

METHODOLOGICAL NOTES

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Generalization of the Leontovich approximation for electromagnetic fields on a dielectric – metal interface

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Abstract. **The Leontovich approximate condition for electromagnetic fields at the dielectric–metal interface, valid for a small surface impedance ζ, is generalized to the case of arbitrary magnitudes of ζ, which provides a broader range of applicability of the impedance approach. The exact boundary condition found is expanded in a series of odd powers of the parameter ζ. Being linear in ζ, the Leontovich condition differs from the exact equation in main order only by terms $\sim \zeta^3$. Thus, in describing wave fields in this approximation, it is not only linear terms that prove to be correct, but also terms of order ζ^2. The accuracy of an impedance approximation turns out to be higher than its developer himself believed. On the basis of the generalization made, the errors of different-order approximations are analyzed for the descriptions of polariton propagation and wave reflection near the interface between an isotropic dielectric and metal submedia. It is shown that in the polariton theory the Leontovich approximation provides a sufficiently high accuracy not only in the infrared range, but also in the whole visible range. In the reflection problem, this approximation is reasonable in most of the visible range within a wide interval of the angles of incidence; however, it is inapplicable when simultaneously the waves are short and the angles of incidence are large. In this domain, the accuracy of the description may be substantially raised only beyond the framework of the Leontovich approximation.**

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1. Leontovich impedance approximation

Substantial features of electromagnetic field structure near the surface of an ordinary metal are connected with the large absolute value of the complex permittivity in metals, $|\varepsilon_{\mathrm{m}}| \gg 1$. This suggests that inside a metal the derivatives of components of an electromagnetic field in the direction normal to its surface are large compared to those in the tangential directions. In these conditions, it follows from the Maxwell equations that in a plane wave inside an isotropic metal near its surface the tangential components of electric and magnetic fields are *approximately* connected by the Leontovich relationship [1]:

$$\mathbf{E}_{\mathrm{t}} + \zeta \mathbf{H}_{\mathrm{t}} \times \mathbf{n} = 0 \,. \tag{1}$$

Here, $\mathbf{n}$ is the unit vector of an outward normal to the metal surface, and $\zeta = \zeta(\omega)$ is the metal surface impedance (see Refs [1–4]). The parameter ζ dependent on the wave frequency ω is uniquely related to the complex permittivity ε_{m}:

$$\zeta(\omega) = \zeta' + \mathrm{i}\zeta'' \equiv \sqrt{\frac{1}{\varepsilon_{\mathrm{m}}(\omega)}} \,, \qquad \zeta' > 0 \,, \qquad \zeta'' < 0 \tag{2}$$

(the permeability μ_{m} of metal is assumed to be equal to unity). The signs of the components ζ' and ζ'' in formula (2) are determined by the energy dissipation conditions inside the metal [2].

According to the standard boundary conditions, the components $\mathbf{E}_{\mathrm{t}}$ and $\mathbf{H}_{\mathrm{t}}$ are continuous at an interface of a metal and a medium; hence, as Leontovich noted [1], approximate relationship (1) which is valid for small impedance can be used as a boundary condition for finding the field outside the metal near its surface.

Boundary condition (1) derived in Ref. [1] is based on the above-given qualitative speculation concerning the structure of an electromagnetic field inside a metal. In Section 3, by using the standard boundary conditions, we will write them out in terms of the impedance which may be arbitrary and not necessarily small. In the exact equation obtained we will single out the part responsible for relationship (1). Then, the remainder will define the degree of accuracy for the impedance approximation. This remainder is presented by a simple function of impedance ζ, which after expanding in an odd-power series of parameter ζ starts with the term $\sim \zeta^3$ (see Section 4). The fact that the correction to condition (1) is proportional to the impedance cubed was already mentioned [1, 3], as applied to the problem of electromagnetic wave reflection from a metal surface. On the other hand, this means that in describing wave field characteristics in the Leontovich approximation, it is not only the linear terms that are correct, but also the quadratic terms of order ζ^2. Thus, the Leontovich impedance approximation corresponds to two orders of the perturbation theory.

As is shown in classical monograph [4], the Leontovich method works perfectly in radiophysics. Nevertheless, in the present work we will be more interested in the optical frequency range. Usually, an analysis of the function $\varepsilon_m(\omega)$ in real metals [5, 6] results in the conclusion that the impedance approximation is well suited for an infrared spectral range and is less applicable to the visible range where the parameter ζ is not so small. However, from the analysis given in Sections 4 and 5 it follows that in real metals the Leontovich approximation is often admissible in the visible spectral range as well. For enhancing the calculation accuracy and expanding the range of applicability of the impedance approximation, we will suggest a generalized impedance approximation.

