

METHODOLOGICAL NOTES

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Interaction of two charged conducting balls: theory and experiment

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Contents

Abstract. Three possible settings are considered for the question of two interacting charged conducting balls: (1) both balls are kept at constant and equal potentials; (2) one ball is charged and isolated, the other is kept at a constant potential with the same sign as the first, and (3) both balls are like charged and isolated. It is shown that fundamentally different problems generally arise here: whereas in the first case the balls always repel, in the second and third cases fairly wide ranges of radius and charge ratios can always be found in which the balls attract each other at close distances. These ranges are identified in the paper. The results are presented of experiments that demonstrate both the repulsion and attraction of like-charged balls. Theory and experiment show satisfactory agreement, both qualitatively and quantitatively.

1. Introduction

The study of the interaction between two charged conducting balls (spheres) has both applied significance [the interaction of charged water drops in the atmosphere (see, for example, Refs [1–3])] and obvious scientific-methodological importance. In particular, the interaction question for two isolated charged conducting balls was thoroughly considered in Ref. [4]. However, it often actually occurs that the balls (or one of them) are connected to a voltage source during the interaction. Experiments (the corresponding results are given in Section 5) showed that in such open ball systems paradoxical (at first glance) effects arose. For example, we repeatedly observed the effect where the like-charged balls which first repelled each other then, upon increasing the potential of one of the balls, began to attract each other, contrary to the expectation that they would be repelled even stronger because the charge of the ball connected to the voltage source also increased while keeping its sign. This and some other effects are interesting not only theoretically, but also practically, because information on the interaction between charged particles is important in technology applications. It is necessary to understand and describe the effects. This is what the present work is devoted to.

Below we shall consider three types of problems concerning the interaction between two charged balls: (1) the balls are connected to the same voltage source and have equal potentials which are kept constant; (2) one ball is connected to a voltage source and the other is isolated and has a charge with the same sign, and (3) both balls are like-charged and isolated. As was mentioned, the last case was considered in Ref. [4]; however, a combined consideration of all the three cases has revealed new features of the interaction. It was also found that the asymptotic case of interacting balls with strongly different radii at a short distance between them corresponds to the case of ball interaction with an infinite conducting plane, which is of high application importance [5, 6]. The data obtained in the experiments performed satisfactorily agree with theoretical results.

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2. Balls with equal potentials

First, let us consider the case of ideal conducting balls (or thin-walled conducting spheres) with equal potentials kept constant due to their connection to the same external voltage source (Fig. 1, switches K_1 and K_2 are closed).

If the ball potentials are kept constant, then the expression for the potential energy of ball interaction can be written in the form [7]

$$W = \frac{1}{2}\left(c_{11}\varphi_1^2 + 2c_{12}\varphi_1\varphi_2 + c_{22}\varphi_2^2\right), \qquad (1)$$

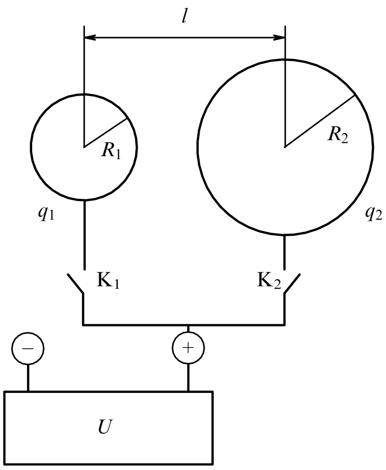

Figure 1. Schematic diagram of the experiment on interaction between charged balls. Different positions of switches K_1 and K_2 correspond to different cases of ball interaction considered in the present paper.

where c_{11}, c_{12}, and c_{22} are the capacity factors, and $\varphi_{1,2}$ are the ball potentials. The expressions for the capacity factors are as follows [7]

$$c_{11} = \frac{1}{k} R_1 R_2 \sinh\beta \sum_{n=1}^{\infty} \left[R_2 \sinh n\beta + R_1 \sinh(n-1)\beta \right]^{-1},$$

$$c_{12} = -\frac{R_1 R_2 \sinh\beta}{kl} \sum_{n=1}^{\infty} (\sinh n\beta)^{-1}, \qquad k = \frac{1}{4\pi\varepsilon_0}, \qquad (2)$$

$$c_{22} = \frac{1}{k} R_1 R_2 \sinh\beta \sum_{n=1}^{\infty} \left[R_1 \sinh n\beta + R_2 \sinh(n-1)\beta \right]^{-1}.$$

Here, ε_0 is the electrodynamic constant, and the parameter β is related to the separation l between ball centers by the formula

$$\cosh\beta = \frac{l^2 - R_1^2 - R_2^2}{2R_1 R_2}. \qquad (3)$$

If the balls are kept at equal potentials by an external voltage source ($\varphi_1 = \varphi_2 = U$), then we have

$$W(l) = \frac{U^2}{2}(c_{11} + 2c_{12} + c_{22}). \qquad (4)$$

The ball interaction force can be found from the expression

$$F_l(l) = \frac{\partial W(l)}{\partial l}. \qquad (5)$$

As was shown [7], in this case of an open system the derivative should be taken with the sign '+'.

