

METHODOLOGICAL NOTES

PACS numbers: 43.25.Nm, 47.10.ad, **47.10.–g**, 47.15.Uv, 47.60.Kz

Submerged Landau jet: exact solutions, their meaning and application

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DOI: 10.3367/UFNe.0180.201001f.0097

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Abstract. Exact hydrodynamic solutions generalizing the Landau submerged jet solution are reviewed. It is shown how exact inviscid solutions can be obtained and how boundary layer viscosity can be included by introducing parabolic coordinates. The use of exact solutions in applied hydrodynamics and acoustics is discussed. A historical perspective on the discovery of a class of exact solutions and on the analysis of their physical meaning is presented.

1. Introduction

Only a few exact solutions of hydrodynamic equations are known for nonlinear flows of viscous fluids. Among them, the Landau solution [1–3] is of particular interest. This solution describes a submerged jet flowing out from a point-like orifice into an unbounded medium. Many who have succeeded in reproducing Landau calculations are delighted with the beauty and hidden symmetry of this problem. Indeed, it successfully simplifies the hopelessly complicated (at first sight) system of nonlinear differential equations with variable coefficients and allows finding a solution that admits an important physical interpretation. Without a doubt, this is one of the most remarkable results in mathematical physics. Exact solutions remain very important even in the era of modern computers. As is well known, it is desirable to test the accuracy of numerical code using an exact solution before applying this code to more complicated problems.

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Received 5 October 2009
Uspekhi Fizicheskikh Nauk **180** (1) 97 – 104 (2010)
DOI: 10.3367/UFNr.0180.201001f.0097
Translated by O V Rudenko; edited by A M Semikhatov

But the Landau solution is a partial solution; therefore, some restrictions on its applicability exist. The most substantial of them is the vanishing of the solution as the viscosity tends to zero. Consequently, the Landau solution is valid for strongly viscous media and does not contain the limit transition to an ideal liquid. The second limitation is that of a zero total flux of mass flowing out of the orifice, meaning that the source transmits only the linear momentum to the medium, while the mass of the jet forms because of the inflow of the surrounding liquid into the paraxial region. Of course, it is meaningless to demand that the unique exact solution describe every possible problem. However, any generalization or any new physical interpretation is valuable for both mathematical physics and applications.

Sometimes, unexpected applications appear. The Landau solution was shown to be an appropriate theory to describe acoustic streaming. Such streams are set in motion by radiation pressure of high-power ultrasound [4]. Applications of exact solutions are discussed in more detail in Sections 5 and 6.

As is shown below, the Landau solution can be improved and somewhat generalized. In particular, exact formulas can be derived describing liquid flows for an arbitrary viscosity. These solutions contain flows of both an ideal liquid and a Landau jet as two limit cases.

2. Mathematical model

The steady-state flow of an incompressible viscous liquid is governed by the Navier–Stokes equations and the continuity equation

$$(\mathbf{u}\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla\,\frac{p}{\rho_0}\,, \quad \operatorname{div}\mathbf{u} = 0\,, \tag{1}$$

where $\mathbf{u}$ is the velocity field, ν is the kinematic viscosity, p is the pressure, and ρ_0 is the density. The flow is assumed to be axially symmetric with respect to the z axis, with its parameters independent of the polar angle. For velocity components $\mathbf{u} = (u \equiv u_r, w \equiv u_z)$ expressed in cylindrical coordinates, Eqns (1) become

$$u\,\frac{\partial u}{\partial r}+w\,\frac{\partial u}{\partial z}-\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\,\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)=-\frac{\partial}{\partial r}\left(\frac{p}{\rho_0}\right), \tag{2}$$

$$u\,\frac{\partial w}{\partial r}+w\,\frac{\partial w}{\partial z}-\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\,\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right)=-\frac{\partial}{\partial z}\left(\frac{p}{\rho_0}\right), \tag{3}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0\,. \tag{4}$$

Equations (2)–(4) are invariant under the transformation

$$u\to\frac{u}{C}\,,\quad w\to\frac{w}{C}\,,\quad p\to\frac{p}{C^2}\,,\quad z\to Cz\,,\quad r\to Cr\,, \tag{5}$$

where C is an arbitrary constant. The invariance of system (2)–(4) under the rescaling of variables in (5) indicates the form of a solution that is also invariant under rescaling (5):

$$u=\frac{1}{z}\,U\!\left(\frac{r}{z}\right),\quad w=\frac{1}{z}\,W\!\left(\frac{r}{z}\right),\quad -\frac{p}{\rho_0}=\frac{1}{z^2}\,P\!\left(\frac{r}{z}\right). \tag{6}$$

The invariant properties of the fluid dynamics equations studied by group theory methods are described in detail in Refs [5–8].

Substituting (6) in partial differential equations (2)–(4) results in a system of ordinary differential equations for the three new unknown functions U, W, and P depending on a single variable $\xi=r/z$:

$$U\,\frac{\mathrm{d}U}{\mathrm{d}\xi}-W\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi U)-\nu\Bigg[(1+\xi^2)\,\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}+\frac{1+4\xi^2}{\xi}\,\frac{\mathrm{d}U}{\mathrm{d}\xi}-\frac{1-2\xi^2}{\xi^2}\,U\Bigg]=\frac{\mathrm{d}P}{\mathrm{d}\xi}\,, \tag{7}$$

$$U\,\frac{\mathrm{d}W}{\mathrm{d}\xi}-W\,\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi W)-\nu\Bigg[(1+\xi^2)\,\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}+\frac{1+4\xi^2}{\xi}\,\frac{\mathrm{d}W}{\mathrm{d}\xi}+2W\Bigg]=-\frac{1}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi^2P)\,, \tag{8}$$

$$\frac{1}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi U)=\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi W)\,. \tag{9}$$

The nonlinear system of three equations (7)–(9) is remarkable because it admits the exact general solution, which can be used for the analysis of jet flows, as well as flows in narrowing (nozzle) and expanding (diffuser) channels.

