Coulomb field with the potential $(e'/4\pi r)\exp(-\eta r)$, then at $\gamma \gg 1$ the leading terms (in this limit) in the transverse components $\theta_A(q)$ of the field EMT differ from (3.8) by the factor

$$C = \frac{\eta r}{1+\eta r}\exp\left(\eta r\right) K_1(\eta r)\,, \tag{3.30}$$

where $K_1(x)$ is the Macdonald function. Then $\Gamma = C\gamma$. The coefficient $C(\eta r)$ decreases monotonically from 1 to 0 as ηr increases, and behaves like $(\pi/2\eta r)^{1/2}$ at $\eta r \gg 1$. The derivation of formula (3.30) assumes that $\eta r \ll \gamma^{3/2}$, $\gamma \gg 1$ which means that it applies to the case $\eta r \gg 1$.

The square-root dependence of the conversion amplitude on the intrinsic scale length η^{-1} of the field at $\eta r \gg 1$ is easy to understand bearing in mind that Γ is proportional to the length l of the conversion region [see (1.14)]. In fact, at $\gamma \gg 1$, the region in which photons are converted into gravitons extends along a line tangent to the charge orbit between the point of tangency to the circle of radius r and the point at which that tangent intersects another circle of the radius $r + \eta^{-1}$ (where both the field and the conversion are considerably weaker); i.e., the conversion region length is

$$l \sim \left[(r+\eta^{-1})^2 - r^2\right]^{1/2}\Big|_{\eta r \gg 1} \approx r\left(\frac{2}{\eta r}\right)^{1/2}. \tag{3.31}$$

We note that the finite relativistic motion of a charge in a Coulomb field can be treated classically if the classical radius of the orbit

$$r = -\frac{ee'}{4\pi m v^2\gamma}\bigg|_{\gamma \gg 1} \approx -\frac{ee'}{4\pi m c^2\gamma}$$

is larger than $\hbar/p$, that is, if the Coulomb center charge exceeds 137. On the other hand, if $|e'/e| > 170$, then the Coulomb field produces pairs and screens itself [19].

We also note that for $\gamma \gg 1$, the charge orbits at a speed close to the speed of light, and hence the conversion region formed by its field outside the orbit moves with a superluminal speed.

As shown in Section 2 of [9], the contribution of the transverse components of the EMT of a material body to GR at $\gamma \gg 1$ is of the same order as the contribution of the current to EMR if Gm^2 is replaced with $e^2/4\pi$. Then, to the order of magnitude, Γ determines the ratio of the transverse components θ_+ and $\theta_\times$ of $\theta_{\alpha\beta}$ to the transverse components t_+ and $t_\times$ of the tensor $t_{\alpha\beta}$. It is interesting to elucidate how this ratio (or Γ) depends on the spatial distribution of the tensor $\theta_{\alpha\beta}(x)$. For this, we consider GR from a nonelectromagnetic system in which the tensor $\theta_{\alpha\beta}(x)$ is confined to a line joining a material particle to the center of rotation.

3.3 Gravitational radiation from a relativistic string with masses at the ends

A relativistic string with masses at the ends [20] is not an electromagnetic system. It can be viewed as a realistic model of a system of two bodies connected by a force field confined to the line joining them. In this case, GR is emitted not only from local sources (the point masses) but also from an extended source, the string. It is interesting to compare the contributions to GR from the two kinds of sources, especially in the ultrarelativistic limit where the string energy is approximately γ times the energy of the masses at its ends.

The system in question is described by the action

$$S = -\mu\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}\mathrm{d}\sigma\left[(\dot{x}x')^2 - \dot{x}^2 x'^2\right]^{1/2} - \sum_{i=1}^{2} m_i\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\left\{-\left[\frac{\mathrm{d}x_\alpha\big(\tau,\sigma_i(\tau)\big)}{\mathrm{d}\tau}\right]^2\right\}^{1/2}, \tag{3.32}$$

where μ is a constant characterizing the string tension, m_1 and m_2 are the masses at the string ends, and $x^\alpha(\tau,\sigma)$ is the 4-vector that parameterizes the world surface of the string. The dot and the prime respectively denote partial derivatives with respect to τ and σ.

We choose an evolution parameter τ coincident with time $t = x^0(\tau,\sigma)$. In this case, action (3.32) takes the form

$$S = \int_{t_1}^{t_2}\mathrm{d}t\left(\int_{\sigma_1}^{\sigma_2}\mathrm{d}\sigma\,\mathcal{L}_{\mathrm{str}} - \sum_{i=1}^{2} m_i\sqrt{1-\dot{\mathbf{x}}_i^2}\right), \tag{3.33}$$
$$\mathcal{L}_{\mathrm{str}} = -\mu\sqrt{\mathbf{x}'^2(1-\dot{\mathbf{x}}^2) + (\dot{\mathbf{x}}\mathbf{x}')^2}\,,\quad \mathbf{x} = \mathbf{x}(t,\sigma)\,,\quad \mathbf{x}_i = \mathbf{x}(t,\sigma_i)\,.$$

The Euler equation describing the string motion is obtained by varying the action S:

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}_{\rm str}}{\partial\dot{\mathbf{x}}}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}_{\rm str}}{\partial\mathbf{x}'}\right)=0\,. \tag{3.34}$$

The equations of motion for the masses at the string ends coincide with the boundary conditions

$$\begin{aligned} &m_1\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\dot{\mathbf{x}}_1}{\sqrt{1-\dot{\mathbf{x}}_1^2}}=-\frac{\partial\mathcal{L}_{\rm str}}{\partial\mathbf{x}'}\,, \qquad \sigma=\sigma_1\,,\\ &m_2\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\dot{\mathbf{x}}_2}{\sqrt{1-\dot{\mathbf{x}}_2^2}}=\frac{\partial\mathcal{L}_{\rm str}}{\partial\mathbf{x}'}\,, \qquad \sigma=\sigma_2\,. \end{aligned} \tag{3.35}$$

These equations have a particular solution

$$\begin{aligned} &x^1(\tau,\sigma)=\sigma\sin\omega\tau\,, \quad x^2(\tau,\sigma)=\sigma\cos\omega\tau\,,\\ &x^3(\tau,\sigma)=0\,, \qquad\qquad x^0(\tau,\sigma)=t=\tau \end{aligned} \tag{3.36}$$

that describes the motion of the string as a straight segment rotating with the angular velocity ω. In (3.36), the parameter τ is chosen to be the coordinate time, while σ is the distance between the point of interest on the string and the center of rotation (with an appropriate sign). Because

$$\begin{aligned} &m_i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\dot{\mathbf{x}}_i}{\sqrt{1-\dot{\mathbf{x}}_i^2}}=-\frac{m_i\omega^2\sigma_i}{\sqrt{1-\sigma_i^2\omega^2}}\,(\sin\omega t,\cos\omega t,0)\,,\\ &\mp\frac{\partial\mathcal{L}_{\rm str}}{\partial\mathbf{x}'}\bigg|_{\sigma_1,\,\sigma_2}=\pm\mu\sqrt{1-\sigma_i^2\omega^2}\,(\sin\omega t,\cos\omega t,0)\,, \end{aligned} \tag{3.37}$$

it follows from the boundary conditions that

$$\sigma_i\omega=\pm\frac{m_i\omega}{2\mu}\mp\sqrt{1+\left(\frac{m_i\omega}{2\mu}\right)^2}\lessgtr 0\,. \tag{3.38}$$

Here and hereinafter, the top and bottom signs correspond to $i=1$ and $i=2$. Because

$$v_i=\mp\sigma_i\omega\,, \qquad i=1,2 \tag{3.39}$$

is the velocity of the mass m_i, the string tension μ can be expressed via the mass and velocity of any of the masses:

$$\frac{\mu}{\omega}=m_i v_i\gamma_i^2\,, \qquad \gamma_i=(1-v_i^2)^{-1/2}\,. \tag{3.40}$$

The system EMT is made up of the EMT $t_{\mu\nu}$ of the masses at the ends of the string and the EMT of the string itself [21]:

$$\begin{aligned} \theta_{\mu\nu}(x)=\mu\int\mathrm{d}\tau\,\mathrm{d}\sigma\,\left[(\dot{x}x')^2-\dot{x}^2x'^2\right]^{-1/2}\delta\big(x-x(\tau,\sigma)\big)\\ \times\left\{x'^2\dot{x}_\mu\dot{x}_\nu+\dot{x}^2x'_\mu x'_\nu-(x'\dot{x})(\dot{x}_\mu x'_\nu+\dot{x}_\nu x'_\mu)\right\}. \end{aligned} \tag{3.41}$$

This expression can be simplified by imposing the gauge condition $x'_\alpha\dot{x}^\alpha=0$. Then

$$\begin{aligned} \theta_{\mu\nu}(x)=\mu\int\mathrm{d}\tau\,\mathrm{d}\sigma\left(-\frac{x'^2}{\dot{x}^2}\right)^{1/2}\delta\big(x-x(\tau,\sigma)\big)\\ \times\left(\dot{x}_\mu\dot{x}_\nu+\frac{\dot{x}^2}{x'^2}\,x'_\mu x'_\nu\right). \end{aligned} \tag{3.42}$$

Using (3.36) and (3.38), we obtain the energy density and the energy of the string:

$$\theta_{00}(x)=\mu\int_{\sigma_1}^{\sigma_2}\frac{\mathrm{d}\sigma}{(1-\omega^2\sigma^2)^{1/2}}\,\delta\big(\mathbf{x}-\mathbf{x}(\tau,\sigma)\big)\,, \tag{3.43}$$

$$\begin{aligned} E^{\rm str}&=\int\mathrm{d}^3x\,\theta_{00}(x)=\frac{\mu}{\omega}\int_{-v_1}^{v_2}\frac{\mathrm{d}x}{(1-x^2)^{1/2}}\\ &=\frac{\mu}{\omega}\,(\arcsin v_2+\arcsin v_1)=\sum_{i=1}^{2}m_i v_i\gamma_i^2\arcsin v_i\,. \end{aligned} \tag{3.44}$$

In the last expression, Eqn (3.40) relating the tension and the velocity v_i at the end of the string loaded with m_i is used. Interestingly, in the ultrarelativistic motion of at least one of the ends, the string energy $E^{\rm str}$ is $\pi\gamma/2$ times the energy $E^{\rm mass}=m_1\gamma_1+m_2\gamma_2$ of the masses at its ends.

We now pass from (3.41) to the Fourier components and use them and formulas (2.48)–(2.50) to construct the transverse components $\theta_A(q)$ describing GR from the string:

$$\begin{aligned} \theta_+(q)&=\sum_n 2\pi\delta(q^0-n\omega)\\ &\times\frac{\mu}{\omega}\int_{-v_1}^{v_2}\frac{\mathrm{d}x}{(1-x^2)^{1/2}}\left\{\left[\left(\frac{n^2}{z^2}-\frac{1}{2}\right)(1+\cos^2\theta)\right.\right.\\ &\left.\left.-\left(x^2-\frac{1}{2}\right)\sin^2\theta\right]J_n(z)-(1+\cos^2\theta)\,\frac{1}{z}\,J'_n(z)\right\}, \end{aligned} \tag{3.45}$$

$$\begin{aligned} \theta_\times(q)&=\sum_n 2\pi\delta(q^0-n\omega)\\ &\times\frac{\mu}{\omega}\,\mathrm{i}\cos\theta\int_{-v_1}^{v_2}\frac{\mathrm{d}x}{(1-x^2)^{1/2}}\,\frac{n}{z}\left[\frac{1}{z}\,J_n(z)-J'_n(z)\right]. \end{aligned} \tag{3.46}$$

Here, $z=|n|x\sin\theta$, $x=\omega\sigma$ is the velocity at the point of the string with the coordinate σ.

We note that the string EMT can be represented by the sum of two terms corresponding to the two halves of the string, i.e., to intervals $\sigma_1\leqslant\sigma<0$ and $0<\sigma\leqslant\sigma_2$. Specifically,

$$\begin{aligned} &\theta_A(q)=\theta_A(q,\,-v_1)+\theta_A(q,v_2)\,,\\ &\theta_A(q,\mp v_{1,2})=\sum_n 2\pi\delta(q^0-n\omega)\,\theta_{An}(q,\mp v_{1,2})\,. \end{aligned} \tag{3.47}$$

Here, the first and the second terms are the contributions to the integral over x from the segments $-v_1\leqslant x<0$ and $0<x\leqslant v_2$.

When the masses are equal, $m_1=m_2$, their velocities are also equal: $v_1=v_2$. In this case, the relation

$$\theta_{An}(q,v)=(-1)^n\,\theta_{An}(q,\,-v) \tag{3.48}$$

leads to the interference between GR from the opposite halves of the string; as a result, the amplitudes $\theta_{An}(q)$ of odd harmonics vanish, while even harmonics have twice the amplitude emitted by each half. Similar interference occurs in GR of the system of two masses at the ends of the string. The transverse components of the EMT in this system consist of the sum of components

$$t_A(q)=t_A(q,\,-v)+t_A(q,v) \tag{3.49}$$

defined by formulas (3.14) and (3.15) with the opposite signs of v.

We also note that the sum $t_A(q,v)+\theta_A(q,v)$ is the GR amplitude with polarization A of an independent object, a string of length r with one end fixed ($\sigma = 0$) and the other loaded with a mass m ($\sigma = r$) and rotating about the fixed one at the angular velocity ω.

Such an object arises naturally in considering a string with masses m_1 and m_2, one of which is much heavier than the other. For example, if the mass m_1 tends to infinity, its distance $r_1 = -\sigma_1$ from the fixed point $\sigma = 0$ and velocity $v_1 = -\sigma_1\omega$ tend to zero:

$$\sigma_1\omega = \frac{m_1\omega}{2\mu} - \sqrt{1 + \left(\frac{m_1\omega}{2\mu}\right)^2} \approx -\frac{\mu}{m_1\omega} \to 0\,, \qquad m_1 \to \infty\,. \tag{3.50}$$

We show in what follows [see (3.51) and (3.52) with mv^2 replaced with $m_1 v_1^2 = \mu^2/m_1\omega^2 \to 0$] that the heavier mass does not emit GR, and it is possible to consider a string with one end fixed at the point $\sigma = 0$. For such a string, the sum $t_A(q, v_2) + \theta_A(q, v_2)$ stands for the GR amplitude with polarization A.

We now analyze the behavior of $\theta_A(q, v)$ in the nonrelativistic and ultrarelativistic limits. For $nv \ll 1$, the expansion of the Bessel functions in (3.45) and (3.46) leads to

$$\theta_{+n}(q, v) \approx \begin{cases} \dfrac{mv^3}{16}(1 - 3\cos^2\theta)\sin\theta\,, & n = 1\,, \\ \dfrac{mv^n(n\sin\theta)^{n-2}}{2^n(n-1)!}(1 + \cos^2\theta)\,, & n \geqslant 2\,, \end{cases} \tag{3.51}$$

$$\theta_{\times n}(q, v) \approx \begin{cases} \mathrm{i}\,\dfrac{mv^3}{16}\sin\theta\cos\theta\,, & n = 1\,, \\ -\mathrm{i}\,\dfrac{mv^n(n\sin\theta)^{n-2}}{2^n(n-1)!}\cos\theta\,, & n \geqslant 2\,. \end{cases} \tag{3.52}$$

These expressions are identical to the nonrelativistic harmonics of the transverse components of the field EMT in the systems considered in Sections 2 and 3.

