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Force acting on a substance in an electromagnetic field

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Contents

Abstract. **This article, in essence, is a continuation of the work by V L Ginzburg and V A Ugarov (*Usp. Fiz. Nauk* 118 175 (1976) [*Sov. Phys. Usp.* 19 94 (1976)]). It is shown that the results given in § 75 of the book *Electrodynamics of Continuous Media* by L D Landau and E M Lifshitz (Moscow: Nauka, 1982, in Russian) and in § 105 of the book *Fundamentals of the Theory of Electricity* by I E Tamm (Moscow: Nauka, 1989, in Russian) unambiguously follow only from the Maxwell equations of macroscopic electrodynamics, the corresponding constitutive equations, and the equations of motion of a substance (the hydrodynamic equations). These results are as follows: (1) the force acting on a unit volume of a motionless substance is given by the sum of the Helmholtz force and the Abraham force; (2) the momentum density of an electromagnetic field is the Umov–Poynting vector divided by c^2, and (3) the stress tensor related to the field coincides in its form with the sum of the stress tensor of the electrostatic field and the stress tensor of the magnetostatic field. Thus, it is proved that the symmetric form of the Abraham tensor stands for the energy–momentum tensor of an electromagnetic field in a motionless medium.**

1. Introduction

The problem of determining the force acting on a substance in an electromagnetic field, as well as the related problem of determining the energy–momentum tensor of an electromagnetic field in a medium, has been discussed in the literature over the course of the whole century since the first work by H Minkowski (1908) and M Abraham (1909). However, a unique solution to these problems has been absent to date; in particular, there is no unambiguous answer to the question of which form of the energy–momentum tensor — Minkowski or Abraham — is correct.

In the publication [1], a consistent approach based only on macroscopic equations such as Maxwell equations, constitutive equations, and equations of motion of a substance (hydrodynamic equations) has been proposed to determine the force acting on a motionless isotropic medium. To derive an expression for the average force acting on a motionless substance in a high-frequency electromagnetic field, the authors of Ref. [1] considered a steadily moving medium and only in the final expression proceeded to the limit of the medium at rest. The rigorous calculation of Ref. [1] leads to the following result: within the limits of a purely macroscopic approach, a unique choice of the energy–momentum tensor of an electro-magnetic field is not obviously possible (though the authors of Ref. [1] prefer the energy–momentum tensor in the Abraham form).

The aim of this article is to show that the ambiguity noted in Ref. [1] can be removed if the approach [1] is not limited only to the steady motion of a medium. As a result, it turns out that the uniquely correct form of the energy–momentum tensor is the Abraham form.

As in Ref. [1], we consider an isotropic centrally symmetric (nongyrotropic) substance and neglect dispersions of the permittivity (ε) and permeability (μ). Force $\bar{\mathbf{f}}$ acting on a unit volume of a motionless substance [1] can usually be derived (see Refs [2, § 75], [3, § 105]) from an equation equivalent to the momentum conservation law in the 'substance plus field' system:

$$\bar{\mathbf{f}} = \bar{\boldsymbol{\sigma}}' - \frac{\partial \bar{\mathbf{g}}}{\partial t}\,, \quad \bar{\sigma}'_i = \frac{\partial \bar{\sigma}_{ij}}{\partial r_j}\,, \tag{1.1}$$

[1] All quantities related to the motionless medium are designated by an over-line.

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where $\bar{\sigma}_{ij}$ is the stress tensor related to the electromagnetic field [2], and $\bar{\mathbf{g}}$ is the momentum density of the electromagnetic field in the medium. It is accepted that $\bar{\sigma}_{ij}$ coincides with the sum of the stress tensor of the electrostatic field and stress tensor of the magnetostatic field:

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ji} = \frac{1}{4\pi}\left[\varepsilon \bar{E}_i \bar{E}_j - \frac{1}{2}\left(\varepsilon - \rho\,\frac{\partial \varepsilon}{\partial \rho}\right)\bar{E}^2 \delta_{ij}\right] + \frac{1}{4\pi} \times \left[\mu \bar{H}_i \bar{H}_j - \frac{1}{2}\left(\mu - \rho\,\frac{\partial \mu}{\partial \rho}\right)\bar{H}^2 \delta_{ij}\right], \tag{1.2}$$

where ρ is the density of the substance.

From Maxwell equations and constitutive equations

$$\bar{\mathbf{D}} = \varepsilon \bar{\mathbf{E}}\,, \quad \bar{\mathbf{B}} = \mu \bar{\mathbf{H}} \tag{1.3}$$

it follows that

$$\bar{\boldsymbol{\sigma}}' = \bar{\mathbf{f}}^{\mathrm{G}} + \frac{1}{4\pi c}\,\frac{\partial}{\partial t}\,(\bar{\mathbf{D}} \times \bar{\mathbf{B}})\,, \tag{1.4}$$

where [3]

$$\bar{\mathbf{f}}^{\mathrm{G}} = \frac{1}{8\pi}\left[\nabla\rho\left(\frac{\partial \varepsilon}{\partial \rho}\,\bar{E}^2 + \frac{\partial \mu}{\partial \rho}\,\bar{H}^2\right) - (\bar{E}^2 \nabla \varepsilon + \bar{H}^2 \nabla \mu)\right]. \tag{1.5}$$

It is also assumed that the field momentum density is given by [4]

$$\bar{\mathbf{g}} = \mathbf{g}^{\mathrm{A}} = \frac{1}{c^2}\,\bar{\mathbf{S}}^{\mathrm{P}} = \frac{1}{4\pi c}\,\bar{\mathbf{E}} \times \bar{\mathbf{H}}\,, \tag{1.6}$$

where $\bar{\mathbf{S}}^{\mathrm{P}}$ is the field energy flux density in a motionless medium (the Umov–Poynting vector).