3. Solutions for an ideal liquid

Setting $\nu=0$ in Eqns (7) and (8) yields

$$U\,\frac{\mathrm{d}U}{\mathrm{d}\xi}-W\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi U)=\frac{\mathrm{d}P}{\mathrm{d}\xi}\,, \tag{10}$$

$$U\,\frac{\mathrm{d}W}{\mathrm{d}\xi}-W\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi W)=-\frac{1}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi^2P)\,, \tag{11}$$

$$\frac{1}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi U)=\frac{\mathrm{d}}{\mathrm{d}\xi}\,(\xi W)\,. \tag{12}$$

Equations (10)–(12) can be solved exactly. First, two independent integrals of system (10)–(12) exist:

$$U^2-(\xi W)^2-2P=C_1\,, \tag{13}$$

$$(U-\xi W)^2-2(1+\xi^2)\,P=C_2\,, \tag{14}$$

where C_1 and C_2 are two arbitrary integration constants. The validity of integral (13) can be easily checked: differentiating it and using continuity equation (12) yields the first equation of motion (10). The validity of the second integral can be checked similarly: differentiating integral (14) and subtracting Eqn (10) from it yields Eqn (11). It is significant that integrals (13) and (14) can be used to check the accuracy of numerical calculations.

It is now convenient to eliminate the variable P from integrals (13) and (14):

$$(1+\xi^2)^2W^2-(\xi U+W)^2=(C_2-C_1)-C_1\xi^2\,. \tag{15}$$

Solving the simplest equation (12) of system (10)–(12) jointly with integral (15) allows deriving the general solution of this system of equations. It is convenient to rewrite Eqn (12) as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big[(1+\xi^2)\,W-(\xi U+W)\big]=\xi W\,. \tag{16}$$

For definiteness, we assume that the right-hand side of (15) is positive, and write it symbolically as

$$f^2=(C_2-C_1)-C_1\xi^2\,. \tag{17}$$

Equation (15) is satisfied identically by the substitution

$$\xi U+W=f\sinh X(\xi)\,,$$

$$(1+\xi^2)\,W=f\cosh X(\xi)\,, \tag{18}$$

where $X(\xi)$ is the new unknown function.

Substitution of (18) in continuity equation (16) transforms it into a linear equation for $\exp(-2X)$:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\exp(-2X)+\left(2\frac{\mathrm{d}}{\mathrm{d}\xi}\ln f-\frac{\xi}{1+\xi^2}\right)\exp(-2X)=\frac{\xi}{1+\xi^2}\,. \tag{19}$$

The solution of Eqn (19) is

$$\exp(-X)=\frac{1}{f}\sqrt{C_3\sqrt{1+\xi^2}-C_1(1+\xi^2)-C_2}\,, \tag{20}$$

where C_3 is the third integration constant. Consequently, the general solution of the system of nonlinear equations (10)–(12) is given by

$$W=\frac{C_3-2C_1\sqrt{1+\xi^2}}{2\sqrt{1+\xi^2}\sqrt{C_3\sqrt{1+\xi^2}-C_1(1+\xi^2)-C_2}}\,, \tag{21}$$

$$U=\frac{2(C_1+C_2)\sqrt{1+\xi^2}-C_3(2+\xi^2)}{2\xi\sqrt{1+\xi^2}\sqrt{C_3\sqrt{1+\xi^2}-C_1(1+\xi^2)-C_2}}\,, \tag{22}$$

$$2P=U^2-(\xi W)^2-C_1\,. \tag{23}$$

We note that this solution is not mentioned even in the most complete reference books (e.g., in [9]).

Constants can be defined differently. For one of the most important groups of problems, the solution must satisfy the boundary conditions induced by the cylindrical symmetry of the flow:

$$U(0)=0\,,\quad 0<|W(0)|<\infty\,. \tag{24}$$

Conditions (24) correspond to a vanishing radial velocity at the axis and a nonzero finite axial velocity at the axis. The boundary conditions are satisfied if the constants are $C_1=C_2=-D^2$, $C_3=-2D^2$. Only one independent constant D remains. The corresponding partial solution following from (21)–(23) is

$$W=\frac{D}{\sqrt{1+\xi^2}}\,,\quad U=\frac{D}{\xi\sqrt{1+\xi^2}}\left(\sqrt{1+\xi^2}-1\right),$$
$$P=\frac{D^2}{\xi^2\sqrt{1+\xi^2}}\left(\sqrt{1+\xi^2}-1\right). \tag{25}$$

It can be easily verified that solution (25) does satisfy boundary conditions (24) as $\xi\to 0$. Away from the axis (as $\xi\to\infty$), both velocity components U and W in (25) tend to zero.