The coincidence is not accidental. It can be shown that the conserved tensor $T_{\alpha\beta}(q)$ satisfies the equation

$$\begin{aligned} &q_i q_j \frac{\partial^2}{\partial q_k \partial q_l} T_{ij}(q) + 2q_i \frac{\partial}{\partial q_l} T_{ik}(q) \\ &\quad + 2q_i \frac{\partial}{\partial q_k} T_{il}(q) + 2T_{kl}(q) \\ &\quad = -q^{0\,2} \int \mathrm{d}^3x\, \exp(-\mathrm{i}\mathbf{qx})\, x_k x_l T^{00}(\mathbf{x}, q^0)\,. \end{aligned} \tag{3.53}$$

For a closed nonrelativistic system, we can approximately set $T^{00}(\mathbf{x}, q^0) \approx t^{00}(\mathbf{x}, q^0)$ in the right-hand side of this equation. Assuming that the system consists of a single point mass moving in a force field, we seek a solution of the resulting equation in the form

$$T_{ij}(q) = t_{ij}(q) + m\int \mathrm{d}t\, \exp(\mathrm{i}q^0 t) f(\mathbf{qx})(\ddot{x}_i x_j + \ddot{x}_j x_i) + \ldots\,, \tag{3.54}$$

where the dots denote terms of type (5.26) (see below) that we do not need. For the function $f(z)$, we then have the equation and the solution

$$zf'(z) + f(z) = \frac{1}{2}\exp(-\mathrm{i}z)\,, \quad f(z) = \frac{1 - \exp(-\mathrm{i}z)}{2\mathrm{i}z}\,. \tag{3.55}$$

Expanding $f(z)$ into a power series in z and assuming the motion to be circular, we arrive at a nonrelativistic expression for all harmonics $T_{ij}(q)$. Their transverse components are presented in Section 2 [see also (3.51) and (3.52)].

For $\gamma \gg 1$, the most essential values of the variables in formulas (3.45) and (3.46) are $n \approx z \sim \gamma^3$, $v - |x| \sim \gamma^{-2}$, and $\alpha \equiv \pi/2 - \theta \sim \gamma^{-1}$. Carrying out the relevant expansions and using (2.6), we obtain

$$\theta_{+n}(q, v) \approx \frac{2m\gamma}{\pi n}\int_1^\infty \frac{\mathrm{d}\xi}{\xi^{1/2}}\,\Phi''(y)\,, \tag{3.56}$$

$$\begin{aligned} &\theta_{\times n}(q, v) \approx \mathrm{i}\,\frac{m\gamma\alpha}{2\pi}\left(\frac{2}{n}\right)^{2/3}\int_1^\infty \frac{\mathrm{d}\xi}{\xi^{1/2}}\,\Phi'(y)\,, \\ &y = \left(\frac{n}{2\gamma^3}\right)^{2/3}(\xi + \gamma^2\alpha^2)\,. \end{aligned} \tag{3.57}$$

These ultrarelstivistic transverse components of the string EMT turn out to be of the same order of magnitude as the transverse components of the EMT of the material body at the string end [cf. (3.14), (3.15) and (3.24), (3.26) with (3.56) and (3.57)].

The reason why the GR from a string at $\gamma \gg 1$ is of the same order as the GR of the mass at its end is as follows. The condition $v - |x| \sim \gamma^{-2}$ implies that radiation is emitted from small segments near the string ends moving at velocities x such that the corresponding Lorentz factor $\gamma(x) \equiv (1 - x^2)^{-1/2}$ is of the order of γ. Although the energy of the string is more than γ times that of the mass, it is distributed over the string such that the energy propagating through the space with a Lorentz factor of the order of γ constitutes only a fraction γ^{-1} of the total energy of the string:

$$\begin{aligned} &\frac{\mu}{\omega}(\arcsin v - \arcsin v')\Big|_{\gamma' \sim \gamma \gg 1} \\ &\quad \approx \frac{\mu}{\omega}\left(\frac{1}{\gamma'} - \frac{1}{\gamma}\right) \sim \frac{\mu}{\omega}\gamma^{-1}\,, \end{aligned} \tag{3.58}$$

i.e., it is precisely of the same order of magnitude as the energy of the mass at the end of the string [see (3.44)].

3.4 Gravitational radiation from a string with unloaded ends

Before proceeding to the limit $m_{1,2} = 0$ (or $v_{1,2} = 1$), we rewrite Eqns (3.45) and (3.46) with $m_1 = m_2 = m$ in the form

$$\begin{aligned} \theta_+ = &\sum_n 2\pi\delta(q^0 - n\omega)\,\frac{\mu}{\omega}\int_{-v}^{v} \frac{\mathrm{d}x}{(1 - x^2)^{1/2}} \\ &\times \left\{\frac{1}{4}(1 + \cos^2\theta)\big[J_{n+2}(z) + J_{n-2}(z)\big] \right. \\ &\left. - \sin^2\theta\left(x^2 - \frac{1}{2}\right)J_n(z)\right\}\,, \end{aligned} \tag{3.59}$$

$$\begin{aligned} \theta_\times = &\sum_n 2\pi\delta(q^0 - n\omega)\,\frac{\mu}{\omega}\,\frac{\mathrm{i}}{4}\cos\theta \\ &\times \int_{-v}^{v} \frac{\mathrm{d}x}{(1 - x^2)^{1/2}}\big[J_{n+2}(z) - J_{n-2}(z)\big]\,. \end{aligned} \tag{3.60}$$

Now, we can pass to the limit $v = 1$ and integrate the Bessel functions with the aid of formula 7.7.2 (11) from Ref. [22]. We thus obtain

$$\theta_+ = \sum_k 2\pi\delta(q^0 - 2k\omega)\,\frac{\pi\mu}{2\omega}\left\{J_{k+1}^2(x) + J_{k-1}^2(x) - \frac{1}{2}\sin^2\theta\left[J_{k+1}(x) + J_{k-1}(x)\right]^2\right\}$$
$$= \sum_k 2\pi\delta(q^0 - 2k\omega)\,\frac{\pi\mu}{\omega}\left\{\cot^2\theta\, J_k^2(x) + J_k'^2(x)\right\}, \quad (3.61)$$

$$\theta_\times = \sum_k 2\pi\delta(q^0 - 2k\omega)\,\frac{\mathrm{i}\pi\mu\cos\theta}{4\omega}\left[J_{k+1}^2(x) - J_{k-1}^2(x)\right]$$
$$= \sum_k 2\pi\delta(q^0 - 2k\omega)\left(-\frac{\mathrm{i}\pi\mu}{\omega}\right)\cot\theta\, J_k(x)\, J_k'(x)\,, \quad (3.62)$$
$$x = k\sin\theta\,.$$

The GR spectrum is given by the combination

$$\frac{1}{2}\,|\theta_+|^2 + 2|\theta_\times|^2 = t\sum_k 2\pi\delta(q^0 - 2k\omega)\left(\frac{\pi\mu}{\omega}\right)^2 \times\left\{\frac{1}{2}\left[\cot^2\theta\, J_k^2(x) + J_k'^2(x)\right]^2 + 2\cot^2\theta\, J_k^2(x)\, J_k'^2(x)\right\}. \quad (3.63)$$

The energy emitted for time $t \gg \omega^{-1}$ is

$$\mathcal{E} = t\, 4\pi G\mu^2 \sum_{k=1}^{\infty} k^2 \int \mathrm{d}\Omega\,\{\ldots\}\,, \quad (3.64)$$

where $\{\ldots\}$ is the expression in braces in (3.63).

We consider the behavior of the terms in this series at $k \gg 1$. In this case, the Airy functions may be used instead of the Bessel functions [see (3.24)]. Bearing in mind that the leading contribution comes from $\cos\theta \sim (2/k)^{1/3}$, we have

$$k^2 \int \mathrm{d}\Omega\,\{\ldots\} \approx \frac{8}{\pi^3 k}\int_0^\infty \frac{\mathrm{d}y}{y^{1/2}}\left[y^2\Phi^4(y) + \Phi'^4(y) + 6y\Phi^2(y)\Phi'^2(y)\right]. \quad (3.65)$$

Series (3.64) hence diverges logarithmically. This divergence is likely to disappear when quantum effects essential for the emission of higher-order harmonics are taken into consideration.

3.5 Polarization amplitudes in the β^2-approximation

The relativistic transverse amplitudes $T_+(q)$ and $T_\times(q)$ of the GR found in the previous sections in three cases — a string with the loaded ends, a charge in a magnetic moment field, and a charge in a Coulomb field — are considerably different. But the leading terms of their harmonics in all three cases coincide in the nonrelativistic approximation at $\beta = v/c \ll 1$ (see the discussion in Sections 2.4, 3.1, and 3.3). These relativistic amplitudes allow just as well finding the next terms of expansion in powers of the small parameter β for all harmonics.

In all three cases, we consider the expansion in β of the main, second harmonic that makes the largest contribution to the amplitudes T_+ and $T_\times$ at $\beta \ll 1$, keeping the terms up to the second order of smallness. We define the amplitude $T_{An}(q)$ of the nth harmonic by the relation

$$T_A(q) = \sum_n 2\pi\delta(q^0 - n\omega)\, T_{An}(q)\,, \qquad T_{An}(q) = t_{An}(q) + \theta_{An}(q)\,, \quad A = +, \times\,, \quad (3.66)$$

and present the expressions for the material t_{A2} and field θ_{A2} constituents of the second harmonic. Naturally, $t_{A2}(q)$ are identical on all three cases:

$$t_{+2}(q) = \frac{1}{2}\, mv^2\gamma\left\{1 - \frac{1}{2}\sin^2\theta - \beta^2\sin^2\theta\right\}, \qquad t_{\times 2}(q) = -\mathrm{i}\,\frac{1}{4}\, mv^2\gamma\cos\theta\left\{1 - \beta^2\sin^2\theta\right\}. \quad (3.67)$$

Conversely, the field constituents $\theta_{A2}(q)$ are different, but only in the terms of the order of β^2.

For a string with a mass,

$$\theta_{+2}^{\mathrm{str}}(q) = \frac{1}{2}\, mv^2\gamma\left\{\left(1 + \frac{2}{3}\,\beta^2\right)\left(1 - \frac{1}{2}\sin^2\theta\right) - \frac{1}{3}\,\beta^2\sin^2\theta\cos^2\theta\right\},$$
$$\theta_{\times 2}^{\mathrm{str}}(q) = -\mathrm{i}\,\frac{1}{4}\, mv^2\gamma\cos\theta\left\{1 + \frac{2}{3}\,\beta^2 - \frac{1}{3}\,\beta^2\sin^2\theta\right\}. \quad (3.68)$$

For a charge in a magnetic moment field,

$$\theta_{+2}^{\mathrm{M}}(q) = \frac{1}{2}\, mv^2\gamma\left\{1 - \frac{1}{2}\sin^2\theta - \frac{1}{3}\,\beta^2\sin^2\theta\cos^2\theta\right\}, \qquad \theta_{\times 2}^{\mathrm{M}}(q) = -\mathrm{i}\,\frac{1}{4}\, mv^2\gamma\cos\theta\left\{1 - \frac{1}{3}\,\beta^2\sin^2\theta\right\}. \quad (3.69)$$

For a charge in a Coulomb field,

$$\theta_{+2}^{\mathrm{C}}(q) = \frac{1}{2}\, mv^2\gamma\left\{\left(1 - \frac{4}{3}\,\beta^2\right)\left(1 - \frac{1}{2}\sin^2\theta\right) - \frac{1}{3}\,\beta^2\sin^2\theta\cos^2\theta\right\},$$
$$\theta_{\times 2}^{\mathrm{C}}(q) = -\mathrm{i}\,\frac{1}{4}\, mv^2\gamma\cos\theta\left\{1 - \frac{4}{3}\,\beta^2 - \frac{1}{3}\,\beta^2\sin^2\theta\right\}. \quad (3.70)$$

For the sum $T_{A2}(q)$, we then have

$$T_{+2}(q) = \frac{1}{2}\, mv^2\gamma\left\{(1 + \delta\beta^2)(1 + \cos^2\theta) - \beta^2\sin^2\theta\left(1 + \frac{1}{3}\cos^2\theta\right)\right\},$$
$$T_{\times 2}(q) = -\mathrm{i}\,\frac{1}{2}\, mv^2\gamma\cos\theta\left\{1 + \delta\beta^2 - \frac{2}{3}\,\beta^2\sin^2\theta\right\}, \quad (3.71)$$

with the parameter $\delta = 1/3,\ 0,\ -2/3$ for the three cases.

The above expressions exhibit the following important properties:

(1) The relativistic correction being excluded, i.e., $\beta = 0$, the field components coincide with the material ones:

$$\theta_{A2}(q) = t_{A2}(q)\,, \qquad A = +, \times\,. \quad (3.72)$$

(2) The expressions do not contain terms linear in β; they are expansions in β^2 if we also recall that $\gamma = 1 + (1/2)\beta^2$ too.

(3) The correction $\delta\beta^2$, different in all three cases, can be taken out of the braces in Eqn (3.71) and made an additional component of the Lorentz factor γ such that $T_{A2}(q)$ assumes the form

$$\begin{aligned} T_{+2}(q) &= \frac{1}{2}\, mv^2(\gamma+\delta\beta^2) \\ &\quad\times\left\{1+\cos^2\theta-\beta^2\sin^2\theta\left(1+\frac{1}{3}\cos^2\theta\right)\right\}, \\ T_{\times 2}(q) &= -\mathrm{i}\,\frac{1}{2}\, mv^2(\gamma+\delta\beta^2)\cos\theta\left\{1-\frac{2}{3}\,\beta^2\sin^2\theta\right\}. \end{aligned} \tag{3.73}$$

Hence, for the three cases considered, the transverse components $T_{A2}(q)$ of the EMT have identical angular dependences but different amplitudes. The difference is due to the terms $\delta\beta^2$ of the same order as the nonrelativistic kinetic energy $(1/2)\beta^2$ of the mass in units of mc^2. Therefore, it is related to the nonlocal properties of the force field holding the mass on the orbit and may be of interest from the experimental standpoint. Also of interest in this context is the GR of masses orbiting their common center. Such a motion can be experienced by clots of dark matter, and their GR might be an important source of information about these objects.

4. Gravitational radiation from masses moving in elliptical orbits

4.1 Movement in the Kepler orbits

We consider the GR of two point masses m_1 and m_2 moving in Kepler orbits around their common center of attraction located in the common focus of each ellipse. By choosing the Cartesian coordinate system with axes 1, 2 in the plane of motion and the origin at the common focus, we can describe the motion of the mass m_1 by the coordinates

$$x_1=\frac{m_2}{m_1+m_2}\, r\cos\psi\,,\qquad x_2=\frac{m_2}{m_1+m_2}\, r\sin\psi\,, \tag{4.1}$$

and the motion of the mass m_2 by the coordinates

$$\xi_1=-\frac{m_1}{m_1+m_2}\, r\cos\psi\,,\qquad \xi_2=-\frac{m_1}{m_1+m_2}\, r\sin\psi\,. \tag{4.2}$$

Here, ψ is the angle between the direction to the mass m_1 and the axis 1 directed from the focus to m_1 as it comes closest to m_2; r is the distance between the masses dependent on the angle ψ,

$$r=\frac{p}{1+e\cos\psi}\,,\qquad p=a(1-e^2)\,. \tag{4.3}$$

The orbits are fully determined by the parameter p, the eccentricity e, and the mass ratio m_1/m_2. Therefore, if $m_1>m_2$, the heavy m_1 and light m_2 respectively move along the small and large ellipses, while the point with the coordinates

$$r_1=r\cos\psi\,,\qquad r_2=r\sin\psi\,, \tag{4.4}$$

moves along an even larger ellipse (see Fig. 1 for $m_1=2m_2$ and $e=1/2$). All three ellipses have the same eccentricity e and their large semiaxes are equal to

$$\frac{m_2}{m_1+m_2}\, a\,,\quad \frac{m_1}{m_1+m_2}\, a\,,\quad a\,. \tag{4.5}$$

The angular velocity of the orbiting masses is described by the equation

$$\dot\psi=A(1+e\cos\psi)^2\,,\qquad A=\sqrt{\frac{G(m_1+m_2)}{p^3}}\,, \tag{4.6}$$

having the solution

$$\begin{aligned} At &= \frac{2}{(1-e^2)^{3/2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\,\tan\frac{\psi}{2}\right) \\ &\quad-\frac{e\sin\psi}{(1-e^2)(1+e\cos\psi)}\,. \end{aligned} \tag{4.7}$$

In specific cases $e=0$ (circle) and $e=1$ (parabola, p is finite and equal to $2r_{\min}$), we have

$$At=\psi\,,\qquad 2At=\tan\frac{\psi}{2}+\frac{1}{3}\tan^3\frac{\psi}{2}\,. \tag{4.8}$$

For elliptical orbits, the period T and the fundamental frequency ω are given by the equation

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{a^3}{G(m_1+m_2)}}\,, \tag{4.9}$$

which corresponds to Kepler's third law stating that the ratio of the squares of the orbital periods of any two planets around the Sun is proportional to the cubes of their major semiaxes.