It is convenient to write out all the parameters in a dimensionless form and express the force in units of the maximum force calculated in the Coulomb approximation — that is, under the assumption that the ball charges reside in their centers, so that all electrical image effects may be neglected. The maximum force is calculated from the balls' touch condition:

$$F_{\rm Cm} = \frac{U^2 R_1 R_2}{k(R_1 + R_2)^2}. \qquad (6)$$

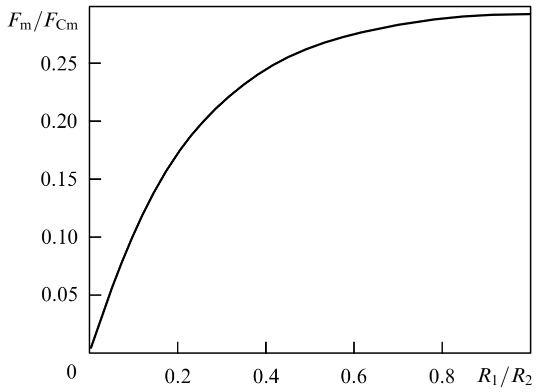

Figure 2. The maximum repulsive force versus the ball radius ratio in the case of equal potentials.

Then we obtain for the force of ball interaction:

$$F_x(x) = F_{\rm Cm} \frac{(R_1 + R_2)^2}{2R_1 R_2} (c'_{11} + 2c'_{12} + c'_{22}),$$
$$x = \frac{l}{R_1 + R_2}. \qquad (7)$$

Here, all the parameters with the dimension of length are expressed in units of $R_1 + R_2$, and the primes denote the derivative with respect to x.

The energy and force were calculated on a computer using Mathcad software. In the computations, 200 terms were retained in sums (2), and in checking computations their number was up to 400. Twice the number of summands does not change the first three significant figures in the calculated results. As expected, at all values of the parameters and all distances between the balls the force is repulsive, falling with the distance between the balls. The maximum repulsive force (in the units of $F_{\rm Cm}$) is shown in Fig. 2 versus the ratio of ball radii. The maximum force realized with similar balls equals $0.296F_{\rm Cm}$ (hereinafter, the numerical results are given within the accuracy of ± 1 in the last digit). This is a self-similar result — that is, the factor 0.296 depends on neither the ball radii nor the charges. In what follows, the self-similarity will be comprehended just in this meaning.

Let us separately consider an important case of similar balls of radius R with equal potentials, which permits an analytical solution. Then, $\varphi_1 = \pm\varphi_2 = \pm U$ and we find

$$W = U^2(c_{11} \pm c_{12}). \qquad (8)$$

Making allowance for the capacity factors given above, we may write

$$W = \frac{U^2 R}{k} \sinh\beta \sum_{n=1}^{\infty} \frac{(-1)^{j_n}}{\sinh n\beta}. \qquad (9)$$

Here, $j_n = n + 1$ if the potentials have the same sign, and $j_n = 0$ in the case of opposite signs. The parameter β is related here to the separation x between the ball centers by the relationship

$$\cosh\beta = \frac{l}{2R} = x.$$

The force acting on each ball is given by

$$F_x = \frac{\partial W}{\partial l} = \frac{U^2 R}{k}\frac{\partial W}{\partial \beta}\frac{\partial \beta}{\partial l}\,.$$

By calculating the derivatives we shall find the final expression for the force in units of $F_{\rm Cm}$:

$$F_x(x) = 2F_{\rm Cm}\sum_{n=1}^{\infty}\left[(-1)^{j_n}(\coth\beta - n\coth n\beta)(\sinh n\beta)^{-1}\right]. \tag{10}$$

In Ref. [4], the asymptotics of the series on the right-hand side of expression (10) for $x \to 1$ ($\beta \ll 1$) was found analytically. In the case of equal-sign potentials, the asymptotic value is $\ln 2/6 \approx 0.116$. Then, the maximum force equals $0.232F_{\rm Cm}$. A numerical calculation in this case yields the high-accuracy asymptotic value of 0.148 [4]; for the force we obtain, respectively, $0.296F_{\rm Cm}$.

For potentials of opposite signs, the asymptotic behavior of the series for approaching balls is expressed as $-1/[2(x-1)]$, $x - 1 \ll 1$ [4]. Correspondingly, the attraction force increases for approaching balls by the asymptotic law

$$F_x(x) = -\frac{F_{\rm Cm}}{x-1}\,, \qquad x - 1 \ll 1\,. \tag{11}$$

This result can also be considered as self-similar.

If the balls are maintained at equal potentials, and electric induction effects are substantial, then their charges are the following [7, 8]:

$$q_{1\rm i} = U(c_{11} + c_{12})\,, \qquad q_{2\rm i} = U(c_{22} + c_{12})\,. \tag{12}$$

In accordance with formulas (8), (9), for similar balls this yields

$$q_{\rm i} = \frac{UR}{k}\left(\sinh\beta\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sinh n\beta}\right). \tag{13}$$

The asymptotics of the expression in parenthesis entering into formula (13) as $x \to 1$ ($\beta \ll 1$) was found in Ref. [4]. The asymptotic value is $\ln 2$ (the sum of an alternating harmonic series). Thus, being charged from the same voltage source, each contacting ball acquires the charge

$$q_{\rm i} = \frac{UR}{k}\ln 2 \approx 0.693\,\frac{UR}{k}\,, \tag{14}$$

i.e., approximately 0.693 of the charge it would acquire if it were charged from the source individually at a large distance from the other ball [8]. This result is also self-similar. If we consider real ball charges [14] and calculate a maximum ball-interaction force in the Coulomb approximation (the charges reside at the ball centers):

$$F_{\rm Ci} = \frac{kq_{\rm i}^2}{4R^2}\,, \tag{15}$$

then the actual force (10) at short distances between the balls is equal to

$$F_x(1.001) \approx \frac{0.296}{(\ln 2)^2}\,F_{\rm Ci} = 0.616F_{\rm Ci}\,. \tag{16}$$

For the force we obtained the same value as that found in Ref. [4] for the case of isolated balls with the charges $q_{\rm i}$.

