

METHODOLOGICAL NOTES

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Hydrodynamic and hydromagnetic stability of the Couette flow

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Abstract. The stability of the dissipative Couette flow is examined in the linear approximation. The onset conditions and flow instability properties are studied, with special emphasis placed on the instability properties in the presence of a magnetic field and density stratification. Theoretical and experimental results on instability parameters are found to be in good agreement (within a few percent).

1. Introduction

The stability of a flow in an annulus between two rotating coaxial cylinders (the Taylor–Couette flow) is a classic problem of hydrodynamic and magnetohydrodynamic stability. Studies of the laminar flow, currently referred to as the Couette flow, trace back to the 19th-century experiments designed to measure fluid viscosities [1–3].

Rayleigh [4] derived a stability criterion [see Section 2, Eqn (26)] for a rotating ideal incompressible fluid under axially symmetric perturbations. Later, it was shown in [5] that the Rayleigh criterion is the necessary and sufficient condition of stability under such perturbations in ideal fluids. Viscosity stabilizes the Couette flow, and the viscous flow that would be unstable according to condition (26) loses stability only if the angular rotation speed (or the Reynolds number) exceeds some critical value; in this case, condition (26) becomes only a sufficient one [6] (see also Ref. [7]). The critical Reynolds numbers theoretically derived by Taylor [8] proved to be in remarkable agreement (within several percent) with his experimental data. Presently, an analytic formula that approximates the Couette flow stability curve for the entire range of governing parameters is known [9].

The success of the theory in accurately reproducing experimental results, as well as the relative simplicity of both theoretical models and experiment, predetermined the great interest in the Taylor–Couette flow, making it a conceptual problem in the theory of hydrodynamic and hydromagnetic stability. Papers devoted to this problem are counted in three- if not four-digit numbers. Many results are summarized in monographs and reviews (see, e.g., Refs [7, 10–13]). An international conference on the Taylor–Couette flow is held every two years; the last one, the 15th in sequence, took place in France in 2007 [14]. It is noteworthy that the agreement between theoretical and experimental results pertaining to the stability of the Couette flow gives support to the so-called global stability theory, which hinges on the presence of boundary conditions. In contrast, results obtained in the framework of a local approach might prove unreliable (i.e., predicting instability for stable flows and vice versa) [15].

In mentioning the successes in studying the Taylor–Couette flow, the existing difficulties must also be mentioned. Admittedly, the Couette flow between a resting inner and a rotating outer cylinders must, according to the Rayleigh stability criterion, be stable under axially symmetric perturbations. But the early experiments in [2, 3] already demonstrated that the stability is lost for a sufficiently fast rotation. This instability was not encountered in the original experiments by Taylor, but subsequent research [16] lent support to Couette's results. The discrepancy between the theory and the experiment has not yet received an exhaustive explanation. The present situation remains ambiguous [17, 18]. In particular,

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the instability of a flow that is stable according to the linear theory can be rooted in nonlinearities and imperfections of the experiment proper: nonalignment of the cylinder axes, roughness of cylinder surfaces, unsteadiness of rotation, and so on. Removing these hampering factors also removes the flow instability [18, 19]. However, the larger the Reynolds number (rotation speed), the higher the constraints on the experimental setup with respect to these imperfections. Nonideal behavior can also emerge from boundary effects stemming from a finite height of the cylinders [20]. They can be neglected for small Reynolds numbers, but become a serious problem for Reynolds numbers of the order of 10^5 or larger.

It is well known (see, e.g., Ref. [21]) that because of the instability, a purely rotational laminar one-dimensional Couette flow evolves into a more complex (but stable) three-dimensional flow with the structure dependent on the relative rotation speed of the cylinders. On the way to well-developed turbulence, the Taylor–Couette flow passes through several such stable states, characterized by an increasingly complex structure, which emerge as the Reynolds number increases. Given this behavior, the instability of a purely rotational Couette flow is commonly referred to as the primary instability of the Taylor–Couette flow.

In this paper, we theoretically study the Couette flow instability (i.e., the primary instability of the Taylor–Couette flow) under the effects of density stratification and a magnetic field. Notwithstanding the long history of the problem, essential progress in this area was achieved only recently; it is barely reflected in the existing review literature on the Taylor–Couette flow. We limit ourselves to the simplest, linear stability theory. This limitation is not essential in our case, because the linear theory already agrees well with experimental data on the primary instability.

In Sections 2–5, general equations describing the system behavior and defining its steady state are given. They are followed by equations describing the stability of the steady state in the linear approximation. As a rule, they can be worked out only by numerical methods, which are standard and are briefly discussed when the linearized equations first appear. Some results for an ideal fluid can be derived analytically. Each section ends with a brief discussion of the results pertaining to the instability of the Couette flow. General conclusions are given in Section 6.

2. The classical Couette flow

We consider a viscous incompressible fluid with a uniform density ρ and dynamic viscosity μ in a gap between two infinitely long coaxial cylinders. The fluid obeys the hydrodynamical equations

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\,\mathbf{U} = -\frac{1}{\rho}\,\nabla P + \nu\Delta\mathbf{U} + \mathbf{g}\,, \tag{1}$$

$$\operatorname{div}\mathbf{U} = 0\,, \tag{2}$$

where P is the pressure, ρ is the density, ν is the kinematic viscosity ($\mu = \rho\nu$), $\mathbf{U}$ is the fluid velocity, and $\mathbf{g}$ is the acceleration due to external forces. In Eqn (1), the contribution of volume viscosity and friction is omitted, as is usually done. Given the geometry of the problem, we consider the cylindrical system of coordinates (R, ϕ, z). In the absence of external forces, system (1), (2) becomes

$$\frac{\partial U_R}{\partial t} + (\mathbf{U}\nabla)\,U_R - \frac{U_\phi^2}{R} = -\frac{1}{\rho}\,\frac{\partial P}{\partial R} + \nu\left(\Delta U_R - \frac{2}{R^2}\,\frac{\partial U_\phi}{\partial \phi} - \frac{U_R}{R^2}\right), \tag{3}$$

$$\frac{\partial U_\phi}{\partial t} + (\mathbf{U}\nabla)\,U_\phi + \frac{U_\phi U_R}{R} = -\frac{1}{\rho R}\,\frac{\partial P}{\partial \phi} + \nu\left(\Delta U_\phi + \frac{2}{R^2}\,\frac{\partial U_R}{\partial \phi} - \frac{U_\phi}{R^2}\right), \tag{4}$$

$$\frac{\partial U_z}{\partial t} + (\mathbf{U}\nabla)\,U_z = -\frac{1}{\rho}\,\frac{\partial P}{\partial z} + \nu\Delta U_z\,, \tag{5}$$

$$\frac{\partial U_R}{\partial R} + \frac{U_R}{R} + \frac{1}{R}\,\frac{\partial U_\phi}{\partial \phi} + \frac{\partial U_z}{\partial z} = 0\,, \tag{6}$$

where

$$(\mathbf{A}\nabla)\,F = A_R\,\frac{\partial F}{\partial R} + \frac{A_\phi}{R}\,\frac{\partial F}{\partial \phi} + A_z\frac{\partial F}{\partial z}\,, \tag{7}$$

$$\Delta F = \frac{\partial^2 F}{\partial R^2} + \frac{1}{R}\,\frac{\partial F}{\partial R} + \frac{1}{R^2}\,\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial z^2}\,. \tag{8}$$

For cylinders rotating with different angular speeds in the general case, system (3)–(6) allows a solution in the form

$$\mathbf{U} = \big(0, R\Omega(R), 0\big)\,, \quad P = P(R)\,, \quad \rho = \rho_0 = \text{const}\,. \tag{9}$$

We note that this form is preserved in the presence of external forces, expressed as $\mathbf{g} = \nabla(\psi(R))$, which simply implies redefining the pressure. For an ideal fluid ($\nu = 0$), the angular velocity is an arbitrary function of the radius satisfying boundary condition. In a viscous fluid, the azimuthal component of momentum equation (4) determines the behavior of the function $\Omega(R)$:

$$U_\phi(R) = R\Omega = a_\Omega R + \frac{b_\Omega}{R}\,, \tag{10}$$

where constants a_Ω and b_Ω are determined from the boundary conditions

$$a_\Omega = \Omega_{\text{in}}\,\frac{\hat{\mu}_\Omega - \hat{\eta}^2}{1 - \hat{\eta}^2}\,, \quad b_\Omega = \Omega_{\text{in}} R_{\text{in}}^2\,\frac{1 - \hat{\mu}_\Omega}{1 - \hat{\eta}^2}\,, \tag{11}$$

$$\hat{\eta} = \frac{R_{\text{in}}}{R_{\text{out}}}\,, \quad \hat{\mu}_\Omega = \frac{\Omega_{\text{out}}}{\Omega_{\text{in}}}\,. \tag{12}$$

Here, R_{in} and R_{out} are the radii and Ω_{in} and Ω_{out} are the angular velocities pertaining to the internal and external cylinders. If the angular velocity is known, the pressure is determined from Eqn (3), which, taking Eqn (9) into account, reduces to

$$\Omega^2 R = \frac{1}{\rho_0}\,\frac{\mathrm{d}P}{\mathrm{d}R}\,. \tag{13}$$

Just the velocity profile obeying Eqn (10) is observed experimentally for a stable, steady Taylor–Couette flow, which is then called the Couette flow. The functional law for the angular velocity profile is therefore fixed and the entire set of possible flows is spanned by the two parameters a_Ω and b_Ω (or $\hat{\eta}$ and $\hat{\mu}_\Omega$), which are fixed by the geometry of the problem (the gap between the cylinders) and the boundary conditions. This simplification is a consequence of the cylindrical geometry and is lacking, for example, in the spherical geometry. For completeness, the existence of a general solution for finite-length cylinders [22] should be mentioned. It obviously depends on the axial coordinate, in addition to the radial one.

We are interested in the stability of solution (9). In this paper, we are limited to exploring the linear stability under infinitesimal perturbations. The perturbed solution is written as

$$u_R(R,\phi,z)\,, \qquad R\Omega(R)+u_\phi(R,\phi,z)\,,$$
$$u_z(R,\phi,z)\,, \qquad P(R)+p(R,\phi,z)\,, \tag{14}$$

where u_R, u_ϕ, u_z, and p are small compared to the respective unperturbed quantities. Linearizing system (3)–(6) with respect to the unperturbed fields, we obtain

$$\frac{\partial u_R}{\partial t}+\Omega\,\frac{\partial u_R}{\partial \phi}-2\Omega u_\phi=-\frac{1}{\rho_0}\,\frac{\partial p}{\partial R}+\nu\left(\Delta u_R-\frac{2}{R^2}\,\frac{\partial u_\phi}{\partial \phi}-\frac{u_R}{R^2}\right), \tag{15}$$

$$\frac{\partial u_\phi}{\partial t}+\Omega\,\frac{\partial u_\phi}{\partial \phi}+\frac{1}{R}\,\frac{\partial}{\partial R}\,(R^2\Omega)\,u_R=-\frac{1}{\rho_0 R}\,\frac{\partial p}{\partial \phi}+\nu\left(\Delta u_\phi+\frac{2}{R^2}\,\frac{\partial u_R}{\partial \phi}-\frac{u_\phi}{R^2}\right), \tag{16}$$

$$\frac{\partial u_z}{\partial t}+\Omega\,\frac{\partial u_z}{\partial \phi}=-\frac{1}{\rho_0}\,\frac{\partial p}{\partial z}+\nu\Delta u_z\,, \tag{17}$$

$$\frac{\partial u_R}{\partial R}+\frac{u_R}{R}+\frac{1}{R}\,\frac{\partial u_\phi}{\partial \phi}+\frac{\partial u_z}{\partial z}=0\,. \tag{18}$$

To set up the problem completely, Eqns (15)–(18) must be complemented by six boundary conditions. The velocity of viscous fluid at a boundary equals that of the boundary, and hence the velocity perturbations are

$$u_R=u_\phi=u_z=0 \tag{19}$$

for both the inner ($R=R_{\text{in}}$) and outer ($R=R_{\text{out}}$) cylinders.

The coefficients of system (15)–(18) depend only on the radial coordinate, and therefore the solution can be written as a sum of normal modes of the form

$$f=f(R)\exp\left[\mathrm{i}(m\phi+kz+\omega t)\right], \tag{20}$$

where f denotes any of the sought variables. Regarding the geometry of the problem, we readily realize that the axial wave number can be any real number, the azimuthal number m can be only an integer, and the increment ω takes an arbitrary complex value. Moreover, without losing the generality, only positive k and m can be considered. Expansion (20) reduces the problem in three dimension to a single dimension:

$$\mathrm{i}(\omega+m\Omega)\,u_R-2\Omega u_\phi=-\frac{1}{\rho_0}\,\frac{\mathrm{d}p}{\mathrm{d}R}+\nu\left[\frac{\mathrm{d}^2u_R}{\mathrm{d}R^2}+\frac{1}{R}\,\frac{\mathrm{d}u_R}{\mathrm{d}R}-\frac{u_R}{R^2}-\left(\frac{m^2}{R^2}+k^2\right)u_R-2\mathrm{i}\,\frac{m}{R^2}u_\phi\right], \tag{21}$$

$$\mathrm{i}(\omega+m\Omega)\,u_\phi+\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}(R^2\Omega)\,u_R=-\frac{\mathrm{i}}{\rho_0}\,\frac{m}{R}\,p+\nu\left[\frac{\mathrm{d}^2u_\phi}{\mathrm{d}R^2}+\frac{1}{R}\,\frac{\mathrm{d}u_\phi}{\mathrm{d}R}-\frac{u_\phi}{R^2}-\left(\frac{m^2}{R^2}+k^2\right)u_\phi+2\mathrm{i}\,\frac{m}{R^2}\,u_R\right], \tag{22}$$

$$\mathrm{i}(\omega+m\Omega)\,u_z=-\frac{\mathrm{i}}{\rho_0}\,kp+\nu\left[\frac{\mathrm{d}^2u_z}{\mathrm{d}R^2}+\frac{1}{R}\,\frac{\mathrm{d}u_z}{\mathrm{d}R}-\left(\frac{m^2}{R^2}+k^2\right)u_z\right], \tag{23}$$

$$\frac{\mathrm{d}u_R}{\mathrm{d}R}+\frac{u_R}{R}+\mathrm{i}\,\frac{m}{R}\,u_\phi+\mathrm{i}ku_z=0\,. \tag{24}$$

If the fluid is ideal ($\nu=0$) and perturbations are axially symmetric ($m=0$), then system (21)–(24) further reduces to a single second-order equation,

$$-\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}(Ru_R)\right]+k^2u_R-\frac{k^2}{\omega^2}\,\frac{1}{R^3}\,\frac{\mathrm{d}}{\mathrm{d}R}(R^2\Omega)^2u_R=0. \tag{25}$$

Equation (25) allows deriving the necessary and sufficient condition for the stability of a rotating fluid. For example, this can be done by invoking the results of the classic Sturm–Liouville theory. Indeed, Eqn (25) with boundary conditions (19) is the classic Sturm–Liouville problem for eigenvalues k^2/ω^2 [7]. Because k is real, the sign of ω^2 coincides with that of k^2/ω^2. According to the general theory, all eigenvalues are positive (and hence the flow is stable) if and only if

$$\frac{1}{R^3}\,\frac{\mathrm{d}}{\mathrm{d}R}\,(R^2\Omega)^2>0 \tag{26}$$

for any point within the interval under consideration.

The conjecture that condition (26) is sufficient for stability was first proved by Rayleigh [4] and bears his name. The exchange method used by Rayleigh is physically very transparent and points to a direct link between criterion (26) and the momentum conservation (see, e.g., Refs [7, 23]). It can be readily shown that for a flow described by Eqn (10), criterion (26) becomes [7]

$$\hat{\mu}_\Omega>\hat{\eta}^2\,. \tag{27}$$

Hence, the ideal Couette flow is stable under axially symmetric perturbations when the angular momentum is an increasing function of the radius everywhere in the annulus between the cylinders. If there are points where the angular momentum decreases with the radius, the flow is unstable under axially symmetric perturbations.

In what follows, the instability evolving in a rotating fluid will be referred to as the *rotational instability* (RI).

The general stability criterion, similar to (26), is still not known for asymmetric perturbations. It can only be argued that (26) is a necessary condition for the stability under asymmetric perturbations [24, 25]. It is nevertheless well known that criterion (26) is insufficient for the stability of an ideal incompressible homogeneous flow under asymmetric perturbations [26]. It was shown in [27] that asymmetric modes might be unstable at the Rayleigh line ($\hat{\mu}_\Omega = \hat{\eta}^2$). However, it was noted in [28] that this result is extremely sensitive to the boundary conditions. The result has been obtained for free boundary conditions, but the unstable modes disappear [29] in the case of the no-slip boundary condition (19).