From formulas (1.1), (1.4), and (1.6) we obtain an expression for the force acting on a motionless substance in an electromagnetic field:

$$\bar{\mathbf{f}} = \bar{\mathbf{f}}^{\mathrm{G}} + \bar{\mathbf{f}}^{\mathrm{A}}\,, \tag{1.7}$$

where the so-called Abraham force is defined as follows:

$$\bar{\mathbf{f}}^{\mathrm{A}} = \frac{\varepsilon\mu - 1}{c^2}\,\frac{\partial \bar{\mathbf{S}}^{\mathrm{P}}}{\partial t} = \frac{\varepsilon\mu - 1}{4\pi c}\,\frac{\partial}{\partial t}\,(\bar{\mathbf{E}} \times \bar{\mathbf{H}})\,. \tag{1.8}$$

We note again that the above derivation of the expression for force (1.7), (1.5), and (1.8) is essentially based on the choice of expression (1.6) for the field momentum.

2. The Ginzburg–Ugarov approach

To obtain the force acting on a substance in an electromagnetic field, an approach was suggested in Ref. [1], which differs from both the standard approach outlined in the Introduction and the approach in a number of other works where the question of force is related to the question of the choice of the energy–momentum 4-tensor of an electromagnetic field. The approach suggested in Ref. [1] consists in the following:

(1) "The energy–momentum tensor in macroscopic electrodynamics is in a sense an auxiliary quantity. The basic quantities are the volume forces as well as the energy density and the energy flux. They are forces that are included in the equations of motion of a medium and can be, in principle, measured" [1, p. 175].

(2) "The general reasons" regarding the choice of expression (1.6) for the field momentum "are not indisputable," and "it is desirable to obtain forces as well as other expressions (for energy density, energy flux, momentum density) by a single method on the basis of the field equations" [1, p. 176].

(3) Since — without using formula (1.6) — equation (1.1), i.e., the momentum conservation law, is not sufficient to obtain the force $\bar{\mathbf{f}}$, still another equation, equivalent to the energy conservation law of the "substance plus field" system, is necessary.

(4) "In a discussion of the energy conservation law it is natural to address moving media, as the force acting on a medium 'works' only for a nonzero speed of the medium" [1, p. 176]. Hence, even if the final goal is to obtain the force $\bar{\mathbf{f}}$ acting on a motionless substance, it is nevertheless necessary to consider a moving medium. This is a very important point in the whole approach [1].

(5) If a medium moves, the corresponding constitutive equations differ from simple equations (1.3) that are valid for a motionless medium only. With an accuracy up to terms $\sim v/c$, where $\mathbf{v}$ is the speed of the medium, the relation between fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ in the fixed (laboratory) reference frame is given by Minkowski equations (see Refs [2, § 76], [3, § 111], [4, § 33], [1, formulas (10), (11), p. 176]):

$$\mathbf{D} = \varepsilon \mathbf{E} + \frac{\varepsilon\mu - 1}{c}\,\mathbf{v} \times \mathbf{H}\,,$$
$$\mathbf{B} = \mu \mathbf{H} - \frac{\varepsilon\mu - 1}{c}\,\mathbf{v} \times \mathbf{E}\,. \tag{2.1}$$

(6) Equality, analogous to Eqn (1.4) and equivalent to the momentum conservation law, results from Maxwell equations (Eqns (6)–(9) in Ref. [1])

$$\operatorname{div} \mathbf{D} = 4\pi\rho^{\mathrm{ext}}\,, \quad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\,\frac{\partial \mathbf{B}}{\partial t}\,, \quad \operatorname{div} \mathbf{B} = 0\,,$$
$$\operatorname{rot} \mathbf{H} = \frac{1}{c}\,\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j}^{\mathrm{ext}}, \tag{2.2}$$

where ρ^{ext} is the density of external (with respect to the substance considered) charges, and $\mathbf{j}^{\mathrm{ext}}$ is the current density generated by them. For this purpose, ρ^{ext} and $\mathbf{j}^{\mathrm{ext}}$ should be expressed in terms of fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ [by using Eqns (2.2)] in the expression for force with which the field acts on external charges (Eqn (1) in Ref. [1]):

$$\mathbf{f}^{\mathrm{ext}} = \rho^{\mathrm{ext}}\mathbf{E} + \frac{1}{c}\,\mathbf{j}^{\mathrm{ext}} \times \mathbf{B}\,. \tag{2.3}$$

As a result, we arrive at the equation

$$\mathbf{f}^{\mathrm{ext}} + \frac{1}{4\pi c}\,\frac{\partial}{\partial t}\,(\mathbf{D} \times \mathbf{B}) = \boldsymbol{\sigma}'^{\mathrm{H}} - \mathbf{f}^1 - \mathbf{f}^2\,, \quad \sigma_i'^{\mathrm{H}} = \frac{\partial \sigma_{ij}^{\mathrm{H}}}{\partial r_j}\,, \tag{2.4}$$

where

$$\sigma_{ij}^{\mathrm{H}} = \sigma_{ji}^{\mathrm{H}} = \frac{1}{8\pi}\Big\{\big[(E_i D_j + E_j D_i) - \mathbf{E}\,\mathbf{D}\delta_{ij}\big] + \big[(H_i B_j + H_j B_i) - \mathbf{H}\,\mathbf{B}\delta_{ij}\big]\Big\}\,, \tag{2.5}$$

[2] For twice repeating indices $i, j, \ldots = x, y, z$, summation is assumed everywhere.

[3] For the electrostatic field, force (1.5) was first obtained by H Helmholtz (see Ref. [2, § 15]).