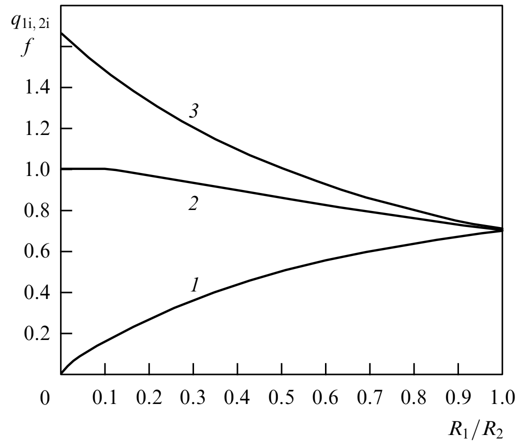

Figure 3. The ball charges versus the radii ratio: (*1*) for a smaller ball, (*2*) for a bigger ball, and (*3*) the factor f [see formula (19)] calculated for various ratios of ball radii.

Returning to the case of balls with different radii, if we calculate the ball charges by formulas (12) in units of UR_1/k, UR_2/k and plot the charge value versus the ratio of ball radii, then we shall obtain the curves marked in Fig. 3 as *1* (a smaller ball) and *2* (a larger ball) [8]. It is clear now why the interaction force for those balls substantially differing in radii tends to zero (see Fig. 2) — the charge of the smaller ball tends to zero (see Fig. 3). Nevertheless, the situation is not so simple. In Ref. [6], the interaction problem was considered for a conducting ball and an infinite plane when they touch each other and are connected to a voltage source. Notably, the ball charge

$$q_{1\rm i} \approx \frac{1}{k}\,1.64R_1^2 E_0 \tag{17}$$

was found, where E_0 is the electric field strength produced by the uniformly charged infinite plane. An infinite plane can be approximately substituted by a ball with very large radius $R_2 \gg R_1$, and formula (17) can then be transformed into

$$q_{1\rm i} = 1.64R_1^2\,\frac{q_{2\rm i}}{R_2^2} = 1.64\,\frac{UR_2}{k}\left(\frac{R_1^2}{R_2^2}\right) = 1.64\,\frac{UR_1}{k}\left(\frac{R_1}{R_2}\right). \tag{18}$$

To obtain this result in the framework of the question of the interaction between two balls, we revert to the first formula in Eqn (12), introduce the notation $\gamma = R_1/R_2$, and expand the expressions for c_{11}, c_{12} in a power series, first, of γ and, second, of β. In the formula for c_{12}, we take $l \approx R_1 + R_2$. Then, we arrive at

$$kc_{11} \approx R_1\sinh\beta\sum_{n=1}^{\infty}\left[\frac{1}{\sinh n\beta} - \gamma\left(\frac{\cosh\beta}{\sinh n\beta} - \coth n\beta\,\frac{\sinh\beta}{\sinh n\beta}\right)\right],$$

$$kc_{12} \approx -R_1(1-\gamma)\sinh\beta\sum_{n=1}^{\infty}\frac{1}{\sinh n\beta}\,.$$

Furthermore, at short distances between the balls, $\beta \ll 1$, assuming $\cosh\beta \approx 1$, $\sinh\beta \approx \beta$, $\sinh n\beta \approx n\beta$, $\coth n\beta \approx 1/n\beta$, after simple transformations we find

$$k(c_{11} + c_{12}) \approx R_1\gamma\sum_{n=1}^{\infty}\frac{1}{n^2} = R_1\left(\frac{R_1}{R_2}\right)\frac{\pi^2}{6} \approx 1.645R_1\left(\frac{R_1}{R_2}\right).$$

Now, taking into account ball charges (12), we obtain the last expression in Eqn (18).

For the appropriate computer calculations with an arbitrary ratio of radii, the first formula in Eqn (12) was written out in the form

$$q_{1\mathrm{i}} = f\,\frac{UR_1}{k}\left(\frac{R_1}{R_2}\right), \qquad f = \frac{k}{R_1}\left(\frac{R_2}{R_1}\right)(c_{11} + c_{12})\,. \tag{19}$$

Here, the capacity factors were calculated by formulas (2) at $l = 1.001(R_1 + R_2)$, which corresponds to almost contacting balls. The resulting tabulated coefficient f for various ratios of ball radii is shown by curve *3* in Fig. 3.

Thus, in the common charging of balls of considerably different sizes, residing closely to each other, the smaller ball acquires the charge that is proportional to its radius squared, the strength of the field produced by the larger ball on its surface, and the coefficient f that varies in the range $\ln 2 \leqslant f(R_1/R_2) \leqslant \pi^2/6$, depending on the ratio of radii.

3. Interaction between balls when the charge of one ball and the potential of the other are preassigned

Let us consider a situation where a conducting ball of radius R_1 is isolated and has the charge $Q_1 > 0$, whereas the other conducting ball of radius R_2 connected by a conductor to the positive pole of a voltage source has the potential U_2 relative to infinity. The distance between the ball centers is $l \geqslant R_1 + R_2$ (in Fig. 1, the key K_2 is closed, and K_1 is open).