For analyzing a viscous Couette flow, it is convenient to bring the equations to dimensionless form. Let $d = R_{\text{out}} - R_{\text{in}}$ be the gap between the cylinders. We take $R_0 = (R_{\text{in}} d)^{1/2}$ to be the unit length, $\Omega_{\text{in}} R_0$ the unit velocity, Ω_{in} the unit frequency, and $\rho_0 \nu \Omega_{\text{in}}$ the unit pressure. The dimensionless number in the problem is the Reynolds number

$$\text{Re} = \frac{\Omega_{\text{in}} R_0^2}{\nu}\,, \tag{28}$$

and the problem parameters are $\hat{\mu}_\Omega$, $\hat{\eta}$, m, k, and ω. With the same notation for dimensionless and dimensional variables, the system of Eqns (21)–(24) is written as

$$\mathrm{i}\text{Re}(\omega + m\Omega)\, u_R - 2\text{Re}\,\Omega u_\phi = -\frac{\mathrm{d}p}{\mathrm{d}R} + \frac{\mathrm{d}^2 u_R}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_R}{\mathrm{d}R} - \frac{u_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_R - 2\mathrm{i}\,\frac{m}{R^2} u_\phi\,, \tag{29}$$

$$\mathrm{i}\text{Re}(\omega + m\Omega)\, u_\phi + \frac{\text{Re}}{R}\,\frac{\mathrm{d}(R^2\Omega)}{\mathrm{d}R}\, u_R = -\mathrm{i}\,\frac{m}{R}\, p + \frac{\mathrm{d}^2 u_\phi}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_\phi}{\mathrm{d}R} - \frac{u_\phi}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_\phi + 2\mathrm{i}\,\frac{m}{R^2} u_R\,, \tag{30}$$

$$\mathrm{i}\text{Re}(\omega + m\Omega)\, u_z = -\mathrm{i}kp + \frac{\mathrm{d}^2 u_z}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_z}{\mathrm{d}R} - \left(\frac{m^2}{R^2} + k^2\right) u_z\,, \tag{31}$$

$$\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R} + \mathrm{i}\,\frac{m}{R} u_\phi + \mathrm{i}k u_z = 0\,. \tag{32}$$

In the general case, system (29)–(32) complemented with boundary conditions (19) is a classic eigenvalue problem of the form

$$\mathcal{L}(Y) = 0\,, \tag{33}$$

where Y denotes the whole set of problem parameters (wave numbers, increment and dimensionless numbers). Generally, $\mathcal{L}$ is complex valued. The imaginary and real parts of $\mathcal{L}$ are simultaneously equal to zero only if all the parameters are eigenvalues.

If a solution exists, the range of parameters where the imaginary part of the increment is positive, $\mathcal{I}\mathrm{m}\,\omega > 0$, corresponds to stable flows, and the range where $\mathcal{I}\mathrm{m}\,\omega < 0$ corresponds to unstable ones. States with $\mathcal{I}\mathrm{m}\,\omega = 0$ correspond to neutrally stable flows (i.e., perturbations neither grow nor decay). The Couette flow is stable at small Reynolds numbers and can lose stability as the Reynolds number increases. Taylor assumed [8] that the flow loses stability for the Reynolds number that is the minimum over neutral regions for all possible values of the parameters (these minimum Reynolds numbers are called critical). We note that in this framework, no statement can be made about the magnitude of the instability increment (the imaginary part of the increment is equal to zero in neutral regions). The instability increment can only be estimated in a nonlinear framework. This aspect makes the linear stability theory for ideal fluids essentially different from that in the viscous case. In an ideal fluid, on the contrary, we can compute the instability increment, but cannot determine the Reynolds number.

The real part of the increment $\mathcal{R}\mathrm{e}\,\omega$ can be equal to or different from zero on the neutral curve in the general case. If $\mathcal{R}\mathrm{e}\,\omega = 0$, then the instability evolves monotonically and the system typically passes into a new stable state as a result of unfolding instability (the process is often referred to as the principle of stability exchange [7]). If the real part of the increment is nonzero, the instability evolves as an oscillatory one.

Typically, problem (33) can only be solved numerically. Three main approaches have been used: 1) Galerkin-type methods, according to which the solution is approximated as a series in basis functions chosen so as to ensure fast convergence (see, e.g., Refs [7, 10]); 2) Runge–Kutta methods, where the solution is reduced, due to the linearity of problem (33), to the sum of solutions of several initial value problems, sought by the Runge–Kutta method (see, e.g., Ref. [30]); 3) the method of finite differences. In all three cases, the problem amounts to computing the determinant of a matrix (generally, a complex-valued one) and finding its zeros. Current practices give preference to the third method. The first one turns out to be rather cumbersome (requires integrations to determine the coefficients of the series) and does not suggest a robust choice for the basis functions. The second method frequently suffers from a strong dependence of the solution on boundary conditions. As a consequence, the solution can only be found when trial initial conditions (which are used to augment the boundary conditions on one of the boundaries to transform the boundary value problem into an initial value one) are fairly close to the true ones. Evidently, a general method allowing the selection of appropriately close trial conditions does not exist.

A typical neutral stability curve for the Couette flow is schematically presented in Fig. 1. According to Fig. 1, the critical Reynolds numbers are minimum when the outer cylinder is at rest ($\hat{\mu}_\Omega = 0$). We note that not only the Reynolds number but also other parameters (k, m, and $\mathcal{R}\mathrm{e}\,\omega$) vary along the curve. For cylinders rotating in the same sense, the axially symmetric mode is always the most unstable and $m = 0$ for $\hat{\mu}_\Omega > 0$. Modes with $m > 0$ become the most unstable [33] only when the cylinders are counter-rotating ($\hat{\mu}_\Omega < 0$). The value of $\hat{\mu}_\Omega$ at which the axially symmetric instability is replaced with the asymmetric one varies as a function of the parameter $\hat{\eta}$.

The asymmetric instability is obviously an oscillatory one, $\mathcal{R}\mathrm{e}\,\omega \neq 0$. As concerns the axially symmetric instability, both the experimental and theoretical results indicated that the principle of stability exchange (i.e., $\mathcal{R}\mathrm{e}\,\omega = 0$) is applicable in this case to the Couette flow. A rigorous proof of this fact was given only recently [32].

When the Reynolds number exceeds the critical value, flows with $\hat{\mu}_\Omega > 0$ evolve from the ground state described by Eqn (10) to another stable state that is periodic in the vertical

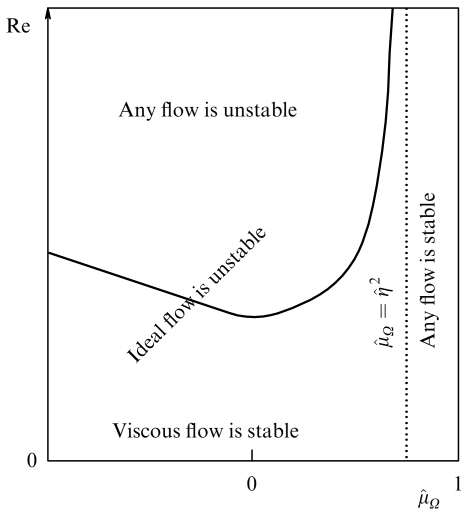


Figure 1. A typical neutral stability curve for a dissipative Couette flow (the solid curve) for a fixed $\hat{\eta}$. A viscous flow is unstable above the curve and stable below it. The dashed line corresponds to the limiting Rayleigh line for the flow described by Eqn (10).

direction and is presently referred to as Taylor vortices. Two vortices rotating in opposite senses then form over a single period. We let δz denote the vortex size in the axial direction; the wave number corresponding to this size is expressed as $k = \pi/\delta z$. Using the unit size R_0 defined above, we obtain

$$\frac{\delta z}{R_{\text{out}} - R_{\text{in}}} = \frac{\pi}{k} \sqrt{\frac{\hat{\eta}}{1-\hat{\eta}}} . \tag{34}$$

The magnitude of k is strongly sensitive to the value of $\hat{\mu}_\Omega$ for $\hat{\eta}$ kept fixed. For a resting outer cylinder ($\hat{\mu}_\Omega = 0$), the Taylor vortices are practically of square form (i.e., the vortex height is nearly equal to the size of the gap between the cylinders). Experimental values for the critical Reynolds number and the axial wave number set by the size of Taylor vortices for the Reynolds numbers slightly in excess of the critical value can be compared to the respective theoretical values. It turns out that the theoretical and experimental values coincide with each other within several percent over virtually the entire interval of parameters $\hat{\eta}$ and $\hat{\mu}_\Omega$.

If the real part of ω is different from zero, it is typically negative (i.e., the instability, if manifested as a wave, propagates against the basic rotation). However, as we see in Section 3, this is not a general statement because many exceptions with a positive real part of ω exist.

3. Flows with nonuniform density

3.1 Radial density stratification

The problem considered in Section 2 allows a natural generalization to the case of nonuniform density. We again consider an incompressible fluid and assume that the density is distributed nonuniformly but the dynamic viscosity is uniform as previously (i.e., is independent of the density). Equations (1) and (2) must be augmented with

$$\frac{\partial \rho}{\partial t} + (\mathbf{U}\nabla)\,\rho = 0 . \tag{35}$$

Systems (1) and (2) together with Eqn (35) again allow a solution of form (9), but for the density depending on the radius, $\rho(R)$, where, as before, $\Omega(R)$ is either an arbitrary function satisfying boundary conditions for an ideal fluid or a function given by Eqn (10) for a viscous fluid. Perturbed state (14) must now be augmented by density perturbations $\rho = \rho_0(R) + \rho(R, \phi, z)$. All the equations of system (21)–(24) except Eqn (21) preserve their form. Equation (21) acquires an additional term related to density perturbations:

$$\begin{aligned} &\mathrm{i}(\omega + m\Omega)\, u_R - 2\Omega u_\phi - \Omega^2 R\,\frac{\rho}{\rho_0} = -\frac{1}{\rho_0}\,\frac{\mathrm{d}p}{\mathrm{d}R} \\ &+\nu\left[\frac{\mathrm{d}^2 u_R}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_R}{\mathrm{d}R} - \frac{u_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_R - 2\mathrm{i}\,\frac{m}{R^2}\,u_\phi\right], \end{aligned} \tag{36}$$

and Eqn (35), after linearization and expansion in normal modes, becomes

$$\mathrm{i}(\omega + m\Omega)\,\rho + \frac{\mathrm{d}\rho_0}{\mathrm{d}R}\,u_R = 0 . \tag{37}$$

We note that in solving system (22)–(24), (36), (37), it must be borne in mind that the kinematic viscosity ν also depends on the radial coordinate because $\nu = \mu/\rho_0(R)$, and μ is taken to be constant. This dependence can be discarded only if the stratification is weak.

For an ideal fluid and axially symmetric perturbations, system of equations (22)–(24), (36), (37) reduces by straightforward manipulations to

$$-\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{\rho_0}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}\,(R u_R)\right] + k^2 u_R - \frac{k^2}{\omega^2}\,\rho_0[\varkappa^2 + N_R^2]\,u_R = 0 , \tag{38}$$

where

$$\varkappa^2 = \frac{1}{R^3}\,\frac{\mathrm{d}}{\mathrm{d}R}\,(\Omega R^2)^2 , \quad N_R^2 = \frac{1}{\rho_0}\,\frac{\mathrm{d}\rho_0}{\mathrm{d}R}\,\Omega^2 R , \tag{39}$$

$\varkappa$ is the epicyclic frequency (the terminology adopted from the literature on astrophysics), and N_R is the buoyancy (Brunt–Väisälä) frequency. In the presence of a radial external force $g_R(R)$, the buoyancy frequency is determined by the joint action of this force and the centrifugal force,

$$N_R^2 = \frac{1}{\rho_0}\,\frac{\mathrm{d}\rho_0}{\mathrm{d}R}\,(\Omega^2 R + g_R) . \tag{40}$$

Equation (38) together with boundary conditions (19) constitutes the classic Sturm–Liouville problem. Therefore, for the stability of an ideal incompressible Couette flow with radial stratification under axially symmetric perturbations, it is necessary and sufficient that the condition [33]

$$\varkappa^2 + N_R^2 > 0 \tag{41}$$

be satisfied. Condition (41) is a generalization of condition (26) to incompressible fluids with radial density stratification. If the fluid is homogeneous, condition (41) reduces to (26), and for a nonrotating fluid, to the Rayleigh–Taylor criterion. A stable density stratification stabilizes ideal flows.

We recall that asymmetric modes can become unstable beyond the Rayleigh limit (see Section 2). This instability, however, is strongly dependent on boundary conditions and seems not to occur for the classical Couette flow. Nevertheless, the results in Section 3.2 demonstrate that a vertical density stratification might have an essential impact on the stability of axially symmetric modes beyond the Rayleigh limit. For a radial stratification, this question is still awaiting an analysis.

The Couette flow with a radial density gradient has not attracted sufficient attention; we can only mention Ref. [34]. Its results, however, lack the generality (computations were carried out only for a resting outer cylinder and weak stratification) sufficient for any conclusions going beyond simple condition (41). Such lack of interest is supposedly related to the fact that even the simplest model of an ideal incompressible fluid agrees well with observations. Besides, experimental data on the Couette flow with radial density stratification in the presence of an additional radial force (to the centrifugal force) are absent to the best of our knowledge. Admittedly, this is linked to the difficulty in creating such a force in the laboratory. We note that a far more complex problem involving buoyancy forces caused by radial temperature gradients has received much more attention (see, e.g., Ref. [35]), but discussing it is beyond the scope of this paper.

3.2 Axial density stratification

We assume the presence of an external force with a uniform acceleration $g_z \mathbf{e}_z$ directed along the cylinder axes (its role is played by the gravity force for vertically aligned cylinders in laboratory conditions). The right-hand side of Eqn (5) then acquires an extra term g_z:

$$\frac{\partial U_z}{\partial t} + (\mathbf{U}\nabla)\, U_z = -\frac{1}{\rho}\,\frac{\partial P}{\partial z} + g_z + \nu \Delta U_z\,. \tag{42}$$

We are interested in the stability of a steady solution of system (3), (4), (6), (35), (42) for a given vertical density stratification $\rho = \rho(z)$. We assume that the steady flow preserves the form it had in the absence of density stratification, $\mathbf{U} = (0, R\Omega(R), 0)$. It then follows that

$$\frac{U_\phi^2}{R} = \frac{1}{\rho}\,\frac{\partial P}{\partial R}\,, \quad \frac{\partial^2 U_\phi}{\partial R^2} + \frac{1}{R}\,\frac{\partial U_\phi}{\partial R} - \frac{U_\phi}{R^2} = 0\,, \quad \frac{1}{\rho}\,\frac{\partial P}{\partial z} = g_z\,. \tag{43}$$

As previously, the second equation of system (43) determines rotation law (10). But differentiating the first of Eqns (43) with respect to z and the third with respect to R, and subtracting the resultant expressions gives

$$R\Omega^2\,\frac{\mathrm{d}\rho}{\mathrm{d}z} = 0\,. \tag{44}$$

According to Eqn (44), the density can depend on the axial coordinate only in the absence of rotation. Therefore, the initial assumption that $\Omega = \Omega(R)$ and $\rho = \rho(z)$ is inconsistent and we must allow a more general functional form of Ω and ρ. The functional form $\Omega = \Omega(R)$ is determined by the boundary conditions (the angular rotation speed of the cylinders is independent of z), and it is therefore nature to assume a more general density distribution, $\rho = \rho(R, z)$. In this case, condition (44) becomes

$$R\Omega^2\,\frac{\partial\rho}{\partial z} - g_z\,\frac{\partial\rho}{\partial R} = 0\,. \tag{45}$$

Hence, even if the initial density distribution in a resting fluid is one-dimensional, $\rho = \rho(z)$, it becomes two-dimensional, $\rho = \rho(R, z)$, under the centrifugal force action. This essentially complicates the problem.