[4] Expression (1.6) for the field momentum density was introduced by Abraham (see Refs [1] and [4, § 35]).

$$\mathbf{f}^1 = -\frac{1}{8\pi}\,\mathrm{rot}(\mathbf{E}\times\mathbf{D}+\mathbf{H}\times\mathbf{B})\,, \tag{2.6}$$

$$f_i^2 = -\frac{1}{8\pi}\left[\left(\mathbf{E}\,\frac{\partial\mathbf{D}}{\partial r_i}-\mathbf{D}\,\frac{\partial\mathbf{E}}{\partial r_i}\right)+\left(\mathbf{H}\,\frac{\partial\mathbf{B}}{\partial r_i}-\mathbf{B}\,\frac{\partial\mathbf{H}}{\partial r_i}\right)\right]. \tag{2.7}$$

By using the symmetric tensor σ_{ij}^{H}, first introduced by H Hertz (see Ref. [4, § 35]), we have written equality (2.4) in a form somewhat different from the form of formula (20a) in Ref. [1]. It is possible to demonstrate that if we use the constitutive equations (1.3) and suppose $\mathbf{f}^{\mathrm{ext}} = 0$ in Eqns (2.5)–(2.7), then equality (2.4) converts to the form (1.4).

(7) An equation equivalent to the energy conservation law can be obtained if in the expression for work $\mathbf{j}^{\mathrm{ext}}\,\mathbf{E}$ executed by the field on external charges, the current $\mathbf{j}^{\mathrm{ext}}$ is expressed in terms of fields with the help of equations (2.2). Then we have (see formula (12) in Ref. [1])

$$\mathbf{j}^{\mathrm{ext}}\mathbf{E} = -\frac{1}{4\pi}\left(\mathbf{E}\,\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\,\frac{\partial\mathbf{B}}{\partial t}\right)-\mathrm{div}\,\mathbf{S}^{\mathrm{P}}\,,$$

$$\mathbf{S}^{\mathrm{P}} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}\,. \tag{2.8}$$

The approach suggested in Ref. [1] is based only on Eqns (2.1) and Eqns (2.4)–(2.8). However, this approach is realized with some restrictions on ε, μ, and velocity $\mathbf{v}$ [1]. In Ref. [1, p. 177], “it is supposed that a change of ε (as well as μ) for a given element of the medium is only connected with the change of the medium density”, namely

$$\varepsilon = \varepsilon(\rho)\,,\quad \mu = \mu(\rho)\,; \tag{2.9}$$

and “the medium velocity is constant in space and in time or, more precisely, the derivatives with respect to $\mathbf{r}$ and t can be neglected everywhere (only div $\mathbf{v}$ is kept),” i.e.

$$\frac{\partial v_i}{\partial r_j} = \frac{1}{3}\,\mathrm{div}\,\mathbf{v}\,\delta_{ij}\,, \tag{2.10}$$

$$\frac{\partial\mathbf{v}}{\partial t} = 0\,. \tag{2.11}$$

Equations (2.1) and (2.4)–(2.8) were investigated in Ref. [1] under conditions (2.9)–(2.11). According to Ref. [1], although “theoretically, the existence of the Abraham force (or, more precisely, a force of this kind) cannot be doubted” [1, p. 181], “conservation laws are unable to determine unequivocally the quantities entering into them” [1, p. 178]), and, more particularly, there are two possibilities: either force $\mathbf{f}$ is determined by formula (1.7) and thus the field momentum is defined by expression (1.6), or the force is given by

$$\mathbf{f} = \mathbf{f}^{\mathrm{G}}, \tag{2.12}$$

and thus the field momentum is [5]

$$\mathbf{g} = \mathbf{g}^{\mathrm{M}} = \frac{1}{4\pi c}\,\mathbf{D}\times\mathbf{B}\,; \tag{2.13}$$

the choice of expression for the field momentum or the force “should be made on the basis of experimental data or some calculations beyond the scope of macroscopic field equations” [1, p. 185].

We are of the opinion that the approach suggested and used in Ref. [1] not only is a ‘more consistent’ one [1, p. 178], but also is the only consistent approach in macroscopic electrodynamics. Therefore, it would still be possible to ‘conform’ with the impossibility — within the limits of this approach — of providing an unambiguous answer to the question of the force acting on a substance, if the force were not expressed only in terms of fields as well as permittivity and permeability (as in absorbing media (see Ref. [2, § 81]). But according to Ref. [1], the force and the field momentum in both possible cases are expressed only in terms of fields and also permittivity ε and permeability μ. This fact, as well as the doubtless importance of this problem, urged us to return to its discussion within the framework of approach [1], i.e., proceeding from equations (2.1) and Eqns (2.4)–(2.8). We repeat the calculations done in Ref. [1] in all the basic points, with one exception only: we do not impose any restrictions on ε, μ, and the medium speed $\mathbf{v}$ (assuming only that $v \ll c$) until the course of calculations forces us to introduce some restrictions (like (2.9)–(2.11) for others) for avoiding internal contradictions in the theory. Thus, we will find out how much the results [1] are related to conditions (2.9)–(2.11) imposed from the very beginning and, if these conditions are not necessary for the approach [1], how the results change when the conditions are waived.

The permittivity and permeability are functions of the density and entropy (or temperature), and also a function of the mixture concentration if the medium is nonuniform in composition, in the reference frame where ε and μ are defined, i.e., in the reference frame with respect to which the substance is at rest. With an accuracy up to terms $\sim v/c$, the density, entropy, temperature, and mixture concentration [6] do not change under the Lorentz transform (see Ref. [4, § 46]). Therefore, we can assume that

$$\varepsilon = \varepsilon(\rho, s, \gamma)\,,\quad \mu = \mu(\rho, s, \gamma)\,, \tag{2.14}$$

where ρ, s, and γ — the density, entropy, and mixture concentration, respectively — are the functions of coordinates $\mathbf{r}$ and time t in the laboratory reference frame.

