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Rotation of the swing plane of Foucault's pendulum and Thomas spin precession: two sides of one coin

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Abstract. Using elementary geometric tools, we apply essentially the same methods to derive expressions for the rotation angle of the swing plane of Foucault's pendulum and the rotation angle of the spin of a relativistic particle moving in a circular orbit (the Thomas precession effect).

1. Introduction

Jean Bernard Léon Foucault conducted his first pendulum experiment in Paris in January 1851, aiming to prove rotation of the Earth by evidently demonstrating the rotation of the swing plane of the pendulum. Originally, the suspension length of the pendulum was 2 m. The next experiment was set up with the suspension length 11 m at the Observatory of Paris. Louis-Napoléon Bonaparte, the first president of the French Republic and nephew of famous French emperor Napoléon I, was informed of Foucault's work and proposed that he conduct an experiment at the Panthéon. The experiment took place on 31 March 1851, with the pendulum bob weight 28 kg suspended under the Panthéon dome by a steel wire 67 m long.

From the standpoint of a ground-based observer, remote stars rotate clockwise and make a complete revolution in 1 sidereal (star) day — 23 hours, 56 minutes, and 4.091 seconds. Aristarchus of Samos, who proposed the first consistent heliocentric system ca. 270 BC, related the observable rotation of stars to the Earth's axial rotation. Similar ideas were expressed by a representative of the Pythagorean school, Philolaus, in the 5th century BC, and by Heraclides in the 4th century BC. If the hypothesis of the Earth's axial rotation is correct, the swing plane of Foucault's pendulum must be retarded due to inertia, rotating with respect to the Earth. If the Earth is stationary, as was believed by the majority of Greek philosophers, including Aristotle and Ptolemy, the swing plane of the pendulum should not rotate.

From the technical standpoint, the experiment with Foucault's pendulum was accessible to all ancient and later civilizations, including the Greek one; however, it was realized only in modern times. For one and a half thousand years, it was considered that the problem of star rotation does not require additional attention; this was related to Aristotle's authority and the success of the Ptolemaic geocentric system, which described (and describes, up to this day) the motions of planets with a high degree of accuracy. The interest in this issue and related discussions were renewed in the 16th century after the work by Nicolaus Copernicus, and mainly were concluded after Johannes Kepler's work at the beginning of the 17th century.

The observed rotation rate of the swing plane of Foucault's pendulum, $\dot{\varphi}_{\rm E} \approx -11^\circ$ per hour, is not equal to zero and is given by $-360^\circ/23.93 \approx -15^\circ$ per hour (the negative sign indicates that the rotation is clockwise).

If an observer connected with the reference frame of remote stars mentally transports Foucault's pendulum along a meridian to the North Pole, keeping a constant angle between the swing plane and the meridian at every moment, then a uniform rotation of the transferred plane with respect to the stars is to be discovered in that reference frame. The respective rotation angles $\varphi_{\rm S}$ and $\varphi_{\rm E}$ for one sidereal day relative to the stars and the initial meridian are connected by $\varphi_{\rm S} = 2\pi + \varphi_{\rm E}$. For small swing angles of the pendulum, in the

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adiabatic approximation,

$$\varphi_{\mathrm{S}} = 2\pi(1 - \cos\vartheta)\,, \tag{1}$$

where ϑ is the polar angle of the pendulum. The adiabatic condition means that the Earth rotation period significantly exceeds the swing period.

The equation for the rotation of the swing plane of Foucault's pendulum, as an illustration of the laws of classical mechanics (see, e.g., [1] and also Section 3 below), is included in university programs for physics faculties.

Foucault's pendulum placed on the North Pole rotates by $\varphi_{\mathrm{E}} = -360^\circ$ per day. On the Equator, the pendulum does not rotate. In handbooks, it can be found that the Panthéon in Paris is on the parallel $\vartheta = 41.15^\circ$ (in geography, the latitude is $\alpha = \pi/2 - \vartheta$). It follows from the above equations that $\dot{\varphi}_{\mathrm{E}} = -11.3^\circ$ per hour, which correspond to observations of Foucault's pendulum and excludes the accompanying 'sky rotation' with high accuracy.

The experiment with Foucault's pendulum gives the first evidence of the Earth's rotation by ground-based means.

In the course of motion on the surface of the Earth, the swing plane of the pendulum, as a consequence of the laws of classical mechanics, remains parallel to itself [2–5]. This surprising fact allows investigating the problem of evolution of Foucault's pendulum by geometric methods.

At the equilibrium point, the velocity of the pendulum bob is in the plane tangent to the Earth surface. The swing plane of the pendulum can be characterized by a vector orthogonal to it. This vector is also in the plane tangent to the Earth surface. For displacements in the absence of external forces and/or torque moments, the tangent vectors undergo parallel transport. For example, a tangent vector of Minkowski space–time $\mathcal{M} = \mathcal{R}^{1,3}$, the 4-velocity remains fixed, i.e., parallel to itself, under a displacement of an inertially moving particle. Vectors tangent to curved surfaces, e.g., to a spherical surface $\mathcal{S}^2$ or to the physical relativistic velocity space, generally change when displaced, remaining inside the tangent space. In the comoving locally Euclidean reference frame, their evolution looks like inertial motion. In the first case, the displacement is in Minkowski space–time. In the last two cases, the displacement is along the spherical surface and in the physical relativistic velocity space.

The geometric basis of an effect currently known as the Thomas spin precession was discovered in 1913 by French mathematician É Borel [6], who described the effect of precession of axes of a rigid body in a circular orbit and pointed out its relation to the noncommutativity of Lorentz transformations. Borel noted an analogy between transformations of vectors on a spherical surface and in the physical relativistic velocity space, and estimated the angle of the circular orbital rotation of rigid body axes in the lowest quadratic order in velocity.

That same year, two young mathematicians from Göttingen, Ludwig Föppl and Percy Daniell [7], derived an exact formula for the precession angle,[1] according to which the coordinate axes of a rigid body for the rest-frame observer turn for one period of a uniform circular motion through the angle

$$\phi_{\mathrm{S}} = 2\pi(1 - \cosh\theta)\,, \tag{2}$$

[1] The paper by Föppl and Daniell was recommended for publication by David Hilbert.

where $\cosh\theta \equiv \gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor, v is the velocity of the body, and c is the speed of light. Nearly at the same time, the relativistic precession was discussed by Ludwig Silbertstein [8].