In real experiments, however, the initial density stratification (without rotation) and the ratio of the centrifugal to gravitational acceleration are both small [36, 37]:

$$\left|\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln z}\right| \ll 1\,, \qquad \left|\frac{R\Omega^2}{g_z}\right| \ll 1\,. \tag{46}$$

With conditions (46), it follows from Eqn (45) that the radial density stratification has an even higher order of smallness than the vertical one. We can then write

$$\rho = \rho_0 + \rho_1(z) + \rho_2(R, z) + \ldots\,, \quad |\rho_1| \ll \rho_0\,, \quad |\rho_2| \ll |\rho_1|\,, \tag{47}$$

where ρ_0 is a uniform reference density. Condition (45) is obviously satisfied in the zeroth approximation; in the first approximation, it takes the form

$$R\Omega^2\,\frac{\mathrm{d}\rho_1}{\mathrm{d}z} - g_z\,\frac{\partial\rho_2}{\partial R} = 0\,. \tag{48}$$

It follows that the steady solution of system (3), (4), (6), (35), (42) constrained by conditions (46) can be expressed as

$$U_R = U_z = 0\,, \quad U_\phi = R\Omega(R)\,,$$

$$P = P_0(z) + P_1(R, z) + \ldots\,, \quad \rho = \rho_0 + \rho_1(z) + \ldots\,, \tag{49}$$

where $\Omega(R)$ is described by Eqns (10) and (11), and P_0 and P_1 ($P_1 \ll P_0$) are determined by the equations

$$\frac{1}{\rho_0}\,\frac{\mathrm{d}P_0}{\mathrm{d}z} = g_z\,, \qquad \frac{U_\phi^2}{R} = \frac{1}{\rho_0}\,\frac{\partial P_1}{\partial R}\,, \qquad \frac{1}{\rho_1}\,\frac{\partial P_1}{\partial z} = g_z\,. \tag{50}$$

We are interested in the stability of steady state (49). In the linear approximation, the perturbed flow is represented as

$$u_R(R, \phi, z)\,, \quad u_\phi(R, \phi, z) + R\Omega(R)\,, \quad u_z(R, \phi, z)\,,$$

$$P_0(z) + P_1(R, z) + p(R, \phi, z)\,, \quad \rho_0 + \rho_1(z) + \rho(R, \phi, z)\,, \tag{51}$$

where the perturbations u_R, u_ϕ, u_z, p, and ρ are small compared to their unperturbed fields.

The linearization of system (3), (4), (6), (35), (42) preserves Eqns (15), (16), and (18); Eqns (42) and (35) are transformed into

$$\frac{\partial u_z}{\partial t} + \Omega\,\frac{\partial u_z}{\partial\phi} = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} + g_z\,\frac{\rho}{\rho_0} + \nu\nabla u_z\,, \tag{52}$$

$$\frac{\partial\rho}{\partial t} + \Omega\,\frac{\partial\rho}{\partial\phi} + \frac{\mathrm{d}\rho_1}{\mathrm{d}z}\,u_z = 0\,, \tag{53}$$

and the linearized system becomes equivalent to the Boussinesq approximation.

If the density stratification is linear, then

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}z} = \mathrm{const}\,, \tag{54}$$

the coefficients of the linear system still depend only on the radius, and the solution can be sought as an expansion in normal modes (20).

To pass to a dimensionless form, we choose $R_0 = (R_{\rm in} d)^{1/2}$ as the unit length ($d = R_{\rm out} - R_{\rm in}$), $\Omega_{\rm in} R_0$ as the unit velocity, $\Omega_{\rm in}$ as the unit frequency, $\rho_0 R_0 \Omega_{\rm in}^2 / g_z$ as the unit density, and $\rho_0 \nu \Omega_{\rm in}$ as the unit pressure. Additionally, we introduce the buoyancy (Brunt–Väisälä) frequency N_z,

$$N_z^2 = \frac{g_z}{\rho_0} \frac{{\rm d}\rho_1}{{\rm d}z} \,. \tag{55}$$

The problem is now characterized by the Reynolds (Re) and Froude (Fr) numbers

$$\mathrm{Re} = \frac{\Omega_{\rm in} R_0^2}{\nu} \,, \qquad \mathrm{Fr} = \frac{\Omega_{\rm in}}{N_z} \,. \tag{56}$$

Keeping the same notation for dimensionless and dimensional fields, we obtain

$$\begin{aligned} &{\rm i}\mathrm{Re}(\omega + m\Omega)\, u_R - 2\mathrm{Re}\,\Omega\, u_\phi = -\frac{{\rm d}p}{{\rm d}R} + \frac{{\rm d}^2 u_R}{{\rm d}R^2} \\ &\quad + \frac{1}{R}\frac{{\rm d}u_R}{{\rm d}R} - \frac{u_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_R - 2{\rm i}\,\frac{m}{R^2}\, u_\phi \,, \end{aligned} \tag{57}$$

$$\begin{aligned} &{\rm i}\mathrm{Re}(\omega + m\Omega)\, u_\phi + \frac{\mathrm{Re}}{R}\frac{{\rm d}(R^2\Omega)}{{\rm d}R}\, u_R = -{\rm i}\,\frac{m}{R}\, p + \frac{{\rm d}^2 u_\phi}{{\rm d}R^2} \\ &\quad + \frac{1}{R}\frac{{\rm d}u_\phi}{{\rm d}R} - \frac{u_\phi}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_\phi + 2{\rm i}\,\frac{m}{R^2}\, u_R \,, \end{aligned} \tag{58}$$

$$\begin{aligned} &{\rm i}\mathrm{Re}(\omega + m\Omega)\, u_z = -{\rm i}kp + \mathrm{Re}\,\rho + \frac{{\rm d}^2 u_z}{{\rm d}R^2} \\ &\quad + \frac{1}{R}\frac{{\rm d}u_z}{{\rm d}R} - \left(\frac{m^2}{R^2} + k^2\right) u_z \,, \end{aligned} \tag{59}$$

$$\frac{{\rm d}u_R}{{\rm d}R} + \frac{u_R}{R} + {\rm i}\,\frac{m}{R}\, u_\phi + {\rm i}k u_z = 0 \,, \tag{60}$$

$${\rm i}(\omega + m\Omega)\,\rho + N_z^2 u_z = 0 \,. \tag{61}$$

The boundary conditions are still given by Eqn (19) for both outer and inner cylinders.

The stability condition for a rotating fluid having an axial density gradient under axially symmetric perturbations has the form [33]

$$\frac{1}{R^3}\frac{{\rm d}(R^2\Omega)^2}{{\rm d}R} + N_z^2 > 0 \,, \tag{62}$$

which translates to Rayleigh criterion (26) for a homogeneous fluid and the Rayleigh–Taylor one for a nonrotating fluid. The Couette flow with an axial density gradient was theoretically considered by Thorpe [38], who conjectured that a stable vertical density gradient stabilizes the flow and reduces the axial size of Taylor vortices. Further theoretical development in Ref. [39] and experiments in Refs [36, 37] for a resting outer cylinder lent support to Thorpe's conclusions.

But the experimental data in Ref. [40] have already demonstrated the existence of a principally new instability developing beyond the Rayleigh limit. Paradoxically, this new instability received proper attention neither from the research community nor from the authors, who failed to emphasize it in the abstract and conclusions, only briefly mentioning it in passing in the text. The only plausible explanation seems to be the lack of sufficient confidence in the accuracy of the experiment.

Only in 2001 did the linear analysis of stability for an ideal Couette flow [41, 42] demonstrate that the sufficient condition for the instability under asymmetric perturbations does not amount to the requirement that the angular momentum be an increasing function of radius. Instead, it constrains the angular velocity magnitude,

$$\frac{{\rm d}\Omega^2}{{\rm d}r} < 0 \,, \tag{63}$$

which shifts the instability bound beyond the Raleigh limit. We note that condition (63) exactly corresponds to that of magnetorotational instability (see Section 4.1, Eqn (97)) and for rotation law (10) takes the form

$$\hat{\mu}_\Omega < 1 \,. \tag{64}$$

Illustrative results in Refs [41, 42] pertaining to nonideal flows have demonstrated that the instability persists beyond the Rayleigh limit ($\hat{\mu}_\Omega = \hat{\eta}^2$).

The analysis in Ref. [43] helped establish the applicability limits (46) of Eqns (57)–(61) used to explore the stability of a Couette flow with vertical density stratification. The numerical results in Ref. [43] have shown that the linear theory in the Boussinesq approximation agrees well with experimental data pertaining to states before the Rayleigh limit at weak stratification (see Fig. 2 in Ref. [43]), as well as to states beyond the Rayleigh limit [40]. Recently, a new and much more elaborate experiment was conducted [44]. Its experimental data and the numerical results in Ref. [43] were found to agree well both before and beyond the Rayleigh limit.

A typical behavior of neutral stability curves (a flow is stable below them and unstable above) for a nonideal Couette flow with stable vertical density stratification is schematically displayed in Fig. 2. In the absence of stable vertical density stratification, the axially symmetric mode (solid curve) is the most unstable. This curve, in accordance with condition (26), does not intersect the Rayleigh limit line ($\hat{\mu}_\Omega = \hat{\eta}^2$). The

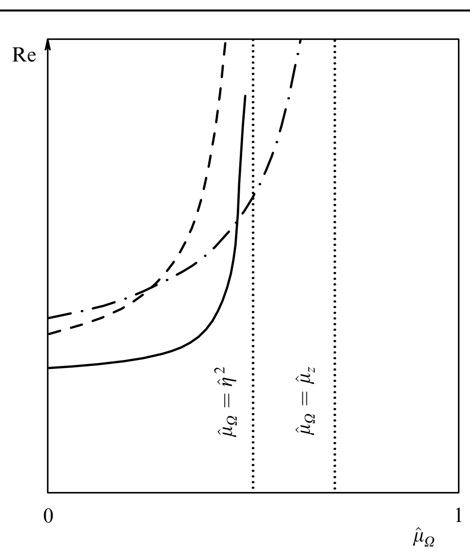


Figure 2. Schematics of neutral stability curves in the presence of axial density stratification and fixed $\hat{\eta}$ for axially symmetric ($m = 0$, the dashed line) and asymmetric ($m = 1$, the dashed-dotted line) perturbations. For comparison, the solid curve corresponds to $m = 0$ in the absence of density stratification.

stable density stratification stabilizes the axially symmetric mode (the dashed curve goes above the solid one), but destabilizes asymmetric modes, which remain unstable even beyond the Rayleigh limit up to some bounding value $\hat{\mu}_\Omega = \hat{\mu}_z$ ($\hat{\eta}^2 < \hat{\mu}_z < 1$). The results of computations in Refs [43, 45] demonstrate that $\hat{\mu}_z$ reaches the largest value for the $m = 1$ mode (the dashed-dotted curve in Fig. 2). The neutral curves for modes with larger m pass above the curve for the $m = 1$ mode.

The results in Refs [43, 45] have additionally shown that the instability criterion for the Couette flow with a vertical density gradient under asymmetric perturbation expressed by Eqn (63) is excessively restrictive. In reality, the bounding value $\hat{\mu}_z$ is strongly sensitive to the gap between the cylinders $\hat{\eta}$ and tends to unity only in the limit of a disappearing gap. Hence, the wider the gap between the cylinders, the smaller is the bounding value $\hat{\mu}_z$.

We recall that asymmetric modes are less stable than the symmetric mode in the uniform Couette flow only if the cylinders rotate in opposite senses [31]. In the presence of stable axial stratification, the asymmetric modes can also be less stable for corotating cylinders.

In the absence of a density gradient, $\mathcal{R}e\,\omega = 0$ and the instability evolves monotonically. Equation (61) shows that in the presence of a density gradient, the neutral stability corresponds to the nonzero real part of the frequency even for the axially symmetric mode, and the instability develops as an oscillatory one. As a rule, $\mathcal{R}e\,\omega < 0$ and the instability is seen as a wave traveling against the rotation.

In summary, the axial density gradient, even being stable, destabilizes the Couette flow and makes it unstable beyond the Rayleigh limit ($\hat{\mu}_\Omega = \hat{\eta}^2$). It is impossible to derive a simple instability criterion in this case due to the asymmetric character of the instability. We say that this is a *stratorotational* instability (SRI).

4. A flow in the presence of a magnetic field

An incompressible homogeneous conducting fluid in the presence of a magnetic field is described by the equations

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\,\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{U} + \frac{1}{c}\,\mathbf{j}\times\mathbf{B}\,, \tag{65}$$

$$\operatorname{div}\mathbf{U} = 0\,, \tag{66}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\operatorname{rot}\mathbf{E}\,, \tag{67}$$

where $\mathbf{j}$ is the electric current density, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and c is the speed of light. If the standard Ohm law is applicable to the fluid,

$$\mathbf{E} + \frac{1}{c}\,\mathbf{U}\times\mathbf{B} = R_{\mathrm{Om}}\,\mathbf{j}\,, \tag{68}$$

where R_{Om} is the specific electric resistance, then neglecting the displacement current

$$\mathbf{j} = \frac{c}{4\pi}\operatorname{rot}\mathbf{B}\,, \tag{69}$$

in the magnetic induction equation, we can write Eqn (67) as

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}(\mathbf{U}\times\mathbf{B}) + \eta\Delta\mathbf{B}\,, \tag{70}$$

where η is the magnetic diffusion coefficient. It is assumed in Eqn (70) that η is uniform and that

$$\operatorname{div}\mathbf{B} = 0\,. \tag{71}$$

In a cylindrical coordinate system, Eqns (65), (66), (70), and (71) become

$$\frac{\partial U_R}{\partial t} + (\mathbf{U}\nabla)\,U_R - \frac{U_\phi^2}{R} - \frac{1}{4\pi\rho}\left((\mathbf{B}\nabla)\,B_R - \frac{B_\phi^2}{R}\right) = -\frac{1}{\rho}\frac{\partial}{\partial R}\left(P + \frac{B^2}{8\pi}\right) + \nu\left(\Delta U_R - \frac{2}{R^2}\frac{\partial U_\phi}{\partial\phi} - \frac{U_R}{R^2}\right), \tag{72}$$

$$\frac{\partial U_\phi}{\partial t} + (\mathbf{U}\nabla)\,U_\phi + \frac{U_\phi U_R}{R} - \frac{1}{4\pi\rho}\left((\mathbf{B}\nabla)\,B_\phi + \frac{B_\phi B_R}{R}\right) = -\frac{1}{\rho R}\frac{\partial}{\partial\phi}\left(P + \frac{B^2}{8\pi}\right) + \nu\left(\Delta U_\phi + \frac{2}{R^2}\frac{\partial U_R}{\partial\phi} - \frac{U_\phi}{R^2}\right), \tag{73}$$

$$\frac{\partial U_z}{\partial t} + (\mathbf{U}\nabla)\,U_z - \frac{1}{4\pi\rho}(\mathbf{B}\nabla)\,B_z = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(P + \frac{B^2}{8\pi}\right) + \nu\Delta U_z, \tag{74}$$

$$\frac{\partial U_R}{\partial R} + \frac{U_R}{R} + \frac{1}{R}\frac{\partial U_\phi}{\partial\phi} + \frac{\partial U_z}{\partial z} = 0, \tag{75}$$

$$\frac{\partial B_R}{\partial t} + (\mathbf{U}\nabla)\,B_R - (\mathbf{B}\nabla)\,U_R = \eta\left(\Delta B_R - \frac{2}{R^2}\frac{\partial B_\phi}{\partial\phi} - \frac{B_R}{R^2}\right), \tag{76}$$

$$\frac{\partial B_\phi}{\partial t} + (\mathbf{U}\nabla)\,B_\phi - (\mathbf{B}\nabla)\,U_\phi + \frac{1}{R}\,(U_\phi B_R - U_R B_\phi) = \eta\left(\Delta B_\phi + \frac{2}{R^2}\frac{\partial B_R}{\partial\phi} - \frac{B_\phi}{R^2}\right), \tag{77}$$

$$\frac{\partial B_z}{\partial t} + (\mathbf{U}\nabla)\,B_z - (\mathbf{B}\nabla)\,U_z = \eta\Delta B_z, \tag{78}$$

$$\frac{\partial B_R}{\partial R} + \frac{B_R}{R} + \frac{1}{R}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z} = 0\,, \tag{79}$$

where $(\mathbf{A}\nabla)\,F$ and ΔF are defined by expressions (7) and (8). We note that only three of the last four equations are independent.

4.1 Uniform axial magnetic field

In the presence of a uniform magnetic field B_0 aligned with the rotation axis, Eqns (72)–(79) allow a steady solution of the general form

$$P = P(R)\,, \qquad \rho = \rho_0 = \text{const}\,, \tag{80}$$

$$U_R(R) = 0\,, \qquad U_\phi(R) = R\Omega(R)\,, \qquad U_z(R) = 0\,, \tag{81}$$

$$B_R(R) = 0\,, \qquad B_\phi(R) = 0\,, \qquad B_z(R) = B_0\,, \tag{82}$$

where, as in the absence of a magnetic field, $\Omega(R)$ is either an arbitrary function of the radius satisfying the boundary conditions for an ideal fluid or is given by Eqn (10) for a nonideal fluid, with the pressure obeying Eqn (13) as previously.

We are interested in the stability of stationary solution (80)–(82). We represent a perturbed solution in the linear approximation as

$$u_R(R,\phi,z)\,, \qquad R\Omega(R)+u_\phi(R,\phi,z)\,, \qquad u_z(R,\phi,z)\,,$$
$$b_R(R,\phi,z)\,, \qquad b_\phi(R,\phi,z)\,, \qquad B_0+b_z(R,\phi,z)\,,$$
$$P(R)+\frac{B_0^2}{8\pi}+p(R,\phi,z)\,, \tag{83}$$

where the perturbations $u_R, u_\phi, u_z, b_R, b_\phi, b_z$, and p are small compared to their unperturbed counterparts (we note that p is the perturbation of the full pressure, with account for the magnetic contribution). The coefficients of linearized system (72)–(79) depend only on the radial coordinate, which allows using expansion in normal modes (20).