Minkowski constitutive equations (2.1) are the direct consequence of the Lorentz transform and therefore are valid, strictly speaking, only when the co-moving reference frame (where the substance is at rest) is inertial, i.e., the substance as a whole moves relative to the laboratory (also inertial) reference frame with a constant velocity $\mathbf{v}(\mathbf{r}, t) = \mathrm{const}$. If the velocity $\mathbf{v} \neq \mathrm{const}$, then the co-moving reference frame is not inertial and the Maxwell equations themselves, a definition of fields in terms of the force acting on a point (probe) charge, and the constitutive equations, generally speaking, significantly differ from the corresponding equations in an inertial reference frame. [7] However, we assume following Refs [1] and [2, § 76] that Minkowski constitutive equations (2.1) valid remain also when the velocity $\mathbf{v}$ is not constant, but is a sufficiently slowly changing function. [8] [It is perhaps necessary to recall once again that our ultimate goal (as in Ref. [1]) is to obtain expressions for the force and the field momentum in a motionless medium.]

[5] This expression for the field momentum density was introduced by Minkowski (see Refs [1], [4, § 35]).

[6] For simplicity we assume that the mixture is two-component.

[7] Regarding microscopic Maxwell equations, see Ref. [5, § 90].

[8] Note that Minkowski equations are used in Ref. [2, § 76] to solve problems related to an electric field induced around a uniformly rotating sphere.

3. Energy conservation law

We start by considering equation (2.8) (Eqn (12) in Ref. [1]). First of all, we should strictly define what we understand as an electromagnetic field force $\mathbf{f}$ acting on a unit volume of a substance. This can be done only by having written the equation of motion of a substance in the presence of a field. For simplicity, we assume that the substance is an ideal liquid. Then, the equation of motion is given by the Euler equation (see Ref. [6, § 2]) with the added force $\mathbf{f}$:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} = \frac{1}{\rho}(-\nabla p + \mathbf{f})\,, \tag{3.1}$$

where $p = p(\rho, s, \gamma)$ is the pressure which would be in the medium in the absence of the field for given (in the presence of the field) values of the density ρ, entropy s, and concentration of the mixture γ (see Ref. [2, § 15]). After such refinement of the sense of pressure p, equation (3.1) fully defines the force $\mathbf{f}$.

Let us find how the energy of a unit volume of the substance changes in time:

$$w^{\mathrm{m}} = \rho\left(\frac{v^2}{2} + \varepsilon^{\mathrm{m}}\right), \tag{3.2}$$

where $\varepsilon^{\mathrm{m}} = \varepsilon^{\mathrm{m}}(\rho, s, \gamma)$ is the internal energy of a unit mass of the substance in the absence of a field. We assume that there are neither thermal nor diffusion fluxes. Then, together with the continuity equation (see Ref. [6, § 1])

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\mathbf{v} = 0\,, \tag{3.3}$$

the following equations are valid (see Ref. [6, §§ 49, 58]):

$$\frac{\partial s}{\partial t} + \mathbf{v}\nabla s = 0\,, \tag{3.4}$$

$$\frac{\partial \gamma}{\partial t} + \mathbf{v}\nabla\gamma = 0\,. \tag{3.5}$$

Repeating the calculations done in Ref. [6, § 6] and also using Eqns (3.1) and (3.3)–(3.5), we obtain

$$\frac{\partial w^{\mathrm{m}}}{\partial t} = -\operatorname{div}\mathbf{S}^{\mathrm{m}} + \mathbf{f}\mathbf{v}\,, \tag{3.6}$$

where the energy flux density of the substance is given by

$$\mathbf{S}^{\mathrm{m}} = \rho\mathbf{v}\left(\frac{v^2}{2} + \varepsilon^{\mathrm{m}} + \frac{p}{\rho}\right). \tag{3.7}$$

The law of conservation of the full energy of the (substance and field) system, taking into account the work of the field on external charges, corresponds to the equation

$$\frac{\partial w^{\mathrm{tot}}}{\partial t} = -\operatorname{div}\mathbf{S}^{\mathrm{tot}} - \mathbf{j}^{\mathrm{ext}}\mathbf{E}\,, \quad w^{\mathrm{tot}} = w^{\mathrm{m}} + w\,,$$
$$\mathbf{S}^{\mathrm{tot}} = \mathbf{S}^{\mathrm{m}} + \mathbf{S}\,, \tag{3.8}$$

where by definition w is the energy density, and $\mathbf{S}$ is the energy flux density of the electromagnetic field. From equations (3.6) and (3.8) we obtain

$$\mathbf{f}\mathbf{v} = -\frac{\partial w}{\partial t} - \operatorname{div}\mathbf{S} - \mathbf{j}^{\mathrm{ext}}\mathbf{E}\,. \tag{3.9}$$

Thus, if the force with which the field acts on the substance is defined according to equation of motion (3.1), then equation (2.8) should be reduced to the form (3.9). We substitute $\mathbf{D}$ and $\mathbf{B}$ from formula (2.1) into (2.8). Derivatives $\partial\varepsilon/\partial t$ and $\partial\mu/\partial t$ we calculate according to formulas (2.14) and (3.3)–(3.5):

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial \varepsilon}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial \varepsilon}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \varepsilon}{\partial \gamma}\frac{\partial \gamma}{\partial t} = -\left(\rho\,\frac{\partial \varepsilon}{\partial \rho}\operatorname{div}\mathbf{v} + \mathbf{v}\nabla\varepsilon\right),$$

$$\frac{\partial \mu}{\partial t} = -\left(\rho\,\frac{\partial \mu}{\partial \rho}\operatorname{div}\mathbf{v} + \mathbf{v}\nabla\mu\right). \tag{3.10}$$