As follows from (4.7), the relation between time t and the angle ψ is rather complicated if $e\neq 0$. Instead of ψ, Lagrange introduced a variable u related to ψ by the expressions

$$\cos\psi=\frac{\cos u-e}{1-e\cos u}\,,\qquad \sin\psi=\frac{\sqrt{1-e^2}\,\sin u}{1-e\cos u}\,. \tag{4.10}$$

The inverse relation between u and ψ is described by the same equations with the substitutions $u\leftrightarrow\psi$, $e\leftrightarrow -e$. Then the time t is related to the distance r and the variable u by the formulas

$$\omega t=u-e\sin u\,,\qquad r=a(1-e\cos u)\,. \tag{4.11}$$

Hence,

$$r_1=a(\cos u-e)\,,\qquad r_2=a\sqrt{1-e^2}\,\sin u\,. \tag{4.12}$$

The dimensionless variables $\tau\equiv\omega t$, ψ, and u were formerly referred to as main, true, and eccentric anomalies. The time t and the angle ψ increase monotonically as u increases; all anomalies coincide at points divisible by π: $\tau=\psi=u=k\pi$.

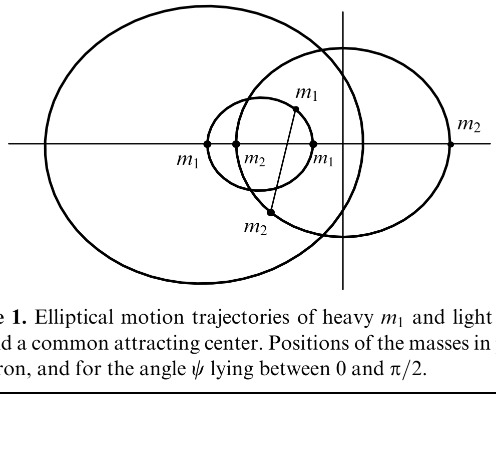

Figure 1. Elliptical motion trajectories of heavy m_1 and light m_2 masses around a common attracting center. Positions of the masses in periastron, apastron, and for the angle ψ lying between 0 and $\pi/2$.

The introduction of the variable u into the description of planetary motion resulted in the appearance of Bessel functions.

The functions r_1 and r_2 describing the elliptical motion, harmonic in u but not in t, are represented by the Fourier series in harmonic functions of time:

$$\sin u = \sum_{n=1}^{\infty} \frac{2}{ne} J_n(ne) \sin n\tau \,,$$

$$\cos u = -\frac{1}{2}\, e + \sum_{n=1}^{\infty} \frac{2}{n} J_n'(ne) \cos n\tau \,.$$

It is assumed in these formulas that the masses m_1 and m_2 orbit counterclockwise. To consider clockwise motion, we must change the sign of time t in Eqns (4.6)–(4.8), which is equivalent to changing the sign of u in Eqns (4.10)–(4.12).

4.2 The trajectory method for calculating gravitational radiation amplitudes

It is known that gravitational radiation is produced by the transverse components of the EMT of the masses m_1 and m_2 and the force field that hold them on the orbits. The conserved EMT satisfies the equation

$$q_i\, q_j\, T_{ij}(q) = \frac{q^{0\,2}}{c^2}\, T_{00}(q) \,, \tag{4.13}$$

in which $q = (\mathbf{q}, q^0)$ is the wave vector and the frequency, temporarily considered independent, and all the components of $T_{\alpha\beta}(q)$ have the dimensions erg s. However, we proceed from the equation derived by differentiating both sides of (4.13) with respect to the components q_k and q_l of the wave vector:

$$\frac{\partial^2}{\partial q_k\, \partial q_l}\, q_i\, q_j\, T_{ij}(q) = \frac{q^{0\,2}}{c^2}\, \frac{\partial^2}{\partial q_k\, \partial q_l}\, T_{00}(q) \,. \tag{4.14}$$

Our main approximation consists in replacing the component $T_{00}(q)$ of the total EMT in the right-hand side of (4.14) by the component $t_{00}(q)$ of the EMT of the masses m_1 and m_2 in the nonrelativistic approximation. We first take $t_{00}(q)$ in the lowest-order approximation and then make it more exact by introducing the first relativistic correction into t_{00}. Finally, we try to add the component $\theta_{00}(q)$ of the force field EMT to $t_{00}(q)$, also in the nonrelativistic limit. The contributions of individual masses to t_{00} being additive, we keep only one of them for simplicity. Replacing $T_{00}(q)$ with

$$t_{00}(q) = mc^2 \int \mathrm{d}t \, \exp\left[\mathrm{i}q^0 t - \mathrm{i}\mathbf{q}\mathbf{x}(t)\right] , \tag{4.15}$$

we obtain

$$\frac{q^{0\,2}}{c^2}\, \frac{\partial^2}{\partial q_k\, \partial q_l}\, t_{00}(q) = -q^{0\,2} m \int \mathrm{d}t \, \exp\left[\mathrm{i}q^0 t - \mathrm{i}\mathbf{q}\mathbf{x}(t)\right] x_k(t) x_l(t) \tag{4.16}$$

in the right-hand side. We now seek the solution $T_{ij}(q)$ of Eqn (4.14) with the approximate right-hand side (4.16) by the substitution

$$T_{ij}(q) = t_{ij}(q) + m \int \mathrm{d}t \, \exp\,(\mathrm{i}q^0 t)\, f(\mathbf{q}\mathbf{x})(\ddot{x}_i x_j + x_i \ddot{x}_j) \,, \tag{4.17}$$

where

$$t_{ij}(q) = m \int \mathrm{d}t \, \exp\left[\mathrm{i}q^0 t - \mathrm{i}\mathbf{q}\mathbf{x}(t)\right] \dot{x}_i(t) \dot{x}_j(t) \tag{4.18}$$

are the transverse components of the EMT of the mass m in the nonrelativistic approximation and $f(z)$ is the sought function. The second term in Eqn (4.17) can be called the spatial part $\theta_{ij}(q)$ of the force field EMT accelerating the mass m. It is easy to show that the contributions $t_{ij}(q)$ and $\theta_{ij}(q)$ coincide in the nonrelativistic approximation.

With the above expressions, the left-hand side of Eqn (4.14) can be represented as

$$\begin{aligned} m \int \mathrm{d}t \, \exp\,(\mathrm{i}q^0 t) \Big\{ & \exp\,(-\mathrm{i}\mathbf{q}\mathbf{x}) \left[2\dot{x}_k \dot{x}_l - 2\mathrm{i}(\mathbf{q}\mathbf{x})(\dot{x}_k x_l + x_k \dot{x}_l) \right. \\ & \left. - x_k x_l (\mathbf{q}\dot{\mathbf{x}})^2\right] + 2 x_k x_l (\mathbf{q}\ddot{\mathbf{x}}) \left[(\mathbf{q}\mathbf{x}) f''(\mathbf{q}\mathbf{x}) + 2 f'(\mathbf{q}\mathbf{x})\right] \\ & + 2(\ddot{x}_k x_l + x_k \ddot{x}_l) \left[(\mathbf{q}\mathbf{x}) f'(\mathbf{q}\mathbf{x}) + f(\mathbf{q}\mathbf{x})\right] \Big\} \,. \end{aligned} \tag{4.19}$$

We now require the function $f(z)$ to satisfy

$$z f'(z) + f(z) = \frac{1}{2} \exp\,(-\mathrm{i}z) \,. \tag{4.20}$$

In this case,

$$z f''(z) + 2 f'(z) = -\frac{\mathrm{i}}{2} \exp\,(-\mathrm{i}z) \,. \tag{4.21}$$

Then the left-hand side of (4.14) becomes

$$\begin{aligned} & m \int \mathrm{d}t \, \exp\,(\mathrm{i}q^0 t - \mathrm{i}\mathbf{q}\mathbf{x}) \left[2\dot{x}_k \dot{x}_l - 2\mathrm{i}(\dot{x}_k x_l + x_k \dot{x}_l)\mathbf{q}\dot{\mathbf{x}} \right. \\ & \quad \left. - x_k x_l (\mathbf{q}\dot{\mathbf{x}})^2 - \mathrm{i} x_k x_l \mathbf{q}\ddot{\mathbf{x}} + \ddot{x}_k x_l + x_k \ddot{x}_l\right] \\ & \quad \equiv m \int \mathrm{d}t \, \exp\,(\mathrm{i}q^0 t) \, \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(x_k x_l \exp\,(-\mathrm{i}\mathbf{q}\mathbf{x})\right) . \end{aligned} \tag{4.22}$$

Hence, if the test function $f(z)$ satisfies (4.20), Eqn (4.14) with the approximate right-hand side (4.16) reduces to the relation between the Fourier transform of the function $g_{kl}(t, \mathbf{q}) \equiv x_k(t) x_l(t) \exp\left[-\mathrm{i}\mathbf{q}\mathbf{x}(t)\right]$ and the Fourier transform of its second derivative with respect to t:

$$\begin{aligned} & m \int \mathrm{d}t \, \exp\,(\mathrm{i}q^0 t) \, \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(x_k x_l \exp\,(-\mathrm{i}\mathbf{q}\mathbf{x})\right) \\ & \quad = -q^{0\,2} m \int \mathrm{d}t \, \exp\,(\mathrm{i}q^0 t) x_k x_l \exp\,(-\mathrm{i}\mathbf{q}\mathbf{x}) \,. \end{aligned} \tag{4.23}$$

Relation (4.23) between the Fourier transform of the function $g(t)$ and its second derivative $g''(t)$ requires that the conditions stipulated by Fikhtengol'ts (see [23], Vol. 3, Pt. 717) be satisfied. In our case, they are satisfied if the function $g(t) \exp\,(-\varepsilon t^2)$ with an infinitesimal parameter ε is considered instead of the periodic function $g(t)$. Then this function remains periodic and simultaneously satisfies Eqn (4.23) over a time interval much longer than the period of $g(t)$ but shorter than $1/\sqrt{\varepsilon}$.

If the function

$$f(z) = \frac{1 - \exp\,(-\mathrm{i}z)}{2\mathrm{i}z} \tag{4.24}$$

is used as the solution of Eqn (4.20), then the tensor $T_{ij}(q)$ defined by formula (4.17) is an exact solution of Eqn (4.14) with the approximate right-hand side (4.16).

For the two masses m_1 and m_2, this tensor becomes

$$T_{ij}(q) = \int \mathrm{d}t \, \exp(\mathrm{i}q^0 t) \Big\{ m_1 \big[\exp(-\mathrm{i}\mathbf{qx})\dot{x}_i\dot{x}_j + f(\mathbf{qx})(\ddot{x}_i x_j + x_i\ddot{x}_j)\big] + m_2\big[\exp(-\mathrm{i}\mathbf{q}\boldsymbol{\xi})\dot{\xi}_i\dot{\xi}_j + f(\mathbf{q}\boldsymbol{\xi})(\ddot{\xi}_i\xi_j + \xi_i\ddot{\xi}_j)\big]\Big\} . \quad (4.25)$$

Assuming the size of a radiation source to be small compared with the wavelength, we expand $\exp(-\mathrm{i}\mathbf{qx})$ and $\exp(-\mathrm{i}\mathbf{q}\boldsymbol{\xi})$ in a Taylor series and use the relation of the coordinates x_i, ξ_i to the coordinates r_i [see (4.1), (4.2), and (4.4)]. Then

$$T_{ij}(q) = \mu \int \mathrm{d}t \, \exp(\mathrm{i}q^0 t) \sum_{n=0}^{\infty} \frac{1}{2}\big[2(n+1)\dot{r}_i\dot{r}_j + \ddot{r}_i r_j + r_i\ddot{r}_j\big] \times \frac{(-\mathrm{i}\mathbf{qr})^n}{(n+1)!} C_n , \quad (4.26)$$

where $\mu = m_1 m_2 (m_1+m_2)^{-1}$ is the reduced mass and

$$C_n = \left(\frac{m_2}{m_1+m_2}\right)^{n+1} + (-1)^n \left(\frac{m_1}{m_1+m_2}\right)^{n+1} , \quad (4.27)$$

and hence

$$C_0 = 1 , \quad C_1 = \frac{m_2 - m_1}{m_1+m_2} , \quad C_2 = \frac{m_2^3 + m_1^3}{(m_1+m_2)^3} . \quad (4.28)$$

Hereafter, we restrict ourselves to the first three terms of the series in (4.26).

We introduce the dimensionless tensors $Q_A(\tau, e)$, $A = ij, ijk, ijkl, \ldots$, periodically dependent on the dimensionless time $\tau = \omega t$ and symmetric in the first two indices ij as well as in the remaining two or more indices:

$$\begin{aligned} &2\dot{r}_i\dot{r}_j + \ddot{r}_i r_j + r_i\ddot{r}_j = v^2 Q_{ij}(\tau, e) , \\ &(4\dot{r}_i\dot{r}_j + \ddot{r}_i r_j + r_i\ddot{r}_j) r_k = a v^2 Q_{ijk}(\tau, e) , \\ &(6\dot{r}_i\dot{r}_j + \ddot{r}_i r_j + r_i\ddot{r}_j) r_k r_l = a^2 v^2 Q_{ijkl}(\tau, e) , \ldots . \end{aligned} \quad (4.29)$$

Here, $v = a\omega$ is the characteristic velocity on the elliptical orbit. Then

$$T_{ij}(q) = \frac{1}{2}\mu v^2 \int \mathrm{d}t \, \exp(\mathrm{i}q^0 t) \Big\{ Q_{ij}(\tau, e) - \mathrm{i}\,\frac{1}{2} C_1 a q_k Q_{ijk}(\tau, e) - \frac{1}{6} C_2 a^2 q_k q_l Q_{ijkl}(\tau, e) + \ldots \Big\} . \quad (4.30)$$

Expanding $Q_A(\tau, e)$ as a Fourier series in $\cos n\tau$ or $\sin n\tau$ depending on the even or odd number of twos in A and denoting the Fourier coefficients by $F_A(n, e)$,

$$\begin{aligned} Q_A(\tau, e) &= \sum_{n \geqslant 0} F_A(n, e) \begin{Bmatrix} \cos n\tau \\ \sin n\tau \end{Bmatrix} , \\ F_A(n, e) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\tau \, Q_A(\tau, e) \begin{Bmatrix} \cos n\tau \\ \sin n\tau \end{Bmatrix} , \end{aligned} \quad (4.31)$$

we obtain

$$T_{ij}(q) = \frac{1}{2}\mu v^2 \sum_{n \geqslant 0} \pi\delta(q^0 - n\omega) \times \left\{ F_{ij}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} - \mathrm{i}\,\frac{1}{2} C_1 a q_k F_{ijk}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} - \frac{1}{6} C_2 a^2 q_k q_l F_{ijkl}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \ldots \right\} \quad (4.32)$$

after integration over t. Here,

$$F_A(n, e)\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

means that $F_A(n, e)$ with an even and odd number of twos in A is respectively multiplied by 1 and i. It is assumed in formula (4.32) that $q^0 > 0$ and that the masses orbit counterclockwise. To consider clockwise motion, it is necessary to change the sign of τ in the functions $Q_A(\tau, e)$. In accordance with expansion (4.31), this results in a change of sign in front of the Fourier coefficients $F_A(n, e)$ with an odd number of twos in A.

Because $|\mathbf{q}| = q^0/c$ for gravitons, $a q_k = n\beta e_k$, where $\beta = v/c$ and e_k are the components of the unit vector $\mathbf{e}$ along the graviton wave vector $\mathbf{q}$:

$$e_1 = \sin\theta\cos\varphi , \quad e_2 = \sin\theta\sin\varphi , \quad e_3 = \cos\theta . \quad (4.33)$$

We recall that Eqn (4.32) contains only the first two components of this vector.

Hence, Eqn (4.32) is the expansion of $T_{ij}(q)$ in powers of the nonrelativistic effective velocity $\beta = v/c \ll 1$ originating from the expansion in the wave vector.