In the early 1920s, Enrico Fermi [9], and later Arthur Walker [10], established a transport rule for vectors to construct preferred reference frames in the general theory of relativity. In the Fermi–Walker transport, vectors tangent to the physical relativistic velocity space behave analogously to the axes of a rigid body in the theory of Borel et al [6–8].

The precession of axes of an accelerated rigid body is known to physicists as the Thomas precession, since Llevellyn Thomas [11] uncovered its fundamental importance for the theory of fine structure of atomic spectra. Thomas based his results on Willem de Sitter's paper on the relativistic precession of the Moon, published in a book by Arthur Eddington [12].

Group-theory aspects of spin precession were introduced to physicists by Wigner [13]. The term 'Wigner rotation' is used not only as a synonym of 'Thomas precession' but also, more generally, as a synonym of the rotation of a rigid body under coordinate transformations.

The history of early studies of Thomas precession is given in [14].

Recently, the geometric nature of the spin precession effect of a relativistic particle has again attracted attention. It was shown in [15] that the rotation angle ϕ_{S} is determined by an integral over the surface limited by the closed trajectory of the particle in the physical relativistic velocity space. This property characterizes parallel transport of vectors in a Riemannian space (see, e.g., [16]). Parallel transport in the relativistic velocity space and Thomas precession were discussed in [17] in detail.

Rotation angles φ_{S} and ϕ_{S} correspond to a geometrical phase that occurs in many areas of physics [4, 5, 18].

The known analogy between rotations and Lorentz transformations is of heuristic value. In particular, we recall that the relativistic velocity addition theorem can be obtained as the composition law for arcs of the great circles on a sphere of imaginary radius in a four-dimensional Euclidean space with one imaginary coordinate (time). By introducing the imaginary coordinate (time), it is possible to transform a hyperboloid of physical relativistic velocities into a sphere of imaginary radius in a four-dimensional Euclidean space.

Both for Foucault's pendulum and for Thomas precession, the surface along which a displacement occurs can therefore be considered a sphere. This suggests that the rotation effects of Foucault's swinging pendulum and Thomas precession, obviously, are geometrically identical; this does not necessarily contradict the different physical natures of these systems.

The aim of this methodological note is to show that expressions for the rotation angles φ_{S} and ϕ_{S} can be obtained by the same method using elementary geometrical tools for parallel transport of vectors over corresponding surfaces. In the first case, this is the Earth's surface, i.e., a spherical surface in the Euclidean space $\mathcal{R}^3$. In the second case, this is the physical relativistic velocity space, i.e., the hyperboloid $u^2 = 1$ in the tangent space $\mathrm{T}_x\mathcal{M}$ of Minkowski space–time.

The obvious tangent-cone geometric method used to derive the main equations in Sections 3 and 4 is often used to illustrate the effect of curvature on the parallel transport of vectors along a spherical surface (see, e.g., Appendix 1 in [1]).

This method was used in [2] and [3] [2] to describe the evolution of Foucault's pendulum. In Section 2, we recall the main principles of parallel transport. In Section 3, based on the consideration of Foucault's pendulum evolution from the dynamic standpoint, we show that the evolution is reduced to parallel transport of the swing plane of the pendulum over a spherical surface, and obtain Eqn (1). In Section 4, the tangent-cone method is generalized to the case of Thomas precession and is used to obtain expression (2).

2. Parallel transport

2.1 Euclidean space

The concept of parallel transport originates in Euclidean geometry. Two vectors are said to be parallel if two straight lines passing through the initial and final points of the vectors are parallel in the sense of the 5th Euclidean postulate, and the vector directions coincide. A continuous transformation that keeps the vector length constant and the vector parallel to itself at every infinitesimal step is called parallel transport. For a vector $\mathbf{A}$ at a point P, there is one and only one vector $\mathbf{A}'$ at a point P' that can be constructed by parallel transport of $\mathbf{A}$ from P to P'.

In a Euclidean space, the vector sum and difference are defined by parallel transport and the triangle rule. The condition for parallel transport can be written as

$$\delta\mathbf{A} = \mathbf{A}' - \mathbf{A} = 0\,, \tag{3}$$

where $\delta\mathbf{A}$ is an infinitesimal displacement. In a Euclidean space, condition (3) also holds for finite displacements.

Parallel transport allows a simple description in analytical geometry. A Cartesian reference frame is defined by a set of base vectors $\mathbf{e}_i$ at a chosen initial point P, which is then parallel transported to other points in space:

$$\delta\mathbf{e}_i = \mathbf{e}_i' - \mathbf{e}_i = 0\,. \tag{4}$$

Contravariant coordinates of a vector $\mathbf{A}$ are fixed by the decomposition $\mathbf{A} = A^i\mathbf{e}_i$, and covariant coordinates are defined by the scalar products $A_i = \mathbf{e}_i\,\mathbf{A}$. Because the basis is orthonormal, $\mathbf{e}_i\,\mathbf{e}_j = \delta_{ij}$, where $\delta_{ij} = \delta^{ij} = \mathrm{diag}\,(1, 1, \ldots, 1)$, we have $A^i = \delta^{ij}A_j = A_i$ and $\mathbf{A}\,\mathbf{B} = A_iB^i$.

In the reference frame thus defined, parallel transport does not change vector coordinates,

$$\delta A_i = \delta(\mathbf{e}_i\,\mathbf{A}) = \delta\mathbf{e}_i\,\mathbf{A} + \mathbf{e}_i\,\delta\mathbf{A} = 0\,, \tag{5}$$

and does not therefore change the scalar product:

$$\delta(\mathbf{A}\,\mathbf{B}) = (\delta A_i)\,B^i + A_i(\delta B^i) = 0\,. \tag{6}$$

2.2 Riemannian space

A Riemannian space is locally Euclidean. At any point, it is possible to find a reference frame where the metric tensor becomes Euclidean, and hence, for any point P' in the vicinity of P, the deviation of the metric tensor from the Euclidean one is small, being of the second order in the distance between P and P'.