Using Eqn (3.10), direct calculations result in Eqn (2.8) in the following form

$$(\mathbf{f}^{\mathrm{G}} + \mathbf{f}^{\mathrm{A}})\,\mathbf{v} = -\frac{\partial}{\partial t}\left[\frac{1}{8\pi}(\varepsilon E^2 + \mu H^2) - \frac{2}{c^2}(\varepsilon\mu - 1)\,\mathbf{v}\mathbf{S}^{\mathrm{P}}\right]$$
$$- \operatorname{div}\left[\mathbf{S}^{\mathrm{P}} - \frac{1}{8\pi}\rho\mathbf{v}\left(\frac{\partial \varepsilon}{\partial \rho}E^2 + \frac{\partial \mu}{\partial \rho}H^2\right)\right] - \mathbf{j}^{\mathrm{ext}}\mathbf{E}\,, \tag{3.11}$$

and the Helmholtz force $\mathbf{f}^{\mathrm{G}}$ and the Abraham force $\mathbf{f}^{\mathrm{A}}$ are defined by formulas (1.5) and (1.8), where the replacements $\bar{\mathbf{E}} \to \mathbf{E}$ and $\bar{\mathbf{H}} \to \mathbf{H}$ should be done. Equation (3.11) has the form required by formula (3.9); the comparison gives

$$\mathbf{f} = \mathbf{f}^{\mathrm{G}} + \mathbf{f}^{\mathrm{A}} + \mathbf{f}^{\perp}\,, \tag{3.12}$$

where $\mathbf{f}^{\perp}$ is the yet unknown force which is perpendicular to the velocity $\mathbf{v}$ and consequently does not contribute to the work $\mathbf{f}\mathbf{v}$; the field energy density is defined as

$$w = \frac{1}{8\pi}(\varepsilon E^2 + \mu H^2) - \frac{2}{c^2}(\varepsilon\mu - 1)\,\mathbf{v}\mathbf{S}^{\mathrm{P}} \tag{3.13}$$

(see formula (19) in Ref. [1]), and

$$\mathbf{S} = \mathbf{S}^{\mathrm{P}} - \frac{1}{8\pi}\rho\mathbf{v}\left(\frac{\partial \varepsilon}{\partial \rho}E^2 + \frac{\partial \mu}{\partial \rho}H^2\right) \tag{3.14}$$

is the field energy flux density (formula (17) in Ref. [1]). We note that for a motionless medium, force (3.12) is reduced to force $\bar{\mathbf{f}}$ (1.7), field energy density (3.13) to the energy density [9] $\bar{w} = 1/8\pi\,(\varepsilon\bar{E}^2 + \mu\bar{H}^2)$, and energy flux density (3.14) to the Umov–Poynting vector $\bar{\mathbf{S}} = \bar{\mathbf{S}}^{\mathrm{P}}$.

4. Momentum and angular momentum conservation laws

Now, consider equation (2.4) (Eqn (20a) in Ref. [1]). Let us find out how the momentum

$$\mathbf{P}^{\mathrm{m}} = \rho\mathbf{v} \tag{4.1}$$

of a unit volume of the substance changes in time. By repeating the calculations performed in Ref. [6, § 7], as well as using Eqns (3.1) and (3.3), we arrive at

$$\frac{\partial P_i^{\mathrm{m}}}{\partial t} = -\frac{\partial \Pi_{ij}^{\mathrm{m}}}{\partial r_j} + f_i\,, \tag{4.2}$$

[9] Energy $\bar{w}$ has an exact thermodynamic meaning (see Ref. [2, § 80]): it is the difference between the values of internal energy in a unit volume of a substance in the presence of a field and in the absence of a field, respectively, for the same ρ, s, and γ.

where the momentum flux density of the substance is given by

$$\Pi_{ij}^{\mathrm{m}} = \Pi_{ji}^{\mathrm{m}} = \rho v_i v_j - \sigma_{ij}^{\mathrm{m}}\,, \text{ and} \tag{4.3}$$

$$\sigma_{ij}^{\mathrm{m}} = \sigma_{ji}^{\mathrm{m}} = -p\delta_{ij} \tag{4.4}$$

is the stress tensor in the absence of the field.

For a system composed of a substance and a field, the full momentum conservation law, taking into account the force with which the field acts on the external charges, corresponds to the equation

$$\frac{\partial P_i^{\mathrm{tot}}}{\partial t} = -\frac{\partial \Pi_{ij}^{\mathrm{tot}}}{\partial r_j} - f_i^{\mathrm{ext}}\,, \quad \text{with} \quad \mathbf{P}^{\mathrm{tot}} = \mathbf{P}^{\mathrm{m}} + \mathbf{g}\,,$$

$$\Pi_{ij}^{\mathrm{tot}} = \rho v_i v_j - \sigma_{ij}^{\mathrm{tot}}\,, \quad \sigma_{ij}^{\mathrm{tot}} = \sigma_{ij}^{\mathrm{m}} + \sigma_{ij}\,, \tag{4.5}$$

where, by definition, $\mathbf{g}$ is the field momentum density, and σ_{ij} is the stress tensor related to the field. From formulas (4.5) and (4.2) we obtain

$$\mathbf{f} = -\frac{\partial \mathbf{g}}{\partial t} + \boldsymbol{\sigma}' - \mathbf{f}^{\mathrm{ext}}, \quad \sigma_i' = \frac{\partial \sigma_{ij}}{\partial r_j}\,. \tag{4.6}$$

It is important to keep in mind that the full stress tensor should certainly be symmetric (see Ref. [2, § 15]): $\sigma_{ij}^{\mathrm{tot}} = \sigma_{ji}^{\mathrm{tot}}$. Thus, taking into account formula (4.4), it follows that

$$\sigma_{ij} = \sigma_{ji} \tag{4.7}$$

as well.