Because the EMT is real in the x-space, its Fourier transform must satisfy the condition

$$T_{ij}(q) = T^*_{ij}(-q) . \quad (4.34)$$

Expression (4.32) obtained for $q^0 > 0$ automatically satisfies this condition if the expression for the time integral

$$\int \mathrm{d}t \, \exp(\mathrm{i}q^0 t) \begin{Bmatrix} \cos n\tau \\ \sin n\tau \end{Bmatrix} = \pi\delta(q^0 - n\omega)\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \pi\delta(q^0 + n\omega)\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} , \quad n \geqslant 0 , \quad (4.35)$$

contains not only the first term but also the second one, valid at $q^0 < 0$. Keeping both terms is equivalent to replacing the sum over positive $n \geqslant 0$ in (4.32) with the sum over all integers $n \gtrless 0$. Such equivalence ensues from the even (odd) dependence on n of the functions $P_A(n)$ with an even (odd) number of twos in A.

Therefore, denoting the expression in braces in (4.32) by $G_{ij}(n, a\mathbf{q})$, it is possible to represent $T_{ij}(q)$ in two different forms:

$$\begin{aligned} T_{ij}(q) &= \frac{1}{2}\mu v^2 \sum_{n \gtrless 0} \pi\delta(q^0 - n\omega) G_{ij}(n, a\mathbf{q}) \\ &= \frac{1}{2}\mu v^2 \sum_{n \geqslant 0} \pi\big[\delta(q^0 - n\omega) G_{ij}(n, a\mathbf{q}) + \delta(q^0 + n\omega) G^*_{ij}(n, -a\mathbf{q})\big] , \end{aligned} \quad (4.36)$$

suitable for any sign of q^0 and satisfying (4.34). However, only the physically interesting case of $q^0 > 0$ is considered in what follows.

4.3 Radiation spectrum and angular distribution

The radiation spectrum and angular distribution are defined by the formula

$$\mathrm{d}\mathcal{E}_{\mathbf{q}} = \frac{8\pi G}{c^2}\left(\frac{1}{2}\big|T_+(q)\big|^2 + 2\big|T_\times(q)\big|^2\right)\frac{\mathrm{d}^3 q}{16\pi^3} , \quad (4.37)$$

where the components T_+ and $T_\times$ are related to the components T_{11}, T_{22}, and T_{12} as

$$T_+ = (\sin^2\varphi - \cos^2\theta\cos^2\varphi)T_{11} - 2\sin\varphi\cos\varphi(1+\cos^2\theta)T_{12} + (\cos^2\varphi - \cos^2\theta\sin^2\varphi)T_{22}\,,$$
$$T_\times = \cos\theta\sin\varphi\cos\varphi(T_{11}-T_{22}) - \cos\theta(\cos^2\varphi - \sin^2\varphi)T_{12}\,. \quad (4.38)$$

We represent these components in a form analogous to (4.36):

$$T_A(q) = \frac{1}{2}\,\mu v^2\sum_{n>0}\pi\delta(q^0 - n\omega)\,G_A(n, a\mathbf{q})\,, \qquad A = +, \times\,.$$

Then

$$\mathrm{d}\mathcal{E}_{\mathbf{q}} = t\,\frac{G(\mu v^2)^2}{16\pi c^2} \times \sum_{n>0}\delta(q^0 - n\omega)\left(\frac{1}{2}\left|G_+(n)\right|^2 + 2\left|G_\times(n)\right|^2\right)\mathrm{d}^3 q\,. \quad (4.39)$$

Here, t is the radiation time that must be much greater than the period $T = 2\pi/\omega$. In this case, $\mathrm{d}\mathcal{E}_{\mathbf{q}}/t$ are the spectral and angular distribution of the mean radiation power.

Simple but cumbersome calculations give the following result for the angular distribution of the nth harmonic:

$$\frac{1}{2}\left|G_+(n)\right|^2 + 2\left|G_\times(n)\right|^2 = \frac{1}{2}|G_{11}|^2(1-e_1^2)^2 + \frac{1}{2}|G_{22}|^2(1-e_2^2)^2 + 2|G_{12}|^2(1-e_1^2)(1-e_2^2) + \mathrm{Re}\,G_{11}G_{22}^*(e_1^2e_2^2 - e_3^2) - 2\,\mathrm{Re}\,G_{11}G_{12}^*e_1e_2(1-e_1^2) - 2\,\mathrm{Re}\,G_{22}G_{12}^*e_1e_2(1-e_2^2)\,. \quad (4.40)$$

We recall that the tensor G_{ij} is defined by the expression in the braces in (4.32), which depends on six independent dimensionless quantities: the harmonic number n, the eccentricity e, the mass ratio m_1/m_2, the velocity $\beta = v/c$, and the angles θ and φ. In angular distribution (4.40), the six bilinear combinations formed from three complex components of G_{ij} are, for $ij = 11$, 22, and 12,

$$\left|G_{ij}(n,\beta)\right|^2 = F_{ij}^2 \pm C_1\beta_n e_2 F_{ij}F_{ij2} + \frac{1}{4}C_1^2\beta_n^2(e_1^2F_{ij1}^2 + e_2^2F_{ij2}^2) - \frac{1}{3}C_2\beta_n^2F_{ij}(e_1^2F_{ij11} + e_2^2F_{ij22})\,. \quad (4.41)$$

Here and hereinafter, $\beta_n = n\beta$, the plus sign in front of the second term corresponds to the indices $ij = 11$ or 22, and the minus sign corresponds to $ij = 12$.

$$\mathrm{Re}\,G_{11}(n,\beta)\,G_{22}^*(n,\beta) = F_{11}F_{22} + \frac{1}{2}\,C_1\beta_n e_2(F_{112}F_{22} + F_{11}F_{222}) + \frac{1}{4}\,C_1^2\beta_n^2(e_1^2F_{111}F_{221} + e_2^2F_{112}F_{222}) - \frac{1}{6}\,C_2\beta_n^2\left[e_1^2(F_{1111}F_{22} + F_{11}F_{2211}) + e_2^2(F_{1122}F_{22} + F_{11}F_{2222})\right]\,. \quad (4.42)$$

For $ii = 11$ or 22,

$$\mathrm{Re}\,G_{ii}(n,\beta)\,G_{12}^*(n,\beta) = \frac{1}{2}\,C_1\beta_n e_1(F_{ii}F_{121} - F_{ii1}F_{12}) + \frac{1}{4}\,C_1^2\beta_n^2 e_1e_2(F_{ii1}F_{122} + F_{ii2}F_{121}) - \frac{1}{3}\,C_2\beta_n^2 e_1e_2(F_{ii}F_{1212} + F_{ii12}F_{12})\,. \quad (4.43)$$

If F_{22}, F_{22k}, and F_{22kl} in the expression for $\mathrm{Re}\,G_{11}G_{22}^*$ are substituted by F_{11}, F_{11k}, and F_{11kl}, the expression turns into $|G_{11}|^2$ as expected. A similar substitution of F_{11}, F_{11k}, and F_{11kl} by F_{22}, F_{22k}, and F_{22kl} converts this expression into $|G_{22}|^2$.

4.4 Angular distribution asymmetry in the case $m_1 \neq m_2$, $e \neq 0$

In the case where $m_1 > m_2$ and $e \neq 0$, the 1-axis extends from the common focus toward the large ellipse along which the smaller mass m_2 moves. In this case, the angular distribution of radiation is asymmetric with respect to both the 2-axis direction and the opposite direction. For the azimuthal angles $\varphi = \pm\pi/2$, with $e_1 = 0$ and $e_2 = \pm\sin\theta$, it follows from (4.40)–(4.43) that

$$\frac{1}{2}|G_+|^2 + 2|G_\times|^2 = \frac{1}{2}\,|G_{11}|^2 + \frac{1}{2}\,|G_{22}|^2\cos^4\theta + 2|G_{12}|^2\cos^2\theta - 2\,\mathrm{Re}\,G_{11}G_{22}^*\cos^2\theta\,. \quad (4.44)$$

In the bilinear combinations of the tensor G_{ij}, the terms linear in $C_1\beta$ have different signs for the angles $\varphi = \pm\pi/2$ whereas the other terms remain unaltered. Therefore, the difference between the angular distributions for the angles $\varphi = \pm\pi/2$ is nonzero:

$$\left(\frac{1}{2}\,|G_+|^2 + 2|G_\times|^2\right)_{\varphi=\pi/2} - \left(\frac{1}{2}|G_+|^2 + 2|G_\times|^2\right)_{\varphi=-\pi/2} = C_1\beta_n\sin\theta\left[F_{11}F_{112} + F_{22}F_{222}\cos^4\theta - 2(F_{11}F_{222} + F_{22}F_{112} + 2F_{12}F_{122})\cos^2\theta\right]\,. \quad (4.45)$$

This remarkable relativistic effect (the parameter $\beta_n = n\beta = nv/c$ contains the speed of light) occurs because at $m_1 > m_2$ and $e \neq 0$, the masses have the highest velocities in the region where they come closest to each other (i.e., near the periastron), such that GR actually forms in this region. At the chosen counterclockwise direction of the mass motion, the velocity of the heavier mass m_1 in periastron is directed along the 2-axis, and the velocity of the light one m_2, opposite to it. Because the velocity and kinetic energy of the light mass are m_1/m_2 times those of the heavy one, the total GR from the two-mass system is largely due to the lighter mass; it is formed in the region where the two masses come closest to each other, and the overall intensity is higher in the direction of the light mass velocity.

It follows from the discussion below formula (4.32) that the Fourier coefficients $F_A(n,e)$ with an odd number of twos in A change sign if the masses moving counterclockwise turn in the opposite direction. In this case, the difference between the angular distributions for the angles $\varphi = \pm\pi/2$ [see (4.45)] also changes sign because the velocity of the lighter mass in the region where it comes closest to its heavy counterpart is directed along the 2-axis, and not opposite to it.

We note that the asymmetry disappears in the case of two equal masses because $C_1 = 0$, and also in the case of zero eccentricity $e = 0$. In the latter case, the coefficients F_{ij} differ from zero only for the quadrupole harmonic $n = 2$, and coefficients F_{ijk}, only for the $n = 1$ and $n = 3$ harmonics.

4.5 The spectrum as a polar angle function. Integral spectrum

Integration of the angular distribution of the nth harmonic (4.40) over φ gives the distribution of its radiation intensity with respect to the angle θ:

$$\int_{-\pi}^{\pi} d\varphi \left(\frac{1}{2}\left|G_+(n)\right|^2 + 2\left|G_\times(n)\right|^2\right)$$
$$= \frac{1}{2} I_1(F_{11}^2 + F_{22}^2) + 2I_2F_{12}^2 + I_3F_{11}F_{22}$$
$$+ \frac{1}{4} C_1^2\beta_n^2\left[\frac{1}{2} I_4(F_{111}^2 + F_{222}^2) + \frac{1}{2} I_5(F_{112}^2 + F_{221}^2)\right.$$
$$+ 2I_6(F_{121}^2 + F_{122}^2) + I_7(F_{111}F_{221} + F_{112}F_{222})$$
$$\left. - 2I_8(F_{111}F_{112} + F_{112}F_{121} + F_{221}F_{122} + F_{222}F_{121})\right]$$
$$- \frac{1}{3} C_2\beta_n^2\left[\frac{1}{2} I_4(F_{11}F_{1111} + F_{22}F_{2222})\right.$$
$$+ \frac{1}{2} I_5(F_{11}F_{1122} + F_{22}F_{2211}) + 2I_6(F_{12}F_{1211} + F_{12}F_{1222})$$
$$+ \frac{1}{2} I_7(F_{1111}F_{22} + F_{11}F_{2211} + F_{1122}F_{22} + F_{11}F_{2222})$$
$$\left. - 2I_8(F_{11}F_{1212} + F_{1112}F_{12} + F_{22}F_{1212} + F_{2212}F_{12})\right]. \quad (4.46)$$

While the Fourier coefficients $F_A(n, e)$ depend on the harmonic number n and the eccentricity e, their θ-dependence is contained in the even-degree polynomials $I_r(s)$ in $s = \sin\theta$:

$$\begin{aligned}
I_1(s) &= 2\pi\left(1 - s^2 + \frac{3}{8} s^4\right), \\
I_2(s) &= 2\pi\left(1 - s^2 + \frac{1}{8} s^4\right), \\
I_3(s) &= 2\pi\left(-1 + s^2 + \frac{1}{8} s^4\right), \\
I_4(s) &= 2\pi\left(\frac{1}{2} s^2 - \frac{3}{4} s^4 + \frac{5}{16} s^6\right), \\
I_5(s) &= 2\pi\left(\frac{1}{2} s^2 - \frac{1}{4} s^4 + \frac{1}{16} s^6\right), \\
I_6(s) &= 2\pi\left(\frac{1}{2} s^2 - \frac{1}{2} s^4 + \frac{1}{16} s^6\right), \\
I_7(s) &= 2\pi\left(-\frac{1}{2} s^2 + \frac{1}{2} s^4 + \frac{1}{16} s^6\right), \\
I_8(s) &= 2\pi\left(\frac{1}{8} s^4 - \frac{1}{16} s^6\right).
\end{aligned} \quad (4.47)$$

We note that the angular distribution of the relativistic correction is determined by five 6th-order polynomials in s vanishing at $\theta = 0$ and π, whereas the angular distribution of the leading, nonrelativistic terms depends on three 4th-order polynomials differing from zero at these points. In other words, the relativistic correction does not affect radiation intensity at $\theta = 0$ and π. Also worthy of note is the interesting symmetry of the expressions in the two square brackets of the relativistic correction. The function in the second square brackets composed of type-$F_{ij}F_{mnkl}$ terms turns into the function in the first brackets if the last index of the four-index coefficient in each of its terms is removed and made the third index of the two-index coefficient, i.e., if $F_{ij}F_{mnkl}$ is replaced with $F_{ijl}F_{mnk}$. Clearly, the inversion $F_{ijl}F_{mnk} \to F_{ij}F_{mnkl}$ changes the function in the first square brackets to the function in the second.

The remaining integration over the angle θ yields

$$\int d\Omega \left(\frac{1}{2}\left|G_+(n)\right|^2 + 2\left|G_\times(n)\right|^2\right)$$
$$= \frac{16\pi}{15}\left\{F_{11}^2 + F_{22}^2 + 3F_{12}^2 - F_{11}F_{22}\right.$$
$$+ \frac{1}{28} C_1^2\beta_n^2\left[F_{111}^2 + F_{222}^2 + 3(F_{112}^2 + F_{221}^2)\right.$$
$$+ 5(F_{121}^2 + F_{122}^2) - F_{111}F_{221} - F_{112}F_{222}$$
$$\left. - 2(F_{112}F_{121} + F_{111}F_{122} + F_{222}F_{121} + F_{221}F_{122})\right]$$
$$- \frac{1}{21} C_2\beta_n^2\left[F_{11}F_{1111} + F_{22}F_{2222}\right.$$
$$+ 3(F_{11}F_{1122} + F_{22}F_{2211}) + 5F_{12}(F_{1211} + F_{1222})$$
$$- \frac{1}{2}(F_{1111}F_{22} + F_{11}F_{2211} + F_{1122}F_{22} + F_{11}F_{2222})$$
$$\left.\left. - 2(F_{11}F_{1212} + F_{1112}F_{12} + F_{22}F_{1212} + F_{2212}F_{12})\right]\right\}. \quad (4.48)$$

Certainly, this spectrum preserves the aforementioned permutation symmetry of expressions in the two square brackets of the relativistic correction.

Using this expression in formula (4.39) and carrying out the remaining integration over $q^2\,dq$ gives the total energy $\mathcal{E}$ emitted during time t,

$$\mathcal{E} = t\,\frac{G(\mu v^2)^2\omega^2}{15c^5}\sum_{n>0}^{\infty} n^2\left\{\ldots\right\}, \quad (4.49)$$

and the radiation power $P = \mathcal{E}/t$. Here, the curly brackets contain the same expression as is enclosed in the braces in Eqn (4.48).