With the known field of flow velocity (25), we can calculate the shape of flow lines:

$$\frac{\mathrm{d}r}{u}=\frac{\mathrm{d}z}{w}\,,\quad \frac{\mathrm{d}r}{\mathrm{d}z}=\frac{U}{W}=\frac{1}{\xi}\left(\sqrt{1+\xi^2}-1\right). \tag{26}$$

The integral of Eqn (26) has the simple form

$$r=r_0\sqrt{1+\frac{2}{r_0}\,z}\,, \tag{27}$$

where r_0 is the radius of a ray tube in the cross section $z=0$. The flow line pattern is shown in Fig. 1 by solid curves for the constant values $r_0=0.5$, 1, 1.5. If a narrowing flow tube modeling a nozzle in which the liquid flows from the right to the left is considered, the constant D in formulas (25) must be negative. For $D>0$, the flow tube expands and emulates a diffuser. In all cases, the normal velocity is zero on the wall, but the tangent velocity is nonzero because the viscosity is not taken into account in solutions (21)–(23) and (25).

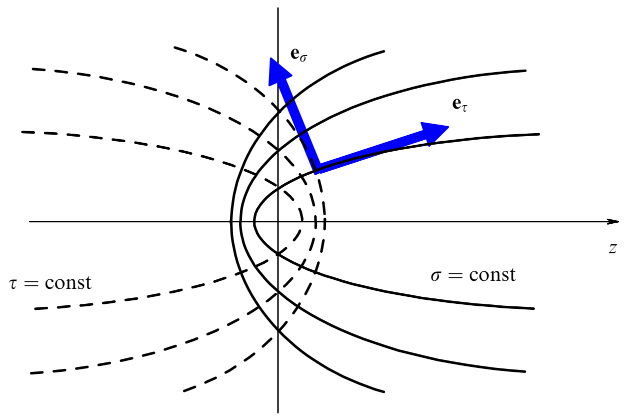

Figure 1. Flow lines (27) (solid curves) and the basis of the parabolic coordinate system (σ, τ) attached to these lines [see formula (58) in Section 5]. Surfaces $\sigma=$ const coincide with flow tubes. Axial sections of $\tau=$ const surfaces are shown by dashed curves.

4. Solutions for a viscous liquid

One particular solution of the system of ordinary differential equations (7)–(9) with account for viscosity is well known:

$$U=2\nu\xi\,\frac{A-\sqrt{1+\xi^2}}{\sqrt{1+\xi^2}\left(A\sqrt{1+\xi^2}-1\right)^2}\,,$$

$$W=2\nu\,\frac{A+A(1+\xi^2)-2\sqrt{1+\xi^2}}{\sqrt{1+\xi^2}\left(A\sqrt{1+\xi^2}-1\right)^2}\,,\quad P=-\frac{2\nu}{\xi}\,U\,. \tag{28}$$

Here, A is a constant. This solution is 'physically correct.' It satisfies boundary conditions (24) on the axis:

$$\frac{U(0)}{2\nu}=\frac{\xi}{A-1}\,,\quad \frac{W(0)}{2\nu}=\frac{2}{A-1}\,. \tag{29}$$

Moreover, flow (28) vanishes far from the axis, as $\xi\to\infty$. Solution (28) was found by Landau as a solution of Eqns (1) written in a spherical coordinate system:

$$-\nu\left[\frac{1}{R}\frac{\partial^2}{\partial R^2}(RU_R)+\frac{1}{R^2}\frac{\partial^2U_R}{\partial\theta^2}+\frac{\cot\theta}{R^2}\frac{\partial U_R}{\partial\theta}\right.$$
$$\left.-\frac{2}{R^2}\frac{\partial U_\theta}{\partial\theta}-\frac{2}{R^2}\,U_R-\frac{2\cot\theta}{R^2}\,U_\theta\right]$$
$$+U_R\frac{\partial U_R}{\partial R}+\frac{U_\theta}{R}\frac{\partial U_R}{\partial\theta}-\frac{U_\theta^2}{R}=-\frac{\partial}{\partial R}\left(\frac{p}{\rho_0}\right), \tag{30}$$

$$-\nu\left[\frac{1}{R}\frac{\partial^2}{\partial R^2}(RU_\theta)+\frac{1}{R^2}\frac{\partial^2U_\theta}{\partial\theta^2}+\frac{\cot\theta}{R^2}\frac{\partial U_\theta}{\partial\theta}\right.$$
$$\left.+\frac{2}{R^2}\frac{\partial U_R}{\partial\theta}-\frac{1}{R^2\sin^2\theta}\,U_\theta\right]$$
$$+U_R\frac{\partial U_\theta}{\partial R}+\frac{U_\theta}{R}\frac{\partial U_\theta}{\partial\theta}+\frac{U_RU_\theta}{R}=-\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{p}{\rho_0}\right), \tag{31}$$

$$\frac{\partial U_R}{\partial R}+\frac{1}{R}\frac{\partial U_\theta}{\partial\theta}+\frac{2}{R}\,U_R+\frac{\cot\theta}{R}\,U_\theta=0\,. \tag{32}$$

Here, the velocity components and the pressure depend on the radius R and azimuth θ and are independent of the polar angle. Evidently, the invariance of Eqns (1) under transformations (5) means that the solution of system (30)–(32) can be sought in another form equivalent to (6):

$$U_R=\frac{\varphi(\theta)}{R}\,,\quad U_\theta=\frac{f(\theta)}{R}\,,\quad -\frac{p}{\rho_0}=\frac{\chi(\theta)}{R^2}\,. \tag{33}$$