The symmetry of the full stress tensor $\sigma_{ij}^{\mathrm{tot}}$ is connected with the law of conservation of the full angular momentum of the (substance and field) system. The angular momentum of the substance and the field in some fixed volume V equals

$$\mathbf{J}^{\mathrm{tot}} = \int_V \mathbf{r} \times \mathbf{P}^{\mathrm{tot}}\,\mathrm{d}V\,. \tag{4.8}$$

Using formulas (4.5), we obtain

$$\frac{\mathrm{d}J_i^{\mathrm{tot}}}{\mathrm{d}t} = e_{ijk} \oint_S r_j (\sigma_{kl}^{\mathrm{tot}} - \rho v_k v_l)\,\mathrm{d}S_l - e_{ijk} \int_V \sigma_{kj}^{\mathrm{tot}}\,\mathrm{d}V - M_i^{\mathrm{ext}}\,, \tag{4.9}$$

where e_{ijk} is the unit, fully antisymmetric pseudotensor of the third rank [5], S is the surface bounding the volume V, and

$$\mathbf{M}^{\mathrm{ext}} = \int_V \mathbf{r} \times \mathbf{f}^{\mathrm{ext}}\,\mathrm{d}V \tag{4.10}$$

is the moment of forces acting from the field side on external charges in volume V. For the moment of forces acting on the substance and the field in volume V to be reduced to a surface integral, the volume integral in formula (4.9) should vanish, i.e., tensor $\sigma_{ij}^{\mathrm{tot}}$ should be symmetric. If the volume V includes the whole substance and the field is absent on the border S, the surface integral in formula (4.9) vanishes. Therefore, if $\sigma_{ij}^{\mathrm{tot}} = \sigma_{ji}^{\mathrm{tot}}$, then, as it should, we arrive at the following equation

$$\frac{\mathrm{d}\mathbf{J}^{\mathrm{tot}}}{\mathrm{d}t} = -\mathbf{M}^{\mathrm{ext}}\,. \tag{4.11}$$

Equation (2.4) should be reduced to the form (4.6) with symmetric tensor σ_{ij} [see expression (4.7)] and with force $\mathbf{f}$ (3.12). To obtain that, it is necessary to properly transform $\mathbf{f}^1$ (2.6) and $\mathbf{f}^2$ (2.7). Using equations (2.1), we find

$$\mathbf{f}^1 = -\frac{1}{2c^2}\,\mathrm{rot}(\varepsilon\mu - 1)\,\mathbf{v} \times \mathbf{S}^{\mathrm{P}} = -\boldsymbol{\sigma}'^1 + \frac{\varepsilon\mu - 1}{c^2}\,(\mathbf{v}\nabla)\,\mathbf{S}^{\mathrm{P}} + \frac{1}{c^2}\,\mathbf{S}^{\mathrm{P}}\,\mathrm{div}(\varepsilon\mu - 1)\,\mathbf{v}\,, \tag{4.12}$$

where

$$\sigma_i'^1 = \frac{\partial \sigma_{ij}^1}{\partial r_j}\,, \quad \sigma_{ij}^1 = \sigma_{ji}^1 = \frac{\varepsilon\mu - 1}{2c^2}\,(v_i S_j^{\mathrm{P}} + v_j S_i^{\mathrm{P}})\,, \tag{4.13}$$

$$f_i^2 = f_i^{\mathrm{G}} - \sigma_i'^2 - \frac{\varepsilon\mu - 1}{c^2}\,\mathbf{v}\,\frac{\partial \mathbf{S}^{\mathrm{P}}}{\partial r_i}\,, \tag{4.14}$$

$$\sigma_i'^2 = \frac{\partial \sigma_{ij}^2}{\partial r_j}\,,$$
$$\sigma_{ij}^2 = \delta_{ij}\left[\frac{1}{8\pi}\,\rho\left(\frac{\partial\varepsilon}{\partial\rho}\,E^2 + \frac{\partial\mu}{\partial\rho}\,H^2\right) - \frac{1}{c^2}\,(\varepsilon\mu - 1)\,\mathbf{v}\mathbf{S}^{\mathrm{P}}\right]. \tag{4.15}$$

Substituting $\mathbf{f}^1$ from Eqn (4.12) and $\mathbf{f}^2$ from Eqn (4.14) into equation (2.4), we reduce it to the form [10]

$$\mathbf{f}^{\mathrm{G}} + \mathbf{f}^3 = -\frac{1}{4\pi c}\,\frac{\partial}{\partial t}\,(\mathbf{D} \times \mathbf{B}) + \boldsymbol{\sigma}'^{\mathrm{H}} + \boldsymbol{\sigma}'^1 + \boldsymbol{\sigma}'^2 - \mathbf{f}^{\mathrm{ext}}, \tag{4.16}$$

where

$$\mathbf{f}^3 = -\frac{1}{c^2}\left[(\varepsilon\mu - 1)\,\mathbf{v} \times \mathrm{rot}\,\mathbf{S}^{\mathrm{P}} + \mathbf{S}^{\mathrm{P}}\,\mathrm{div}(1 - \varepsilon\mu)\,\mathbf{v}\right]. \tag{4.17}$$

Force $\mathbf{f}^3$ (4.17) can be represented in another form; taking into account relationship (3.10), we have

$$\frac{\partial\varepsilon\mu}{\partial t} = -\left[\mathrm{div}(\varepsilon\mu - 1)\,\mathbf{v} + \left(\rho\,\frac{\partial\varepsilon\mu}{\partial\rho} + 1 - \varepsilon\mu\right)\mathrm{div}\,\mathbf{v}\right], \tag{4.18}$$

so that

$$\mathbf{f}^3 = -\frac{1}{c^2}\,\frac{\partial\varepsilon\mu}{\partial t}\,\mathbf{S}^{\mathrm{P}} + \mathbf{f}^{\perp} + \mathbf{f}^{\parallel}\,, \tag{4.19}$$

where the force perpendicular to the velocity is given by

$$\mathbf{f}^{\perp} = -\frac{\varepsilon\mu - 1}{c^2}\,\mathbf{v} \times \mathrm{rot}\,\mathbf{S}^{\mathrm{P}} + \frac{1}{c^2}\left(\varepsilon\mu - 1 - \rho\,\frac{\partial\varepsilon\mu}{\partial\rho}\right)\left[\mathbf{S}^{\mathrm{P}} - \mathbf{v}\,\frac{\mathbf{v}\mathbf{S}^{\mathrm{P}}}{v^2}\right]\mathrm{div}\,\mathbf{v}\,, \tag{4.20}$$

and the force parallel to the velocity is defined as

$$\mathbf{f}^{\parallel} = \frac{1}{c^2}\,\mathbf{v}\left(\varepsilon\mu - 1 - \rho\,\frac{\partial\varepsilon\mu}{\partial\rho}\right)\frac{(\mathbf{S}^{\mathrm{P}}\mathbf{v})}{v^2}\,\mathrm{div}\,\mathbf{v}\,. \tag{4.21}$$