In a local Euclidean reference frame originating at a point P, trajectories passing through P, $\mathbf{x} = \mathbf{v}l + O(l^3)$, where l is the infinitesimal distance from P, define elements of lines or geodesics. [3] A curve is called geodesic if each of its infinitesimal elements is an element of a geodesic. In a Euclidean space, straight lines define the shortest distances between two points. Correspondingly, in a Riemannian space, elements of geodesics define the shortest distances between nearby points, and a geodesic, between any points in the vicinity of a curve connecting them.

In a local Euclidean reference frame, parallel transport is defined by Eqn (3) in the first order in the displacement. Generally, parallel transport is defined in relation to a transformation along a curve, where a vector is parallel transported for each infinitesimal step at every point. Parallel transport preserves scalar products, lengths, and angles between vectors locally and therefore along the whole curve.

For the parallel transport along a straight line of a Euclidean space, the angle between a vector $\mathbf{A}$ and the vector tangent to the trajectory remains fixed. This property holds locally in a Riemannian space. Together with the requirement of a fixed length, it uniquely defines the parallel transport of a vector along a geodesic in a two-dimensional Riemannian space. In higher-dimensional spaces, there exists an additional freedom of rotation of $\mathbf{A}$ around a vector tangent to the trajectory vector. For parallel transport in a Euclidean space and locally in a Riemannian space, $\mathbf{A}$ remains in the initial plane spanned by $\mathbf{A}$ and a vector $\mathbf{v}$ tangent to the trajectory at the point P. This property forbids any rotations and uniquely defines parallel transport in Riemannian spaces of higher dimensions. [4]

2.3 Riemannian space as a hypersurface of Euclidean space

A Riemannian space can be embedded into a Euclidean space $\mathcal{E}$ of a higher dimension and considered as a hypersurface $\Sigma \subset \mathcal{E}$. A Cartesian reference frame on the hyperplane $\Pi(\mathrm{P})$ tangent to Σ at a point P defines a locally Euclidean reference frame Σ at P. In the vicinity of $\mathrm{P} \in \Sigma$, metric relations and algebraic operations on vectors belonging to $\Pi(\mathrm{P})$ and Σ coincide in the first order in the distance from P. This allows operating with geometrical objects in a Riemannian space locally in the same way as in the Euclidean space.

For parallel transport in the vicinity of $\mathrm{P} \in \Sigma$, vectors belonging to the tangent space $\mathrm{T_P}\Sigma$ satisfy Eqns (3) and (4).

Vectors of a Euclidean space $\mathcal{E}$ have components orthogonal to $\mathrm{T_P}\Sigma$. The conditions for parallel transport do not limit a change of these components. For any extension of the definition, a parallel transport of a vector Σ in $\mathbf{A} \in \mathrm{T_P}\mathcal{E}$ is

[2] In 1954, R L Mills and C N Young were the first to introduce fields with non-Abelian gauge symmetry groups into elementary particle physics.

[3] The trajectory of a test particle passing through the center of mass of a freely falling lift (point P) deviates from a straight line in the order $O(l^3)$, where l is the distance from P; the particle gravitational acceleration is of the order $O(l)$. The absence of $O(l^2)$ terms in the equation for the trajectory implies that in the infinitesimal neighborhood of P, the motion is inertial. An external force and acceleration are equal to zero at P according to Newton's second law.

[4] It is also possible to consider geodesics originating at Π_2 in the hypersurface Π_2 spanned by $\mathbf{A}$ and a vector $\mathbf{v}$ tangent to the trajectory. Such geodesics form a two-dimensional hypersurface Σ_2 tangent to Π_2 at the point P. For parallel transport from P to $\mathrm{P}' \in \Sigma_2$, a geodesic is entirely in Σ_2, and the vector $\mathbf{A}$ remains tangent to Σ_2. This property forbids rotations of the vector and leads to the above result for $\mathbf{A}'$ at P' with the accuracy $O(l^2)$. In the limit $l \to 0$, the parallel transport is well defined.

not the parallel transport in $\mathcal{E}$ in general. From the standpoint of $\mathcal{E}$, a precession of $\mathbf{A}$ occurs.

We let $\mathbf{e}_i$ denote basis vectors of $\mathrm{T_P}\Sigma$. The same vectors are basis vectors of $\Pi(\mathrm{P})$. In the infinitesimal vicinity of $\mathrm{P} \in \Sigma$, parallel transport conditions (3) and (4) for vectors $\mathbf{A} \in \mathrm{T_P}\mathcal{E}$ can be written as

$$\mathbf{e}_i\,\delta\mathbf{A} = 0\,, \tag{7}$$

$$\mathbf{e}_i\,\delta\mathbf{e}_k = 0\,. \tag{8}$$

In the first order, the components of $\mathbf{A}$ tangent to Σ do not change. Equation (8) shows that Christoffel symbols become zero in the locally Euclidean reference frame.

The Fermi–Walker equation [9, 10, 19] also fixes the change of the components of $\mathbf{A}$ normal to $\mathrm{T_P}\Sigma$. In the problems of Foucault's pendulum and Thomas precession, the normal components are zero. In this case, Eqns (7) are kinematically complete. For vectors in $\mathrm{T_P}\Sigma$, the Fermi–Walker equation is equivalent to Eqn (7).

Parallel transport from P to P′ inside a hypersurface in the case where P′ is near P can be regarded as the corresponding parallel transport of a vector inside $\Pi(\mathrm{P})$ with a subsequent projection on $\Pi(\mathrm{P}')$ (Fig. 1). The result is independent of the point in the intersection of $\Pi(\mathrm{P})$ and $\Pi(\mathrm{P}')$ at which the projection is made. Because parallel transport is defined for infinitesimal displacements, it is defined for the whole trajectory.