Equations (30)–(32) with ansatz (33) reduce to the system of nonlinear ordinary differential equations:

$$\nu(\varphi''+\varphi'\cot\theta)-f\varphi'+\varphi^2+f^2=2\chi\,,$$
$$\nu\varphi'-ff'+\chi'=0\,, \tag{34}$$
$$f'+f\cot\theta+\varphi=0\,.$$

The prime here indicates the θ derivative. The second equation of system (34) can be integrated and allows eliminating the variable χ from the first equation:

$$\nu(\varphi'' + \varphi' \cot\theta) - f\varphi' + \varphi^2 + 2\nu\varphi = A_1\,,$$
$$f' + f\cot\theta + \varphi = 0\,. \quad (35)$$

Landau set the constant A_1 equal to zero. However, even for $A_1 \neq 0$, system (35) can be integrated twice. This can be done after subtracting φ times the second equation in (35) from the first equation. After this, system (35) simplifies to

$$\nu(\varphi'\sin\theta)' - (f\varphi\sin\theta)' + 2\nu\varphi\sin\theta = A_1\sin\theta\,,$$
$$\varphi\sin\theta = -(f\sin\theta)'\,. \quad (36)$$

It is now convenient to introduce the new variables

$$\Phi = \varphi\sin\theta\,, \quad F = f\sin\theta\,. \quad (37)$$

The second equation in (36) then takes the simple form $\Phi = -F'$, which allows reducing the first equation in system (36) to the first-order equation [10, 11]

$$2\nu(F'\sin\theta - 2F\cos\theta) - F^2 = -A_1\cos^2\theta + 2A_2\cos\theta + 2A_3\,. \quad (38)$$

Solving Eqn (38), we express velocity components (33) in terms of F,

$$U_R = \frac{1}{R}\frac{\mathrm{d}F}{\mathrm{d}\zeta}\,, \quad U_\theta = \frac{1}{R}\frac{F}{\sin\theta}\,, \quad \zeta \equiv \cos\theta\,, \quad (39)$$

and derive the equation for flow lines:

$$\frac{\mathrm{d}R}{RU_R} = \frac{\mathrm{d}\theta}{U_\theta}\,, \quad \frac{\mathrm{d}R}{R} = \frac{\Phi}{F}\,\mathrm{d}\theta = -\,\mathrm{d}\ln F\,. \quad (40)$$

Integrating (40) gives the simple law

$$R(\theta) = \frac{B}{F(\theta)}\,, \quad (41)$$

where different constants B correspond to different flow lines.

Consequently, the function F must be determined to analyze the structure of the stream. This can be done by solving Eqn (38) directly or after its linearization by means of the substitution

$$F = 2\nu(1-\zeta^2)\,\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln V\,, \quad \zeta \equiv \cos\theta\,. \quad (42)$$

The corresponding linear second-order equation is

$$4\nu^2(1-\zeta^2)^2\,\frac{\mathrm{d}^2V}{\mathrm{d}\zeta^2} = (A_1\zeta^2 - 2A_2\zeta - 2A_3)\,V\,. \quad (43)$$

Its general solution is expressed in terms of hypergeometric functions [12].

The Landau solution corresponds to all three constants being equal to zero in the right-hand side of (43) or (38). For example, integrating Eqn (43) at $A_1 = A_2 = A_3 = 0$ and performing inverse transformations yields

$$V = G(\zeta - A)\,, \; F = 2\nu\,\frac{1-\zeta^2}{\zeta - A}\,, \; f = -2\nu\,\frac{\sin\theta}{A - \cos\theta}\,. \quad (44)$$

where G and A are constants. The last formula coincides with formula (23,18) in book [3].

We consider the ideal medium first. In this case, $\nu = 0$ in Eqn (38) and

$$F = -\sqrt{A_1\cos^2\theta - 2A_2\cos\theta - 2A_3}\,,$$
$$U_R = \frac{1}{R}\,\frac{A_2 - A_1\cos\theta}{\sqrt{A_1\cos^2\theta - 2A_2\cos\theta - 2A_3}}\,,$$
$$U_\theta = -\frac{\sqrt{A_1\cos^2\theta - 2A_2\cos\theta - 2A_3}}{R\sin\theta}\,. \quad (45)$$

The minus sign is chosen in the solution for the F function to reduce this particular solution to form (21)–(23) derived above. In fact, by passing from spherical coordinates to cylindrical ones, for example, for the axial velocity

$$u_z = \frac{W(\xi)}{z} = U_R\cos\theta - U_\theta\sin\theta\,, \quad (46)$$

where $R = z\sqrt{1+\xi^2}$ and $\sin\theta = \xi/\sqrt{1+\xi^2}$, we obtain

$$W = -\frac{1}{\sqrt{1+\xi^2}}\,\frac{A_2 + 2A_3\sqrt{1+\xi^2}}{\sqrt{A_1 - 2A_2\sqrt{1+\xi^2} - 2A_3(1+\xi^2)}}\,. \quad (47)$$

Solution (47) differs from another form of solution in (21) only in the notation for the constants: $A_1 = -C_2$, $-2A_2 = C_3$, and $2A_3 = C_1$. Consequently, Eqn (38) allows obtaining the general solution for a nonviscous medium.