In order to find the force of ball interaction we can first write out their potential energy in the form

$$W = \frac{1}{2}(Q_1\varphi_1 + q_2 U_2)\,. \tag{20}$$

Now, we express the potential of the first ball in terms of its charge, and the charge of the other ball in terms of its potential. For this purpose we write [7]

$$Q_1 = c_{11}\varphi_1 + c_{12}U_2\,, \qquad U_2 = s_{22}q_2 + s_{12}Q_1\,. \tag{21}$$

Hence follows

$$\varphi_1 = \frac{Q_1}{c_{11}} - \frac{U_2 c_{12}}{c_{11}}\,, \qquad q_2 = \frac{U_2}{s_{22}} - \frac{Q_1 s_{12}}{s_{22}}\,. \tag{22}$$

Here, s_{ik} are the so-called potential coefficients. Since the capacity coefficients can be explicitly expressed, we may express the potential coefficients in terms of capacity coefficients [7]:

$$s_{11} = \frac{c_{22}}{c_{11}c_{22} - c_{12}^2}\,, \qquad s_{12} = -\frac{c_{12}}{c_{11}c_{22} - c_{12}^2}\,, \qquad s_{22} = \frac{s_{11}c_{11}}{c_{22}}\,. \tag{23}$$

Let us write out the potential energy in units of $W_{\mathrm{Cm}} = Q_1 U_2 R_2/(R_1 + R_2)$ in the form

$$W(l) = \frac{W_{\mathrm{Cm}}}{2b}\left[\left(\frac{1}{\alpha_1 c_{11}} - \frac{c_{12}}{c_{11}}\right) + \left(\frac{\alpha_1}{s_{22}} - \frac{s_{12}}{s_{22}}\right)\right],$$
$$\alpha_1 = \frac{U_2(R_1 + R_2)}{kQ_1}\,, \qquad b = \frac{R_2}{R_1 + R_2}\,. \tag{24}$$

Here, all the quantities with the dimension of length are measured in units of $R_1 + R_2$. The interaction force will be calculated by the formula

$$F_l(l) = \frac{1}{2}\left(-Q_1\,\frac{\partial\varphi_1}{\partial l} + U_2\,\frac{\partial q_2}{\partial l}\right). \tag{25}$$

Then, in view of formulas (21) and (22), we obtain in the dimensionless variables that

$$F_x(x) = \frac{F_{\mathrm{Cm}}}{2b}\left[\left(\frac{c'_{11}}{\alpha_1 c_{11}^2} + \frac{c'_{12}c_{11} - c'_{11}c_{12}}{c_{11}^2}\right) - \left(\frac{\alpha_1 s'_{22}}{s_{22}^2} + \frac{s'_{12}s_{22} - s'_{22}s_{12}}{s_{22}^2}\right)\right], \tag{26}$$

$$F_{\mathrm{Cm}} = \frac{Q_1 U_2 R_2}{(R_1 + R_2)^2}\,.$$

Primes stand for the derivatives with respect to $x = l/(R_1 + R_2)$.

First, let us consider the case of similar balls with $R_1 = R_2 = R$, which at a short distance $(x \to 1)$ have equal potentials U_2 and equal charges $q_{2\mathrm{i}} = Q_1 = \ln 2\, U_2 R/k$. Next, the parameter $\alpha_1 = 2/\ln 2 \approx 2.89$. In this case, the force calculation by Eqn (26) in Mathcad with 200 terms retained in the sums yields the force which is repulsive over all the range of definition $x > 1$. The force reaches a maximum for approaching balls $(x \to 1)$, which is $F_x(1.001) \approx 0.426 F_{\mathrm{Cm}}$. If this force is expressed in units of the maximum force calculated in the Coulomb approximation with the real charge $q_{2\mathrm{i}}$, then we again arrive at the self-similar result of Ref. [4]:

$$F_x(1.001) \approx \frac{0.426}{\ln 2}\,\frac{kQ_1 q_{2\mathrm{i}}}{(R_1 + R_2)^2} \approx 0.616 F_{\mathrm{Ci}}\,.$$

This finding allows one to consider this solution as a self-similar one as well.

The interaction force between two similar balls remains repulsive in character over the whole range of definition $x \geqslant 1$ only under the condition $2.56 < \alpha_1 < 3.28$ (that is, when the ball potentials are equal or close at short distances between the balls); if the condition fails, a domain of ball attraction arises near the point of balls' contact.

Consider now the calculated results for balls with different radii. Assume, for example, $R_2/R_1 = 2$. It turns out in this case that at relatively small $\alpha_1 < 5.11$, and short distances between the balls there is a domain in which the force is attractive (Fig. 4, curve *4*, $\alpha_1 = 3$). For $5.11 < \alpha_1 < 7.44$, the force is repulsive over the whole range of definition. It takes the maximum value of $0.541 F_{\mathrm{Cm}}$ when the balls are close to each other and have equal potentials. This occurs at $\alpha_1 = 6.12$ [the potential of the first ball was calculated by formula (22)]. The appropriate dependence of the force on the separation between the ball centers is illustrated by curve *2* in Fig. 4 (the curve *1* corresponds to the Coulomb law $1/x^2$). For $\alpha_1 > 7.44$, the domain of attractive force arises again (curve *5*, $\alpha_1 = 25$) near the point of balls' contact.

In Fig. 5 plotted at a semi-logarithmic scale, the parameter domains are marked in which the ball interaction force has a repulsive character over the whole range of definition $x \geqslant 1$. Domain 1 corresponds to α_1 and the case where the charge of one ball and the potential of the other are preassigned. Curve *3* in Fig. 4 is appropriate to the boundary value of the parameter $\alpha_1 = 5.11$, which corresponds to the lower bound of the domain 1 in Fig. 5. In this case, the force near the point

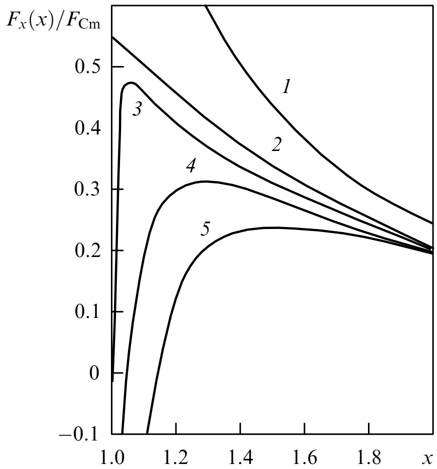

Figure 4. The calculated interaction force for two balls of different radii at various values of the parameter α_1. Curve *1* corresponds to the Coulomb law $1/x^2$; curves *2*, *3*, *4*, and *5* refer to $\alpha_1 = 6.12$, $\alpha_1 = 5.11$, $\alpha_1 = 3$, and $\alpha_1 = 25$, respectively.