Equations (7) and (8) lead to the conclusion that if the basis vectors and the tangent vector $\mathbf{A}$ are parallel transported simultaneously along some curve, then coordinates of $\mathbf{A}$ in the local basis remain invariant. This property follows from Eqns (7) and (8) and the vector decomposition $\mathbf{A} = A^i\mathbf{e}_i$, i.e.,

$$\delta(\mathbf{e}_i\,\mathbf{A}) = \delta\mathbf{e}_i\,\mathbf{A} + \mathbf{e}_i\,\delta\mathbf{A} = A^k\mathbf{e}_k\,\delta\mathbf{e}_i + \mathbf{e}_i\,\delta\mathbf{A} = 0\,. \tag{9}$$

Vectors $\mathbf{e}_\vartheta$ and $\mathbf{e}_\varphi$ forming a spherical basis in $\mathcal{S}^2$ are related by parallel transport along a meridian for a fixed azimuthal angle. Therefore, the parallel transport of $\mathbf{A}$ along a meridian does not change the local coordinates of $\mathbf{A}$. In particular, the orientation of the swing plane of Foucault's pendulum relative to remote stars can be naturally (although not uniquely) defined by the parallel transport of the swing plane of the pendulum along a meridian into a local reference frame of an observer on the North Pole.

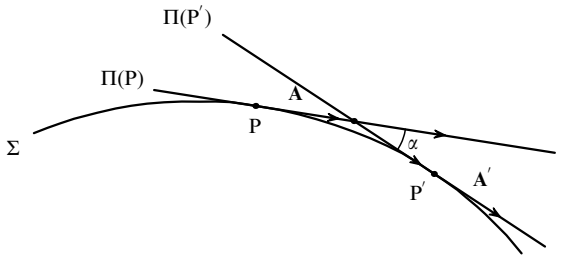

Figure 1. Parallel transport of a vector $\mathbf{A}$ between two nearby points P and P′ along a hypersurface Σ. Here, $\Pi(\mathrm{P})$ and $\Pi(\mathrm{P}')$ are hyperplanes tangent to Σ at the points P and P′. The vector $\mathbf{A}$ is parallel transported (in the sense of Euclidean geometry) inside $\Pi(\mathrm{P})$ to the intersection of $\Pi(\mathrm{P})$ and $\Pi(\mathrm{P}')$ and is then projected on $\Pi(\mathrm{P}')$. At the final stage, $\mathbf{A}$ is parallel transported in the sense of Euclidean geometry inside $\Pi(\mathrm{P}')$ to a point P′. The distance between P and P′ is small, of the first order in the angle α, and the change in the vector length $\delta|\mathbf{A}| = (1 - \cos\alpha)|\mathbf{A}|$ is small, of the second order in α. In the continuum limit, the length $|\mathbf{A}|$ remains constant.

The synchronization of reference frames in the special theory of relativity assumes that the basis vectors of frames S′ are obtained by boost transformations of the basis vectors of some preferred reference frame S. The statements that the one-parametric family of boosts in a two-dimensional plane defines some geodesic in the hyperboloid $u^2 = 1$ of relativistic velocities and that the boost of basis vectors is a parallel transport along a geodesic are proved in the Appendix. As a consequence, we note that parallel transport of a polarization 4-vector a along a geodesic does not change the local coordinates of a.

The general mathematical formalism necessary to describe Riemannian spaces can be found in [1, 16, 19]. A sphere embedded in a three-dimensional Euclidean space $\mathcal{R}^3$ and a hyperboloid of physical relativistic velocities embedded in the relativistic velocity space $\mathrm{T}_x\mathcal{M}$ are still simple enough and allow using elementary geometric methods.

3. Rotation of the swing plane of Foucault's pendulum

3.1 Dynamic conditions

We focus on those dynamical aspects of the evolution of Foucault's pendulum that are closely related to the problem geometry (see also [2–5, 18]).

The pendulum suspension center moves along a circular trajectory. The Coriolis force due to rotation of the Earth acts on the pendulum. The reaction force of the pendulum suspension resists gravity. [5]

Gravity is directed towards the center of the Earth and the direction of the reaction force depends on the bob position. The radial components of the gravity force and reaction force are mutually compensated. The reaction force creates a tangent (in the small-angle approximation) component to the Earth surface that tends to return the pendulum to the equilibrium position.

The Coriolis force appears in the pendulum equation of motion due to rotation of the reference frame connected to the Earth,

$$\mathbf{F}_\mathrm{C} = 2m\mathbf{v} \times \mathbf{\Omega}\,, \tag{10}$$

where m and $\mathbf{v}$ are the mass and velocity of the pendulum bob, and $\mathbf{\Omega}$ is the Earth rotation frequency. The velocity $\mathbf{v}$ is tangent to the surface. At the equator, both $\mathbf{v}$ and $\mathbf{\Omega}$ belong to the tangent plane, and therefore $\mathbf{F}_\mathrm{C}$ is parallel to the free-fall acceleration. As a result, the Coriolis force does not create a torque. If the pendulum moves along the equator, its swing plane does not rotate relative to the direction of motion. This remains valid for the pendulum motion along any great circle of a sphere.

The problem of dynamic evolution of the pendulum can now be reformulated as a pure geometric problem of parallel transport of the pendulum swing plane. The pendulum trajectory at a fixed ϑ can be approximated by a set of great circle arcs of the sphere. Along every such arc, the pendulum preserves its state because the swing plane does not rotate

[5] The centrifugal force removes a vector orthogonal to the swing plane of the pendulum from the space tangent to the Earth's surface. This effect leads to a perturbation of the free-fall acceleration $\mathbf{g}$. It can be neglected with the accuracy $\mathbf{\Omega}^2 R \sin\vartheta/|\mathbf{g}| \sim 0.003$, where $|\mathbf{\Omega}| = 2\pi/23.93$ per hour, $R = 6371$ km is the radius of the Earth, $\sin\vartheta \sim 1$, and $|\mathbf{g}| = 9.81$ m s^{-2}.

relative to the arc direction. The pendulum swing occurs by inertia. However, relative to the circle of a polar angle ϑ, the swing plane does rotate because of the non-Euclidean spherical geometry of the surface. In the continuum limit, where the arc lengths vanish, the initial trajectory defined by the polar angle ϑ is restored and the rotation angle of the pendulum swing plane is obtained.

Earth's oblateness is $f \approx 1/300$ ($f = (a-c)/a$, where a is the equatorial radius and b is the polar radius). We regard the Earth as an ideal sphere and neglect small deviations from the corresponding law of parallel transport connected with that ellipsoidal form.