As was shown above, the particular solution satisfying boundary conditions (24) for an axially symmetric flow of an ideal liquid can be derived in the case where the constants are equal to $C_1 = C_2 = -D^2$, $C_3 = -2D^2$. It is interesting to analyze this case for a viscous liquid as well. We set $A_1 = D^2$, $A_2 = D^2$, and $2A_3 = -D^2$ in equation (43) and reduce it to the form

$$(1+\zeta)^2\,\frac{\mathrm{d}^2V}{\mathrm{d}\zeta^2} = \frac{D^2}{4\nu^2}\,V\,. \quad (48)$$

Equation (48) is reduced to an autonomous equation by the change of the variable $1+\zeta = \exp(t)$. Solving it and performing transformation (42), we find the corresponding solution of Eqn (38):

$$F = \nu(1-\zeta)\left[1 + \gamma\,\frac{(1+\zeta)^\gamma + G}{(1+\zeta)^\gamma - G}\right], \quad \gamma \equiv \sqrt{1 + \frac{D^2}{\nu^2}}\,. \quad (49)$$

Here, γ is a constant depending on the ratio D/ν, which has the meaning the Reynolds number, and G is the integration constant. In the limit case $\nu \to 0$, the previous solution (25) for an ideal medium follows from (49). In the opposite limit case of small Reynolds numbers ($\nu \to \infty$, $\gamma \to 1$), Landau solution (44) can be derived with the constant $G = A + 1$. The intermediate case of arbitrary Reynolds numbers is the most interesting, of course.

It is important to note that generalized (in comparison with the Landau solution) equation (49) can describe a nonzero mass flux from a point-like orifice. Indeed, the mass stream through a spherical surface of radius R equals

$$Q = 2\pi\rho R^2\int_0^\pi U_R\sin\theta\,\mathrm{d}\theta = 2\pi\rho R\big[F(0) - F(\pi)\big]\,. \quad (50)$$

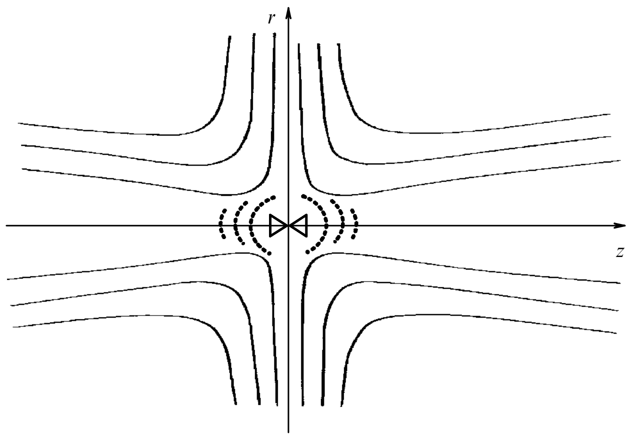

Figure 2. Flow lines of a 'double-sided' submerged jet or of an acoustic stream caused by two counter propagating ultrasonic waves. Transducers are schematically shown by triangles, and the fronts of waves traveling from the origin of coordinates, by dashed lines.

It follows from here that the mass flux for Landau solution (44) is zero because $F(0) = F(\pi) = 0$. In other cases, this conclusion is incorrect. For example, it follows from (49) for an ideal liquid that $F = D(1 - \zeta)$ and the difference $F(0) - F(\pi)$ in formula (50) is $2D$. For an arbitrary viscosity, Eqns (49) and (50) imply that

$$Q = 4\pi\rho R\nu\left(\sqrt{1 + \frac{D^2}{\nu^2}} - 1\right). \tag{51}$$

But Eqn (51) is derived under the assumption of the absence of singularities of the integrand in (50) in the range $0 < \theta \leqslant \pi$. This assumption is not valid for all values of constants in solution (49).

An interesting structure of the stream forms if the constants are related as $G = -(\gamma + 1)/(\gamma - 1)$. In this case, flow lines (41) are shown in Fig. 2. For the right half-space, flow lines arrive from infinity (at $\theta = \pi/2$) and recede to infinity (at $\theta = 0$). Integral (50) is zero in the limits $0 < \theta < \pi/2$; in other words, sources of mass are absent inside this range. Mirror-reflecting the flow pattern with respect to the vertical axis does not give a jump in the velocity. A double-sided type of stream jet forms as a result of such a mirror reflection (Fig. 2, $\gamma = 2$). By analogy with the Landau jet interpretation, we can say that the flow shown in Fig. 2 can occur if the point-like source transmits to the medium equal fluxes of linear momentum directed along the positive and negative sides of the z axis. At the same time, the mass inflow is zero in this case.

We note that flow lines (41) recede to the infinity at $\theta = 0$, irrespective of the constants G and γ in solution (49). An exception is the case $G = 2^\gamma$, where the stream line intersects the z axis at the point $R_1 = B/(4\nu)$. The second point of intersection is $R_2 = B/[2\nu(\gamma - 1)]$. Consequently, in the case $G = 2^\gamma$, loops of flow lines are formed. If in addition $\gamma = 3$, the stream has a constant total mass. However, sources and sinks of mass distributed along the z axis appear in this area.

5. Applications to hydrodynamics

The Landau submerged jet solution is often said to be a beautiful mathematical result unrelated to real applied problems. In this section and in Section 6, some applications of exact solutions to hydrodynamics and acoustics are pointed out.

One of the problems is related to the optimization of operation regimes of advanced eco-technology devices known as high-speed water jet cutting devices [13]. A water jet flowing out with velocities 200 – 1000 m s^{-1} is formed inside a narrowing nozzle. At its input, a pressure 200–500 MPa is applied. The optimum form of the nozzle must minimize the pressure loss and increase the threshold of the laminar–turbulent flow transition. The turbulent regime creates undesirable output fluctuations. The growth of instability decreases the distance from the nozzle section to the cutting object at which the jet breaks into pieces and its effective operation is impossible.