If only the first four terms are kept in the braces, the equation for the nth harmonic power

$$P(n) = \frac{G(\mu v^2)^2\omega^2}{15c^5}\, n^2(F_{11}^2 + F_{22}^2 + 3F_{12}^2 - F_{11}F_{22}) \quad (4.50)$$

exactly coincides with $P(n)$ obtained in [24] [formulas (19) and (20)]. In comparing, it must be kept in mind that $4J_n(ne) = F_{11} + F_{22}$, $n \neq 0$. The Fourier coefficients $F_A(n, e)$, $A = ij,\ ijk,\ ijkl$, and their main properties are presented in the Appendix.

Peters and Mathews [24] draw attention to the fact that the mean radiation power $P(n)$ summed over all harmonics is a steeply growing function of e as $e \to 1$ (see formula (16) in [24]). As a matter of fact, the limit $e \to 1$ at a constant a is nonphysical. In this case, the maximum and minimum speeds of each mass on elliptical orbits respectively tend to ∞ and 0.

For the mass m_2, for example,

$$v_{\max} = \frac{m_1}{m_1+m_2}\sqrt{\frac{G(m_1+m_2)(1+e)}{a(1-e)}}\Bigg|_{e\to1} \to \infty\,,$$
$$v_{\min} = \frac{m_1}{m_1+m_2}\sqrt{\frac{G(m_1+m_2)(1-e)}{a(1+e)}}\Bigg|_{e\to1} \to 0\,. \tag{4.51}$$

To avoid conflict with the theory of relativity, the transition to the limit $e \to 1$ must be performed at a constant $p = a(1-e^2)$. In this case, the ellipse turns into a parabola, the mean power $\langle P\rangle$ tends to zero (because $T\to\infty$), and the total radiation energy for time T is finite, being actually equal to

$$\mathcal{E} = \frac{64\pi}{5}\left(\frac{G\mu^2}{p}\right)\left(\frac{v_*}{c}\right)^5\left(1+e^2\,\frac{73}{24}+e^4\,\frac{37}{96}\right), \tag{4.52}$$

and tending to the finite energy

$$\mathcal{E} = \frac{170\pi}{3}\left(\frac{G\mu^2}{p}\right)\left(\frac{v_*}{c}\right)^5, \qquad v_* = \sqrt{\frac{G(m_1+m_2)}{p}} \tag{4.53}$$

emitted from both parabolic orbits of the masses m_1 and m_2. Here, v_* is the effective orbital velocity, and the maximum velocity does not exceed $2v_*$. Evidently, the nonrelativistic consideration is valid for $v_* \ll c$. At $v_* = c$, the parameter $p = G(m_1+m_2)/c^2$ becomes twice the gravitational (Schwarzschild) radius of the two-mass system, and the emitted energy $\mathcal{E}$ is two orders of magnitude higher than the interaction energy of the two masses. We note that at $e\to1$ and finite p, the fundamental frequency ω tends to zero, whereas the maximum angular frequency $\dot\psi_{\max}$ remains finite, and their ratio

$$\frac{\dot\psi_{\max}}{\omega} = \sqrt{\frac{1+e}{(1-e)^3}}$$

determines a maximum in the harmonic distribution over n.

4.6 The improved trajectory method

As mentioned in Section 4.2, Eqn (4.32) is the expansion of $T_{ij}(q)$ in powers of the nonrelativistic velocity $\beta = v/c \ll 1$ and, at the same time, in powers of the wave vector $\mathbf{q}$. But keeping terms of the order of β^2 in the braces of (4.32), we should refine the approximate expression (4.16) used on the right-hand side of Eqn (4.14). We replace $t_{00}(q)$ by its exact expression

$$t_{00}(q) = mc^2\int \mathrm{d}t\,\exp\left[\mathrm{i}q^0 t - \mathrm{i}\mathbf{q}\mathbf{x}(t)\right]\gamma(t)\,, \tag{4.54}$$

in which, however, we keep only the lower-order term in the expansion of the Lorentz factor $\gamma(t)$:

$$\gamma(t) \approx 1 + \frac{1}{2}\,\beta^2(t)\,. \tag{4.55}$$

This operation modifies the right-hand side of (4.14), which takes form (4.16) with the γ-factor in the integrand,

$$\frac{q^{0\,2}}{c^2}\,\frac{\partial^2}{\partial q_k\,\partial q_l}\,t_{00}(q) = -q^{0\,2}m\int \mathrm{d}t\,\exp\left[\mathrm{i}q^0t - \mathrm{i}\mathbf{q}\mathbf{x}(t)\right]\gamma(t)x_k(t)x_l(t)\,, \tag{4.56}$$

and the solution of (4.14) for $T_{ij}(q)$ becomes

$$T_{ij}(q) = m\int \mathrm{d}t\,\exp(\mathrm{i}q^0t)\left\{\gamma\dot x_i\dot x_j\exp(-\mathrm{i}\mathbf{q}\mathbf{x}) + \left[\frac{1}{2}\,\gamma(\ddot x_ix_j + x_i\ddot x_j) + \dot\gamma(\dot x_ix_j + x_i\dot x_j)\right]\frac{\exp(-\mathrm{i}\mathbf{q}\mathbf{x})-1}{(-\mathrm{i}\mathbf{q}\mathbf{x})} + \ddot\gamma x_ix_j\,\frac{\exp(-\mathrm{i}\mathbf{q}\mathbf{x})-1+\mathrm{i}\mathbf{q}\mathbf{x}}{(-\mathrm{i}\mathbf{q}\mathbf{x})^2}\right\} \tag{4.57}$$

[cf. (4.17) and (4.18)]. Indeed, Eqn (4.14) for such a tensor with the modified right-hand side (4.56) reduces to the relation between the Fourier transform of the function $\gamma(t)x_k(t)x_l(t)\exp[-\mathrm{i}\mathbf{q}\mathbf{x}(t)]$ and the Fourier transform of its second derivative with respect to t regardless of whether the approximate (4.55) or exact [cf. (4.23)] value of the γ-factor is used.

Again using the smallness of the ratio of the radiative system dimensions to the wavelength, i.e., the smallness of $\mathbf{q}\mathbf{x}$, we can represent the expression in the curly brackets with up to the terms of the order of $v^2\beta^2$. Then

$$T_{ij}(q) = \frac{1}{2}\,m\int \mathrm{d}t\,\exp(\mathrm{i}q^0t) \times\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\gamma x_ix_j) - \frac{1}{2}\,\mathrm{i}\mathbf{q}\mathbf{x}(4\dot x_i\dot x_j + \ddot x_ix_j + x_i\ddot x_j) - \frac{1}{6}(\mathbf{q}\mathbf{x})^2(6\dot x_i\dot x_j + \ddot x_ix_j + x_i\ddot x_j)\right\}. \tag{4.58}$$

As expected, the γ-factor occurs only in the part independent of the wave vector $\mathbf{q}$ since the $\mathbf{q}$-dependent terms are of the orders $v^2\beta$ and $v^2\beta^2$. As a result, the tensor $T_{ij}(q)$ for a two-mass system is represented in form (4.32) with F_{ij} replaced by $\tilde F_{ij}$,

$$\tilde F_{ij}(n,e) = F_{ij}(n,e) + \frac{1}{2}\,C_2\beta^2 f_{ij}(n,e)\,, \tag{4.59}$$

where $\beta = v/c = a\omega/c \ll 1$ and the Fourier coefficient is

$$f_{ij}(n,e) = -\frac{n^2}{\pi}\int_{-\pi}^{\pi}\mathrm{d}\tau\,\frac{r_ir_j(\dot r_1^2+\dot r_2^2)}{a^2v^2}\begin{Bmatrix}\cos n\tau\\ \sin n\tau\end{Bmatrix}. \tag{4.60}$$

The explicit expressions for all the Fourier coefficients and their properties are presented in the Appendix. Using them, we can write the explicit expressions for polarization amplitudes

$$T_{An}(q) = \frac{1}{4}\,\mu v^2 G_A(n,q)\,, \qquad A = +,\times\,,$$

of the nth harmonic obtained by the trajectory method in the β^2 approximation without considering the θ_{00} component of the force field EMT. In the particular case where $e=0$ and $m_2 \ll m_1$, with $C_2 = 1$, we have the second harmonic amplitudes

$$T^{\mathrm{trj}}_{+2}(q) = \frac{1}{2}\,\mu v^2\gamma\left\{1+\cos^2\theta - \beta^2\sin^2\theta\left(1+\frac{1}{3}\cos^2\theta\right)\right\}\exp(2\mathrm{i}\varphi)\,,$$
$$T^{\mathrm{trj}}_{\times2}(q) = -\mathrm{i}\,\frac{1}{2}\,\mu v^2\gamma\cos\theta\left\{1-\frac{2}{3}\,\beta^2\sin^2\theta\right\}\exp(2\mathrm{i}\varphi) \tag{4.61}$$

[cf. (3.73), where $\varphi = 0$]. These amplitudes coincide with the GR amplitudes of a charge in a magnetic moment field. It appears that the nonlocal properties of this field manifest themselves, by virtue of its strong falloff, in the GR amplitudes in the terms of a higher order than β^2.

We next discuss the replacement of F_{ij} with $\tilde{F}_{ij}$ in the terms bilinear in F_{ij}, such as those in Eqns (4.41), (4.42), (4.46), (4.48) and (4.49). This substitution means that

$$\begin{aligned} F_{ij}^2 &\to F_{ij}^2 + C_2\beta^2 F_{ij} f_{ij}\,, \\ F_{11}F_{22} &\to F_{11}F_{22} + \frac{1}{2}\,C_2\beta^2(F_{11}f_{22} + F_{22}f_{11})\,. \end{aligned} \tag{4.62}$$

The above expressions contain additional terms of the order of β^2 linear in $f_{ij}(n,e)$. The replacement of (4.59) is not needed in the terms linear in F_{ij} but bilinear in the Fourier coefficients (such as $F_{ij}F_A$, $A = mnk, mnkl$) contained in expressions (4.41)–(4.43), (4.45), (4.46), (4.48), and (4.49) because it would lead to excess accuracy, i.e., the appearance of order-β^3 and β^4 terms.

To summarize, the trajectory method permits finding the transverse GR amplitudes with terms $\sim \beta^2$ using only the $t_{00}(q)$ component of the mass EMT. It is known, however, from three examples of the mass motion along the same circumference but under the action of different force fields that these fields are responsible for different order-β^2 additions to the Lorentz factor determining the amplitudes of the transverse components T_+ and $T_\times$ [see (3.73)]. Because γ is the kinetic energy of the mass in units of mc^2, an addition of the same order as the nonrelativistic kinetic energy $(1/2)\beta^2$ may be regarded as the effective energy of a force field contributing to GR. An example of taking this energy into account by the trajectory method for a massive string is given below using both the component $t_{00}(q)$ and the component $\theta_{00}(q)$ of the force field EMT in the right-hand side of Eqn (4.14)

4.7 An example of taking the force field energy into account

We apply the trajectory method to the calculation of GR by a string of length r with one end fixed and the other loaded with a mass m. We then add the component $\theta_{00}(q,v)$ of the string EMT to the $t_{00}(q)$ component of the mass EMT [using the same notation as in (3.47)]. According to Section 3.3,

$$\begin{aligned} \theta_{00}(q,v) &= \mu \int \mathrm{d}t\, \exp(\mathrm{i}q^0 t) \\ &\times \int_0^r \frac{\mathrm{d}\sigma}{\sqrt{1-(\omega\sigma/c)^2}} \exp\left[-\mathrm{i}\mathbf{qx}(\tau,\sigma)\right] \\ &= \frac{mv^2\gamma^2}{\beta} \int \mathrm{d}t\, \exp(\mathrm{i}q^0 t) \int_0^{\arcsin\beta} \mathrm{d}\alpha\, \exp\left[-\mathrm{i}\,\frac{\sin\alpha}{\beta}\,\mathbf{qr}(t)\right]. \end{aligned} \tag{4.63}$$

The last formula is derived by passing from the variable σ to the dimensionless variable α,

$$\sigma = \frac{\sin\alpha}{\beta}\,r\,, \tag{4.64}$$

such that the exponent in the integral over α takes the form

$$-\mathrm{i}\mathbf{qx}(\tau,\sigma) = -\mathrm{i}\,\frac{\sin\alpha}{\beta}\,\mathbf{qr}(t)\,, \tag{4.65}$$

where $r_1 = r\cos\omega t$ and $r_2 = r\sin\omega t$ are the coordinates of the mass-loaded end of the string. We also used relation (3.40) between the string tension and the velocity $v = \omega r$ of its end, assuming the string rotates counterclockwise.

The additional term in the right-hand side of (4.14) takes the form

$$\begin{aligned} \frac{q^{0\,2}}{c^2}\,\frac{\partial^2}{\partial q_k\,\partial q_l}\,\theta_{00}(q,v) &= -q^{0\,2} m\beta\gamma^2 \int \mathrm{d}t\, \exp(\mathrm{i}q^0 t) \\ \times \int_0^{\arcsin\beta} \mathrm{d}\alpha \left(\frac{\sin\alpha}{\beta}\right)^2 &\exp\left(-\mathrm{i}\,\frac{\sin\alpha}{\beta}\,\mathbf{qr}\right) r_k(t)\, r_l(t)\,. \end{aligned} \tag{4.66}$$

Up to now, no assumptions have been made regarding the magnitude of the velocity, i.e., the parameter $\beta = v/c$.

We use the trajectory method to calculate the radiation amplitudes $T_+(q)$ and $T_\times(q)$ in the nonrelativistic approximation, taking account of order-β^2 relativistic corrections. We consider Eqn (4.66) in this approximation. Because $\mathbf{qr} \sim \beta$, keeping the first three terms in the expansion of the exponential and the calculating the respective integrals over α,

$$\int_0^{\arcsin\beta} \mathrm{d}\alpha \left(\frac{\sin\alpha}{\beta}\right)^k = \frac{\beta}{k+1}\left(1 + \frac{k+1}{2(k+3)}\,\beta^2 + \ldots\right), \qquad k = 2,3,4\,,$$

leads to the following expression for additional term (4.66):

$$\begin{aligned} &\frac{q^{0\,2}}{c^2}\,\frac{\partial^2}{\partial q_k\,\partial q_l}\,\theta^{00}(q,v) \\ &= -q^{0\,2} m\beta^2\gamma^2\,\frac{1}{3}\int \mathrm{d}t\, \exp(\mathrm{i}q^0 t)\, r_k(t)\, r_l(t) \\ &\times \left[1 + \frac{3}{10}\,\beta^2 - \mathrm{i}\,\frac{3}{4}\,\mathbf{qr} + \frac{3}{10}\,(\mathbf{qr})^2\right]. \end{aligned} \tag{4.67}$$

Its comparison with the main expression in the right-hand side of Eqn (4.14), including the $t^{00}(q)$ component,

$$\begin{aligned} &\frac{q^{0\,2}}{c^2}\,\frac{\partial^2}{\partial q_k\,\partial q_l}\,t^{00}(q) \\ &= -q^{0\,2} m\gamma \int \mathrm{d}t\, \exp(\mathrm{i}q^0 t)\, r_k(t)\, r_l(t) \left[1 - \mathrm{i}\mathbf{qr} - \frac{1}{2}\,(\mathbf{qr})^2\right], \end{aligned} \tag{4.68}$$

shows that expression (4.67) is $(1/3)\beta^2$ times Eqn (4.68). In other words, it is obtained with an excess accuracy, which implies that $\gamma = 1$ and all bracketed terms except 1 can be omitted. Because the addition to amplitudes (4.61) (obtained by the trajectory method with the use of the $t^{00}(q)$ component of the matter EMT alone) is equivalent to replacing the Lorentz factor γ in these amplitudes with the factor $\gamma + (1/3)\beta^2$, we have

$$\gamma \to \gamma + \frac{1}{3}\,\beta^2 = 1 + \frac{1}{2}\,\beta^2 + \frac{1}{3}\,\beta^2 = 1 + \frac{5}{6}\,\beta^2\,. \tag{4.69}$$

As a result, the amplitudes exactly coincide with the string amplitudes in (3.73), $\delta = 1/3$. The addition of $(1/3)\beta^2$ to $\gamma = 1 + (1/2)\beta^2$, i.e., to the kinetic energy of the mass in units of mc^2, plays the role of the effective energy of the force field involved in GR.