3.2 The tangent-cone method in the problem of Foucault's pendulum

In Fig. 2, the Northern Hemisphere of the Earth is covered by a cone with apex B. The cone touches the Earth surface at the parallel set by the polar angle $\vartheta = \angle \mathrm{BOA}$. The vector (orthogonal to the swing plane) moves together with the pendulum along the circle. At each point of the trajectory, the vector is in the tangent space of the sphere and the cone. Assuming the sphere radius to be equal to unity, we obtain

$$\mathbf{OA} = (\mathbf{n}\sin\vartheta, \cos\vartheta)\,, \tag{11}$$

$$\mathbf{CA} = (\mathbf{n}\sin\vartheta, 0)\,, \tag{12}$$

$$\mathbf{BA} = \left(\mathbf{n}\sin\vartheta, -\frac{\sin^2\vartheta}{\cos\vartheta}\right), \tag{13}$$

where $\mathbf{n} = (\cos\varphi, \sin\varphi)$ is the unit vector in the equatorial plane. To find the vector $\mathbf{BA}$, we write $\mathbf{BA} = (\mathbf{n}\sin\theta, z)$ and fix z using the orthogonality condition

$$\mathbf{BA}\,\mathbf{OA} = 0\,. \tag{14}$$

Figure 2. The Earth is shown as an ideal sphere centered at a point O, with the radius $|\mathbf{OA}|$. Foucault's pendulum is placed at a point A on the circle of the polar angle $\vartheta = \angle \mathrm{BOA}$. For one sidereal day, the pendulum completes one revolution around the Earth. Parallel transport of the swing plane of the pendulum can be simply presented as parallel transport of a vector normal to the plane. This vector belongs to the tangent space of the sphere as well as of the conical surface (looking like a Vietnamese hat) with the apex at B; the cone touches the Earth surface at the latitude of the pendulum. The point C is the center of the circle along which the pendulum moves. The points B, C, and O belong to the Earth's rotation axis.

The metric on the cone $\mathcal{R}^3$ induced from the Euclidean space is Euclidean. Indeed, we can choose a reference frame (ρ, φ) on the cone, where ρ is the distance (Euclidean in the sense of $\mathcal{R}^3$) from the point B, and φ is the azimuthal angle defined above. The infinitesimal distance between two points on the cone is

$$\mathrm{d}l^2 = \mathrm{d}\rho^2 + \cos^2\vartheta\rho^2\,\mathrm{d}\varphi^2\,. \tag{15}$$

By changing the variable $\varphi \to \varphi/\cos\vartheta$ (where we recall that the angle ϑ is constant), we can obtain the result that the metric tensor on the cone becomes Euclidean in the polar coordinate frame.

Thus, it is possible to cut the cone along the line BA, unfold it, and place it on a plane, as shown in Fig. 3. The metric in the unfolded cone remains Euclidean, and the distances between points on the cone and angles do not change.

Parallel transport in a Euclidean space is simple and evident. It is shown in Fig. 3 for a vector originating at a point A and directed along the meridian.

The pendulum rotates with the Earth counterclockwise. The swing plane of the pendulum rotates clockwise in the direction of the rotation of the stars. In the reference frame related to the Earth, the rotation angle φ_{E} is negative. Its value is determined by the ratio of the arc length AA' along the pendulum path, $2\pi|\mathbf{CA}|$, to the radius $|\mathbf{BA}|$ of the circle shown in Fig. 3. The arc length AA' is equal to the length of the circle in Fig. 2 with the center at C and the radius $|\mathbf{CA}|$, where $|\mathbf{BA}|$ is the slant height of the cone. The vector lengths defined by Eqns (12) and (13) are $|\mathbf{CA}| = \sin\vartheta$ and $|\mathbf{BA}| = \tan\vartheta$. Therefore,

$$\varphi_{\mathrm{E}} = -\frac{2\pi|\mathbf{CA}|}{|\mathbf{BA}|} = -2\pi\cos\vartheta\,. \tag{16}$$

The rotation angle relative to remote stars is given by Eqn (1).

Equation (16), obtained for the Northern Hemisphere, where $\vartheta \leqslant \pi/2$, is also valid for the Southern Hemisphere. To

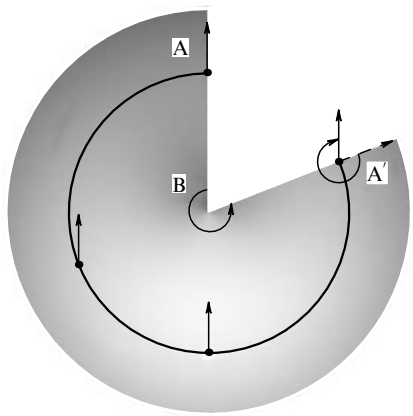

Figure 3. Parallel transport of a vector that is orthogonal to the swing plane of a pendulum. The pendulum, located initially at a point A, rotates with the Earth counterclockwise and reappears at the point A in one complete revolution. In the plane, however, the pendulum arrives at a point $\mathrm{A}' \neq \mathrm{A}$ that is physically the same as A. The path in the plane is not closed. The angle φ_{E} shown by the oriented arc around point A', $|\varphi_{\mathrm{E}}| < 2\pi$, determines the rotation angle of the swing plane of the pendulum in the reference frame related to the Earth. Points A and B are the same as in Fig. 2.

see that, we should mirror reflect Fig. 2 relative to the equatorial plane and repeat the argument.

4. Thomas precession

Figure 4 shows a three-dimensional projection of the relativistic velocity space and the embedded hyperboloid $u^2 = 1$ of physical relativistic velocities. The scalar product is calculated using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. A particle with a 4-velocity A is initially at the point $u = (\gamma, \gamma\mathbf{v}/c)$ of the hyperboloid.

4.1 The tangent space of the physical relativistic velocity space

4.1.1 The polarization vector. The axial polarization 4-vector is determined by a three-dimensional axial vector $\mathbf{a}$ in the rest frame of a particle as the expected value $\hat{\mathbf{s}}/s$, where $\hat{\mathbf{s}}$ is the spin operator and s is the particle spin. Because transformations of three-dimensional vectors under Lorentz transformations are ill-defined, the polarization of a relativistic particle should be described by a four-dimensional vector. In the rest frame, it can be defined as

$$a = (0, \mathbf{a}) \,, \tag{17}$$

where $\mathbf{a}^2 = -a^2 = 1$ for pure states and $\mathbf{a}^2 = -a^2 < 1$ for mixed states.