To bring the equations to dimensionless form, we take $R_0=[R_{\rm in}(R_{\rm out}-R_{\rm in})]^{1/2}$ as the unit length, $\Omega_{\rm in}$ as the unit frequency, η/R_0 as the unit perturbed velocity, B_0 as the unit magnetic field (both perturbed and unperturbed), and $\rho_0\nu\eta/R_0^2$ as the unit pressure.

Linearizing system (72)–(79), expanding the unknown fields in normal modes (20), and keeping the notation used for dimensional fields for their dimensionless counterparts, we obtain

$$\begin{aligned}&{\rm iRe}(\omega+m\Omega)\,u_R-2{\rm Re}\,\Omega u_\phi-{\rm iHa}^2kb_R\\&\quad=-\frac{{\rm d}p}{{\rm d}R}+\frac{{\rm d}^2u_R}{{\rm d}R^2}+\frac{1}{R}\,\frac{{\rm d}u_R}{{\rm d}R}-\frac{u_R}{R^2}\\&\quad-\left(\frac{m^2}{R^2}+k^2\right)u_R-2{\rm i}\,\frac{m}{R^2}\,u_\phi\,,\end{aligned} \tag{84}$$

$$\begin{aligned}&{\rm iRe}(\omega+m\Omega)\,u_\phi+\frac{\rm Re}{R}\,\frac{{\rm d}(R^2\Omega)}{{\rm d}R}\,u_R-{\rm iHa}^2kb_\phi\\&\quad=-{\rm i}\,\frac{m}{R}\,p+\frac{{\rm d}^2u_\phi}{{\rm d}R^2}+\frac{1}{R}\,\frac{{\rm d}u_\phi}{{\rm d}R}-\frac{u_\phi}{R^2}\\&\quad-\left(\frac{m^2}{R^2}+k^2\right)u_\phi+2{\rm i}\,\frac{m}{R^2}\,u_R\,,\end{aligned} \tag{85}$$

$$\begin{aligned}&{\rm iRe}(\omega+m\Omega)\,u_z-{\rm iHa}^2kb_z=-{\rm i}kp+\frac{{\rm d}^2u_z}{{\rm d}R^2}+\frac{1}{R}\,\frac{{\rm d}u_z}{{\rm d}R}\\&\quad-\left(\frac{m^2}{R^2}+k^2\right)u_z\,,\end{aligned} \tag{86}$$

$$\frac{{\rm d}u_R}{{\rm d}R}+\frac{u_R}{R}+{\rm i}\,\frac{m}{R}\,u_\phi+{\rm i}ku_z=0\,, \tag{87}$$

$$\begin{aligned}&{\rm iRe\,Pm}(\omega+m\Omega)\,b_R-{\rm i}ku_R=\frac{{\rm d}^2b_R}{{\rm d}R^2}+\frac{1}{R}\,\frac{{\rm d}b_R}{{\rm d}R}-\frac{b_R}{R^2}\\&\quad-\left(\frac{m^2}{R^2}+k^2\right)b_R-2{\rm i}\,\frac{m}{R^2}b_\phi\,,\end{aligned} \tag{88}$$

$$\begin{aligned}&{\rm iRe\,Pm}(\omega+m\Omega)\,b_\phi-{\rm Re\,Pm}R\,\frac{{\rm d}\Omega}{{\rm d}R}\,b_R-{\rm i}ku_\phi\\&=\frac{{\rm d}^2b_\phi}{{\rm d}R^2}+\frac{1}{R}\,\frac{{\rm d}b_\phi}{{\rm d}R}-\frac{b_\phi}{R^2}-\left(\frac{m^2}{R^2}+k^2\right)b_\phi+2{\rm i}\,\frac{m}{R^2}b_R,\end{aligned} \tag{89}$$

$$\begin{aligned}&{\rm iRe\,Pm}(\omega+m\Omega)\,b_z-{\rm i}ku_z\\&\quad=\frac{{\rm d}^2b_z}{{\rm d}R^2}+\frac{1}{R}\,\frac{{\rm d}b_z}{{\rm d}R}-\left(\frac{m^2}{R^2}+k^2\right)b_z,\end{aligned} \tag{90}$$

$$\frac{{\rm d}b_R}{{\rm d}R}+\frac{b_R}{R}+{\rm i}\,\frac{m}{R}\,b_\phi+{\rm i}kb_z=0\,, \tag{91}$$

where the magnetic Prandtl number, the Hartmann number, and the Reynolds number,

$${\rm Pm}=\frac{\nu}{\eta}\,, \qquad {\rm Ha}=\frac{B_0R_0}{\sqrt{4\pi\rho_0\nu\eta}}\,, \qquad {\rm Re}=\frac{\Omega_{\rm in}R_0^2}{\nu}\,, \tag{92}$$

are the dimensionless problem numbers.

The boundary conditions for the velocity are set by expressions (19) on both the inner and outer cylinders. The boundary conditions for the magnetic field depend on the conducting properties of the cylinder material. In theoretical considerations, it is commonly assumed that the cylinders are either ideal conductors or dielectrics. For an ideal conductor, the normal component of the magnetic field and the tangential component of the electric field vanish at the boundary,

$$b_R=0\,, \qquad \frac{{\rm d}b_\phi}{{\rm d}R}+\frac{b_\phi}{R}=0\,. \tag{93}$$

Boundary conditions (93) apply for the inner as well as outer cylinders.

For an ideal dielectric (i.e., a dielectric with the magnetic permeability equal to unity), the internal magnetic field must coincide with the external one at the boundary. In this case, the absence of the normal component of the electric current on the boundary immediately yields

$$b_\phi=\frac{m}{kR}\,b_z\,. \tag{94}$$

This conditions also holds for both the inner and the outer boundaries. From the solution of the potential equation $\Delta\psi=0$ (where $\mathbf{B}=\nabla\psi$) in cylindrical coordinates, we have

$$b_R+\frac{{\rm i}b_z}{I_m(kR)}\left(\frac{m}{kR}\,I_m(kR)+I_{m+1}(kR)\right)=0 \tag{95}$$

at $R=R_{\rm in}$ and

$$b_R+\frac{{\rm i}b_z}{K_m(kR)}\left(\frac{m}{kR}\,K_m(kR)-K_{m+1}(kR)\right)=0 \tag{96}$$

at $R=R_{\rm out}$, where I_m and K_m are the modified Bessel functions, with the respective finite limits as $R\to 0$ and $R\to\infty$.

Obviously, the axial uniform magnetic field is stable per se. However, already in 1959 Velikhov [46] showed that it destabilizes the Couette flow and the condition of an ideal flow stability under axially symmetric perturbations becomes

$$\frac{{\rm d}\Omega^2}{{\rm d}R}>0\,, \tag{97}$$

which corresponds to the inequality $\hat\mu_\Omega>1$ for rotation law (10). In the absence of a magnetic field, the flow is stable under axially symmetric perturbations for $\hat\mu_\Omega>\hat\eta^2$, and therefore the main manifestation of the destabilizing action of the axial uniform magnetic field is the destabilization of flows with $\hat\eta^2<\hat\mu_\Omega<1$. This instability is conventionally referred to as the *magnetorotational instability* (MRI). A surge of interest in the MRI followed the recognition of its importance as a plausible mechanism responsible for the turbulence in accretion disks [47–49]. It is noteworthy that the instability brought about by the azimuthal magnetic field (see Section 4.2) can even be more important for understanding the nature of turbulence in accretion disks. We emphasize that the MRI for accretion disks was either derived in the framework of the local approach neglecting the boundary conditions or obtained by numerical simulations. In both cases, the reliability of the results causes serious doubts. The difficulties of the local approach were briefly mentioned in the Introduction (as regards accretion disks, see also Ref. [50]). As concerns numerical simulations, it can be argued that they are carried out for parameters that are very far from those pertaining to typical accretion disks. Accordingly, the question on the relevance of the MRI to the accretion disks remains open.

It is somewhat counterintuitive that criterion (97) does not involve the magnetic field and does not reduce to Rayleigh criterion (26) in the limit of small magnetic fields. The paradox, as noted by Velikhov [46], is caused by treating the fluid as an ideal one. In an ideal fluid, the magnetic field is frozen in the fluid and becomes perturbed when a small volume of the fluid is perturbed. In this case, the perturbed part of the magnetic field line remains connected with the unperturbed one that is rotating with the unperturbed angular velocity. Correspondingly, the perturbed volume of the fluid preserves the angular velocity instead of the angular momentum (as would be the case in the absence of the field), and criterion (26) is replaced by (97).

In the presence of dissipative processes, if the magnetic field manages to diffuse out of the perturbed volume (or degrades there) as the instability is evolving, then it ceases to influence the development of the instability. Recalling that the magnetic field decay time is inversely proportional to the magnetic diffusion coefficient η, while the instability development time is inversely proportional to the kinematic viscosity ν, we expect that the MRI would depend on the magnetic Prandtl number Pm [see Eqn (92)]. For small Pm, the magnetic field does not have a chance to diffuse out of the perturbed volume only under fast rotation (large Reynolds numbers) and the MRI can be manifested only at large Reynolds numbers (see below). In general, the larger the magnetic Prandtl number, the smaller are the Reynolds numbers allowing an MRI.

It is worth mentioning that in experiments on magnetized Couette flows [51–54], liquid metals with very low Pm (of the order of 10^{-5} or smaller) are used. And indeed, the MRI has not been observed experimentally thus far. The results in Ref. [55], as argued by its authors, demonstrate the MRI for the spherical geometry. But the initial flow state (in the absence of a magnetic field) was already unstable (turbulent), and therefore the instability observed is not the true initial flow instability. This does not permit identifying the observed instability with the MRI. Instead, we can speak about the instability of a magnetic field under the action of turbulent flow.

Chandrasekhar's computations for a nonideal fluid [7] conducted in the limit of small Pm for a hydrodynamically unstable flow with $\hat{\mu}_\Omega < \hat{\eta}^2$ have demonstrated, in concert with experiments, that the magnetic field only stabilizes the flow. Subsequent computations [56–59] carried out in the same approximation of small Pm confirmed the stabilizing effect of the magnetic field. On this background, the analysis in Ref. [60] remained unnoticed. The presence of the MRI for a hydrodynamically unstable flow ($\hat{\mu}_\Omega < \hat{\eta}^2$) at Pm ~ 1 was actually demonstrated in [60].

It was shown in [61] that for hydrodynamically unstable flows in the approximation of small Pm and a narrow gap ($1 - \hat{\eta} \ll 1$), the magnetorotational instability does disappear.

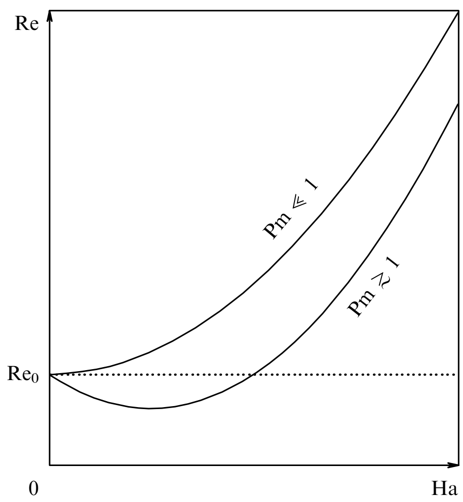


Figure 3. Schematics of neutral stability curves for a hydrodynamically unstable Couette flow ($\hat{\mu}_\Omega < \hat{\eta}^2$) in the presence of a uniform axial magnetic field.

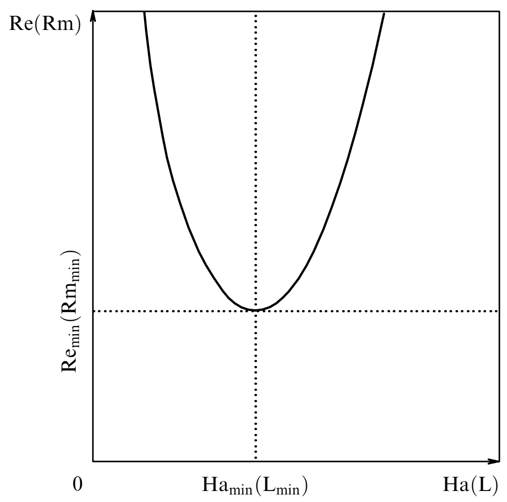
Numerical simulations in Ref. [62] generalize this result to an arbitrary gap between the cylinders. They show that the value Pm $= 0.25$ is the minimum for which the magnetorotational instability is still observed for flows with $0 < \hat{\mu}_\Omega < \hat{\eta}^2$.

The results in Ref. [62] are illustrated in Fig. 3, which sketches the behavior of neutral stability curves (flows are stable below the curve and unstable above it) for a hydrodynamically unstable Couette flow ($\hat{\mu}_\Omega < \hat{\eta}^2$). It can be seen that the flow is unstable in the absence of the magnetic field for Re $>$ Re$_0$, because the neutral stability curves intersect the ordinate axis at Re$_0$, the critical Reynolds number in the absence of a magnetic field. For small Pm, the magnetic field only suppresses the instability (the critical Reynolds numbers increase with an increase in the Hartmann number). But for Pm $\geqslant 1$, the MRI is generated for weak magnetic fields (and the flow becomes unstable for Reynolds numbers smaller than Re$_0$). We note that the neutral stability curve becomes universal for sufficiently small Pm (Pm $< 10^{-2}$). In contrast, it loses universality for large Pm. Its minimum displaces as a function of Pm and becomes deeper as Pm increases (although it does not reach zero because the magnetic field is stable per se).

We return to the main effect of the MRI, the destabilization of flows with $\hat{\eta}^2 < \hat{\mu}_\Omega < 1$. A typical neutral stability curve for such flows is plotted in Fig. 4. Because the flow is hydrodynamically stable, the curve does not intersect the ordinate axis. Its main feature is the presence of a minimum for a certain value of the magnetic field (because the flow is stable in the absence of a magnetic field and a strong field stabilizes the flow). The results in Ref. [62] have demonstrated that the coordinates of this minimum in the (L, Rm) plane are constant if the magnetic Prandtl number is sufficiently small (Pm $< 10^{-2}$). Here,

$$\mathrm{L} = \frac{B_0 R_0}{(4\pi\rho_0)^{1/2}\eta}\,, \qquad \mathrm{Rm} = \frac{\Omega_{\mathrm{in}} R_0^2}{\eta} \tag{98}$$

are the Lundquist number and the magnetic Reynolds number. For small Pm, viscous dissipation processes, which are proportional to ν, proceed much more slowly than those of Joule dissipation, which are proportional to η. Conse-



Figurte 4. Schematics of a neutral stability curve for a hydrodynamically stable Couette flow ($\hat{\eta}^2 < \hat{\mu}_\Omega < 1$) in the presence of a uniform axial magnetic field.

quently, the dimensionless numbers Rm and L are indeed better suited for describing flows with small Pm. We mention that the neutral stability curve in Fig. 4 becomes universal in the Rm and L variables (it is independent of Pm) if $\mathrm{Pm} < 10^{-2}$.

The results in Ref. [62] are derived using exact equations (84)–(91). Computations for very small Pm values encounter nontrivial technical difficulties (as is almost always the case with numerical computations when parameters become either very large or very small). These difficulties can be avoided by using approximate equations. In this case, instead of the Chandrasekhar approximation, which assumes only the smallness of Pm, a more elaborate approach must be used that also takes the smallness of terms proportional to the kinematic viscosity into account [61].

Is it possible to observe an MRI in experiments with liquid metals for flows with $\hat{\eta}^2 < \hat{\mu}_\Omega < 1$? The results in Refs [62, 63] suggest a rather negative answer to this question. Indeed, for small Pm, we have

$$\mathrm{Re} \sim \mathrm{Pm}^{-1}\,, \tag{99}$$

which gives Reynolds numbers about 10^6 or higher for $\mathrm{Pm} \sim 10^{-5}$. As noted in the Introduction, the instability at such large Reynolds numbers is observed even for theoretically stable flows, which is presumably linked to the nonideal character of the experiment. The relative accuracy required to remove the nonideal features must be of the order of or smaller than Pm. Such accuracy remains beyond the reach of present-day research. Therefore, observing an MRI in experiments with liquid metals is a challenging problem because instabilities triggered by the nonideal character of setups can dominate the MRI proper. Experiments with plasma might offer a more straightforward perspective [64].