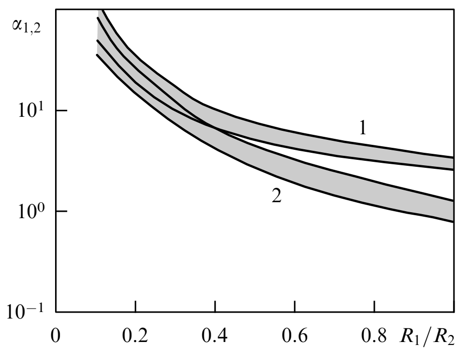

Figure 5. The parameter domains are shaded in which the interaction force has a repulsive character at short distances between the balls. Domain 1 (α_1) corresponds to the case where the bigger ball is connected to a voltage source, while the smaller one is isolated and has a certain charge. Domain 2 (α_2) corresponds to the case where both balls have certain charges and are isolated.

of balls' contact $x = 1$ is close to zero; it reaches the maximum equal to $0.469 F_{Cm}$ at $x \approx 1.08$.

4. Isolated balls with preassigned charges

In this case, the balls are preliminarily charged to certain quantities $q_1 > 0$, $q_2 > 0$, and both switches in Fig. 1 are open. The potential energy and ball interaction force are calculated under this conditions by the formulas [4]

$$W(x) = \frac{kq_1q_2}{R_1 + R_2} \frac{\alpha_2^2 c_{11} - 2\alpha_2 c_{12} + c_{22}}{2\alpha_2(c_{11}c_{22} - c_{12}^2)} = \frac{kq_1q_2}{R_1 + R_2} V(x) , \tag{27}$$

$$\alpha_2 = \frac{q_2}{q_1} ,$$

$$F_x(x) = -F_{Cm} \frac{\partial V(x)}{\partial x} , \qquad F_{Cm} = \frac{kq_1q_2}{(R_1 + R_2)^2} . \tag{28}$$

If the balls are similar, $R_1 = R_2$, and equally charged, $q_1 = q_2$, we have a self-similar result for their repulsive force, and at short distances between the balls [4] we find $F_x(1.001) \approx 0.616 F_{Cm}$. The calculation of the force according to formulas (28) shows that the force remains repulsive over the whole range of definition $x \geqslant 1$ only for $1/1.24 < q_2/q_1 < 1.24$. At other ratios of ball charges, a domain of attraction arises at short distances between the balls.

In the case of balls with different radii, the situation is similar to that given in Sections 2 and 3: at equal or close ball potentials near the point of balls' contact the force is repulsive over the whole range of definition. For example, at $R_2/R_1 = 2$ the potentials near the point of balls' contact ($x = 1.001$) are equal within an accuracy of up to three significant figures at $q_2/q_1 = 3.47$, and the repulsive force is $F_x(1.001) \approx 0.633$, reducing with an increase in the distance between the balls. Here, the ball potentials were controlled by the formulas

$$\varphi_1 = s_{11}q_1 + s_{12}q_2 , \qquad \varphi_2 = s_{22}q_2 + s_{12}q_1 . \tag{29}$$

In Fig. 5, the domain of parameters 2 is shaded, in which the ball interaction force has a repulsive character over the whole range of its definition $x \geqslant 1$. Note that this domain has not been mentioned in Ref. [4].

The existence of such domains (1 and 2 in Fig. 5) can be explained physically as follows: if the charge of one ball is relatively small, the determining role belongs to the interaction of the greater charge with its own image in the more weakly charged ball. This interaction is attractive; thus, the top and bottom parts of the plane ($\alpha = q_2/q_1$, R_1/R_2) are domains of attraction. It is this circumstance that explains ball attraction upon increasing the potential (charge) on one of the balls mentioned in the Introduction.

In Fig. 6, the maximum repulsive force is depicted for equal potentials of balls near the point of their contact versus the radii ratio. Curve *1* corresponds to one isolated ball having a charge and the other being maintained at a constant potential (this case was considered in Section 3). Curve *2* corresponds to isolated charged balls. The fact that at greater ratios of their radii the distinction between these cases vanishes is clear: at a greater ratio R_2/R_1 (and smaller R_1/R_2), the bigger ball being connected to a voltage source acquires a charge close to U_2R_2/k [8] (recall that at $R_2 = R_1$ it acquires a charge of only $0.693 U_2R_2/k$). Moreover, if the interaction force is measured in units of

$$F_{Ci} = \frac{kq_{1i}q_{2i}}{(R_1 + R_2)^2} ,$$

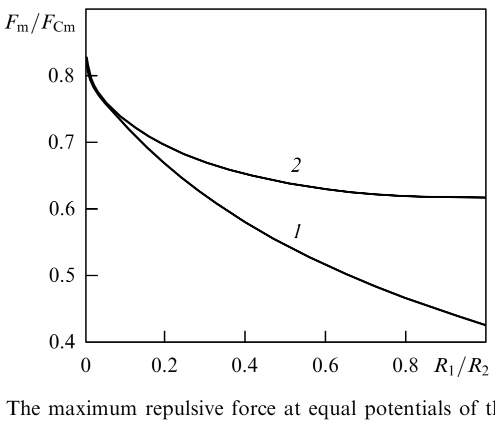

Figure 6. The maximum repulsive force at equal potentials of the balls versus the ratio of their radii, when one ball is charged and isolated and the other ball has a constant potential (curve *1*), or when both balls are charged and isolated (curve *2*). The limiting case of infinitesimal ratio of ball radii corresponds to ball interaction with an infinite plane [6].

where $q_{1i} = U(c_{11} + c_{12})$ and $q_{2i} = U(c_{22} + c_{12})$ are the charges acquired by the balls from the voltage source (here, the capacity factors should be calculated at the point of ball contact; in the calculations we took $x = 1.001$), then both the curve in Fig. 2 and curve *1* in Fig. 6 would coincide with curve *2* in Fig. 6.