In the rest frame $u = (1, \mathbf{0})$, therefore,

$$ua = 0 \,. \tag{18}$$

The scalar product is Lorentz invariant, and hence Eqn (18) is valid in all inertial frames.

This statement can be complemented by a more general one. The equivalence principle means that all phenomena in a comoving local inertial frame occur in the same way as in a global inertial frame. Representation (17) and condition (18) are therefore also valid in comoving inertial reference frames of accelerated particles. In such frames, the allowed changes of a are limited by rotations of the spatial component of the vector.

It is possible to arrive at Eqn (18) differently. Using the angular momentum tensor and the 4-momentum $p_l = mu_l$, the Pauli–Lubanski vector is constructed as

$$J^i = \frac{1}{m}\, \varepsilon^{ijkl} M_{jk} p_l \,. \tag{19}$$

This vector is proportional to the polarization 4-vector $J^i = sa^i$. To see this, it is sufficient to choose the reference frame where $p = (m, \mathbf{0})$. In this frame, the particle is at rest and its orbital momentum is zero; and the tensor M_{jk}, related to its spin only, defines the spatial part of the vector J^i. Equation (18) is valid because $p = mu$ and the tensor ε^{ijkl} is totally antisymmetric ($\varepsilon_{0123} = +1$).

From the geometrical standpoint, Eqn (18) means that a belongs to the tangent space of the hyperboloid $u^2 = 1$ embedded in the relativistic velocity space $\mathrm{T}_x\,\mathcal{M}$. Furthermore, $\mathrm{T}_x\,\mathcal{M}$ is the tangent space to $\mathcal{M}$ at point $x \in \mathcal{M}$.

Parallel transport of the polarization vector is used in two senses. First, it is used in the sense of parallel transport in $\mathcal{M}$, e.g., along a spiral particle trajectory in Minkowski space–time. Second, it is used in the sense of parallel transport in $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$, e.g., along the circle of a fixed γ on the hyperboloid of 4-velocities for some $x \in \mathcal{M}$.

The polarization vector implicitly depends on the particle position $x \in \mathcal{M}$ and velocity $u \in \mathrm{T}_x\mathcal{M}|_{u^2=1}$. Vectors a and a' of two particles at points x and x' can be matched when the particle 4-velocities are the same, $u = u'$. Two observers at points x and x' moving with the same velocities $u = u'$ belong to the same inertial reference frame, up to some rotation. The second observer turns the axes of his frame in the direction of the axes of the first observer's frame. Then the observers communicate the coordinates a and a'.

Exactly the same result can be obtained by parallel transporting the vectors a and a' in Minkowski space–time as if these vectors belonged to the space–time $\mathrm{T}_x\,\mathcal{M}$, although according to Eqn (18), they belong to the tangent space $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$.

For $x = x'$ and $u \neq u'$, the polarization vectors are matched by using the scheme of parallel transport in the physical relativistic velocity space along the geodesic connecting points u and u'.

For $x = x'$ and $u \neq u'$, the vector a' is parallel transported $(x', u') \to (x, u')$ in $\mathcal{M}$, then it is parallel transported $(x, u') \to (x, u)$ along the geodesic in $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$, and after that a' matches a. Because parallel transport in $\mathcal{M}$ does not change the vector coordinates, and, hence, parallel transports in $\mathcal{M}$ and $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$ commute, the result is independent of the order of the operations.

Thus, the polarization vectors can be regarded as vectors $\mathrm{T}_x\mathcal{M}|_{u^2=1}$ for any chosen value of x, e.g., $x = (0, \mathbf{0})$, and can be characterized by the 4-velocity $u \in T_x\mathcal{M}|_{u^2=1}$ only.

As a result, an observer based in some reference frame, e.g., at the origin, is able to analyze and make consistent conclusions on the character of the spin precession for a particle moving with any velocity and acceleration.

In quantum mechanics, the uncertainty relation does not allow measuring positions and velocities simultaneously. This restriction does not create difficulties because particle localization is not important and only transport in the velocity space contributes to the precession.

4.1.2 Angular momentum of a mechanical top. The angular momentum $\mathbf{L}$ of a mechanical top is a three-dimensional axial vector in the rest frame. The same arguments as for the polarization vector lead in a relativistically invariant way to a 4-vector characterizing the angular momentum of a mechanical top. As a result, we come to the representation $J = (0, \mathbf{L})$ in the rest frame and to the conclusion that the 4-vector J is tangent to the physical relativistic velocity space, i.e., $uJ = 0$. In view of the noted similarity, a mechanical top is often regarded as a mechanical image of a spinning electron.

Under the action of an external force applied to the center of mass in a chosen direction, the mechanical top is parallel transported from one inertial reference frame to another. For this reason, the mechanical top is convenient to represent coordinate axes of inertial reference frames.

4.1.3 Particle 4-acceleration. Taking the derivative of both sides of the equation $u^2 = 1$, we obtain $wu = 0$. The acceleration $w = \mathrm{d}u/\mathrm{d}s$ (as well as the polarization) belongs to the tangent space $\mathrm{T}_x\mathcal{M}|_{u^2=1}$.

4.2 The tangent-cone method in the problem of Thomas precession

We consider a cone tangent to the hyperboloid at a point A and along a fixed-γ circle, i.e., along the particle trajectory in the relativistic velocity space. In the circular orbit, the tangent

Figure 4. Three-dimensional projection of the relativistic velocity space. The set of relativistic velocities forms a hyperboloid with $u_0 = +\sqrt{1+\mathbf{u}^2}$, where $\mathbf{u} = \gamma\mathbf{v}/c$, and $\mathbf{v}$ is the three-dimensional velocity. The vector $\mathbf{OA} = (\gamma, \mathbf{n}\gamma v/c)$ specifies the initial 4-velocity of a particle in a circular orbit of constant γ. The vector **BA** is orthogonal to the vector **OA** and hence belongs to the space tangent to the hyperboloid. The cone with the apex B (looking like a flipped Vietnamese hat) touches the hyperboloid at the point A as well as the whole circular orbit with the center C. The point O is the apex of the light cone. Points O, B, and C belong to the same axis.

spaces of the hyperboloid and the cone coincide. The polarization vector a belongs to both tangent spaces.