The left-hand side (LHS) of Eqn (4.16) still essentially differs from the force $\mathbf{f}$, for which we have obtained (from the energy conservation law) the above expression (3.12) with the Abraham force. Hence, there are two possibilities for further

[10] The corresponding equation (24) is given in Ref. [1] for an already motionless medium.

calculations [1]: either leaving Eqn (3.11) unchanged, to transform equation (4.16) with $\mathbf{f}^3$ (4.19)–(4.21), so that the Abraham force $\mathbf{f}^{\mathrm{A}}$ appears on its LHS or, leaving Eqn (4.16) unchanged, to transform equation (3.11) in order to exclude the Abraham force $\mathbf{f}^{\mathrm{A}}$ from it. In this section, we proceed with the first possibility.

We add to both sides of equation (4.16) the Abraham force ([see formula (1.8)]:

$$\mathbf{f}^{\mathrm{A}} = \frac{\varepsilon\mu - 1}{c^2} \frac{\partial \mathbf{S}^{\mathrm{P}}}{\partial t} = \frac{1}{c^2} \left[\frac{\partial}{\partial t} (\varepsilon\mu - 1) \mathbf{S}^{\mathrm{P}} - \frac{\partial \varepsilon\mu}{\partial t} \mathbf{S}^{\mathrm{P}} \right]. \quad (4.22)$$

Taking into account expressions (4.19)–(4.21), we finally obtain

$$\mathbf{f} + \mathbf{f}^{\parallel} = -\frac{1}{4\pi c} \frac{\partial}{\partial t} \left[\mathbf{D} \times \mathbf{B} - (\varepsilon\mu - 1) \mathbf{E} \times \mathbf{H} \right] + \boldsymbol{\sigma}'^{\mathrm{H}} + \boldsymbol{\sigma}'^{1} + \boldsymbol{\sigma}'^{2} - \mathbf{f}^{\mathrm{ext}}, \quad (4.23)$$

where $\mathbf{f}$ is the force (3.12) obtained from the energy conservation law; its component $\mathbf{f}^{\perp}$ is determined by formula (4.20). Comparing the right-hand sides of Eqns (4.6) and (4.23), we find that the electromagnetic field momentum density is given by

$$\mathbf{g} = \frac{1}{4\pi c} \left[\mathbf{D} \times \mathbf{B} - (\varepsilon\mu - 1) \mathbf{E} \times \mathbf{H} \right], \quad (4.24)$$

and the stress tensor [see formulas (2.5), (4.13), and (4.15)] is equal to

$$\sigma_{ij} = \sigma_{ji} = \sigma_{ij}^{\mathrm{H}} + \sigma_{ij}^{1} + \sigma_{ij}^{2}. \quad (4.25)$$

The LHS of Eqn (4.23) contains an additional term $\mathbf{f}^{\parallel}$, in comparison with force (3.12), which hence should be negligible as compared to $\mathbf{f}$:

$$|\mathbf{f}^{\parallel}| \ll |\mathbf{f}^{\mathrm{G}}|, \quad |\mathbf{f}^{\mathrm{A}}|. \quad (4.26)$$

Inequality (4.26) is the only condition for which all the quantities — the field energy density w (3.13), the field energy flux density $\mathbf{S}$ (3.14), the field momentum flux density $\mathbf{g}$ (4.24), the density of the force $\mathbf{f}$ (3.12) acting on the substance in the field, and the symmetric stress tensor σ_{ij} (4.25) — are consistently defined (with an accuracy of $\sim v/c$) in macroscopic electrodynamics. It is necessary to show only that these definitions are unique. As for condition (4.26), we have $\mathbf{f}^{\parallel} = 0$ (except in the obvious case when $\mathbf{v} \perp \mathbf{S}^{\mathrm{P}}$), when the medium can be considered as incompressible ($\operatorname{div} \mathbf{v} = 0$ [2, § 76]) and when the medium is a sufficiently rarefied gas (then $\mu = 1$, $\rho \partial \varepsilon / \partial \rho = \varepsilon - 1$ [2, § 15]).

5. Uniqueness of expressions for the force and the field momentum

We now make use of the second possibility: without changing equation (4.16) with $\mathbf{f}^3$ from formula (4.19), we transform equation (3.11) by subtracting from both its sides the work of the Abraham force (4.22):

$$\mathbf{f}^{\mathrm{A}} \mathbf{v} = \frac{1}{c^2} \left[\frac{\partial}{\partial t} (\varepsilon\mu - 1) \mathbf{v} \mathbf{S}^{\mathrm{P}} - \frac{\partial \varepsilon\mu}{\partial t} \mathbf{v} \mathbf{S}^{\mathrm{P}} - (\varepsilon\mu - 1) \mathbf{S}^{\mathrm{P}} \frac{\partial \mathbf{v}}{\partial t} \right]. \quad (5.1)$$