Based on numerical simulations, it is argued in [65] that on the Rayleigh line ($\hat{\mu} = \hat{\eta}^2$), the dependence of the critical Reynolds number on Pm follows not Eqn (99) but the much shallower law

$$\mathrm{Re} \sim \mathrm{Pm}^{-1/2}\,. \tag{100}$$

This result can be derived analytically [63] by taking into account that $a_\Omega = 0$ [see Eqn (11)] in expression (10) for the angular velocity. Critical Reynolds numbers corresponding to law (100) are markedly lower ($\sim 10^4$) than those for law (99). However, the results in Ref. [63] demonstrate that the change in critical Reynolds numbers at small Pm occurs as a jump in the vicinity of the Rayleigh line (in practice, this happens in a very narrow range with the width of the order of Pm), which strongly hampers experimental efforts (in particular, the rotation of the cylinders must be maintained with a very high relative accuracy, no worse than the order of Pm). It was noted in [66] that if critical Reynolds numbers are taken not for the optimal magnetic field (when they are minimal) but for some fixed magnetic field, then the jump might be not so sharp. For strong magnetic fields, indeed, the jump is more gradual, but the Reynolds numbers themselves become large, preserving the problems noted above.

As regards the MRI structure, we remark that similarly to the flow in the absence of a magnetic field [31], the most unstable mode for cylinders rotating in the same sense in weak magnetic fields is the axially symmetric ($m = 0$) monotonic ($\mathcal{R}\mathrm{e}\,\omega = 0$) mode. The behavior in other cases depends on boundary conditions and the magnitude of Pm. For insulated boundaries, the axially symmetric mode continues to be the most unstable as Ha increases. The situation changes for conducting boundaries. First, if Pm is less than ~ 1, a critical Hartmann number $\mathrm{Ha_{cr}}$ exists. Above it, for $\mathrm{Ha} > \mathrm{Ha_{cr}}$, the asymmetric ($m = 1$) mode becomes the most unstable [62, 67]. The replacement of the axially symmetric instability with the asymmetric one at $\mathrm{Ha} > \mathrm{Ha_{cr}}$ and for conducting boundaries was first predicted in Ref. [59] for the infinitely narrow gap and $\mathrm{Pm} = 0$. Nevertheless, the question calls for further research because, according to Ref. [63], and in contrast to the results in Ref. [59], an oscillatory ($\omega \neq 0$) axially symmetric mode may be most unstable. We note that oscillatory axially symmetric instability of Couette flows for ideally conducting cylinders at large Ha was first obtained in Ref. [56].

As already noted, the axial wave number k corresponding to the critical Reynolds number defines the axial size of Taylor vortices arising as the result of the instability. In the presence of a vertical magnetic field, cells are stretched along it and the axial size of vortices increases accordingly [62, 63].

As follows from calculations and experimental data, the axially symmetric instability is commonly monotonic ($\omega = 0$). However, a rigorous proof of this proposition is lacking thus far. For strong magnetic fields and conducting boundaries, the axially symmetric instability, as noted previously, can become oscillatory ($\mathcal{R}\mathrm{e}\,\omega \neq 0$). The asymmetric instability is always oscillatory. The results in Ref. [63] have shown that $\mathcal{R}\mathrm{e}\,\omega > 0$ and the vortices rotate in the same sense as the main flow.

4.2 Azimuthal magnetic field

In the presence of an azimuthal magnetic field, Eqns (72)–(79) allow a steady solution of the general form

$$P = P(R)\,, \qquad \rho = \rho_0 = \mathrm{const}\,, \tag{101}$$

$$U_R = 0\,, \qquad U_\phi = R\Omega(R)\,, \qquad U_z = 0\,, \tag{102}$$

$$B_R = 0\,, \qquad B_\phi = B_\phi(R)\,, \qquad B_z = 0\,, \tag{103}$$

where, as previously, $\Omega(R)$ and $B_\phi(R)$ are arbitrary functions of the radius satisfying the boundary conditions for an ideal fluid. For a viscous fluid, the angular velocity is determined by Eqn (10) as usual, while the expression for the magnetic field follows from Eqn (77),

$$B_\phi(R) = a_B R + \frac{b_B}{R}\,, \tag{104}$$

where constants a_B and b_B are determined from the boundary conditions as

$$a_B = \frac{B_{\rm in}}{R_{\rm in}}\,\frac{\hat\eta(\hat\mu_B - \hat\eta)}{1-\hat\eta^2}\,, \qquad b_B = B_{\rm in}\,R_{\rm in}\,\frac{1-\hat\mu_B\hat\eta}{1-\hat\eta^2}\,. \tag{105}$$

Here,

$$\hat\mu_B = \frac{B_{\rm out}}{B_{\rm in}}\,, \tag{106}$$

$\hat\eta$ is given by Eqn (12), and $B_{\rm in}$ and $B_{\rm out}$ are the azimuthal magnetic fields on the inner and outer cylinders. We note that the first term in the right-hand side of Eqn (104) corresponds to the uniform current density j_z,

$$j_z = \frac{c}{2\pi}\,a_B\,, \tag{107}$$

and the second term is current-free. The pressure is determined from Eqn (72), which now takes the form

$$\Omega^2 R = \frac{1}{\rho_0}\,\frac{\rm d}{{\rm d}R}\left(P + \frac{B_\phi^2}{8\pi}\right) - \frac{1}{4\pi\rho_0}\,\frac{B_\phi^2}{R}\,. \tag{108}$$

To explore the stability of steady solution (101)–(103) in the linear approximation, the perturbed solution is represented in the form

$$u_R(R,\phi,z)\,, \qquad R\Omega(R) + u_\phi(R,\phi,z)\,, \qquad u_z(R,\phi,z)\,,$$

$$b_R(R,\phi,z)\,, \qquad B_\phi(R) + b_\phi(R,\phi,z)\,, \quad b_z(R,\phi,z)\,,$$

$$P(R) + \frac{B_\phi^2(R)}{8\pi} + p(R,\phi,z)\,, \tag{109}$$

where the perturbations u_R, u_ϕ, u_z, b_R, b_ϕ, b_z, and p are small compared with the respective unperturbed fields. The coefficients of linearized system (72)–(79) depend only on the radial coordinate, which allows using the expansion in normal modes (20).

To make the equations dimensionless, we choose $R_0 = [R_{\rm in}(R_{\rm out} - R_{\rm in})]^{1/2}$ as the unit length, $\Omega_{\rm in}$ as the unit frequency, η/R_0 as the unit perturbed velocity, $B_{\rm in}$ as the unit magnetic field (perturbed and unperturbed), and $\rho_0\nu\eta/R_0^2$ as the unit pressure.

Linearizing system (72)–(79), expanding in normal modes (20), and keeping the same notation for dimensionless and dimensional fields, we obtain

$$\begin{aligned}&{\rm i}{\rm Re}(\omega + m\Omega)\,u_R - 2{\rm Re}\,\Omega u_\phi - {\rm i}{\rm Ha}^2\,\frac{m}{R}\,B_\phi b_R \\ &\quad + 2{\rm Ha}^2\,\frac{B_\phi}{R}\,b_\phi = -\frac{{\rm d}p}{{\rm d}R} + \frac{{\rm d}^2u_R}{{\rm d}R^2} + \frac{1}{R}\,\frac{{\rm d}u_R}{{\rm d}R} - \frac{u_R}{R^2} \\ &\quad - \left(\frac{m^2}{R^2} + k^2\right)u_R - 2{\rm i}\,\frac{m}{R^2}\,u_\phi\,,\end{aligned} \tag{110}$$

$$\begin{aligned}&{\rm i}{\rm Re}(\omega + m\Omega)\,u_\phi + \frac{\rm Re}{R}\,\frac{{\rm d}(R^2\Omega)}{{\rm d}R}\,u_R - \frac{{\rm Ha}^2}{R}\,\frac{{\rm d}(RB_\phi)}{{\rm d}R} \\ &\quad - {\rm i}{\rm Ha}^2\,\frac{m}{R}\,B_\phi b_\phi = -{\rm i}\,\frac{m}{R}\,p + \frac{{\rm d}^2u_\phi}{{\rm d}R^2} + \frac{1}{R}\,\frac{{\rm d}u_\phi}{{\rm d}R} - \frac{u_\phi}{R^2} \\ &\quad - \left(\frac{m^2}{R^2} + k^2\right)u_\phi + 2{\rm i}\,\frac{m}{R^2}u_R\,,\end{aligned} \tag{111}$$

$$\begin{aligned}&{\rm i}{\rm Re}(\omega + m\Omega)\,u_z - {\rm i}{\rm Ha}^2\frac{m}{R}\,B_\phi b_z = -{\rm i}kp + \frac{{\rm d}^2u_z}{{\rm d}R^2} \\ &\quad + \frac{1}{R}\,\frac{{\rm d}u_z}{{\rm d}R} - \left(\frac{m^2}{R^2} + k^2\right)u_z\,,\end{aligned} \tag{112}$$

$$\frac{{\rm d}u_R}{{\rm d}R} + \frac{u_R}{R} + {\rm i}\,\frac{m}{R}\,u_\phi + {\rm i}ku_z = 0\,, \tag{113}$$

$$\begin{aligned}&{\rm i}{\rm Re}\,{\rm Pm}(\omega + m\Omega)\,b_R - {\rm i}\,\frac{m}{R}\,B_\phi u_R = \frac{{\rm d}^2b_R}{{\rm d}R^2} + \frac{1}{R}\,\frac{{\rm d}b_R}{{\rm d}R} \\ &\quad - \frac{b_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right)b_R - 2{\rm i}\,\frac{m}{R^2}\,b_\phi\,,\end{aligned} \tag{114}$$

$$\begin{aligned}&{\rm i}{\rm Re}\,{\rm Pm}(\omega + m\Omega)\,b_\phi - {\rm Re}\,{\rm Pm}R\,\frac{{\rm d}\Omega}{{\rm d}R}\,b_R \\ &\quad + R\frac{\rm d}{{\rm d}R}\left(\frac{B_\phi}{R}\right)u_R - {\rm i}\,\frac{m}{R}\,B_\phi u_\phi = \frac{{\rm d}^2b_\phi}{{\rm d}R^2} + \frac{1}{R}\,\frac{{\rm d}b_\phi}{{\rm d}R} \\ &\quad - \frac{b_\phi}{R^2} - \left(\frac{m^2}{R^2} + k^2\right)b_\phi + 2{\rm i}\,\frac{m}{R^2}\,b_R\,,\end{aligned} \tag{115}$$

$$\begin{aligned}&{\rm i}{\rm Re}\,{\rm Pm}(\omega + m\Omega)\,b_z - {\rm i}\,\frac{m}{R}\,B_\phi u_z = \frac{{\rm d}^2b_z}{{\rm d}R^2} + \frac{1}{R}\,\frac{{\rm d}b_z}{{\rm d}R} \\ &\quad - \left(\frac{m^2}{R^2} + k^2\right)b_z\,,\end{aligned} \tag{116}$$

$$\frac{{\rm d}b_R}{{\rm d}R} + \frac{b_R}{R} + {\rm i}\,\frac{m}{R}\,b_\phi + {\rm i}kb_z = 0\,, \tag{117}$$

where

$${\rm Pm} = \frac{\nu}{\eta}\,, \quad {\rm Ha} = \frac{B_{\rm in}R_0}{\sqrt{4\pi\rho_0\nu\eta}}\,, \quad {\rm Re} = \frac{\Omega_{\rm in}\,R_0^2}{\nu} \tag{118}$$

(we note that the Hartmann number Ha here differs from that introduced in Section 4.1)

The boundary conditions for the velocity are determined by Eqns (19). For the magnetic field, the boundary conditions are expressed by Eqn (93) for ideally conducting cylinders and by (94) and (95) or (96) for nonconducting ones.

The necessary and sufficient condition of stability under axially symmetric perturbations for an ideal rotating fluid in the presence of an azimuthal magnetic field was derived in [68]:

$$\frac{1}{R^3}\,\frac{\rm d}{{\rm d}R}\,(R^2\Omega)^2 - \frac{R}{4\pi\rho_0}\,\frac{\rm d}{{\rm d}R}\left(\frac{B_\phi}{R}\right)^2 > 0\,. \tag{119}$$

Condition (119) can readily be recovered by reducing the problem to a second-order equation, in analogy to the procedure outlined in Section 2. The first term, as should be expected, corresponds to Rayleigh condition (26). According to Eqn (119), the azimuthal magnetic field, in contrast to the uniform axial magnetic field, can be unstable even in the absence of rotation (as is well known from the theory of plasma pinch stability; see, e.g., Ref. [69]). We say that the rotation is stable under axially symmetric perturbations if it is stable according to the Rayleigh condition (the first term in Eqn (119) is positive), and unstable otherwise. Similarly, the azimuthal magnetic field is said to be stable under axially symmetric perturbations if the second term (with regard for its sign) in Eqn (119) is positive, and unstable if it is negative. Correspondingly, according to Eqn (26), any unstable (stable) rotation $\Omega(R)$ can be stabilized (destabilized) by the azimuthal magnetic field $B_\phi(R)$ with the suitable structure and amplitude. Obviously, the converse is also true and an unstable (stable) azimuthal magnetic field can be stabilized (destabilized) by choosing rotation.

Using expression (104) for the azimuthal magnetic field, it is straightforward to show that the magnetic field is stable if

$$0 \leqslant \hat{\mu}_B \leqslant \frac{1}{\hat{\eta}} \equiv \hat{\mu}_0 \,, \tag{120}$$

and unstable for all other $\hat{\mu}_B$. We recall that the rotation is stable for $\hat{\mu}_\Omega > \hat{\eta}^2$. According to Eqn (119), if $\hat{\mu}_B$ lies outside interval (120), then for any rotation law (arbitrary $\hat{\mu}_\Omega$), there exists a critical value of the magnetic field above which the flow loses stability. Evidently, the critical value of the magnetic field is equal to zero for unstable rotations and increases with an increase in the angular velocity for stable rotations.

It seems rather unexpected that the question of stability of the Couette flow in the presence of an azimuthal magnetic field has not received attention. In particular, we are not aware of any experimental results. Moreover, until recently, only a single theoretical study [70] dealt with the realistic dissipative Couette flow in a current-free magnetic field ($\hat{\mu}_B = \hat{\eta}$ or $a_B = 0$). Because this $\hat{\mu}_B$ belongs to interval (120), it is not surprising that only stabilization of the flow by the azimuthal magnetic field was found in [70].

The stability of nonideal Couette flows under axially symmetric perturbations in the presence of an azimuthal magnetic field was recently analyzed theoretically in [71, 72] for arbitrary $\hat{\mu}_B$ and for both conducting and insulating boundaries. For simplicity, the authors restricted themselves to the case of monotonic ($\mathcal{R}\mathrm{e}\,\omega = 0$) axially symmetric perturbations, which are typically most unstable. The results in Refs [71, 72] have demonstrated that the stability of a dissipative Couette flow under axially symmetric perturbations agrees with ideal criterion (119). In this case, dissipative processes stabilize the flow and two critical numbers emerge: the Reynolds and the Hartmann number. The dissipative flow with $\hat{\mu}_\Omega < \hat{\eta}^2$ actually loses stability only if the rotation is fast enough (large Reynolds number), and the azimuthal magnetic field with $\hat{\mu}_B < 0$ or $\hat{\mu}_B > 1/\hat{\eta}$ actually loses stability only for a sufficiently large amplitude (large Hartmann number). In the general case, a flow characterized by unstable rotation and an unstable magnetic field is stable only for subcritical Reynolds and Hartmann numbers (see curve *1* in Fig. 5). The critical Reynolds number is maximal in the absence of a magnetic field and decreases to zero at some finite Hartmann number. In turn, the critical Hartman number is maximum in the absence of rotation and decreases to zero at some finite Reynolds number. In this way, the combination of unstable rotation and an unstable magnetic field enhances the instability of the Couette flow in the general case. An exception occurs for the simplest solution of the magnetohydrodynamic equations,

$$\mathbf{V} = \pm \frac{\mathbf{B}}{(4\pi\rho)^{1/2}} \,. \tag{121}$$

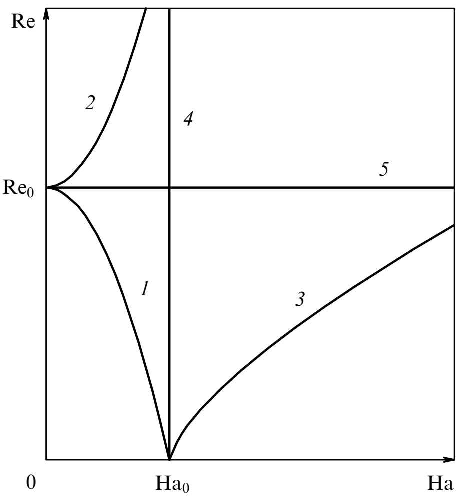


Figure 5. Schematics of the curves of neutral stability under axially symmetric perturbations for an unstable rotation and unstable magnetic field (curve *1*), unstable rotation and stable magnetic field (curve *2*), stable rotation and unstable magnetic field (curve *3*), neutral rotation and unstable magnetic field (curve *4*), and unstable rotation and neutral magnetic field (curve *5*). For curves *1*, *2*, and *5*, the flow is stable (unstable) below (above) the curves. For curve *4*, the flow is unstable (stable) to the right (left) of the curve. Finally, for curve *3*, the flow is unstable (stable) below (above) the curve.