Let B be the cone apex. The vectors in Fig. 4 are

$$\mathbf{OA} = \left(\gamma,\ \frac{\mathbf{n}\gamma v}{c}\right), \tag{20}$$

$$\mathbf{CA} = \left(0,\ \frac{\mathbf{n}\gamma v}{c}\right), \tag{21}$$

$$\mathbf{BA} = \left(\frac{\gamma v^2}{c^2}\,,\ \frac{\mathbf{n}\gamma v}{c}\right), \tag{22}$$

where $\mathbf{n}$ is the unit vector in the plane of the particle rotation. We can fix **BA** by writing $\mathbf{BA} = (w, \mathbf{n}\gamma v/c)$ and finding w from the orthogonality condition

$$\mathbf{BA}\,\mathbf{OA} = 0\,. \tag{23}$$

Equations (20)–(22) are similar to Eqns (11)–(13).

The vector **BA** is tangent to the hyperboloid. Under the parallel transport, its rotation angle coincides with the rotation angle of the polarization vector.

The metric induced on the cone can be obtained as follows. The cone points are characterized by the vectors

$$\mathbf{BX} = \rho\left(\frac{v}{c}\,,\ \mathbf{n}\right), \tag{24}$$

where $\mathbf{n} = (\cos\phi, \sin\phi, 0)$ lies in the rotation plane (u_x, u_y). The vector **BX** is obtained by stretching **BA** and rotating it around the u_0 axis. An infinitesimal variation of **BX** tangent to the cone surface can be written as

$$\mathrm{d}\mathbf{BX} = \frac{\partial\mathbf{BX}}{\partial\rho}\,\mathrm{d}\rho + \frac{\partial\mathbf{BX}}{\partial\phi}\,\mathrm{d}\phi = \mathrm{d}\rho\left(\frac{v}{c}\,,\ \mathbf{n}\right) + \rho(0, \mathrm{d}\mathbf{n})\,, \tag{25}$$

where $\mathrm{d}\mathbf{n} = (-\sin\phi, \cos\phi, 0)\,\mathrm{d}\phi$, $\mathbf{n}\,\mathrm{d}\mathbf{n} = 0$.

The length of a 4-vector is defined by the scalar product

$$|\mathbf{XY}| = \sqrt{-\mathbf{XY}\,\mathbf{XY}}\,. \tag{26}$$

The infinitesimal distance between two points on the cone is given by

$$\mathrm{d}l^2 \equiv -\mathrm{d}\mathbf{BX}\,\mathrm{d}\mathbf{BX} = \frac{\mathrm{d}\rho^2}{\gamma^2} + \rho^2\mathrm{d}\phi^2\,. \tag{27}$$

To bring $\mathrm{d}l^2$ to Euclidean form, we rescale $\phi \to \phi/\gamma$ and $\rho \to \gamma\rho$ and obtain

$$\mathrm{d}l^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2\,. \tag{28}$$

The metric induced on the cone from the Minkowski space–time is therefore Euclidean. Equation (28) defines it in the polar reference frame.

The cone in Fig. 4 can therefore be cut along BA, unfolded preserving distances and angles, and placed on a plane as shown in Fig. 5.

The particle rotates counterclockwise, while the polarization vector rotates clockwise. In the comoving reference frame, the rotation angle ϕ_{E} is negative. Its value is determined by the ratio of the length of the arc AA′ along the particle trajectory, i.e., $2\pi|\mathbf{CA}|$, and the circle radius $|\mathbf{BA}|$. The trajectory is a circle of a fixed latitude on the hyperboloid of physical relativistic velocities, as shown in Fig. 4. The length of the circle is determined by the radius $|\mathbf{CA}|$.

In the plane of Fig. 5, the angle corresponding to the arc AA′ along the trajectory exceeds 2π. The distances between points A, B and A, C can be obtained using Eqns (21) and (22):

$$|\mathbf{CA}| = \frac{\gamma v}{c}\,, \tag{29}$$

$$|\mathbf{BA}| = \frac{v}{c}\,. \tag{30}$$

CA and **BA** are spatial vectors, and their lengths are real. Finally, we obtain

$$\phi_{\mathrm{E}} = -\frac{2\pi|\mathbf{CA}|}{|\mathbf{BA}|} = -2\pi\gamma\,. \tag{31}$$

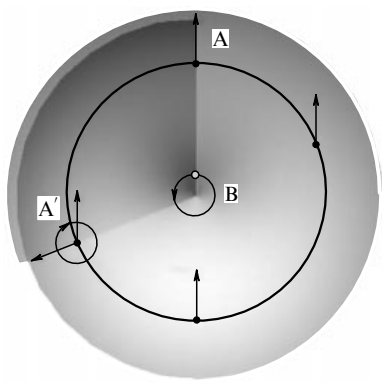

Figure 5. The tangent cone with the apex B shown in Fig. 4 is cut along the segment BA and unfolded in the plane. The point A belongs to the cone and the hyperboloid. For one revolution of a particle from A to A′ along a circular orbit, the polarization vector rotates by angle $|\phi_{\mathrm{E}}| > 2\pi$ in the comoving frame. Points A (= A′) and B are the same as in Fig. 4.

In an inertial reference frame where the origin $x = y = z = 0$ coincides with the axis of a spiral trajectory in $\mathcal{M}$, the rotation angle $\phi_{\mathrm{S}} = 2\pi + \phi_{\mathrm{E}}$ is given by Eqn (2).

In the space $\mathrm{T}_x\,\mathcal{M}$, such a reference frame is in the vertex of the hyperboloid $u = (1, \mathbf{0})$. From the standpoint of the problem of Foucault's pendulum, it can be considered an analogue of a local reference frame of an observer on the North Pole.