The relativistic GR amplitudes considered previously for three cases (strings, a charge in a magnetic moment field, and a charge in a Coulomb field) in the β^2-approximation differed only by the order-β^2 terms added to the Lorentz factor $\gamma = 1 + (1/2)\beta^2$:

$$T_{+2}(q) = \frac{1}{2}\, mv^2(\gamma + \delta\beta^2) \times \left\{1 + \cos^2\theta - \beta^2 \sin^2\theta\left(1 + \frac{1}{3}\cos^2\theta\right)\right\},$$
$$T_{\times 2}(q) = -\mathrm{i}\,\frac{1}{2}\, mv^2(\gamma + \delta\beta^2)\cos\theta\left\{1 - \frac{2}{3}\,\beta^2\sin^2\theta\right\}, \tag{4.70}$$

i.e., by the respective values of the parameter δ: 1/3, 0, and $-2/3$. This means that the nonlocality of different force fields driving a point mass along the same trajectory is poorly manifested in the amplitudes T_+ and $T_\times$ (only in the terms $\sim \beta^2$). At the same time, they have a similar local action on the particle, as is revealed by the trajectory method where only the local component $t^{00}(q)$ of the matter EMT is taken into account and the total EMT is kept.

Thus, the trajectory method allows finding the leading terms of the amplitudes $T_+(q)$ and $T_\times(q)$ of all harmonics. For a quadrupole-like harmonic, relativistic corrections $\sim \beta^2$ reflect both local and nonlocal effects of the force field. The former are derived exactly by the trajectory method, and the latter require the θ^{00} component of the force field EMT to be used in the right-hand side of Eqn (4.14).

For a motion along elliptical trajectories, this method allows finding the amplitudes T_+ and $T_\times$ for all harmonics in the β^2-approximation using the t_{00} component of the mass EMT in the right-hand side of Eqn (4.14); nonlocal effects of the force field are taken into account, as in the case of circular orbits, in the form of order-β^2 additions to the leading Fourier coefficients $F_{ij}(n, e)$, unrelated to the graviton wave vector $\mathbf{q}$.

5. Gravitational radiation by a charge passing through a Coulomb field and a magnetic moment field

5.1 The passage of a charge through a Coulomb field

The trajectory followed by a charge e as it passes a Coulomb center with the charge e' is a planar curve that can be described by the parametric equations

$$x_1(\xi) = r(\xi)\cos\varphi(\xi), \qquad x_2(\xi) = r(\xi)\sin\varphi(\xi),$$
$$t(\xi) = \frac{b}{(1-\gamma^{-2})^{1/2}}\left(\sinh\xi + \frac{\varkappa}{\gamma^2}\,\xi\right), \tag{5.1}$$

where r is the distance between the charge and the center, φ is the angle of deflection from the symmetry axis (1-axis), and t is the time:

$$r(\xi) = a + b\cosh\xi,$$
$$\varphi(\xi) = \frac{1}{(1-v^2)^{1/2}}\arcsin\frac{(1-\varkappa^2)^{1/2}\sinh\xi}{\cosh\xi + \varkappa}. \tag{5.2}$$

These formulas are derived by the method described in [6] (see par. 39).

The charge motion is characterized by three independent parameters: the dimensional impact parameter β, the dimensionless Lorentz factor γ of the charge at infinity, and the dimensionless ratio

$$v = \frac{\alpha}{Mc}, \tag{5.3}$$

equal to the ratio of the product of charges $\alpha = ee'/4\pi$ to the particle angular momentum M times the speed of light; all the remaining parameters are their functions:

$$a = \frac{v}{(1-\gamma^{-2})^{1/2}}\,\beta, \quad b = \beta\left(1 + \frac{v^2}{\gamma^2 - 1}\right)^{1/2}, \quad \varkappa = \frac{a}{b}. \tag{5.4}$$

We consider the case of repulsion, $v > 0$. The extension to the case of attraction is rather simple. The final Eqn (5.31) holds for either sign of v.

The scattering angle is given by

$$\chi = \pi - \left[\varphi(\infty) - \varphi(-\infty)\right] = \pi - \frac{2}{(1-v^2)^{1/2}}\arccos\varkappa. \tag{5.5}$$

We first consider the EMR spectrum of the charge. We characterize the direction of the wave vector $\mathbf{q}$ by the angle δ it makes with the $(1,2)$ plane and the angle ψ between the $(\mathbf{q}, 3)$ plane and the 2-axis:

$$\mathbf{q} = |\mathbf{q}|(\cos\delta\sin\psi,\ \cos\delta\cos\psi,\ \sin\delta). \tag{5.6}$$

Using the conditions $q^\alpha j_\alpha(q) = 0$ of current conservation and $q^2 = 0$, we then have the EMR spectrum

$$|j_\alpha(q)|^2 = (1 - \cos^2\delta\sin^2\psi)\,|j_1|^2 + (1 - \cos^2\delta\cos^2\psi)\,|j_2|^2 - 2\cos^2\delta\sin\psi\cos\psi\,\mathrm{Re}\; j_1 j_2^*. \tag{5.7}$$

The current density components are defined by the integrals

$$j_1(q) = eb\int_{-\infty}^{\infty}\mathrm{d}\xi\,\exp\left[-\mathrm{i}f(\xi)\right] \times \left[\sinh\xi\cos\varphi(\xi) - \left(\frac{1-\varkappa^2}{1-v^2}\right)^{1/2}\sin\varphi(\xi)\right], \tag{5.8}$$

$$j_2(q) = eb\int_{-\infty}^{\infty}\mathrm{d}\xi\,\exp\left[-\mathrm{i}f(\xi)\right] \times \left[\sinh\xi\sin\varphi(\xi) + \left(\frac{1-\varkappa^2}{1-v^2}\right)^{1/2}\cos\varphi(\xi)\right], \tag{5.9}$$

$$f(\xi) = q_1 r(\xi)\cos\varphi(\xi) + q_2 r(\xi)\sin\varphi(\xi) - q^0 b(1-\gamma^{-2})^{-1/2}\left(\sinh\xi + \frac{\varkappa}{\gamma^2}\,\xi\right). \tag{5.10}$$

It what follows, we confine ourselves to the ultrarelativistic case $\gamma \gg 1$ and require that the parameter v be of the order of γ^{-1} or smaller, i.e.,

$$v \lesssim \frac{1}{\gamma} \ll 1. \tag{5.11}$$

Then $\varkappa \approx v \ll 1$, and up to the terms of the fourth order of smallness in the parameters that occur in condition (5.11), we have

$$\left(\frac{1-\varkappa^2}{1-v^2}\right)^{1/2} \approx 1, \qquad b \approx \beta.$$

Moreover, both the scattering angle χ and the effective values of the angles δ and ψ are small:

$$\chi \approx 2v\,; \qquad |\delta|,\, |\psi| \sim \frac{1}{\gamma} \ll 1\,. \tag{5.12}$$

The relevant expansion in formulas (5.1) and (5.8)–(5.10) leads to

$$x_1(\xi) = \beta\left[1 + v\cosh\xi - \frac{1}{2}\,v^2\sinh\xi\arcsin\left(\tanh\xi\right) + \ldots\right], \tag{5.13}$$

$$x_2(\xi) = \beta\left[\sinh\xi - \frac{1}{2}\,v^2\sinh\xi + \frac{1}{2}\,v^2\arcsin\left(\tanh\xi\right) + \ldots\right],$$

$$j_1 \approx e\beta vS\,, \qquad j_2 \approx e\beta C\,, \tag{5.14}$$

$$\left|\,j_\alpha(q)\right|^2 \approx e^2\beta^2\left[v^2|S|^2 + (\delta^2+\psi^2)|C|^2 - 2v\psi\,\mathrm{Re}\,(SC^*)\right], \tag{5.15}$$

where

$$(S, C) = \int_{-\infty}^{\infty} \mathrm{d}\xi\,(\sinh\xi,\, \cosh\xi)\exp\left[-\mathrm{i}f(\xi)\right], \tag{5.16}$$

$$f(\xi) = \eta - z\sinh\xi + w\,\cosh\xi + s\arcsin\left(\tanh\xi\right), \tag{5.17}$$

$$\eta = \beta q^0\psi\,, \qquad z = \frac{1}{2}\,\beta q^0\left(\delta^2+\psi^2+v^2+\frac{1}{\gamma^2}\right), \qquad w = \beta q^0 v\psi\,, \qquad s = \frac{1}{2}\,\beta q^0 v^2\,. \tag{5.18}$$

We note that $|\,j_\alpha(q)|^2$ is an odd function of ψ. This follows from the fact that a change in the sign of ψ and the complex conjugation result in reversing the sign of $S(\psi)$, leaving $C(\psi)$ unaltered:

$$S^*(-\psi) = -S(\psi)\,, \qquad C^*(-\psi) = C(\psi)\,,$$

which in turn follows from the property $f(\xi,\psi) = -f(-\xi,-\psi)$. The evenness of the square of the current in ψ implies the equality of the EMR intensities emitted along the initial and final directions.

Because the charge deflection angle χ is so small that the conditions (5.11) and (5.12) are satisfied, the Coulomb field effectively acts on the charge only over a distance of the order of the impact parameter $\beta \approx b$. Consequently, $\xi \sim 1$ are effective [see (5.1), (5.2), (5.13)]. It then follows from (5.17) that z, w, $s \sim 1$ in the integrals S and C are effective and, accordingly, $q^0 \sim \beta^{-1}\gamma^2$ (cf. par. 77 in [6]).

The integrals S and C cannot be expressed through the known special functions. However, at the deflection angle $\chi \ll \gamma^{-1}$,

$$S \approx 2\mathrm{i}\exp\left(-\mathrm{i}\eta\right)K_1(z)\,, \qquad C \approx 2\mathrm{i}\,\frac{w}{z}\exp\left(-\mathrm{i}\eta\right)K_1(z)\,,$$
$$\chi \ll \gamma^{-1} \ll 1\,. \tag{5.19}$$

In this case, the EMR spectrum is given by the equation

$$\left|\,j_\alpha(q)\right|^2 = 4e^2v^2\beta^2K_1^2(z)\left[1 - \frac{4\psi^2}{\gamma^2(\delta^2+\psi^2+1/\gamma^2)^2}\right], \tag{5.20}$$

and the total energy emitted during the passage is

$$\mathcal{E}_{EM} = \int\frac{\mathrm{d}^3q}{16\pi^3}\left|\,j_\alpha(q)\right|^2 = \frac{\pi e^4e'^2\gamma^2}{4(4\pi)^3m^2\beta^3}\,, \tag{5.21}$$

in agreement with the equation in Problem 1 of par. 73 in [6].

We now turn to the calculation of the GR spectrum. It is determined by two transverse components of the total EMT, which can be very simply expressed through the EMT components in a coordinate system K' whose $3'$-axis lies in the same direction as $\mathbf{q}$ [see (2.47), (2.48)]. The transition to K' from the coordinate system K under consideration, where the trajectory lies in the plane $(1, 2)$ symmetrically with respect to the 1-axis and the radiation wave vector is characterized by the angles δ and ψ [see (5.6)], can be accomplished through two spatial rotations. One is the rotation $K \to K''$ about the 3-axis by ψ, which puts $\mathbf{q}$ in the $(3 \equiv 3'', 2'')$ plane of the intermediate coordinate system K''. The other is the rotation $K'' \to K'$ about the $1''$-axis through the angle $\theta = \pi/2 - \delta$, such that the vector $\mathbf{q}$ lies along the $3'$-axis of the K' system. The expression of the EMT components $T'_{ij}(q')$ entering (2.47) and (2.48) in the K' system in terms of the components $T_{ij}(q)$ in K yields

$$\begin{aligned} T_+(q) &= (\cos^2\psi - \sin^2\delta\sin^2\psi)T_{11} \\ &\quad -2\sin\psi\cos\psi(1+\sin^2\delta)T_{12} + (\sin^2\psi - \sin^2\delta\cos^2\psi)T_{22} \\ &\quad + 2\sin\delta\cos\delta(\sin\psi T_{13} + \cos\psi T_{23}) - \cos^2\delta T_{33}\,, \end{aligned} \tag{5.22}$$

$$\begin{aligned} T_\times(q) &= \sin\delta\sin\psi\cos\psi(T_{11} - T_{22}) \\ &\quad + \sin\delta(\cos^2\psi - \sin^2\psi)T_{12} - \cos\delta(\cos\psi T_{13} - \sin\psi T_{23})\,. \end{aligned} \tag{5.23}$$

Formulas (1.3), (2.13), (2.39), and (2.40) in Sections 1 and 2 are used to construct the spatial components $T_{ij} = t_{ij} + \theta_{ij}$. Then

$$t_{ij}(q) = m\int_{-\infty}^{\infty}\mathrm{d}\xi\,x'_ix'_j(t'^2 - \mathbf{x}'^2)^{-1/2}\exp\left[-\mathrm{i}f(\xi)\right], \qquad i, j = 1, 2\,, \tag{5.24}$$

$$\begin{aligned} \theta_{ij}(q) &= -\mathrm{i}\alpha|\mathbf{q}|\int_{-\infty}^{\infty}\mathrm{d}\xi\,\exp\left[-\mathrm{i}f(\xi)\right] \\ &\quad\times\int_0^1\mathrm{d}u\,\exp\left[\mathrm{i}u\big(\mathbf{qx} + |\mathbf{q}|r\big)\right] \\ &\quad\times\left\{\frac{x_ix'_j + x_jx'_i}{2r} + t'\,\frac{x_ix_j}{r^2}\left(u + \frac{\mathrm{i}}{|\mathbf{q}|r}\right)\right\} + \ldots, \end{aligned} \tag{5.25}$$

where the prime denotes the derivative with respect to ξ. The dots in (5.25) denote the terms of the form

$$(q_ib_j + q_jb_i)A\,, \quad \delta_{ij}B\,, \tag{5.26}$$

where A and B are rotation-invariant functions that depend on $\mathbf{q}$ and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ characterizing the charge trajectory as a whole, the direction of its axis of symmetry, and the tangent to the trajectory's apex; $\mathbf{b}$ is one of the vectors $\mathbf{q}$, $\mathbf{e}_1$, $\mathbf{e}_2$.

In the case of the $K \to K'$ rotation, the terms in (5.26) turn into

$$(q'_ib'_j + q'_jb'_i)A\,, \quad \delta_{ij}B \tag{5.27}$$

and transverse EMT components (2.48) make no contribution because $q'_1 = q'_2 = 0$. Therefore, neither the terms in (5.26) nor the components T_{13}, T_{23}, T_{33} have to be calculated, as was first observed in [25].

To further analyze the ultrarelativistic case [to be precise, (5.11)], we introduce the relevant expressions for the components t_{ij} and θ_{ij}, $i, j = 1, 2$:

$$t_{11} \approx \alpha v \int_{-\infty}^{\infty} \mathrm{d}\xi\, \tanh\xi \sinh\xi \exp\left[-\mathrm{i}f(\xi)\right], \qquad (5.28)$$

$$t_{12} \approx \alpha S, \qquad t_{22} \approx \frac{\alpha}{v}\, C,$$

$$\theta_{11} \approx \alpha v \int_{-\infty}^{\infty} \mathrm{d}\xi\, \tanh\xi \exp\left[-\xi - \mathrm{i}f(\xi)\right],$$

$$\theta_{12} \approx \frac{1}{2}\,\alpha \int_{-\infty}^{\infty} \mathrm{d}\xi\, \exp\left[-\xi - \mathrm{i}f(\xi)\right], \qquad (5.29)$$

$$\theta_{22} \approx \alpha \int_{-\infty}^{\infty} \mathrm{d}\xi\, \sinh\xi \exp\left[-\xi - \mathrm{i}f(\xi)\right].$$

Here, $f(\xi)$ is given by (5.17) and the relation $m\gamma\beta v \approx \alpha$ is used. It can be seen that the components θ_{ij} are of the order of t_{ij} with the exception of $\theta_{22} \sim vt_{22} \ll t_{22}$. For the components T_{ij}, we obtain

$$T_{11} \approx \alpha v S, \qquad T_{12} \approx \frac{1}{2}\,\alpha(S + C), \qquad T_{22} \approx \frac{\alpha}{v}\, C. \qquad (5.30)$$

Substituting these components in (5.22) and (5.23), omitting all the terms containing T_{13}, T_{23}, and T_{33}, and taking into account that δ and ψ are small angles [see (5.12)], we find the GR spectrum

$$8\pi G\left(\frac{1}{2}\,|T_+|^2 + 2|T_\times|^2\right) = \frac{4\pi G m^2 \gamma^2}{e^2}\left[(v - \psi)^2 + \delta^2\right]\left|j_\alpha(q)\right|^2. \qquad (5.31)$$

Here, the $|j_\alpha(q)|^2$ spectrum is given by formula (5.15).