It is well known that such a flow is stable for an ideal fluid (see, e.g., Ref. [7]). The stability of the Couette flow obeying Eqn (121) can be established straightforwardly by substituting Eqn (121) in condition (119). For an ideal Couette flow where the rotation and the magnetic field are arbitrary functions of the radius, it is then easy to choose the radial dependence such that both the rotation and magnetic field are unstable, but their combination linked by condition (121) results in a stable flow [73]. In this case, the rotation and the magnetic field are stable at some locations and unstable at others (obviously, the locations where the rotation is unstable are those where the magnetic field is stable, and vice versa). For a dissipative Couette flow with the angular velocity and magnetic fields specified by Eqns (10) and (104), the compensation of instabilities, as shown in Ref. [73], occurs for axially symmetric perturbations, for cylinders rotating in opposite directions, $\hat{\mu}_\Omega < 0$, with the magnetic field changing sign between the cylinders, $\hat{\mu}_B < 0$, and if the condition $\hat{\eta}|\hat{\mu}_B| \leqslant |\hat{\mu}_\Omega|$ holds. In this case, the flow is stable in some vicinity of the line that corresponds to the stability of the ideal Couette flow.

The stability of flows with unstable rotation and a stable magnetic field is illustrated by curve *2* in Fig. 5, and flows with stable rotation and an unstable magnetic field, by curve *3*. Predictably, the stable magnetic field stabilizes the flow and the critical Reynolds numbers increase with the Hartmann number. Similarly, stable rotation stabilizes the unstable magnetic field, and critical Hartmann numbers increase with the Reynolds number. Noteworthy is a special feature of the combination of an unstable magnetic field and stable rotation: the flow is unstable at small Reynolds numbers and stable when the Reynolds number is large. This happens because this instability, in its essence, is that of a magnetic field, not of rotation. We call it the *pinch instability* (PI).

In addition to stable and unstable solutions, a neutrally stable rotation (at $\hat{\mu}_\Omega = \hat{\eta}^2$) and a neutrally stable magnetic field (at $\hat{\mu}_B = 1/\hat{\eta}$) also exist for axially symmetric perturbations. They have no respective impact on the critical Hartmann and Reynolds numbers (curves *4* and *5* in Fig. 5). We note that curves *1*, *2*, and *5* correspond to the same value of $\hat{\mu}_\Omega$ (because the critical Reynolds numbers coincide for Ha = 0), but to different values of $\hat{\mu}_B$; curves *1*, *3*, and *4* correspond, accordingly, to the same $\hat{\mu}_B$ but different $\hat{\mu}_\Omega$.

The computation results indicate that the critical Reynolds numbers for axially symmetric ($m = 0$) and monotonic ($\mathcal{R}\mathrm{e}\,\omega = 0$) perturbations are independent of the magnetic Prandtl number. This result can be readily obtained analytically [71]. As a consequence, Ha_0 is also independent of Pm. It is straightforward to realize that Ha_0, being the number at which the critical Reynolds number vanishes, is independent of the flow parameters ($\hat{\mu}_\Omega$).

Boundary conditions, in general, do not influence the behavior described above on the qualitative level, although quantitatively, the critical numbers are naturally sensitive to the boundary conditions. Nevertheless, there are some peculiarities brought about by the boundary conditions. For conducting cylinders and flows with $-1 < \hat{\mu}_B < 0$, the axial wave number corresponding to the minimum Hartmann number is equal to zero and one-dimensional perturbations are the most unstable. For dielectric cylinders, two-dimensional perturbations are always the most unstable [71]. In addition, for conducting cylinders, there exists a parasitic solution that hampers the analysis under some circumstances [72].

Thus far, we have been discussing the case of axially symmetric perturbations. However, it is well known that for a plasma, pinch perturbations with $m = 1$ are the most unstable (see, e.g., Ref. [69]). In the absence of rotation, the necessary and sufficient condition for the stability of an ideal fluid in an azimuthal magnetic field under asymmetric adiabatic perturbations (actually, those with $m = 1$, because perturbations with larger m are more stable) was obtained for cylindrical geometry in [74] (see also Ref. [75]):

$$-\frac{\mathrm{d}}{\mathrm{d}R}\,(RB_\phi^2) > 0\,. \qquad (122)$$

For a magnetic field given by (104), it can be shown that condition (122) holds if

$$0 \leqslant \hat{\mu}_B \leqslant \frac{4\hat{\eta}(1-\hat{\eta}^2)}{3-2\hat{\eta}^2-\hat{\eta}^4} \equiv \hat{\mu}_1\,. \qquad (123)$$

It is easy to obtain that $\hat{\mu}_1 < \hat{\mu}_0$ for $0 < \hat{\eta} < 1$, and therefore interval (123) is always narrower than interval (120). In this sense, the Couette flow with an azimuthal magnetic field is less stable under $m = 1$ perturbations than under axially symmetric perturbations. We recall that the current-free magnetic field ($\hat{\mu}_B = \hat{\eta}$) corresponds to parameters inside interval (123) is stable under both axially symmetric and asymmetric perturbations. Moreover, in the case of a fluid filling the entire volume inside the outer radius, we have $\hat{\eta} = 0$ for a pinch, and interval (123) degenerates. Correspondingly, the perturbations with $m = 1$ are always unstable for a pinch (in the absence of rotation). This property is certainly well known (see, e.g., Ref. [69]).

The stability of the Couette flows with an azimuthal magnetic field under asymmetric perturbations was studied in [76]. The results demonstrate, as in the case of axially symmetric perturbations, that magnetic fields can be subdivided into stable and unstable with respect to asymmetric perturbations according to ideal criterion (122). Accordingly, a magnetic field is referred to as stable under asymmetric perturbations if $\hat{\mu}_B$ belongs to interval (123) and unstable otherwise. As before, the unstable magnetic field loses its stability in a dissipative fluid only when the Hartmann number exceeds some critical value Ha_1.

Unfortunately, as mentioned above, a criterion that would allow labeling rotations as stable or unstable under asymmetric perturbations has not yet been proposed. Nevertheless, we can always decide about the stability of a flow by performing direct numerical simulations. As in other cases, viscosity stabilizes the flow and a critical Reynolds number Re_1 exists above which an unstable flow loses stability. Obviously, the critical Hartmann numbers are higher when $\hat{\mu}_B$ lies closer to stability interval (123). Critical Reynolds numbers for asymmetric perturbations, in contrast to the case of axially symmetric perturbations, depend on Pm. However, the critical Hartmann numbers depend only weakly on Pm for insulating boundaries and lack such a dependence for conducting cylinders. We emphasize that critical Hartmann numbers were numerically found only for two modes, $m = 0$ and $m = 1$ [76].

As already argued in Section 2, the instability of a Couette flow in the absence of a magnetic field is axially symmetric for cylinders rotating in the same sense and becomes asymmetric when they are counterrotating. However, in the presence of an azimuthal magnetic field, stability interval (123) for asymmetric perturbations is narrower than interval (120) related to axially symmetric perturbations. Thus, as $\hat{\mu}_B$ increases, the magnetic field loses stability, first with respect to the $m = 1$ mode (there exists only the Hartmann number Ha_1). As $\hat{\mu}_B$ increases further, the axially symmetric mode also loses stability (and the second Hartmann number Ha_0 arises). The rotation makes this arrangement more complicated, and the only true conjecture is that the instability has either the $m = 0$ or the $m = 1$ mode. Which of them is most unstable is strongly sensitive to the parameter choice (the modes can change, for instance, as a function of the magnetic field strength). To summarize, it seems difficult to find a rigorous selection criterion without turning to direct numerical computations [76].

For asymmetric perturbations, various combinations of stable and unstable rotation and the magnetic field typically behave similarly to the axially symmetric case. For example, a stable magnetic field stabilizes the Couette flow, and critical Reynolds numbers (if they exist) increase with an increase in the magnetic field strength.

There is an important distinction between axially symmetric and asymmetric perturbations, however. It is manifested in considering a combination of stable rotation and a magnetic field. A flow with the rotation and a magnetic field stable under axially symmetric perturbations remains stable. But a flow with rotation and the azimuthal magnetic field stable under asymmetric perturbations might become unstable. The instability is analogous to the MRI for a homogeneous axial magnetic field and was apparently first described in Ref. [77], although the authors did not identify it with a combined instability of the two stable components. It is referred to as the *azimuthal magnetorotational instability* (AMRI) in what follows.

The neutral stability curve for the AMRI was first computed in Ref. [78] (see also Refs [45, 79]). Its typical shape is schematically presented in Fig. 6. The main distinction between the AMRI and MRI lies in the existence of a minimum Hartmann number above which the instability

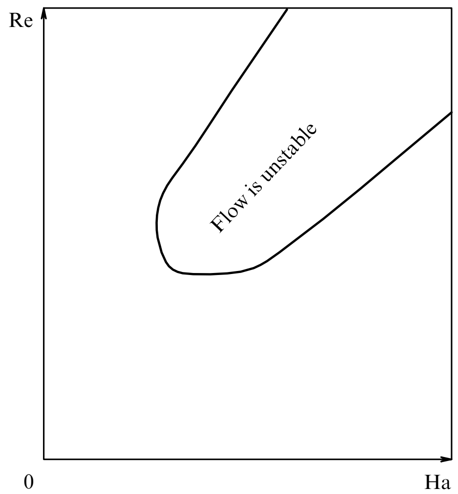


Figure 6. Schematic of the neutral stability curve for the $m = 1$ mode for stable rotation and a stable azimuthal magnetic field.

evolves. For the MRI, the left branch asymptotically tends to the ordinate axis without intersecting it (see Fig. 4). For the AMRI, the asymptote is tilted to the ordinate axis. As a result, for every Hartmann number greater than the critical one, there exist two Reynolds numbers, the minimum and maximum ones, that bound the instability interval. The flow is stable for small and large Reynolds numbers, but unstable for intermediate Reynolds numbers (Fig. 6).

We mention that the AMRI also exists for an unstable magnetic field, and is then combined with the PI yielding a complex resulting pattern. Depending on the parameters, the existence regions of both instabilities can be either well separated or intersecting [78].

As a conclusion to the discussion here, we list some open questions that are essential for understanding the Couette flow in the presence of an azimuthal magnetic field. As already mentioned, the simplest solution of the magnetohydrodynamics equations is stable [7]. This implies compensation of instabilities for axially symmetric perturbations [73]. Similarly, compensation must also exist for asymmetric perturbations. An important question concerns the constraints on the existence of the AMRI. For an ideal fluid, the AMRI is predicted for flows with the angular velocity decreasing with radius [77]. It would be desirable to generalize this result to dissipative fluids. The results in Refs [78, 79] allow assuming that the neutral stability curve for the AMRI tends to a universal curve in coordinates (L, Rm) for $\mathrm{Pm} \ll 1$. It would be interesting to confirm this result by computations performed for smaller values of Pm.

4.3 Helical (axial + azimuthal) magnetic field

In the presence of a helical magnetic field $\mathbf{B} = (0, B_\phi, B_z)$, Eqns (72)–(79) allow a steady solution of the general form

$$P = P(R)\,, \qquad \rho = \rho_0 = \text{const}\,, \tag{124}$$

$$U_R = 0\,, \qquad U_\phi = R\Omega(R)\,, \qquad U_z = 0\,, \tag{125}$$

$$B_R = 0\,, \qquad B_\phi = B_\phi(R)\,, \qquad B_z = B_z(R)\,, \tag{126}$$

where $\Omega(R)$, $B_\phi(R)$, and $B_z(R)$ are arbitrary functions of the radius satisfying boundary conditions in the case of an ideal fluid. For a dissipative fluid, the angular velocity is given by Eqn (10), the azimuthal magnetic field is given by Eqn (104), and B_z is determined from Eqn (78):

$$B_z(R) = a_z + b_z \ln(R)\,, \tag{127}$$

where

$$a_z = B_{z\,\mathrm{in}} \left(1 - \frac{1-\hat\mu_z}{\ln\hat\eta} \ln R_{\mathrm{in}}\right), \quad b_z = B_{z\,\mathrm{in}} \frac{1-\hat\mu_z}{\ln\hat\eta}\,, \tag{128}$$

$$\hat\mu_z = \frac{B_{z\,\mathrm{out}}}{B_{z\,\mathrm{in}}}\,, \tag{129}$$

and $B_{z\,\mathrm{in}}$ and $B_{z\,\mathrm{out}}$ are the axial magnetic fields at the inner and outer cylinders.

We note that the first term in Eqn (127) is current free, while the second term corresponds to the azimuthal current density

$$j_\phi = \frac{c}{4\pi} \frac{b_z}{R}\,. \tag{130}$$

The pressure is, as before, determined from Eqn (72), which now takes the form

$$\Omega^2 R = \frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}R}\left(P + \frac{B_\phi^2 + B_z^2}{8\pi}\right) - \frac{1}{4\pi\rho} \frac{B_\phi^2}{R}\,. \tag{131}$$

We are interested in the stability of steady solution (124)–(126). In the linear approximation, the perturbed solution is represented in the form

$$u_R(R,\phi,z)\,, \quad R\Omega(R) + u_\phi(R,\phi,z)\,, \quad u_z(R,\phi,z)\,,$$

$$b_R(R,\phi,z)\,, \quad B_\phi(R) + b_\phi(R,\phi,z)\,, \quad B_z(R) + b_z(R,\phi,z)\,,$$

$$P(R) + \frac{B_\phi^2(R) + B_z^2(R)}{8\pi} + p(R,\phi,z)\,, \tag{132}$$

where the perturbations u_R, u_ϕ, u_z, b_R, b_ϕ, b_z, and p are small compared with their unperturbed fields. The coefficients of linearized system (72)–(79) depend solely on the radial coordinate, which allows normal mode expansion (20).

To make the equations dimensionless, we use $R_0 = [R_{\mathrm{in}}(R_{\mathrm{out}} - R_{\mathrm{in}})]^{1/2}$ as the unit length, Ω_{in} as the unit frequency, η/R_0 as the unit perturbed velocity, $B_{z\,\mathrm{in}}$ as the unit magnetic field (both perturbed and unperturbed), and $\rho_0 \nu\eta/R_0^2$ as the unit pressure.