Meridians on a sphere are formed by a trace of the point $(0, 0, 1)$ under rotation in the planes $(0, 0, 1)$ and $(\cos\varphi, \sin\varphi, 0)$, where φ is the azimuthal angle numbering the meridians. Analogous to meridians are orbits in $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$ formed by a trace of the point $u = (1, \mathbf{0})$ for boost transformations in the planes $(1, \mathbf{0})$ and $(0, \mathbf{n})$, where $\mathbf{n}$ is the unit vector numbering the orbits. Orbits of the point $u = (1, \mathbf{0})$ are geodesics as well as meridians on a sphere (see the Appendix).

We note that Eqn (1) can be transformed to Eqn (2) by the substitution $\vartheta \to \mathrm{i}\theta$, which formally corresponds to the replacement of rotations by Lorentz transformations.

The centrifugal and reaction forces keep a particle at rest in the comoving frame. The precession rate cannot be estimated without knowing the Lorentz nature of the reaction force.

The circular orbit of a particle can be approximated by a set of small segments of geodesics on the hyperboloid of velocities. Such segments corresponding to boost transformations are similar to the arcs of the great circles of a sphere. Parallel transport of a polarization vector along each of these segments does not lead to rotation locally. However, relative to a fixed-γ trajectory, the polarization vector turns because of the non-Euclidean nature of the physical space of relativistic velocities. In the continuum limit, the segments of geodesics reproduce the particle trajectory. Parallel transport along the trajectory gives the Thomas (i.e., the universal geometric) component of spin precession.

The force of reaction related to potentials that are scalar under the Lorentz group does not influence spin precession. In this case, Thomas precession is the only effect. This fact tells us about the absence of torque in scalar potentials.

In potentials that are vectors under the Lorentz group, an external torque leading to the well-known Larmor precession acts on the spin of the particles. This case is characteristic of electrons in atoms [11, 20], and antiprotons and hyperons in exotic atoms [21–23].

In [23], in particular, it was shown that the rate of spin precession in the Bargmann–Michel–Telegdi equation [24] consists of two parts; the first is caused by the Thomas precession, and the second by the Larmor precession. Due to the purely kinematic nature, the Thomas precession influences spectroscopy and static characteristics of nuclei [20] and hadrons [25, 26].

5. Conclusion

The tangent-cone method, used for illustration of vector rotation for parallel transport over the surface of a sphere and, in particular, for the description of rotation of the swing plane of Foucault's pendulum, is generalized to the case of Thomas precession of spin of a relativistic particle moving in a circular orbit.

In the problems of Foucault's pendulum and Thomas precession, vectors characterizing the system state are not influenced by an external rotating moment and evolve by inertia, undergoing parallel transport. We have used the close analogy between the parallel transport over the surface of a sphere in the three-dimensional Euclidean space and over the surface of the hyperboloid $u^2 = 1$ in the four-dimensional space of relativistic velocities. In both cases, the evolution is reduced to parallel transport in the usual Euclidean space represented by the tangent cone surface.

Thus, the basic equation for the Thomas spin precession of a relativistic particle moving in a circular orbit can be obtained by elementary geometric constructions.

Appendix. Geodesics in the relativistic velocity space

Here, we prove two statements made at the end of Section 2.3.

1. The points $u, u' \in \mathrm{T}_x\mathcal{M}|_{u^2=1}$ describe two inertial reference frames S and S$'$. In the frame S, $u = (1, \mathbf{0})$. We write u' as

$$u' = (\cosh\theta, \mathbf{n}\sinh\theta)\,, \tag{A.1}$$

where $\mathbf{v} = c\mathbf{n}\tanh\theta$ is the velocity of S$'$ in the frame S, and $\mathbf{n}$ is the unit vector. Reference frames S and S$'$ are related by a boost in the plane (u, u').

The metric induced in $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$ is determined by the interval $\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}u$ and in variables (A1) is given by

$$\mathrm{d}s^2 = -\mathrm{d}\theta^2 - \sinh^2\theta\,\mathrm{d}\mathbf{n}^2\,. \tag{A.2}$$

The interval between two points on $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$ is negative, furthermore, $-\mathrm{d}s^2 \geqslant \mathrm{d}\theta^2$. Thus, we find

$$\int_u^{u'} \sqrt{-\mathrm{d}s^2} \geqslant \theta\,. \tag{A.3}$$

Any deviations from curve (A.1) that θ-connects points u and u' for fixed $\mathbf{n}$ increase the distance between u and u'.

Hence, the set of 4-velocities (A.1) with constant $\mathbf{n}$ determines a geodesic on $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$. On the other hand, this geodesic is an orbit formed by a trace of the point $u = (1, \mathbf{0})$ for boost transformations in the plane of vectors $(1, \mathbf{0})$ and $(0, \mathbf{n})$.

2. Basis vectors of the tangent space $\mathrm{T}_x\,\mathcal{M}|_{u^2=1}$ can be chosen as

$$e_\theta = \frac{\partial u'}{\partial\theta} = (\sinh\theta, \mathbf{e}_r\cosh\theta)\,, \tag{A.4}$$

$$e_\vartheta = \frac{1}{\sinh\theta}\,\frac{\partial u'}{\partial\vartheta} = (0, \mathbf{e}_\vartheta)\,, \tag{A.5}$$

$$e_\varphi = \frac{1}{\sinh\theta\sin\vartheta}\,\frac{\partial u'}{\partial\varphi} = (0, \mathbf{e}_\varphi)\,, \tag{A.6}$$

where $\mathbf{e}_r = \mathbf{n} \equiv (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, $\mathbf{e}_\vartheta$, $\mathbf{e}_\varphi$ are the basis vectors of the spherical reference frame in $\mathcal{R}^3$,

$$\mathbf{e}_\vartheta = \frac{\partial\mathbf{n}}{\partial\vartheta}\,, \tag{A.7}$$

$$\mathbf{e}_\varphi = \frac{1}{\sin\vartheta}\,\frac{\partial\mathbf{n}}{\partial\varphi}\,. \tag{A.8}$$

We note that $e_\alpha e_\beta = -\delta_{\alpha\beta}$ along the geodesic.

For a displacement $\theta \to \theta + \delta\theta$ corresponding to a boost in the direction $\mathbf{n}$, basis vectors (A.4)–(A.6) change. Their variations satisfy the conditions

$$e_\alpha\,\delta e_\beta = 0\,, \tag{A.9}$$

which are the parallel transport conditions according to Eqn (8).

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