The maximum Coulomb force for $R_1 \ll R_2$ can be represented in the form

$$F_{\mathrm{Cm}} = \frac{kq_1q_2}{(R_1+R_2)^2} = q_1\,\frac{kq_2}{R_2^2(1+R_1/R_2)^2} \approx q_1E_2\,, \qquad (30)$$

where $E_2 = kq_2/R_2^2$ is the field strength on the surface of the bigger ball in the ideal case of an absence of the electric image effects. In Ref. [6], it was established that a touching conducting ball and an infinite conducting plane connected to a voltage source interact with force $F_x = 0.832qE_0$, where E_0 is the electric field strength near the uniformly charged plane, and q is the charge acquired by the ball. From Fig. 6 one can see that at small R_1/R_2, i.e., when the radius of one ball is much greater than that of the other ball and the bigger ball can actually be substituted by a plane, both the curves asymptotically approach 0.832 (in particular, at $R_1/R_2 = 0.01$ the calculations yield $F_x = 0.826q_1E_2$).

5. Experiments

A photograph of the experimental setup is displayed in Fig. 7. A principal element of the setup is the electrostatic dynamometer constituting the physical pendulum whose plumb possesses an electric charge. The installation involves, in addition to the dynamometer, a high-voltage power supply and a conducting ball installed on an isolating pole capable of moving in the plane of pendulum oscillations in the horizontal and vertical directions. The dynamometer itself comprises the foam ball *1* covered by aluminum foil, polyethylene rod *2* connected to the ball, steel pivot pin *3* with bearings from glass beads, polyethylene holder *4*, pointer *5*, and scale *6*. The electrostatic dynamometer is mounted on the ebonite rod *7*. The guide plate *8* made from plexiglass, over which the ebonite pillar *9* moves, is placed on the table under the dynamometer. The lower part of the pillar has a slit providing its vertical motion. At the end of the pillar, the steel nickel-plated ball *10* is fixed and connected to a terminal of the high-voltage source with a wire enclosed in a high-voltage insulating sheath. The other terminal of the source is grounded. The parameters of the electrostatic dynamometer are as follows: the distance from the center of mass of the moving system to its axis of rotation is $\mathrm{CO} = a = 12.2$ mm, the distance from the ball center to the axis of rotation is $b = 56.0$ mm, the distance from the axis of rotation to the scale is $c = 100$ mm, and the pendulum mass is $m = 0.76$ g (see Fig. 8). The equilibrium condition for the moving system of the dynamometer (pendulum) with an axis of rotation is the equality between the moments of force:

$$Fb\cos\gamma = mga\sin\gamma\,;$$

hence, the horizontal force acting on the dynamometer ball is

$$F = \frac{mga}{b}\tan\gamma = \left(\frac{mga}{bc}\right)z = Kz\,, \qquad (31)$$

where K is the dynamometer sensitivity for a millimeter scale, $K = 0.016\ \mathrm{mN\,mm^{-1}}$. At a distance of 2 m from the setup, just against the dynamometer, a digital camera was placed for obtaining a sharp image of the balls and millimeter scale divisions. A transistor converter with voltage multiplication was used as the high-voltage source which provided an adjustable potential difference in the limits from 0 to 50 kV. The voltage was measured by a high-voltage voltmeter comprising a microammeter with a limit of 60 μA, and a resistor $r_1 = 400$ MΩ connected to it in series. In addition to the microammeter, an M-838 multimeter was utilized, which operated in the voltmeter mode with an internal resistance of 1 MΩ. The internal resistance of the high-voltage source was measured by the method of two loads with the resistances $r_1 = 400$ MΩ and $r_2 = 200$ MΩ. High voltage was also obtained from a power supply (see Fig. 7) with included digital voltmeter which provided an adjustable potential difference in the limits from 0 to 30 kV. The standard

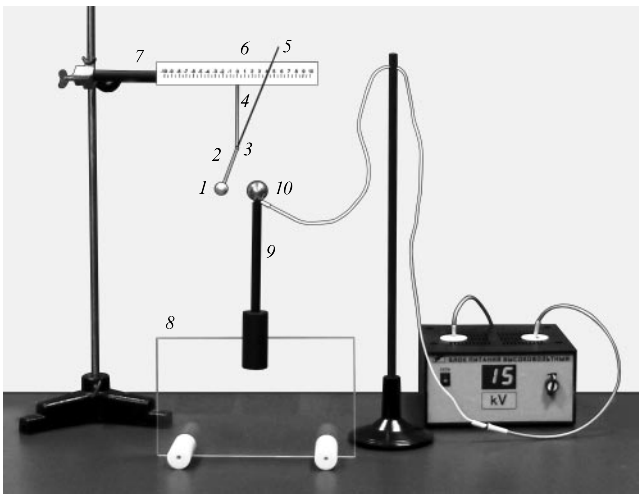

Figure 7. The experimental setup: (*1*) the foam ball covered by aluminum foil, (*2*) the polyethylene rod, (*3*) the steel pivot pin, (*4*) the polyethylene holder, (*5*) the pointer, (*6*) the scale, (*7*) the ebonite rod, (*8*) the guide plate, (*9*) the ebonite pillar, (*10*) the steel nickel-plated ball connected to a pole of the high-voltage source.