Therefore, when an ultrarelativistic charge transverses a Coulomb field and the deflection angle is $\chi \lesssim \gamma^{-1}$, the GR spectrum is proportional to the EMR spectrum, while the conversion amplitude

$$\Gamma = \gamma\left[(v - \psi)^2 + \delta^2\right]^{1/2}, \qquad (5.32)$$

depends on the direction of the wave vector and the orbital parameters.

We note that Γ exhibits no symmetry as ψ changes monotonically from the maximum at $\psi = -v$ to the minimum at $\psi = v$, meaning that GR emitted along the initial charge direction is more intense than that emitted along the final direction of the charge for both attraction and repulsion (Fig. 2). The reason may be that EMR emitted along the initial direction is converted into gravitons over a greater length than EMR emitted along the final direction of the charge.

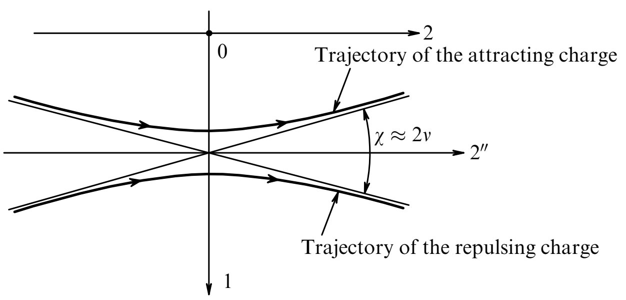

Figure 2. The trajectory of a charge in a Coulomb field of attraction and repulsion with identical impact parameters illustrating formula (5.32).

Because v and the effective values of the angles ψ, δ are constrained by conditions (5.11) and (5.12), it follows that $\Gamma \sim 1$, in agreement with (1.15) (see the Introduction).

While the proportionality of the GR and EMR spectra is due to ultrarelativism, the decrease in Γ to $\Gamma \sim 1$ can be attributed to the reduction of the region where GR forms to the region in which EMR is produced. As a result, both GR sources, the local EMT $t_{\alpha\beta}$ of the material body and the nonlocal EMT $\theta_{\alpha\beta}$ of the proper and external electromagnetic fields, make contributions of the same order of magnitude.

For $v \ll \gamma^{-1} \ll 1$, substituting expression (5.20) in (5.31) gives the GR spectrum found in this approximation in [26]. In formula (17.20) of Ref. [26], the direction of vector $\mathbf{q}$ is defined by the angles θ and φ:

$$\mathbf{q} = |\mathbf{q}|(\sin\theta\cos\varphi, \cos\theta, \sin\theta\sin\varphi),$$

rather than by ψ and δ [see (5.6)], but the smallness of all angles, except φ, simplifies the relations: $\psi = \theta\cos\varphi$, $\delta = \theta\sin\varphi$. Integration of the spectrum over the wave vector $\mathbf{q}$ gives the total GR energy $\mathcal{E}_G$ differing from $\mathcal{E}_{EM}$ in (5.21) by the factor $4\pi G m^2/e^2$ as previously noted in [27].

In the foregoing, we assumed the charge motion to be ultrarelativistic, i.e., $\gamma \gg 1$. In a more general but still relativistic case, when the velocity of motion is not small and $\gamma - 1 \gtrsim 1$, the factor $|j_\alpha(q)|^2$ is not separated from the expression for the GR spectrum [see (1.5)]. Nevertheless, in this case too, the EMR spectrum may be of interest both in and of itself and as a tool for the assessment of GR.

Most calculations can be made as described before, but the parameters $\varkappa$ and v are significantly different even if they remain of the same order of magnitude: $\varkappa \approx v/\upsilon$, where $\upsilon = \upsilon_\infty$ is the charge velocity at infinity. The coordinates $x_{1,2}(\xi)$, unlike (5.13), now depend on three rather than two parameters:

$$x_1(\xi) = b\left[1 + \varkappa\cosh\xi - \frac{1}{2}\,v^2\sinh\xi\arcsin(\tanh\xi) + \ldots\right],$$

$$x_2(\xi) = b\left[\sinh\xi - \frac{1}{2}\,\varkappa^2\sinh\xi + \frac{1}{2}\,v^2\arcsin(\tanh\xi) + \ldots\right]. \qquad (5.33)$$

The current density squared contains two Macdonald functions instead of one in (5.20):

$$\left|j_\alpha(q)\right|^2 = 4e^2a^2\left\{\left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1 - \upsilon\cos\theta)^2}\right]K_1^2(z) + \frac{1}{\gamma^4\upsilon^2}\left[\frac{2}{1 - \upsilon\cos\theta} - 1 - \frac{1}{\gamma^2(1 - \upsilon\cos\theta)^2}\right]K_0^2(z)\right\}. \qquad (5.34)$$

The argument z of the Macdonald functions and the direction $\mathbf{n}$ of the wave vector are given by

$$z = \frac{bq^0}{\upsilon}(1 - \upsilon\cos\theta),$$
$$\mathbf{n} = \frac{\mathbf{q}}{q^0} = (\sin\theta\cos\varphi, \cos\theta, \sin\theta\sin\varphi). \qquad (5.35)$$

The EMR spectrum defined by formula (1.1) permits integrating over the frequency $q^0 = |\mathbf{q}|$. The arising angular distribution of EMR has the form

$$\frac{\mathrm{d}\mathcal{E}^{\mathrm{EM}}_{\mathbf{n}}}{\mathrm{d}\Omega} = \frac{e^2a^2}{4\pi\cdot 32}\left(\frac{v}{b}\right)^3\left\{\frac{3}{(1-v\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-v\cos\theta)^2}\right]\right.$$
$$\left. + \frac{1}{\gamma^4v^2}\left[\frac{2}{(1-v\cos\theta)^4}-\frac{1}{(1-v\cos\theta)^3}-\frac{1}{\gamma^2(1-v\cos\theta)^5}\right]\right\}. \tag{5.36}$$

The invariance of the square of the current density under the reflection $\varphi \to \pi - \varphi$ implies the identity between the EMR spectra emitted along the initial and final directions of the charge motion. Integration over the angles yields the total radiation energy

$$\mathcal{E}^{\mathrm{EM}} = \frac{\pi e^4 e'^2\gamma^2}{4(4\pi)^3 m^2\beta^3 v}\left(1+\frac{1}{3\gamma^2}\right), \tag{5.37}$$

which can be found in a problem in [6, par. 73]. However, unlike Ref. [6], we use Heaviside units in which the charge e is related to e_{G} in Gaussian units by the equality $e = \sqrt{4\pi}\, e_{\mathrm{G}}$, let the impact parameter be denoted by β, and assume the speed of light $c = 1$.

With the known square of the current density, it is possible to estimate the GR of the charge passing the Coulomb center by formula

$$\mathrm{d}\mathcal{E}_{\mathbf{q}} = A^2\left|j_\alpha(q)\right|^2\frac{\mathrm{d}^3q}{16\pi^3}. \tag{5.38}$$

Here, A is the photon-to-graviton conversion amplitude over the entire photon path. It can be found by integrating the conversion amplitude at the dx segment of the photon path,

$$\mathrm{d}A = \frac{\sqrt{4\pi G}}{c^2}\,E_\perp(x)\,\mathrm{d}x, \tag{5.39}$$

where $E_\perp(x)$ is the field transverse to the photon wave vector at the point with coordinate x along the photon path.

For a photon emitted at a minimal distance β from the Coulomb field, we have

$$A = \frac{\sqrt{4\pi G}}{c^2}\,\frac{e'}{4\pi\beta}. \tag{5.40}$$

We express A in terms of the deflection angle $\chi \approx 2\varkappa \ll 1$ using the relations

$$\frac{e'}{4\pi\beta} = \frac{pcv}{e}, \qquad v \approx \frac{v}{c}\,\varkappa \approx \frac{v}{2c}\,\chi.$$

Then

$$A \approx \sqrt{\frac{4\pi Gm^2}{e^2}}\left(\frac{v}{c}\right)^2\gamma\chi. \tag{5.41}$$

5.2 The passage of a charge through a magnetic moment field

We consider a charge passing through the field of a magnetic moment in the equatorial plane. The charge motion is characterized by the same parameters β and γ as the motion of a charge in a Coulomb field. The role of the dimensionless interaction parameter $\nu = \alpha/Mc$ is played by the parameter

$$\nu = -\frac{e\mathfrak{M}}{4\pi M\beta c}, \tag{5.42}$$

where $\mathfrak{M}$ is the magnetic moment directed along axis 3. Solutions of the equations of motion are sought by iterations over this parameter, which is assumed to be small. In the zeroth approximation, a particle moves parallel to the 2-axis. Two iterations yield

$$\begin{aligned}
x_1(\xi) &= \beta\left[1+\nu\cosh\xi-\frac{3}{2}\,\nu^2\sinh\xi\arcsin(\tanh\xi)+\ldots\right],\\
x_2(\xi) &= \beta\left[\sinh\xi+\frac{1}{2}\,\nu^2\arcsin(\tanh\xi)+\ldots\right],\\
t(\xi) &= \frac{\beta}{(1-\gamma^{-2})^{1/2}}\left(1+\frac{1}{2}\,\nu^2\right)\sinh\xi+\ldots,\\
r(\xi) &= \beta(\cosh\xi+\nu+\ldots).
\end{aligned} \tag{5.43}$$

At small ν, the deflection angle is given by $\chi \approx 2\nu$ [cf. (5.12)]. Although solutions (5.43) of the equations of motion differ from (5.13) and (5.1) by terms of the order of ν^2, the function $f(\xi) = q_\alpha x^\alpha(\xi)$ coincides with (5.17) over the effective range of $\mathbf{q}$ if condition (5.11) is satisfied. In the prefactors of the expressions for j_1, j_2 for the electromagnetic current and t_{11}, t_{12}, t_{22} for the material body EMT, the second-order terms in ν, γ^{-1} can be neglected. Therefore, $|j_\alpha(q)|^2$ and t_{ij} are the same as in (5.15) and (5.28). Calculations show that the transverse components θ_+ and $\theta_\times$ of the field EMT coincide with those made up of components (5.29) in the Coulomb case. In other words, the GR spectrum of a charge passing through a magnetic moment field is given, under the condition $\nu \lesssim \gamma^{-1} \ll 1$, by the same formula (5.31) as the GR spectrum of a charge traversing a Coulomb field.

In a more general case of relativistic motion of the charge, when $\gamma - 1 \gtrsim 1$ and the parameter ν is still small, $\nu \ll 1$, the time coordinate $t(\xi)$ in (5.43) differs from $t(\xi)$ for a Coulomb field by the absence of the term $(\varkappa/\gamma^2)\xi$ [see (5.1)]. As a result, the squared current density becomes

$$\left|j_\alpha(q)\right|^2 = 4e^2b^2\nu^2\left[1-\frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-v\cos\theta)^2}\right]K_1^2(z), \tag{5.44}$$

where z and $\mathbf{n}$ are the same as in (5.35).

The integration of the EMR spectrum defined by (1.1) over the frequency q^0 leads to the angular distribution

$$\frac{\mathrm{d}\mathcal{E}^{\mathrm{EM}}_{\mathbf{n}}}{\mathrm{d}\Omega} = \frac{3e^4\mathfrak{M}^2 v}{32(4\pi)^3m^2b^5\gamma^2}\left[\frac{1}{(1-v\cos\theta)^3}-\frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1-v\cos\theta)^5}\right]. \tag{5.45}$$

Integration of this distribution over the angles gives the total energy emitted by a charge as it passes through a magnetic moment field,

$$\mathcal{E}^{\mathrm{EM}} = \frac{\pi e^4\mathfrak{M}^2\gamma^2 v}{4(4\pi)^3m^2b^5}. \tag{5.46}$$

This expression is to be compared with energy (5.37) emitted by a charge in a Coulomb field under the same conditions.

For $\gamma \gg 1$, the difference from the Coulomb case disappears, with $(e\mathfrak{M}/b)^2$ corresponding to $(ee')^2$.

The intensity of GR by a charge passing through a magnetic moment field is estimated as in the Coulomb case, i.e., EMR spectrum (1.1) is multiplied by the square of the respective photon-to-graviton conversion amplitude. This amplitude is defined by the same formula (5.39) containing the $H_{\perp}(x)$ component of the magnetic field instead of the electric one. As a result,

$$A = \sqrt{\frac{4\pi G m^2}{e^2}} \left(\frac{v}{c}\right)^2 \gamma\chi \,,$$

where χ is the scattering angle. This expression coincides with (5.41) and is defined to an order of magnitude by the root factor. Hence, the passage of a relativistic charge through the fields being considered is not accompanied by the enhancement of GR [27].

6. Discussion

In the electromagnetic systems considered in Sections 2 and 3, the charge motion is caused by external fields with a nonlocal EMT $\theta_{\mu\nu}$ making substantial contributions to GR of the system. Therefore, the classical GR of the system, unlike its EMR, may be a source of information about its internal structure. Moreover, for ultrarelativistic systems, the contribution from a nonlocal source $\theta_{\mu\nu}$ is greater than that from a local one by the factor γ^2 and the GR spectrum at $\gamma \gg 1$ is proportional to the EMR spectrum. An argument in favor of the universal character of this relation is the quasi-plane-wave nature of the external field in the rest frame of an ultrarelativistic charge. For a plane-wave field, relation (5.1) is exact, irrespective of the velocity of the charge, as shown in [9] for a linearly or circularly polarized monochromatic field; this seems to be the case for a more general plane-wave field as well.

The proportionality between the GR and EMR spectra and the order of the quantity $\Gamma \sim \gamma$ at $\gamma \gg 1$ are closely related to the fact that of the two sources of GR, the local EMT $t_{\alpha\beta}$ of the material body and the nonlocal EMT $\theta_{\alpha\beta}$ of the proper and external electromagnetic fields, just the latter becomes predominant in the ultrarelativistic limit. This means that GR is produced as the charge emits a virtual or real photon at a segment of the trajectory γ times smaller than its curvature radius r. This photon is then converted into a graviton as it interacts with a quantum of the external field over the path of the order of l, which is the extension of the external field in the photon propagation direction.

When the field EMT dominates over the EMT of the material body, the current $j(q - k_{\mathrm{ef}})$ entering (1.12) is almost on the mass shell and the conversion amplitude Γ is given by (1.14). If the EMTs of the field and the material body are of the same order, this estimate of Γ remains true if the components of the currents $j(q - k_{\mathrm{ef}})$ and $j(q)$ transverse to $\mathbf{q}$ are comparable in magnitude. But if the transverse components of the current $j(q - k_{\mathrm{ef}})$ are much greater than those of $j(q)$, as is the case when the trajectory of the charge is almost a straight line through the field ($\chi \ll \gamma^{-1} \ll 1$), then $\Gamma \sim 1$ [see (1.15)].

The use of spatial components of the conserved current and conserved EMT for the description of GR and EMR spectra is convenient not only for the assessment of contributions of the field and matter EMT constituents to the GR spectrum. The polarization amplitudes formed by the spatial components directly determine the GR and EMR spectra and their computation significantly facilitates evaluating these spectra [see (1.10), (1.11)].

We emphasize that because of the nonconservation of the tensors $t_{\mu\nu}$ and $\theta_{\mu\nu}$ taken separately, their contributions to GR in the relativistically invariant form can hardly be considered self-contained. For example, the quantity $t_{\mu\nu}t^{\mu\nu *} - (1/2)|t^{\mu}_{\mu}|^2$ is negative for the charge motion in a circular orbit [28]. But if the mixed and temporal components of the total EMT $T_{\mu\nu}$ are expressed via its spatial component with the aid of the conservation law (see formula (10.4.14) in [8]), the terms of GR intensity quadratic in the spatial components t_{ij} make a positive contribution of the material tensor, those quadratic in θ_{ij} make a positive contribution of the field tensor, and the terms bilinear in t_{ij} and θ_{kl} determine the interference contribution.