Linearizing system (72)–(79), expanding the unknown fields in normal modes (20), and keeping the same notation for dimensional and dimensionless fields, we obtain

$$\begin{aligned} &\mathrm{iRe}(\omega + m\Omega)\, u_R - 2\mathrm{Re}\,\Omega u_\phi - \mathrm{iHa}^2\alpha\, \frac{m}{R}\, B_\phi b_R \\ &\quad + 2\alpha\mathrm{Ha}^2 \frac{B_\phi}{R}\, b_\phi - \mathrm{iHa}^2 k B_z b_R = -\frac{\mathrm{d}p}{\mathrm{d}R} + \frac{\mathrm{d}^2 u_R}{\mathrm{d}R^2} \\ &\quad + \frac{1}{R} \frac{\mathrm{d}u_R}{\mathrm{d}R} - \frac{u_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_R - 2\mathrm{i}\, \frac{m}{R^2}\, u_\phi\,, \end{aligned} \tag{133}$$

$$\begin{aligned} &\mathrm{iRe}(\omega + m\Omega)\, u_\phi + \frac{\mathrm{Re}}{R} \frac{\mathrm{d}(R^2\Omega)}{\mathrm{d}R}\, u_R - \alpha\, \frac{\mathrm{Ha}^2}{R} \frac{\mathrm{d}(RB_\phi)}{\mathrm{d}R} \\ &\quad - \mathrm{iHa}^2\alpha\, \frac{m}{R}\, B_\phi b_\phi - \mathrm{iHa}^2 k B_z b_\phi = -\mathrm{i}\, \frac{m}{R}\, p + \frac{\mathrm{d}^2 u_\phi}{\mathrm{d}R^2} \\ &\quad + \frac{1}{R} \frac{\mathrm{d}u_\phi}{\mathrm{d}R} - \frac{u_\phi}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) u_\phi + 2\mathrm{i}\, \frac{m}{R^2}\, u_R\,, \end{aligned} \tag{134}$$

$$\mathrm{iRe}(\omega + m\Omega)\,u_z - \mathrm{iHa}^2\alpha\,\frac{m}{R}\,B_\phi b_z - \mathrm{Ha}^2\,\frac{\mathrm{d}B_z}{\mathrm{d}R}\,b_R$$
$$- \mathrm{iHa}^2 k B_z b_z = -\mathrm{i}kp + \frac{\mathrm{d}^2 u_z}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_z}{\mathrm{d}R} - \left(\frac{m^2}{R^2} + k^2\right) u_z\,, \tag{135}$$

$$\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R} + \mathrm{i}\,\frac{m}{R}\,u_\phi + \mathrm{i}ku_z = 0\,, \tag{136}$$

$$\mathrm{iRe\,Pm}(\omega + m\Omega)\,b_R - \mathrm{i}kB_z u_R - \mathrm{i}\alpha\,\frac{m}{R}\,B_\phi u_R = \frac{\mathrm{d}^2 b_R}{\mathrm{d}R^2}$$
$$+ \frac{1}{R}\,\frac{\mathrm{d}b_R}{\mathrm{d}R} - \frac{b_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) b_R - 2\mathrm{i}\,\frac{m}{R^2}\,b_\phi\,, \tag{137}$$

$$\mathrm{iRe\,Pm}(\omega + m\Omega)\,b_\phi - \mathrm{Re\,Pm}R\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\,b_R + \alpha R\,\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{B_\phi}{R}\right) u_R$$
$$- \mathrm{i}\alpha\,\frac{m}{R}\,B_\phi u_\phi - \mathrm{i}kB_z u_\phi = \frac{\mathrm{d}^2 b_\phi}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}b_\phi}{\mathrm{d}R} - \frac{b_\phi}{R^2}$$
$$- \left(\frac{m^2}{R^2} + k^2\right) b_\phi + 2\mathrm{i}\,\frac{m}{R^2}\,b_R\,, \tag{138}$$

$$\mathrm{iRe\,Pm}(\omega + m\Omega)\,b_z + \frac{\mathrm{d}B_z}{\mathrm{d}R}\,u_R - \mathrm{i}\alpha\,\frac{m}{R}\,B_\phi u_z - \mathrm{i}kB_z u_z$$
$$= \frac{\mathrm{d}^2 b_z}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}b_z}{\mathrm{d}R} - \left(\frac{m^2}{R^2} + k^2\right) b_z\,, \tag{139}$$

$$\frac{\mathrm{d}b_R}{\mathrm{d}R} + \frac{b_R}{R} + \mathrm{i}\,\frac{m}{R}\,b_\phi + \mathrm{i}kb_z = 0\,. \tag{140}$$

The dimensionless numbers of the problem are once again the Prandtl number Pm, the Hartmann number Ha, defined here differently from Sections 2, 3, 4.1, and 4.2, the Reynolds number Re, and, additionally, the ratio of the azimuthal and axial magnetic fields on the inner cylinder,

$$\mathrm{Pm} = \frac{\nu}{\eta}\,,\quad \mathrm{Ha} = \frac{B_{z\,\mathrm{in}} R_0}{\sqrt{4\pi\rho_0 \nu\eta}}\,,\quad \mathrm{Re} = \frac{\Omega_{\mathrm{in}} R_0^2}{\nu}\,,\quad \alpha = \frac{B_{\mathrm{in}}}{B_{z\,\mathrm{in}}}\,. \tag{141}$$

The boundary conditions for the velocity are given by expressions (19) at both the inner and outer cylinders. The magnetic field must satisfy either boundary conditions (93) for conducting cylinders or those specified by Eqns (94), (95), and (96) for insulating ones.

The essential distinction of this case from those considered above lies in its other symmetry type. As a result, when studying the Couette flow in a helical magnetic field, we must analyze perturbations with $m > 0$ as well as with $m < 0$. Equivalently, we can restrict ourselves to perturbations with $m > 0$, but allow magnetic field configurations with $\alpha > 0$ as well as with $\alpha < 0$.

Unfortunately, a detailed analysis of the Couette flow stability in the presence of a helical magnetic field is not yet available. To explore the stability of the magnetic field proper, the so-called energy principle is frequently used [80]. It was generalized in Ref. [81] to allow the presence of a steady flow. But the results in Refs [80, 81] are too general and do not allow formulating any simple stability condition. In discussing the stability of the helical magnetic field per se, we note Refs [82, 83]. It is shown in [82] that the magnetic field is unstable if

$$\int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}} B_\phi^2 R\,\mathrm{d}R > 2\int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}} B_z^2 R\,\mathrm{d}R\,. \tag{142}$$

Three different stability conditions for a helical magnetic field and an ideal conducting plasma are derived in [83]. The first is a sufficient stability condition, the second is a necessary condition, and the third is a necessary and sufficient condition of stability. Discouragingly, the most relevant, necessary and sufficient condition depends on the zeros of a solution of the Euler–Lagrange equation (a second-order equation analogous to Eqn (25) but with a helical magnetic field), which are not known in the general form (although they can be found in concrete cases). This hampers its practical use. We note that both the sufficient and necessary conditions were generalized to the case where radial forces are acting [74]. For axially symmetric perturbations in an incompressible fluid, the sufficient stability condition was generalized to the case of rotating fluid in [26]. The necessary stability condition in the absence of rotation was derived in [84]; it is applicable to magnetic field configurations containing a singular point where $mB_\phi - kB_z = 0$. The condition in [84] was generalized in [85] to the case of a moving fluid (see also Ref. [86]).

The stability of a dissipative pinch was explored in Ref. [87]. In the case of a pinch, in contrast to the Couette flow, the fluid fills the entire volume confined by the outer pinch radius ($R_{\mathrm{in}} = \hat{\eta} = 0$); according to Eqns (11) and (105), the angular velocity is then constant, $\Omega = \mathrm{const}$ ($b_\Omega = 0$), while in the azimuthal magnetic field, according to Eqns (104) and (105), only the first term is preserved ($b_B = 0$), pertaining to a constant axial current. The important difference between a pinch and a Couette flow is the presence of a free, not fixed, outer boundary.

In the presence of a uniform axial field B_0, the problem admits an analytic solution expressed in terms of Bessel functions. In particular, it is shown in Ref. [87] that in the limit of vanishing viscosity (as mentioned above, this limit agrees well with experiments performed with liquid metals), perturbations with $m = -1$ and the axial wave numbers such that $B_\phi(R_{\mathrm{out}})/R_{\mathrm{out}} + kB_0 = 0$ are always unstable (see also Ref. [88]). We recall that the instability, as discussed, depends on the sign of the azimuthal number. It turns out that perturbations with $m < 0$ are less stable (or, equivalently, the magnetic field configurations with $\alpha < 0$ are more prone to instability).

We mention the results in [89], where a stability criterion for a compressible fluid differing from that in [84] was derived.

As regards the true Couette flow, it was recently shown that adding a current-free azimuthal magnetic field (which is itself stable) to a uniform axial magnetic field destabilizes the Couette flow with respect to axially symmetric perturbations [90, 91]. In the presence of an axial magnetic field, the Couette flow loses stability beyond the Rayleigh line. But the critical Reynolds numbers are very large at small magnetic Prandtl numbers typical of experiments with liquid metals (see Section 4.1). It turns out [90, 91] that adding a current-free azimuthal magnetic field to the axial one markedly reduces the magnitude of critical Reynolds numbers (from $\sim 10^6$ to $\sim 10^4$).

4.4 A magnetic field and the Hall currents

If Hall currents are significant in the fluid, Ohm's law (68) becomes

$$\mathbf{E}+\frac{1}{c}\,\mathbf{U}\times\mathbf{B}=R_{\mathrm{Om}}\,\mathbf{j}+R_{\mathrm{H}}\,\mathbf{j}\times\mathbf{b}\,, \tag{143}$$

where R_{H} is the specific Hall resistance and $\mathbf{b}$ is the unit vector along the magnetic field. With the Hall currents taken into account, Eqn (67) is written as

$$\frac{\partial\mathbf{B}}{\partial t}=\mathrm{rot}(\mathbf{u}\times\mathbf{B})+\eta\Delta\mathbf{B}-\beta\,\mathrm{rot}(\mathrm{rot}\,\mathbf{B}\times\mathbf{B})\,, \tag{144}$$

where βB is the Hall diffusion coefficient. Here and hereafter, it is assumed that η and β are independent of the coordinates.

It is a straightforward exercise to verify that all three magnetic field configurations given by Eqns (80)–(82), (101)–(103), and (124)–(126) also satisfy the magnetohydrodynamic equations with Hall currents (65), (66), and (144) taken into account. In this section, we do not write equations for particular cases of uniform axial and azimuthal magnetic fields (they readily follow from the general equations written below upon setting $B_z=0$ or $B_\phi=0$, respectively), and directly write general equations valid in the presence of both fields. After linearization, expansion in normal modes, and normalization, Eqns (133)–(136) and (140) retain their form, while Eqns (137)–(139) become

$$\begin{aligned}
&\mathrm{i}\,\mathrm{Re}\,\mathrm{Pm}(\omega+m\Omega)\,b_R-\mathrm{i}kB_zu_R-\mathrm{i}\alpha\frac{m}{R}\,B_\phi u_R\\
&=\frac{\mathrm{d}^2b_R}{\mathrm{d}R^2}+\frac{1}{R}\,\frac{\mathrm{d}b_R}{\mathrm{d}R}-\frac{b_R}{R^2}-\left(\frac{m^2}{R^2}+k^2\right)b_R-2\mathrm{i}\,\frac{m}{R^2}\,b_\phi\\
&-\alpha_{\mathrm{H}}\Bigg(\mathrm{i}\,\frac{m}{R}\,\frac{\mathrm{d}B_z}{\mathrm{d}R}\,b_R-\alpha\,\frac{m^2}{R^2}\,B_\phi b_z-k\,\frac{m}{R}\,B_zb_z\\
&-\mathrm{i}k\,\frac{\mathrm{d}B_\phi}{\mathrm{d}R}\,b_R+\alpha k\,\frac{m}{R}\,B_\phi b_\phi+k^2B_zb_\phi-\mathrm{i}\alpha k\,\frac{B_\phi}{R}\,b_R\Bigg),
\end{aligned} \tag{145}$$

$$\begin{aligned}
&\mathrm{i}\,\mathrm{Re}\,\mathrm{Pm}(\omega+m\Omega)\,b_\phi-\mathrm{Re}\,\mathrm{Pm}R\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\,b_R\\
&+\alpha R\,\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{B_\phi}{R}\right)u_R-\mathrm{i}\alpha\,\frac{m}{R}\,B_\phi u_\phi-\mathrm{i}kB_zu_\phi\\
&=\frac{\mathrm{d}^2b_\phi}{\mathrm{d}R^2}+\frac{1}{R}\,\frac{\mathrm{d}b_\phi}{\mathrm{d}R}-\frac{b_\phi}{R^2}-\left(\frac{m^2}{R^2}+k^2\right)b_\phi+2\mathrm{i}\,\frac{m}{R^2}\,b_R\\
&-\alpha_{\mathrm{H}}\Bigg(-\alpha k\,\frac{m}{R}\,B_\phi b_R-k^2B_zb_R-\frac{\mathrm{d}B_z}{\mathrm{d}R}\,\frac{\mathrm{d}b_R}{\mathrm{d}R}\\
&-\frac{\mathrm{d}^2B_z}{\mathrm{d}R^2}\,b_R+\mathrm{i}\alpha\,\frac{m}{R}\,\frac{B_\phi}{R}\,b_z-\mathrm{i}\alpha\,\frac{m}{R}\,\frac{\mathrm{d}B_\phi}{\mathrm{d}R}\,b_z\\
&-\mathrm{i}\alpha\,\frac{m}{R}\,B_\phi\,\frac{\mathrm{d}b_z}{\mathrm{d}R}-\mathrm{i}k\,\frac{\mathrm{d}B_z}{\mathrm{d}R}\,b_z-\mathrm{i}kB_z\,\frac{\mathrm{d}b_z}{\mathrm{d}R}-\mathrm{i}\alpha\,\frac{2}{R}\,kB_\phi b_\phi\Bigg),
\end{aligned} \tag{146}$$

$$\begin{aligned}
&\mathrm{i}\,\mathrm{Re}\,\mathrm{Pm}(\omega+m\Omega)\,b_z+\frac{\mathrm{d}B_z}{\mathrm{d}R}\,u_R-\mathrm{i}\alpha\,\frac{m}{R}\,B_\phi u_z-\mathrm{i}kB_zu_z\\
&=\frac{\mathrm{d}^2b_z}{\mathrm{d}R^2}+\frac{1}{R}\,\frac{\mathrm{d}b_z}{\mathrm{d}R}-\left(\frac{m^2}{R^2}+k^2\right)b_z\\
&-\alpha_{\mathrm{H}}\Bigg[\alpha\,\frac{\mathrm{d}B_\phi}{\mathrm{d}R}\left(\frac{2}{R}\,b_R+\frac{\mathrm{d}b_R}{\mathrm{d}R}+\mathrm{i}\,\frac{m}{R}\,b_\phi\right)+\alpha\,\frac{\mathrm{d}^2B_\phi}{\mathrm{d}R^2}\,b_R\\
&+\alpha B_\phi\left(\mathrm{i}\,\frac{m}{R}\,\frac{\mathrm{d}b_\phi}{\mathrm{d}R}+\frac{m^2}{R^2}\,b_R+\frac{1}{R}\,\frac{\mathrm{d}b_R}{\mathrm{d}R}+2\mathrm{i}\,\frac{m}{R^2}\,b_\phi\right)\\
&+B_z\left(\mathrm{i}k\,\frac{b_\phi}{R}+\mathrm{i}k\,\frac{\mathrm{d}b_\phi}{\mathrm{d}R}+k\frac{m}{R}\,b_R\right)+\mathrm{i}k\,\frac{\mathrm{d}B_z}{\mathrm{d}R}\,b_\phi\Bigg],
\end{aligned} \tag{147}$$

where α_{H} is the Hall parameter (the ratio of the Hall magnetic diffusion time to the Ohmic dissipation time of the magnetic field),

$$\alpha_{\mathrm{H}}=\frac{\beta B_{z\,\mathrm{in}}}{\eta}\,. \tag{148}$$

The dimensionless numbers in (145)–(147) are given by (141); the terms proportional to α_{H} owe their existence to the Hall effect.

We recall that similarly to the case of a helical magnetic field, both the $m>0$ and $m<0$ possibilities must be considered.

For an ideal dielectric boundary, the magnetic field at the fluid boundary must coincide with the external one in the presence of the Hall effect; hence, conditions (95) and (96) preserve their form.

For conducting boundaries, in the first order in the perturbations, the requirement that tangential components of the electric field vanish gives

$$\frac{\mathrm{d}b_\phi}{\mathrm{d}R}+\frac{b_\phi}{R}=\mathrm{i}\alpha_{\mathrm{H}}B_\phi\left(kb_\phi-\frac{m}{R}\,b_z\right). \tag{149}$$

This condition is, as before, applicable to the outer and inner cylinders. The condition requiring that the normal component of the magnetic field vanish at the boundary ($b_R=0$) does not change.

The influence of Hall currents on the stability of a Couette flow in the presence of a uniform vertical magnetic field was considered in [92]. Without Hall currents, the direction of the magnetic field is irrelevant: only the magnitude matters (Eqns (84)–(91) depend only on Ha^2). The Hall currents make the instability sensitive to the magnitude and the direction of the magnetic field. Moreover, in the presence of the Hall effect, a Couette flow in a uniform axial magnetic field becomes unstable for any rotation law (critical Reynolds numbers exist for any $\hat{\mu}_\Omega$). The destabilization of rotation with an increasing angular velocity under the action of the Hall effect was first discovered for accretion disks [93]. Flows that are stable even under the MRI ($\hat{\mu}_\Omega>1$) are destabilized by the magnetic field directed against the rotation vector (if the Hall resistance sign is positive). The Hall effect also destabilizes flows that were unstable without it, for instance, the flow with a resting outer cylinder. In this case, the critical Reynolds numbers decrease dramatically. Nonetheless, estimates of the magnetic field magnitude conductive to the *Hall magnetorotational instability* (HMRI) in experiments with liquid metals predict excessively strong magnetic fields ($\sim 10^7$ G), which creates serious obstacles to their practical implementation. The HRMI, however, might prove to be important in astrophysical applications (especially for weakly ionized accretion disks [93–96]). We emphasize that even there, the development of HMRI demands that the magnetic fields be sufficiently strong [92, 97].

It is shown in [98] that the Hall effect can also essentially affect the pinch instability of the azimuthal magnetic field. A consequence of this instability can be, for example, a different magnitude of the azimuthal magnetic field in the different hemispheres of a neutron star. In addition, the computation results in Ref. [98] show that under the action of the Hall effect, the AMRI may unfold for arbitrary rotation laws, similarly to the MRI. Once again, the laboratory study of these effects is impeded by the limitation on the magnitude of the magnetic field needed for their manifestation.