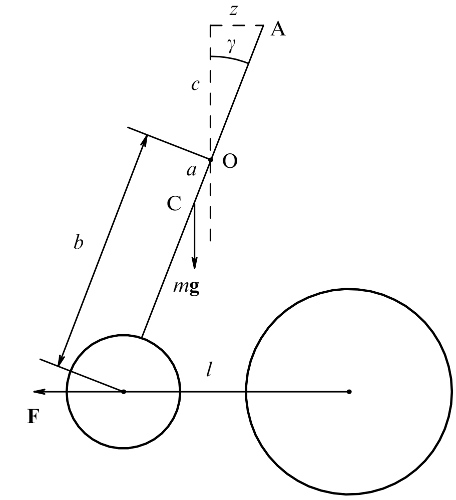

Figure 8. A schematics of an electrostatic dynamometer. (For the calculation of the force acting on a movable ball in equilibrium.)

voltmeter of this unit was tested by the high-voltage meter; the reading coincidence was satisfactory within the accuracy of two significant figures.

In the experiments, the force of electrostatic interaction was studied for two conducting balls, which were charged by touching a pole having certain potential U_2, as a function of the distance l between the ball centers. The radii of the dynamometer ball and the steel ball were, respectively, $R_1 = 8.5$ mm and $R_2 = 13.5$ mm.

The experiment was performed in the following way: (1) the negative pole of the high-voltage source was grounded, and the positive pole was connected to the steel ball on the pillar; (2) the voltage was increased to the required value by adjusting the high-voltage power supply; (3) by moving the holder over the guide, the steel ball moved closer to the dynamometer ball, until the distance between their nontouching surfaces became on the order of or shorter than 1 mm; (4) by moving the holder in the vertical direction, we adjusted the ball centers to equal heights and this position of all elements was photographed; (5) the steel ball moved away stepwise from the dynamometer ball, keeping the horizontal position of the line connecting their centers, and each step was photographed; (6) the experiment was finished as the dynamometer pointer approached the zero position.

The set of photographs obtained were stored in a computer. The required magnification and scale were determined. With this aim, the length of the electrostatic dynamometer scale image was measured on the screen in arbitrary units (a.u.). For example, if 10 cm = 13.2 a.u., the scale is $M = 7.56$ mm a.u.$^{-1}$. The distance l between the ball centers was measured in arbitrary units on the screen and expressed in millimeters by using the scale. The pointer shift in millimeters was found from the screen image of the dynamometer scale; then, the force of electrostatic interaction between the balls was calculated by utilizing the known sensitivity K. These data were used for plotting the force versus inverse dimensionless distance squared between the ball centers, $x = l/(R_1 + R_2)$ (Fig. 9). From the experimental plot one may conclude that at sufficiently long distances between the balls the Coulomb law holds true (the straight line in Fig. 9). If the balls are separated by a short distance, the force of electrostatic interaction is less than follows from the Coulomb law for the charges concentrated at the ball centers.

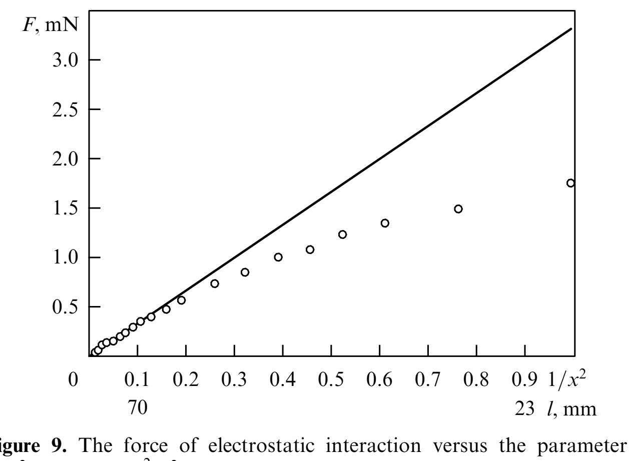

Figure 9. The force of electrostatic interaction versus the parameter $1/x^2 = (R_1 + R_2)^2/l^2$. Circles denote experimental data. The voltage is 15 kV. Below the abscissa axis, two distances between the ball centers are given in millimeters for reference.

For qualitative estimates, a certain separation between the ball centers on the rectilinear part of the plot (see Fig. 9) and the corresponding value of the interaction force were chosen. For example, the separation $l = 81$ mm, which is almost ten times longer than the radius of the smaller ball, corresponds to the force $F \approx 0.24$ mN measured by the dynamometer. After the balls, the bigger of which was maintained at the potential $U_2 = 15$ kV, come into contact in the experiment, the charge of the smaller ball, as follows from the theory and from the curve *1* in Fig. 3, equals $q_{1i} = 0.564\, U_2 R_1/k = 7.99$ nC for the radii ratio $R_1/R_2 = 8.5/13.5 = 0.629$. The charge of the larger ball connected to the high-voltage source and, hence, having a constant potential relative to the ground is $q_2 = U_2 R_2/k = 22.5$ nC at large distances between the balls. Then, the force of ball interaction calculated by the Coulomb law in the range of its validity constitutes $F = k q_{1i} q_2/l^2 \approx 0.25$ mN, which agrees with the experimental value within the accuracy of 4%. Thus, the electrostatic dynamometer measures the electrostatic force of ball interaction with a sufficient accuracy, and the voltmeters also sufficiently accurately measure the potential of the larger ball relative to the ground.