From this standpoint, formula (2.4) defines the force field-independent contribution of the material body tensor moving uniformly in a circle. In the ultrarelativistic limit and the effective region of $\mathbf{q}$, this contribution, similar to EMR, forms over a trajectory segment smaller than the local curvature radius r; it is described for this segment by the asymptotic equations in Section 2.1. For the contact forces considered in Section 2.1, GR is determined by the contribution from the matter tensor alone.

For the electromagnetic systems considered in Sections 2, 3, and 5, the contribution of the tensor θ_{ij} to the GR spectrum is comparable with that of t_{ij}; at $\gamma \gg 1$, it is γ^2 times greater than the latter and is proportional to the EMR spectrum.

The dependence of the amplitude Γ on characteristics of the force field, the wave vector direction, and the motion parameters of a massive particle, especially its Lorentz factor, is governed by the behavior of both the charge proper field and the external field over a relatively large region. For $\chi \ll 1$, the position of photon emission is not localized, which makes it difficult to account quantitatively for the conversion of photons into gravitons. However, the knowledge of the qualitative behavior of Γ is quite enough to assess the GR spectrum because the properties of the EMR $|j_{\alpha}(q)|^2$ spectrum are regarded as known.

For nonelectromagnetic systems, it is interesting to qualitatively compare the transverse components θ_{+}, $\theta_{\times}$ and t_{+}, $t_{\times}$. The contributions of these components to the GR for a rotating mass-loaded relativistic string turned out to be of the same order, although the string energy is $\pi\gamma/2$ times the energy of the masses at its ends.

We note that the contribution of the EMT of a material body to GR in a circular motion is described by the exact expression (2.4). For a body in arbitrary ultrarelativistic motion, the contribution of its EMT to GR in the effective region of $\mathbf{q}$ is estimated as

$$8\pi G\left(\frac{1}{2}\,|t_{+}|^2 + 2|t_{\times}|^2\right) \sim \frac{4\pi G m^2}{e^2}\,\left|j_{\alpha}(q)\right|^2 . \qquad (6.1)$$

For a circular motion, the lower-order harmonics in this estimate have an additional factor γ^2 in the right-hand side, radiation is formed over the entire orbit, and order-of-unity radiation angles are effective. Therefore, the transition from the components j_i and t_{ij} to the transverse components is not accompanied by a γ- and γ^2-fold reduction, as is the case with the components in the effective range of $\mathbf{q}$.

The present results are equally applicable to a bunch of charged particles whose size is small compared with the radiation wavelength, and hence radiation at this wavelength is coherent. Because the wavelength of the main radiation by an individual particle is γ^3 times shorter than the wavelength of the first harmonic, the bunch emits radiation coherently in the lower-order harmonics and incoherently in the harmonics with $n \sim \gamma^3$.

A system of two masses moving in elliptical orbits about their common center of inertia may be an essential compact object of dark matter. The GR of such a system is a valuable source of information about its structure. The trajectory method described in Section 4 for calculating the transverse components of the conserved EMT of a two-mass system is based on the approximate knowledge of its time components. Using this generalization of the Einstein method for the calculation of quadrupole radiation permits determining the polarization amplitudes of all GR harmonics in the long-wave approximation and corrections to them due to the relativistic motion of masses and the force field outside the orbit.

We are delighted to have the opportunity to publish this article in the issue dedicated to the memory of V L Ginzburg. It may be thought that he would have found Section 4 the most interesting. With this in mind, we preserved the explicit notation c for the speed of light, as he used to do in his papers. Also, he might have remarked that we should introduce the lower-order relativistic corrections into elliptical trajectories too. We do know, Vitaly Lazarevich, where and how such corrections may appear, but not all of them at once.

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7. Appendix. Fourier coefficients $F_A(n,e)$ and $f_{ij}(n,e)$

All Fourier coefficients are expressed through the Bessel functions $J_n(ne)$ and their combinations

$$S_{ns}(ne) = J_{n-s}(ne) - J_{n+s}(ne)\,,$$
$$C_{ns}(ne) = J_{n-s}(ne) + J_{n+s}(ne)\,,$$

where n is the harmonic number and $s = 1,\ 2,\ 3,\ 4$. These functions have the following properties:

(1) $S_{ns}(ne) = 0$ at $n = 0,\ s \geqslant 0,\ C_{ns}(ne) = 0$ at $n = 0,\ s \geqslant 1$, but $C_{00}(0) = 2$;

(2) $C_{ns}(ne)$ are even and $S_{ns}(ne)$ are odd functions of n;

(3) under the change of sign e, $S_{ns}(-ne) = (-1)^{n-s} S_{ns}(ne)$ and $C_{ns}(-ne) = (-1)^{n-s} C_{ns}(ne)$.

$$F_{11} = -n(S_{n2} - 2eS_{n1})\,,$$
$$F_{12} = -\sqrt{1-e^2}\,n(C_{n2} - 2J_n)\,,$$
$$F_{22} = (1-e^2)\,nS_{n2}\,.$$
$$f_{11} = -n^2\left[(1-e^2)(C_{n2} - eC_{n1}) - \frac{1}{n}\,S_{n2}\right],$$
$$f_{12} = -\sqrt{1-e^2}\,n^2\left[\left(1 - \frac{1}{2}\,e^2\right)S_{n2} - eS_{n1} - \frac{1}{n}\,C_{n2}\right],$$
$$f_{22} = (1-e^2)\,n^2\left[C_{n2} - eC_{n1} - \frac{1}{n}\,S_{n2}\right].$$
$$F_{111} = -2n\left[\frac{1}{4}\,S_{n3} - eS_{n2} + \left(\frac{1}{4} + e^2\right)S_{n1}\right],$$
$$F_{112} = 2\sqrt{1-e^2}\,(eS_{n2} + S_{n1}) - F_{222}\,,$$
$$F_{121} = 2\sqrt{1-e^2}\,(eS_{n2} - S_{n1}) - F_{222}\,,$$
$$F_{122} = 2(eC_{n2} - C_{n1}) - F_{111}\,,$$
$$F_{221} = 2\left[eC_{n2} + (1-2e^2)C_{n1}\right] - F_{111}\,,$$
$$F_{222} = \frac{1}{2}\,(1-e^2)^{3/2} n(C_{n3} - C_{n1})\,.$$
$$F_{1111} = -n\left[\frac{1}{4}\,S_{n4} - \frac{3}{2}\,eS_{n3} + \left(\frac{1}{2} + 3e^2\right)S_{n2} - e\left(\frac{3}{2} + 2e^2\right)S_{n1}\right],$$
$$F_{1112} = -\sqrt{1-e^2}\,n\left[\frac{1}{4}\,C_{n4} - \frac{3}{4}\,eC_{n3} - \frac{1}{2}\,C_{n2} + e(3-2e^2)C_{n1}\right],$$
$$F_{1211} = -\sqrt{1-e^2}\,n\left[\frac{1}{4}\,C_{n4} - \frac{3}{2}\,eC_{n3} + (1+3e^2)C_{n2} - e\left(\frac{15}{4} - e^2\right)C_{n1}\right],$$
$$F_{1122} = (1-e^2)\left(4J_n + n\left[\frac{1}{4}\,S_{n4} - \frac{3}{2}\,S_{n2} + 2eS_{n1}\right]\right),$$
$$F_{1212} = (1-e^2)\left(-2J_n + n\left[\frac{1}{4}\,S_{n4} - \frac{3}{4}\,eS_{n3} + \frac{5}{4}\,eS_{n1}\right]\right),$$
$$F_{2211} = (1-e^2)\left(4J_n + n\left[\frac{1}{4}\,S_{n4} - \frac{3}{2}\,eS_{n3} + 3\left(\frac{1}{2} + e^2\right)S_{n2} - \frac{11}{2}\,eS_{n1}\right]\right),$$
$$F_{1222} = (1-e^2)^{3/2} n\left[\frac{1}{4}\,C_{n4} - C_{n2} + \frac{3}{2}\,J_n\right],$$
$$F_{2212} = (1-e^2)^{3/2} n\left[\frac{1}{4}\,C_{n4} - \frac{3}{4}\,eC_{n3} + \frac{1}{2}\,C_{n2}\right],$$
$$F_{2222} = -(1-e^2)^2 n\left(\frac{1}{4}\,S_{n4} - \frac{1}{2}\,S_{n2}\right).$$

Fourier coefficients $F_A(n,e)$ are either even or odd functions of n,

$$F_A(-n,e) = \pm F_A(n,e)\,,$$

depending on the even or odd number of twos in the combined index A. When the sign of e is changed, $F_A(n,e)$ behave as even (odd) functions e depending on the evenness (oddness) of the sum of n and the number of indices in A, i.e.,

$$F_A(n,-e) = (-1)^n F_A(n,e) \quad \text{for} \quad A = ij,\ ijkl\,,$$
$$F_A(n,-e) = (-1)^{n+1} F_A(n,e) \quad \text{for} \quad A = ijk\,.$$

This means that the coefficients $F_{ij}(n,e)$ and $F_{ijkl}(n,e)$ are expanded in even powers of e,

$$F_A(n,e) = a_0 + a_2 e^2 + a_4 e^4 + \ldots\,,$$

if n is even, and in odd powers of e,

$$F_A(n,e) = a_1 e + a_3 e^3 + \ldots\,,$$

if n is odd. In contrast, the coefficients $F_{ijk}(n,e)$ are expanded in even powers of e if n is odd and in odd powers of e if n is even.

For $e = 0$, the following coefficients are nonzero:

$$F_{11}(n) = f_{11}(n) = -2\delta_{n2}\,, \qquad F_{1111}(n) = -\delta_{n2} - \delta_{n4}\,,$$

$$F_{12}(n) = f_{12}(n) = -2\delta_{n2}\,, \qquad F_{1112}(n) = \delta_{n2} - \delta_{n4}\,,$$

$$F_{22}(n) = f_{22}(n) = 2\delta_{n2}\,. \qquad F_{1211}(n) = -2\delta_{n2} - \delta_{n4}\,,$$

$$F_{111}(n) = -\frac{1}{2}\,\delta_{n1} - \frac{3}{2}\,\delta_{n3}\,, \qquad F_{1122}(n) = 4\delta_{n0} - 3\delta_{n2} + \delta_{n4}\,,$$

$$F_{112}(n) = \frac{5}{2}\,\delta_{n1} - \frac{3}{2}\,\delta_{n3}\,, \qquad F_{1212}(n) = -2\delta_{n0} + \delta_{n4}\,,$$

$$F_{121}(n) = -\frac{3}{2}\,\delta_{n1} - \frac{3}{2}\,\delta_{n3}\,, \qquad F_{2211}(n) = 4\delta_{n0} + 3\delta_{n2} + \delta_{n4}\,,$$

$$F_{122}(n) = -\frac{3}{2}\,\delta_{n1} + \frac{3}{2}\,\delta_{n3}\,, \qquad F_{1222}(n) = -2\delta_{n2} + \delta_{n4}\,,$$

$$F_{221}(n) = \frac{5}{2}\,\delta_{n1} + \frac{3}{2}\,\delta_{n3}\,, \qquad F_{2212}(n) = \delta_{n2} + \delta_{n4}\,,$$

$$F_{222}(n) = -\frac{1}{2}\,\delta_{n1} + \frac{3}{2}\,\delta_{n3}\,. \qquad F_{2222}(n) = \delta_{n2} - \delta_{n4}\,.$$

References

1. Zakharov V D *Gravitatsionnye Volny v Teorii Tyagoteniya Einshteina* (Gravitational Waves in Einstein's Theory) (Moscow: Nauka, 1972) [Translated into English (New York: Halsted Press, 1973)]
2. Misner C W, Thorne K S, Wheeler J A *Gravitation* (San Francisco: W.H. Freeman, 1973) [Translated into Russian (Moscow: Mir, 1977)]
3. Braginsky V B, Caves C M, Thorne K S *Phys. Rev. D* **15** 2047 (1977)
4. Diambrini Palazzi G, Fargion D *Phys. Lett. B* **197** 302 (1987)
5. Melissinos A C *Nuovo Cimento B* **62** 190 (1981)
6. Landau L D, Lifshitz E M *Teoriya Polya* (The Classical Theory of Fields) (Moscow: Nauka, 1988) [Translated into English (Oxford: Butterworth-Heinemann, 1994)]
7. Schwinger J *Particles, Sources, and Fields* (Reading, Mass.: Addison-Wesley, 1970) [Translated into Russian (Moscow: Mir, 1973)]
8. Weinberg S *Gravitation and Cosmology* (New York: Wiley, 1972) [Translated into Russian (Moscow: Mir, 1975)]
9. Nikishov A I, Ritus V I *Zh. Eksp. Teor. Fiz.* **96** 1547 (1989) [*Sov. Phys. JETP* **69** 876 (1989)]
10. Gal'tsov D V, Grats Yu V *Zh. Eksp. Teor. Fiz.* **68** 777 (1975) [*Sov. Phys. JETP* **41** 387 (1976)]
11. Nikishov A I, Ritus V I *Zh. Eksp. Teor. Fiz.* **98** 1151 (1990) [*Sov. Phys. JETP* **71** 643 (1990)]
12. Gertsenshtein M E *Zh. Eksp. Teor. Fiz.* **41** 113 (1961) [*Sov. Phys. JETP* **14** 84 (1962)]
13. Pustovoit V I, Gertsenshtein M E *Zh. Eksp. Teor. Fiz.* **42** 163 (1962) [*Sov. Phys. JETP* **15** 116 (1962)]
14. Weber J "Gravitation and light", in *Gravitation and Relativity* (Eds H Chiu, W F Hoffmann) (New York: W.A. Benjamin, 1964) Ch. 5 [Translated into Russian (Moscow: Mir, 1965) p. 374]
15. Nikishov A I, Ritus V I *Zh. Eksp. Teor. Fiz.* **46** 776 (1964) [*Sov. Phys. JETP* **19** 529 (1964)]
16. Ritus V I *Tr. Fiz. Inst. Akad. Nauk* **111** 5 (1979) [*J. Sov. Laser Res.* **6** 497 (1985)]
17. Mitskevich N V *Fizicheskie Polya v Obshchei Teorii Otnositel'nosti* (Physical Fields in the General Theory of Relativity) (Moscow: Nauka, 1969)
18. Sushkov O P, Khriplovich I B *Zh. Eksp. Teor. Fiz.* **66** 3 (1974) [*Sov. Phys. JETP* **39** 1 (1974)]
19. Zel'dovich Ya B, Popov V S *Usp. Fiz. Nauk* **105** 403 (1971) [*Sov. Phys. Usp.* **14** 673 (1972)]
20. Barbashov B M, Nesterenko V V *Model' Relyativistskoi Struny v Fizike Adronov* (The Relativistic String Model in Hadron Physics) (Moscow: Energoatomizdat, 1987) [Translated into English: *Introduction to the Relativistic String Theory* (Singapore: World Scientific, 1990)]
21. Turok N *Nucl. Phys. B* **242** 520 (1984)
22. Bateman H, Erdélyi A *Higher Transcendental Functions* Vol. 2 (New York: McGraw-Hill, 1954) [Translated into Russian (Moscow: Nauka, 1966)]
23. Fichtenholz G M *Kurs Differentsial'nogo i Integral'nogo Ischisleniya* (The Course of Differential and Integrational Calculus) Vol. 3 (Moscow: Nauka, 1969) [Translated into English: *Infinite Series: Rudiments* (New York: Gordon and Breach, 1970)]
24. Peters P C, Mathews J *Phys. Rev.* **131** 435 (1963)
25. Peters° *Phys. Rev. D* **5** 2476 (1972)
26. Gal'tsov D V, Grats Yu V, Petukhov V I *Izluchenie Gravitatsionnykh Voln Elektrodinamicheskimi Sistemami* (Emission of Gravitational Waves by Electrodynamic Systems) (Moscow: Izd. MGU, 1984)
27. Peters° *Phys. Rev. D* **8** 4628 (1973)
28. Ritus V I *Zh. Eksp. Teor. Fiz.* **80** 1288 (1981) [*Sov. Phys. JETP* **53** 659 (1981)]