Solution (25) for an ideal liquid flow is a good approximation for a stream with a finite Reynolds number in a high-pressure chamber far from its wall. If the shape of the nozzle coincides with the shape of the flow tube in (27), the walls cannot significantly influence the stream. We describe below how to take the viscosity in a thin boundary layer into account. Evidently, solutions for jets of strongly viscous liquids are inapplicable to this problem; for such solutions, it is incorrect to set the tangential component of the velocity equal to zero at the wall.

We rewrite solution (25) using the original notation:

$$u = \frac{D}{r}\left(1 - \frac{z}{\sqrt{r^2 + z^2}}\right), \quad w = \frac{D}{\sqrt{r^2 + z^2}},$$

$$\frac{p}{p_0} = 1 - \frac{\rho}{p_0}\frac{D^2}{r^2}\left(1 - \frac{z}{\sqrt{r^2 + z^2}}\right), \tag{52}$$

where p_0 is the pressure at $z \to \infty$. The constant D must be negative for converging flows (directed toward smaller values of z; see Fig. 1). This constant is proportional to the complete mass stream through the cross section of the nozzle,

$$Q = -\rho\int_0^{r(z)} w 2\pi r\,\mathrm{d}r = 2\pi\rho r_0|D|, \tag{53}$$

where $r(z)$ is the radius of the stream tube (27) congruous with the nozzle. It follows that the flux in (53) is finite and has the same magnitude in any cross section because the liquid is incompressible. Let the positions of the output and input cross sections of the nozzle be marked as z_1 and z_2. Evidently, the singular point $z = 0$ cannot be located between these limits, and therefore $0 < z_1 < z < z_2$. Let the corresponding radii of the cross sections be r_1 and r_2.

The coordinate of the output cross section is determined from the condition that the pressure p in (52) be equal to the atmospheric pressure or approximately equal to zero because the pressure inside the chamber is very high:

$$p(0, z_1) = 0, z_1 = |D|\sqrt{\frac{\rho}{2p_0}}. \tag{54}$$

The input pressure at $z_2 = z_1 + L$, where L is the length of the nozzle, is close to the pressure at infinity; therefore,

$$p(0, z_2) = p_0(1 - \kappa^2), \quad z_2 = |D|\sqrt{\frac{\rho}{2p_0}}\,\frac{1}{\kappa}, \tag{55}$$

where $\kappa^2 \ll 1$ is an unknown auxiliary constant. Simple transformations of formulas (52)–(56) lead to the following solution for the input pressure creating the necessary mass consumption at the given geometric parameters of the nozzle:

$$p_0 = \frac{Q^2}{2\pi^2 \rho} (r_1^2 - r_0^2)^{-2}, \quad \kappa = \frac{r_1^2 - r_0^2}{r_2^2 - r_0^2}. \tag{56}$$

The parameter of stream line (27), congruent with the wall, is given by

$$r_0 = \frac{r_2^2 - r_1^2}{2L},$$

and the coordinates of the cross sections are

$$z_1 = L \frac{r_1^2 - r_0^2}{r_2^2 - r_1^2}, \quad z_2 = L \frac{r_2^2 - r_0^2}{r_2^2 - r_1^2}.$$

It can be shown that this theory is valid only if the condition

$$2Lr_1 > r_2^2 - r_1^2 \tag{57}$$

is satisfied. It follows from formula (57) that the chamber must be sufficiently long. For example, at typical radii $r_1 = 0.125$ mm and $r_2 = 1.6$ mm, the length must be $L > 10.2$ mm. In real devices, condition (57) holds. Tests have demonstrated that the jet flowing out jams at smaller chamber lengths.

The comparison of the theory with the parameters of machines manufactured by Water Jet Sweden AB (Ronneby) showed that the nozzle design deduced empirically is close to the calculated parameters. For one of the machines with $r_1 = 0.125$ mm, $r_2 = 1.6$ mm, and $L = 20$ mm, the fluid consumption is $Q = 0.020$ kg s^{-1}, the pressure is $p_0 = 3500$ bar, and $\kappa = 0.0045$. The theory demonstrates good correspondence: $p_0 = 3200$ bar. The small discrepancy can be connected with a measurement error, with weak leakage in the pumping system, with a difference in the shape of the real nozzle from the ideal one, or with an unaccounted viscosity near the wall.

One more boundary condition appears for a viscous liquid. In addition to the condition of 'impermeability,' which means that the normal velocity projection is zero, the tangential projection must also be zero. To take the viscosity into account, a rather elegant approach can be used. We note that the flow tube described by (27) and, consequently, the shape of the nozzle is a paraboloid of revolution. This allows introducing curvilinear orthogonal coordinates, namely, confocal parabolic coordinates (σ, τ), as is shown in Fig. 1. The new and old cylindrical coordinates are related as [14]

$$\begin{aligned} &r = \sqrt{x^2 + y^2} = \sigma\tau, \quad z = \frac{1}{2}(\tau^2 - \sigma^2), \\ &u = \frac{\sigma u_\tau + \tau u_\sigma}{\sqrt{\sigma^2 + \tau^2}}, \quad w = \frac{\tau u_\tau - \sigma u_\sigma}{\sqrt{\sigma^2 + \tau^2}}. \end{aligned} \tag{58}$$