As a result, instead of Eqn (3.11) we obtain the following equation

$$\mathbf{f}^{\mathrm{M}} \mathbf{v} - \frac{\varepsilon\mu - 1}{c^2} \mathbf{S}^{\mathrm{P}} \frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial w^{\mathrm{M}}}{\partial t} - \operatorname{div} \mathbf{S} - \mathbf{j}^{\mathrm{ext}} \mathbf{E}, \quad (5.2)$$

where

$$\mathbf{f}^{\mathrm{M}} = \mathbf{f}^{\mathrm{G}} + \mathbf{f}^{\perp} - \frac{1}{c^2} \mathbf{S}^{\mathrm{P}} \frac{\partial \varepsilon\mu}{\partial t}, \quad (5.3)$$

$$w^{\mathrm{M}} = w + \frac{\varepsilon\mu - 1}{c^2} \mathbf{v} \mathbf{S}^{\mathrm{P}} = \frac{1}{8\pi} (\mathbf{E} \mathbf{D} + \mathbf{H} \mathbf{B}), \quad (5.4)$$

with w and $\mathbf{S}$ being defined by formulas (3.13) and (3.14). Thus, equation (4.16), taking into account expressions (4.19) and (5.3), has the following form

$$\mathbf{f}^{\mathrm{M}} + \mathbf{f}^{\parallel} = -\frac{1}{4\pi c} \frac{\partial (\mathbf{D} \times \mathbf{B})}{\partial t} + \boldsymbol{\sigma}' - \mathbf{f}^{\mathrm{ext}}, \quad (5.5)$$

where $\sigma_i' = \partial \sigma_{ij} / \partial r_j$, while the stress tensor σ_{ij} and the force $\mathbf{f}^{\parallel}$ parallel to the velocity are still determined by Eqns (4.25) and (4.21), respectively. According to formula (3.9), equation (5.2) has the necessary form only if the second term on its LHS can be neglected:

$$\left| \frac{\varepsilon\mu - 1}{c^2} \mathbf{S}^{\mathrm{P}} \frac{\partial \mathbf{v}}{\partial t} \right| \ll |\mathbf{f}^{\mathrm{M}} \mathbf{v}|. \quad (5.6)$$

Condition (5.6) is clearly equivalent to condition (2.11), except for the obvious case when $\mathbf{S}^{\mathrm{P}} \perp \partial \mathbf{v} / \partial t$. If, together with condition (4.26), condition (5.6) also takes place, then, in agreement with Ref. [1], w^{M} (5.4) (along with w (3.13)) can be considered as the field energy density; $\mathbf{g}^{\mathrm{M}}$ [along with $\mathbf{g}$ (4.24)] can be considered as the momentum density: $\mathbf{g}^{\mathrm{M}} = \mathbf{D} \times \mathbf{B} / 4\pi c$, and $\mathbf{f}^{\mathrm{M}}$ [along with $\mathbf{f}$ (3.12)] can be considered as the force: $\mathbf{f}^{\mathrm{M}} = \mathbf{f}^{\mathrm{G}}$. If not limited by only stationary motion of the medium [retracting condition (2.11) or (5.6)], then, with only one condition (4.26), all the quantities are uniquely and consistently defined in accordance with the results of Section 4.

6. Conclusion

According to Ref. [1], the force acting on a substance in an electromagnetic field, as well as other quantities proportional to the field squared (energy, energy flux, field momentum), should be obtained uniformly on the basis of Maxwell equations, the corresponding constitutive equations, and the equations of motion of the medium. This approach has been realized in Ref. [1] for an isotropic centrally symmetric (nongyrotropic) medium by neglecting dispersion of the permittivity ε and permeability μ and, in addition to that, by imposing further restrictions on ε, μ, and the medium's velocity $\mathbf{v}(\mathbf{r}, t)$ [see formulas (2.9)–(2.11)]. Authors of Ref. [1] have shown that within the limits of their approach (in our opinion, it is the only consistent approach in macroscopic electrodynamics), when conditions (2.9)–(2.11) are satisfied, it is obviously not possible to unambiguously define the force acting on the substance and the field momentum: in the limit when the medium's motion can be neglected, either expression for the force includes the Abraham force (1.7) and then the field momentum is the Umov–Poynting vector (1.6) divided by c^2, or the expression for the force does not include the Abraham force [see relationship (2.12)] and then the field momentum is $\mathbf{g}^{\mathrm{M}}$ [see formula (2.13)].

It would be interesting to find out to what extent conditions (2.9)–(2.11) are necessary for approach [1] itself, and if they are not necessary and it is possible to retract them, how far the result concerning the ambiguity of the force and the field momentum definitions is related to these conditions. In our article, it is shown that not one of conditions (2.9)–(2.11) appears necessary. The only condition which is necessary to satisfy in order to avoid internal contradiction in approach [1] is condition (4.26) on $\operatorname{div} \mathbf{v}$ [see formula (4.21)], which is not reduced to any one of conditions (2.9)–(2.11).

Conditions (2.9) and (2.10) are not essential, i.e., retracting them does not change result [1] on the impossibility of an unambiguous definition of the force and the field momentum. Condition (2.11) is essential in this sense: if condition (2.11) is accepted, i.e., we are limited only by stationary motions of the substance, the choice of two possible expressions for the field momentum — (1.6) or (2.13) — and, accordingly, for the force — (1.7) or (2.12) — becomes really impossible, in agreement with Ref. [1]. If condition (2.11) is retracted, the second possibility, namely, relationships (2.12) and (2.13) (the Abraham force being absent), is excluded, as it leads to internal contradictions in approach [1]. *Thus, if remaining within the scope of the macroscopic theory we would like to obtain the force acting in an electromagnetic field on a substance at rest, we should nevertheless consider a moving* (*and moving nonstationary*) *medium* [1]. *In this case, formulas* (1.6) *and* (1.7) *take place for the field momentum and the force; these formulas coincide with the corresponding formulas given in Refs* [2, § 75] *and* [3, § 105]. As for the energy–momentum 4-tensor of an electromagnetic field in a motionless medium, the Abraham symmetric form is valid for it (see Eqns (32) and (33) in Refs [1] and [4, § 35]).

After the present article was submitted (October, 2008), a paper [7] by V G Veselago was submitted to (December 2008) and published in *Physics-Uspekhi* where, in particular, the author also considers a problem discussed in Refs [1–4, 8, 9], as well as in the present article. The solution to this problem in Ref. [7] differs from all known solutions; in the abstract of article [7] it is formulated that: "it is pointed out that the energy–momentum tensor in the Abraham form as a matter of fact is not a tensor as it is not relativistic-invariant," while in the text of the paper it is written only that this statement "is easily proved by direct calculation." We do not agree with this statement and we hope to discuss this problem in a future paper.

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