5. A flow with a nonuniform density and a magnetic field

An incompressible inhomogeneous fluid in the presence of a magnetic field is described by the equations

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\,\mathbf{U} = -\frac{1}{\rho}\,\nabla P + \mathbf{g} + \nu\Delta\mathbf{U} + \frac{1}{c}\,\mathbf{j}\times\mathbf{B}\,, \qquad (150)$$

$$\operatorname{div}\mathbf{U} = 0\,, \qquad \operatorname{div}\mathbf{B} = 0\,, \qquad (151)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}(\mathbf{U}\times\mathbf{B}) + \eta\Delta\mathbf{B}\,, \qquad (152)$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{U}\nabla)\rho = 0\,. \qquad (153)$$

5.1 Radial density stratification and an azimuthal magnetic field

Equations (150)–(153) admit the solution

$$U_R = 0\,, \quad U_\phi = R\Omega(R)\,, \quad U_z = 0\,, \qquad (154)$$

$$B_R = 0\,, \quad B_\phi = B_\phi(R)\,, \quad B_z = 0\,, \qquad (155)$$

$$\rho = \rho_0(R)\,, \quad P = P_0(R)\,, \qquad (156)$$

where $\Omega(R)$ and $B_\phi(R)$ are, as everywhere above, either arbitrary functions satisfying boundary conditions in the case of an ideal fluid or functions defined by expressions (10) and (104) for a dissipative fluid; the pressure P_0 is defined by Eqn (108).

It can be readily shown that for an ideal fluid, and if perturbations are axially symmetric, linearized system (150)–(153), similarly to those in the case with solely radial density stratification (see Section 3.1) or solely an azimuthal magnetic field (see Section 4.2), reduces to the second-order equation

$$-\frac{\mathrm{d}}{\mathrm{d}R}\left[\rho_0\,\frac{1}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}\left(Ru_R\right)\right] + k^2\rho_0 u_R$$
$$-\rho_0\,\frac{k^2}{\omega^2}\left[\frac{1}{R^3}\,\frac{\mathrm{d}}{\mathrm{d}R}(R^2\Omega)^2 - \frac{R}{4\pi\rho_0}\,\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{B_\phi}{R}\right)^2\right.$$
$$\left. + (g_R + \Omega^2 R)\,\frac{1}{\rho_0}\,\frac{\mathrm{d}\rho_0}{\mathrm{d}R}\right] = 0\,. \qquad (157)$$

As before, equation (157) with boundary conditions (19) is a classic Sturm–Liouville problem. Accordingly, for the stability of the Couette flow under axially symmetric perturbations, it is necessary and sufficient that

$$\frac{1}{R^3}\,\frac{\mathrm{d}}{\mathrm{d}R}\,(R^2\Omega)^2 - \frac{R}{4\pi\rho_0}\,\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{B_\phi}{R}\right)^2$$
$$+ (g_R + \Omega^2 R)\,\frac{1}{\rho_0}\,\frac{\mathrm{d}\rho_0}{\mathrm{d}R} > 0\,. \qquad (158)$$

Condition (158) is a generalization of Michael condition (119) to the case of an inhomogeneous incompressible fluid. A more detailed analysis of the stability of solution (154)–(156) has not been performed for the Couette flow. We note that, as in the absence of a magnetic field, the kinematic viscosity is in the general case a function of the radial coordinate (it can be considered uniform only in the approximation of weak stratification).

For completeness, we note that in the absence of rotation, the necessary and sufficient condition of the stability of the azimuthal magnetic field under both axially symmetric and asymmetric perturbations in the presence of radial as well as axial stratification is known [75]. In the presence of stratification, it was shown for axially symmetric perturbations that conditions generalizing those derived in Ref. [75] are only sufficient for stability [99].

5.2 Axial density stratification and an azimuthal magnetic field

It was shown in Section 3.2 that in the presence of vertical density stratification, system (150)–(153) allows a solution for the angular velocity in form (10) only in the limit of slow rotation and small density stratification (46). It can be easily verified that the same holds in the presence of an azimuthal magnetic field that depends only on the radial coordinate. Thus, system (150)–(153) under constraints (46) allows the solution

$$U_R = 0\,, \quad U_\phi = R\Omega(R)\,, \quad U_z = 0\,, \qquad (159)$$

$$B_R = 0\,, \quad B_\phi = B_\phi(R)\,, \quad B_z = 0\,, \qquad (160)$$

$$P = P_0(z) + P_1(R,z)\,, \quad \rho = \rho_0 + \rho_1(z)\,, \qquad (161)$$

where ρ_0 is a uniform reference density, P is the total pressure (with the magnetic part), $|P_1/P_0| \ll 1$, $\rho_1/\rho_0 \ll 1$, and $\Omega(R)$ and $B_\phi(R)$ are, as above, either arbitrary functions satisfying boundary conditions for an ideal fluid or the functions defined by expressions (10) and (104) for a dissipative fluid.

We consider the linear stability of the solution of Eqns (159)–(161) using a perturbed solution in the form $\mathbf{U}+\mathbf{u}(R,\phi,z)$, $\mathbf{B}+\mathbf{b}(R,\phi,z)$, $\rho_0+\rho_1(z)+\rho(R,\phi,z)$, and $P_0(z)+P_1(R,z)+B_\phi^2/8\pi+p(R,\phi,z)$. With conditions (46), linearized system (150)–(153) takes the classic form of the Boussinesq approximation with coefficients depending only on the radius. Consequently, we can use expansion (20) in normal modes. Performing linearization, expansion in normal modes, and normalization (the units selected for that are given in Sections 3.2 and 4.2), we transform system (150)–(153) to the form (as everywhere above, the same notation is used for dimensional and dimensionless variables)

$$\mathrm{iRe}(\omega + m\Omega)\,u_R - 2\mathrm{Re}\,\Omega u_\phi - \mathrm{iHa}^2\,\frac{m}{R}\,B_\phi b_R$$
$$+ 2\mathrm{Ha}^2\,\frac{B_\phi}{R}\,b_\phi = -\frac{\mathrm{d}p}{\mathrm{d}R} + \frac{\mathrm{d}^2u_R}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_R}{\mathrm{d}R} - \frac{u_R}{R^2}$$
$$-\left(\frac{m^2}{R^2}+k^2\right)u_R - 2\mathrm{i}\,\frac{m}{R^2}\,u_\phi\,, \qquad (162)$$

$$\mathrm{iRe}(\omega + m\Omega)\,u_\phi + \frac{\mathrm{Re}}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}\,(R^2\Omega)\,u_R - \frac{\mathrm{Ha}^2}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}(RB_\phi)$$
$$-\mathrm{iHa}^2\,\frac{m}{R}\,B_\phi b_\phi = -\mathrm{i}\,\frac{m}{R}\,p + \frac{\mathrm{d}^2u_\phi}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_\phi}{\mathrm{d}R} - \frac{u_\phi}{R^2}$$
$$-\left(\frac{m^2}{R^2}+k^2\right)u_\phi + 2\mathrm{i}\,\frac{m}{R^2}\,u_R\,, \qquad (163)$$

$$\mathrm{iRe}(\omega + m\Omega)\,u_z - \mathrm{iHa}^2\,\frac{m}{R}\,B_\phi b_z = -\mathrm{i}kp + \mathrm{Re}\,\rho$$
$$+ \frac{\mathrm{d}^2u_z}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}u_z}{\mathrm{d}R} - \left(\frac{m^2}{R^2}+k^2\right)u_z\,, \qquad (164)$$

$$\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R} + \mathrm{i}\,\frac{m}{R}\,u_\phi + \mathrm{i}ku_z = 0\,, \tag{165}$$

$$\mathrm{iRe\,Pm}(\omega + m\Omega)\,b_R - \mathrm{i}\,\frac{m}{R}\,B_\phi u_R = \frac{\mathrm{d}^2 b_R}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}b_R}{\mathrm{d}R} - \frac{b_R}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) b_R - 2\mathrm{i}\,\frac{m}{R^2}\,b_\phi\,, \tag{166}$$

$$\mathrm{iRe\,Pm}(\omega + m\Omega)\,b_\phi - \mathrm{Re\,Pm}R\,\frac{\mathrm{d}\Omega}{\mathrm{d}R}\,b_R + R\,\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{B_\phi}{R}\right)u_R - \mathrm{i}\,\frac{m}{R}\,B_\phi u_\phi = \frac{\mathrm{d}^2 b_\phi}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}b_\phi}{\mathrm{d}R} - \frac{b_\phi}{R^2} - \left(\frac{m^2}{R^2} + k^2\right) b_\phi + 2\mathrm{i}\,\frac{m}{R^2}\,b_R\,, \tag{167}$$

$$\mathrm{iRe\,Pm}(\omega + m\Omega)\,b_z - \mathrm{i}\,\frac{m}{R}\,B_\phi u_z = \frac{\mathrm{d}^2 b_z}{\mathrm{d}R^2} + \frac{1}{R}\,\frac{\mathrm{d}b_z}{\mathrm{d}R} - \left(\frac{m^2}{R^2} + k^2\right) b_z\,, \tag{168}$$

$$\frac{\mathrm{d}b_R}{\mathrm{d}R} + \frac{b_R}{R} + \mathrm{i}\,\frac{m}{R}\,b_\phi + \mathrm{i}kb_z = 0\,, \tag{169}$$

$$\mathrm{i}(\omega + m\Omega)\rho + N_z^2 u_z = 0\,. \tag{170}$$

The dimensionless numbers in the problem — the magnetic Prandtl number Pm, Hartmann number Ha, Reynolds number Re, and Froude number Fr — are defined as

$$\mathrm{Pm} = \frac{\nu}{\eta}\,, \quad \mathrm{Ha} = \frac{B_{\mathrm{in}} R_0}{\sqrt{4\pi\rho_0 \nu\eta}}\,, \quad \mathrm{Re} = \frac{\Omega_{\mathrm{in}} R_0^2}{\nu}\,, \quad \mathrm{Fr} = \frac{\Omega_{\mathrm{in}}}{N_z}\,, \tag{171}$$

where N_z is Brunt–Väiälä frequency (55).

The general stability condition for an ideal Couette flow in the presence of axial density stratification and an azimuthal magnetic field is unknown. For a dissipative Couette flow, this problem was first considered in Ref. [45]. Three cases were treated separately: 1) the azimuthal magnetorotational instability, when the magnetic field and the rotation are unstable separately; 2) the magnetic field is stable under axially symmetric perturbations but unstable under perturbations with $m = 1$ (one critical Hartmann number); 3) the magnetic field is unstable under perturbation with $m = 0$ as well as $m = 1$.

Briefly, the stability of a Couette flow in this case exhibits stability properties combining those pertaining to the case with density stratification but without the magnetic field and the case with a magnetic field and a uniform density, which creates an intricate resulting picture.

In the first case, we can speak about the influence of axial density stratification on the AMRI as well as the influence of the magnetic field on the SRI. The influence of density stratification largely amounts to destabilizing the rotation with respect to axially symmetric perturbations, and the rotation stable for a uniform density can become unstable in the presence of axial density stratification. We recall that the AMRI exists for arbitrary rotation with the angular velocity decreasing in absolute value. The SRI exists in a narrower interval (depending on the gap between the cylinders). An azimuthal magnetic field destabilizes flows with decreasing angular velocity also in the presence of a stable density stratification. The stable density stratification makes critical Reynolds and Hartmann numbers larger. The influence of the magnetic field on the SRI is similar to its influence on the RI. For small Pm, the magnetic field suppresses the instability (critical Reynolds numbers increase), but for $\mathrm{Pm} > 1$, the magnetic field facilitates the instability if the magnitude of the magnetic field is not too large (critical Reynolds numbers are lower with the magnetic field than without it).

The most significant feature of the effect of the stable axial density stratification on an unstable magnetic field is the essential expansion of the stability regions for flows characterized by a combination of unstable rotation and an unstable magnetic field.

6. Conclusions

The results described in this review demonstrate that despite more than a century of history of studying the Couette flow stability, we are still far from fully understanding it. This is even more so in the presence of additional factors such as nonuniform density distribution and a magnetic field. These factors have been selected not arbitrarily but because of their omnipresence in laboratory conditions as well as in natural conditions related, for instance, to astrophysical objects.

We briefly summarize the main results pertaining to the instability of the Couette flow.

The classical ideal Couette flow is stable under axially symmetric perturbations if the angular momentum magnitude is an increasing function of radius, Eqn (26). The exhaustive criterion for asymmetric perturbations has not yet been established. But both theory and experiment suggest that for cylinders spinning in the same direction, the axially symmetric mode is the most unstable. This instability is monotonic (the real part of the instability increment is equal to zero) and is called the *rotational instability*. The viscosity stabilizes the RI and a viscous Couette flow that would be unstable according to criterion (26) actually loses its stability only if the rotation is sufficiently fast (large Reynolds numbers). The RI shows good correspondence between theory and experiment. For completeness, we remark that theoretical results predict an asymmetric instability beyond the Rayleigh limit, although not for rigid boundaries, as for the Couette flow, but for free ones.

A stable vertical density stratification stabilizes the axially symmetric mode but destabilizes asymmetric modes, which become more unstable even for cylinders rotating in the same sense. The most unstable in this case is the $m = 1$ mode, while the instability ceases to be monotonic and becomes oscillatory. Moreover, this instability persists even beyond the Rayleigh limit. It is called the *stratorotational instability*. The existence boundary of the SRI is between Rayleigh limit (26) and classical magnetorotational instability limit (97) and is strongly sensitive to the size of the gap between the cylinders — the instability limit decreases as the gap increases. Such behavior is suggestive of a strong sensitivity to boundary conditions. Exploring this question requires going beyond the framework of classical Couette flows. The results that follow are ambiguous, however. On the one hand, there are results indicating that the SRI disappears in the absence of rigid boundaries [100]; on the other hand, there are results showing that the SRI is also preserved for soft boundaries when boundary stresses are absent [101]. As we have already noted, good agreement is also observed for the Couette flow between theoretical and experimental data on the SRI [44].

A uniform axial magnetic field, being stable per se, nevertheless destabilizes the Couette flow. This instability, termed *magnetorotational*, is historically the best known Couette flow instability except, perhaps, the RI proper. It is manifested for both stable and unstable flows in the absence of a magnetic field. For flows unstable in the absence of a magnetic field, the critical Reynolds numbers may decrease in the presence of a field. Still, this property is strongly dependent on the magnetic Prandtl number and is manifested only if $\mathrm{Pm} \sim 1$. The main manifestation of the MRI, however, is the destabilization of flows having an angular velocity whose absolute value decreases with radius, and a change of the instability criterion from (26) to (97). The critical numbers of this instability are strongly influenced by the magnetic Prandtl number, which entail difficulties when it comes to laboratory experiments. As a result, the MRI has not as yet been observed experimentally.

The azimuthal magnetic field complicates the problem because it can be unstable by itself (in the absence of rotation). This is the so-called *pinch instability*. The combination of an azimuthal magnetic field and rotation creates an intricate picture of the interaction between the RI and PI. Depending on the parameters, the most unstable is either the $m = 0$ or the $m = 1$ mode. Noteworthy are two interesting facts pertaining to this case. The first is the compensation of instabilities when a combination of unstable rotation and an unstable magnetic field can result in a stable flow. The second is the opposite phenomenon, when a combination of stable rotation and a stable magnetic field excite the so-called *azimuthal magnetorotational instability*. The characteristics of the AMRI in practice resemble those of the MRI, including the limits within which it exists (it destabilizes flows as the magnitude of angular velocity decreases), except for the AMRI being asymmetric ($m = 1$). Experimental implementation of interesting regimes in the presence of an azimuthal magnetic field requires maintaining currents inside the fluid, which is apparently problematic to a certain degree.

The situation with a helical magnetic field is even more complex. The only result worth mentioning is an essential decrease in critical Reynolds numbers under the joint action of a uniform vertical magnetic field and a current-free azimuthal magnetic field (being stable itself). The experiments in Refs [102–104] are interpreted by their authors as confirmation of the *helical magnetorotational instability*. This interpretation is questioned in Ref. [105, 106] and the observed instability is explained as a result of transient amplification of modes excited in boundary layers, which have no relation to global instability. The question therefore remains open.

Theoretically, the Hall effect is most promising from the standpoint of exploring the instability. The analysis for a uniform axial magnetic field has shown that this effect leads to destabilization of flows with any law of rotation. The main obstacle on the road to its laboratory implementation as well as its manifestations in natural conditions is the need for extremely strong magnetic fields. We note that the sensitivity of the stability of rotating magnetized fluid to modifications of the Ohm law seems to be, a universal feature. It was shown in Ref. [107] that the modification of the Ohm law in the presence of a dust component also leads to destabilization of rotation with an arbitrary angular velocity profile.

The study of the Couette flow stability with a nonuniform density and a magnetic field is admittedly only in its earlier phase. But the first results have already demonstrated an exceptionally complex picture of the SRI interaction with the AMRI and PI.

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