In the parabolic coordinates, Eqns (1) take the form

$$\begin{aligned} &u_\tau \frac{\partial u_\tau}{\partial \tau} + u_\sigma \frac{\partial u_\tau}{\partial \sigma} + \frac{\sigma u_\tau - \tau u_\sigma}{\sigma^2 + \tau^2} u_\sigma = -\frac{\partial}{\partial \tau} \frac{p}{\rho_0} \\ &\quad + \frac{\nu}{\sigma\tau} \frac{\partial}{\partial \sigma} \left[\frac{\sigma\tau}{\sqrt{\sigma^2 + \tau^2}} \left(\frac{\partial u_\tau}{\partial \sigma} - \frac{\partial u_\sigma}{\partial \tau} + \frac{\sigma u_\tau - \tau u_\sigma}{\sigma^2 + \tau^2} \right) \right], \\ &u_\sigma \frac{\partial u_\sigma}{\partial \sigma} + u_\tau \frac{\partial u_\sigma}{\partial \tau} + \frac{\tau u_\sigma - \sigma u_\tau}{\sigma^2 + \tau^2} u_\tau = -\frac{\partial}{\partial \sigma} \frac{p}{\rho_0} \\ &\quad - \frac{\nu}{\sigma\tau} \frac{\partial}{\partial \tau} \left[\frac{\sigma\tau}{\sqrt{\sigma^2 + \tau^2}} \left(\frac{\partial u_\tau}{\partial \sigma} - \frac{\partial u_\sigma}{\partial \tau} + \frac{\sigma u_\tau - \tau u_\sigma}{\sigma^2 + \tau^2} \right) \right], \\ &\frac{1}{\sigma} \frac{\partial}{\partial \sigma} (\sigma u_\sigma) + \frac{1}{\tau} \frac{\partial}{\partial \tau} (\tau u_\tau) + \frac{\sigma u_\sigma + \tau u_\tau}{\sigma^2 + \tau^2} = 0. \end{aligned} \tag{59}$$

Solution (52) transforms, in accordance with (58), into the simple form

$$u_\sigma = 0, \quad u_\tau = \frac{2D}{\tau\sqrt{\sigma^2 + \tau^2}}. \tag{60}$$

As can be easily verified, this solution satisfies Eqns (59) with zero viscosity abd the boundary condition $u_\sigma = 0$. To generalize result (60) and derive a solution satisfying the second boundary condition, one has to solve Eqns (59) numerically or pass to the boundary layer approximation. In the simplified equations of the Prandtl type, solution (60) is just the limit function to which the viscous solution tends at large distances from the wall.

In concluding this section, we note once again that a singular solution was used here to describe a real flow. Evidently, pieces of other solutions without singularities can be used in describing flows in tubes of varying cross sections.

6. Applications to acoustics

The fact that a Landau jet imparts linear momentum to the medium but does not introduce mass allows describing the acoustically induced streams in terms of Landau-type solutions. Acoustic streams were observed by scientists as far back as Faraday [15]; the streams have now acquired special applied significance, in particular for industrial [16] and medical [17] diagnostics. We have in mind streams caused by the absorption of acoustic waves propagating in a homogeneous medium. In this case, the linear momentum transported by the wave passes into the medium, setting it in motion. For a description of acoustic streams, hydrodynamic equations (1) are used; an additional term is inserted into the right-hand side of the equation of motion to describe the radiation force **F** [18]. This force depends on the time-averaged (over the period) quadratic combinations of variables describing the acoustic field [4].

The self-similar Landau solution corresponds to an acoustic stream if the wave is strongly absorbed at distances much less than the typical length of variation of the stream structure. The constant A in solution (44) is shown in book [4] to be related to the wave intensity I as

$$A\left[1 + \frac{4}{3(A^2 - 1)} - \frac{A}{2} \ln \frac{A+1}{A-1}\right] = \frac{S}{16\pi\nu^2} \frac{I}{c_0 \rho_0}, \tag{61}$$

where S is the cross section area of the acoustic beam and c_0 is the speed of sound.

A second possibility exists when a self-similar substitution transforms the partial differential equations into a system of ordinary ones. This case relates to streams caused by the nonlinear absorption of a spherical sawtooth-shaped wave. Here, the radial component of the radiation force is nonzero, being proportional to R^3 [4]. Using the notation

$$F_R(R, \theta) = \frac{Y}{R^3} \frac{d^2\Psi}{d\zeta^2}, \quad \zeta = \cos\theta, \tag{62}$$

we can perform the same transformations as were performed in Section 4 and derive the Riccati-type equation

$$\begin{aligned} &2\nu\left[(1 - \zeta^2) \frac{dF}{d\zeta} + 2\zeta F\right] + F^2 \\ &\quad = A_1\zeta^2 - 2A_2\zeta - 2A_3 - 2Y\Psi(\zeta). \end{aligned} \tag{63}$$

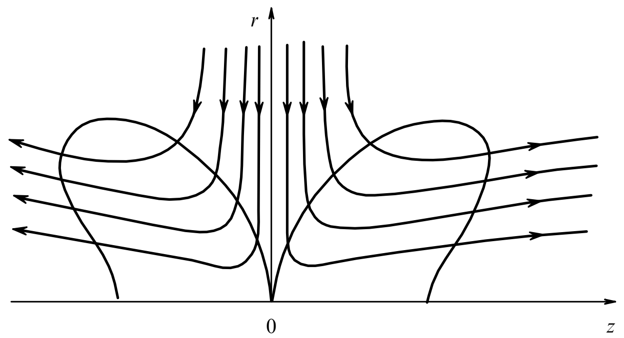

Figure 3. An acoustic stream induced by a radiation force with a directivity pattern in the form of a 'butterfly.'