To compare the theoretical and experimental results qualitatively and quantitatively at various operating voltages, the curves of force dependence on the inverse distance squared were reduced to those in dimensionless variables in the following way. Instead of the interaction force established experimentally, we used the ratio F/F_{Cm}, where F_{Cm} is a maximum value of the Coulomb force found graphically at $x = 1$, assuming the charges of touching balls are concentrated at the ball centers (for the curve in Fig. 9 we have $F_{\mathrm{Cm}} \approx 3.35$ mN). The curves obtained in this way are demonstrated in Fig. 10, in which the experimental results processed in the same way at a voltage of 10 kV are also presented. Straight line *1* corresponds to the Coulomb law of interaction, and curve *2* is the interaction force calculated by formula (26) at the experimental parameters $R_1/R_2 = 0.629$, and $\alpha_1 = U_2(R_1 + R_2)/(k q_{1i}) = 4.59$. In this case, one obtains

$$F_{\mathrm{Cm}} = \frac{q_{1i} U_2 R_2}{(R_1 + R_2)^2}\,.$$

It is evident that the experimental results fit the theoretical curve *2* reasonably well.

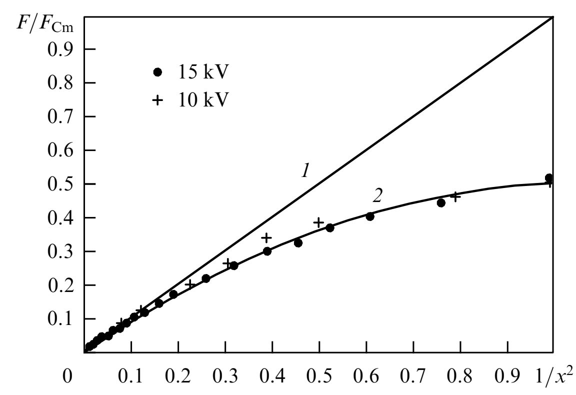

Figure 10. The normalized force of ball interaction versus the inverse distanced squared between the ball centers. Straight line *1* corresponds to the Coulomb law, and curve *2* is the calculation by formula (26).

Attraction of like-charged balls was observed in two cases, one of which was mentioned above. Namely, initially the balls touching each other were charged from a power supply at a certain voltage, which resulted in the smaller ball deflecting through a certain angle and being maintained in equilibrium in this position. Then the larger ball was slightly moved aside, so that the distance between the balls increased; simultaneously, the voltage on the larger ball was raised by approximately a factor of 1.5 and ball repulsion changed to attraction and they came into contact. Then, the smaller ball was again repelled, deflecting in this case through a greater angle. The second situation with ball attraction was observed when the charge of a ball, conversely, was reduced. To this end, the balls were first charged likely, similarly to the first case. Then, the distance between them was slightly enlarged and the larger ball was disconnected from the voltage source. In a short time (on the order of several seconds), repulsion changed to attraction. The reason was that one ball lost its charge sooner than the other and their potentials began to differ substantially. According to the above theory, by increasing or reducing the charge of one of the balls, we leave the narrow range of ball repulsion (see Fig. 5), getting higher or lower to the range of ball attraction.

It should be noted that finite air conductivity may play a determining role in some experiments. For example, in the tests described in Ref. [9] with a ball and disk connected to the same voltage source, ball nonrepulsion from the disk (stiction to disk) was observed at voltages of up to 10 kV. Probably, such manifestation of electric image effects occurred due to weak currents flowing from sharp disk edges, which provided a potential difference between the ball and disk.

6. Conclusion

The investigation performed shows that the character of ball interaction in the framework of the three problem statements considered is different in the general case. If the interacting balls are connected to the same terminal of a voltage source, they would always repel each other. In the other two cases considered (one ball is charged and then isolated, the other ball is connected to a voltage source; both balls are charged likely and then isolated), one can always find the range of parameters in which the balls would be attracted at short distances. An exception is the case where initially the potentials of balls separated by a short distance are equal due to either the choice of parameters or the contact of balls. In this case, balls removed from each other would always be repelled (if the charge loss due to air conductivity is eliminated), the repulsion force would fall with increasing the distance between the balls and tend to the Coulomb force. Such a scenario of ball interaction is realized independently of whether the balls (or one of them) remain connected to the voltage source or not. This situation was realized experimentally (see Figs 9 and 10). According to the theory and experiment, the actual force of ball interaction when they are drawn together is less than the force calculated in the Coulomb approximation (the point charges at the ball centers). In particular, this force is $jkq_{1i}q_{2i}/(R_1+R_2)^2$, $j<1$ if the balls touch. Here, q_{1i} and q_{2i} are the charges acquired by the balls. In experiments, this deviation from the Coulomb law is definitely observed with a sufficiently high accuracy (see Figs 9 and 10). However, already at the distance $l = 2(R_1+R_2)$ between the ball centers the Coulomb approximation is well fulfilled: an actual force differs from that calculated in the Coulomb approximation by approximately 6%. With a reduction (increase) in the ratio of ball radii, the factor j monotonically rises from $j = 0.616$ (similar balls) to $j = 0.832$ in the asymptotic 'ball–infinite conducting plane' case. Here, the asymptotics found for j coincides with the known solution to the problem upon interaction of a like-charged ball and plane [6].

On the radius ratio–charge ratio parameter plane, the domain of repulsive force for like-charged balls occupies a relatively narrow range (see Fig. 5), which expands with a reduction in the ratio of radii. Inside this range, the ball potentials are either equal or close. As soon as the ball potentials begin to noticeably differ, the force acting on the balls becomes attractive in character (for example, at $R_1/R_2 = 0.5$ the ball potential difference of 4% is sufficient). This theoretical conclusion is qualitatively confirmed by experiments.

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