Similarly, substitution (42) can be used to linearize Eqn (63) and derive an equation generalizing Eqn (43):

$$4\nu^2(1-\zeta^2)^2\,\frac{\mathrm{d}^2 V}{\mathrm{d}\zeta^2} = \big(A_1\zeta^2 - 2A_2\zeta - 2A_3 - 2Y\Psi(\zeta)\big)\,V\,. \tag{64}$$

The constant Y in formulas (62)–(64) depends on the intensity of the sound at the axis of the beam, on the frequency, and on the linear and nonlinear parameters of the medium.

One of the simplest solutions of this problem describes flow lines (41) of the form

$$R(\theta) = \frac{B}{\sin^2\theta\cos\theta}\,. \tag{65}$$

Stream (65) can be realized at the constant value $A_1 = 12\nu^2$ with the following dependence of the radiation force on the polar angle:

$$F_R(R,\theta) = \frac{72\nu^2}{R^3}\cos^2\theta\left(1-\frac{5}{6}\cos^2\theta\right). \tag{66}$$

Flow lines (65) and directivity pattern (66) are shown in Fig. 3.

One more group of 'acoustical' problems is connected with control of jet. Sonic impact on jets in nozzles and diffusers can affect their instability and, consequently, the conditions for their practical application. The use of exact solutions of hydrodynamic equations offers the principal possibility of individual study of other nonlinearities responsible for 'sound–sound' (in moving media) and 'sound–stream' interactions [4].

7. About the history of the problem

The history of the Landau submerged jet problem is not simple. Three nearly independent lines of investigation can be traced. The first group is made up of papers largely belonging to the realm of mathematics and mechanics [10–12]. The second group comprises papers written by theoretical physicists [1–3, 19]. The third group is work published outside Russia [20–23].

In 1934, Slezkin [10] proceeded from a nonlinear fourth-order equation for the flow function written in cylindrical coordinates. The system of hydrodynamic equations for viscous incompressible liquid (1) is known to reduce to such a single equation. In turn, this partial differential equation was reduced to an ordinary differential equation that was successfully integrated three times. As a result, a Riccati-type first-order equation (38) was derived. Thereafter, that equation was linearized and reduced to form (43). The priority work [10] is printed on less than two journal pages and does not contain solutions of the derived equations or, consequently, their physical analysis. In book [11] published in 1955 as a textbook for university students, Slezkin again concentrates mainly on mathematical transformations (11, § 12, pp. 150–154), but simultaneously notes that "the simplest solution of the differential equation... was obtained by Landau and was interpreted as a solution corresponding to submerged jet."

In 1944, Landau published his famous study [1], where he restricted himself to the simplest (from the mathematical standpoint) solution, which nevertheless carried a very important physical meaning. That solution was analyzed in detail. It is reproduced today in many textbooks, including those used in the course of theoretical physics [3]. Of course, Landau was not familiar with the work by Slezkin at that time. The journal *Scientific notes of Moscow University* was difficult to access, even in the 1940s, and it was not among the scientific journals popular in the physical community. Later, reference [10] was given in book [3].

In 1950, Yatseev published paper [12], following Yu B Rumer's suggestion. In that paper, the same equations were derived as in [10]. For a linearized equation like (43), the general solution was written in explicit form in terms of hypergeometric functions. Components of the tensor of the momentum flux density were expressed in terms of the integration constants, and equations for flow lines were derived. The Landau solution was derived, as was one new solution, which, according to the author, "has no physical meaning." There is a footnote: "After the manuscript was submitted for publication, I received a message of L D Landau concerning the work by Slezkin [10] in which its author came to the same equation but in a different way."

In 1952, Rumer himself turned to the problem of a submerged jet [19]. His goal was to eliminate the defect of the Landau solution given by the zero mass of a liquid flowing out of an orifice. For that, an approximate solution of Eqns (30)–(32) was sought in the form of a $1/R$ power series expansion. Terms of the order of $1/R^2$ were actually taken into account, which are corrections to the exact representation (33). After solving the new equations, the Landau solution was modified and the flux of mass was calculated.

In 1951–1952, Squire published papers [20, 21] where the procedure for reducing Eqns (1) to a linear equation was described again. He derived and analyzed the Landau solution and took the temperature distribution in the stream into account; it is important for hot jets formed inside a combustion chamber to flow out. Is seems that the studies by Slezkin, Landau, Yatseev, and Rumer were unknown to him, because corresponding references are absent in papers [20, 21] and in subsequent review [22].

In book [23] published recently, this history is stated as follows: "Yatseev (1950) derived the general solution to the equation derived first by Slezkin (1934). Application to the round jet is described by Landau (1944) and after that in more detail by Squire (1951, 1952, 1955)."

We are thankful to A G Kulikovskii, S V Nesterov, A D Polyanin, V V Pukhnachev, and A V Uvarov for the helpful discussions and suggestions. This work was supported by the Swedish Knowledge Foundation, Water Jet Sweden AB, GE Fanuc Automation CNC Nordic AB and Blekinge

Institute of Technology, and the Russian Foundation for Basic Research.

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