2. Wave fields near the interface between a crystal and an isotropic medium in terms of impedance

Let us consider wave fields near the interface between a crystal (anisotropic dielectric) and an isotropic medium. For the sake of generality, we now make no assumptions concerning the value of the permittivity ε_m for the isotropic medium. In the subsequent discussion, however, we will come to a real metal by setting this quantity complex-valued and large in modulus.

By choosing the coordinate x-axis along the wave propagation direction in the interface plane, and the y-axis along normal $\mathbf{n}$ to it (see Fig. 1a), we can present a stationary electromagnetic field in the form

$$\begin{pmatrix} \mathbf{E}(x,y,t) \\ \mathbf{H}(x,y,t) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(y) \\ \mathbf{H}(y) \end{pmatrix} \exp\left[\mathrm{i}\omega\left(\frac{1}{c}\, nx - t\right)\right]. \tag{3}$$

Here, the wavenumber $k = k_x$ is substituted by the dimensionless refraction parameter $n = k/k_0$, where $k_0 = \omega/c$, c is the speed of light, and t is time. Parameter n characterizes the reduced phase velocity of wave field propagation in a medium: $v = c/n = \omega/k$. Wave field (3) in this case can be of two types. It describes either polaritons (the wave fields localized along the interface, see Fig. 1b) or a stationary field arising as the result of the reflection of a plane bulk wave in the crystal from the interface (see Fig. 1c).

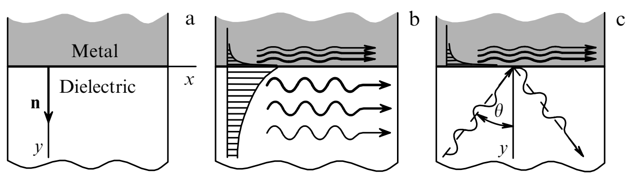

Figure 1. Dielectric–metal structure in the attached coordinate system (a), the wave fields arising in the structure for surface polariton propagation (b), and in bulk wave reflection in a dielectric (c).

In the first case (for polaritons), the vector amplitudes in formula (3) can be written out in the form

$$\begin{pmatrix} \mathbf{E}(y) \\ \mathbf{H}(y) \end{pmatrix} = \begin{cases} C_{\mathrm{I}} \begin{pmatrix} \mathbf{E}_{\mathrm{I}}^0 \\ \mathbf{H}_{\mathrm{I}}^0 \end{pmatrix} \exp\left(\mathrm{i}p_{\mathrm{I}}' - p_{\mathrm{I}}''\right) ky \\ \quad + C_{\mathrm{II}} \begin{pmatrix} \mathbf{E}_{\mathrm{II}}^0 \\ \mathbf{H}_{\mathrm{II}}^0 \end{pmatrix} \exp\left(\mathrm{i}p_{\mathrm{II}}' - p_{\mathrm{II}}''\right) ky\,, & y \geqslant 0\,, \\ \left[C_{\mathrm{TM}} \begin{pmatrix} \mathbf{E}_{\mathrm{TM}}^0 \\ \mathbf{H}_{\mathrm{TM}}^0 \end{pmatrix} + C_{\mathrm{TE}} \begin{pmatrix} \mathbf{E}_{\mathrm{TE}}^0 \\ \mathbf{H}_{\mathrm{TE}}^0 \end{pmatrix} \right] \\ \quad \times \exp\left[\left(\mathrm{i}p_{\mathrm{m}}' + p_{\mathrm{m}}''\right) ky\right], & y \leqslant 0\,. \end{cases} \tag{4}$$

Here, C_α are the amplitude coefficients. Indices I and II refer to two independent wave branches in the crystal. Subscript TM indicates that the *magnetic* field in the corresponding mode is orthogonal to the sagittal plane xy in which the wave vectors of all partial waves reside. Similarly, subscript TE reveals that the *electric* field in this mode is orthogonal to the same sagittal plane. The complex parameters $p_\alpha = p_\alpha' + \mathrm{i}p_\alpha''$ ($\alpha = \mathrm{I}, \mathrm{II}$) and $p_{\mathrm{m}} = p_{\mathrm{TM}} = p_{\mathrm{TE}} = p_{\mathrm{m}}' - \mathrm{i}p_{\mathrm{m}}''$ are responsible for localization of the wave field (4) near the interface $y = 0$. Localization arises under the condition $p_\alpha'' > 0$ ($\alpha = \mathrm{I}, \mathrm{II}, \mathrm{m}$). It is assumed that the conditions mentioned hold true. These inequalities can be easily realized if we substitute an isotropic medium for the crystal.

In the second case (the reflection problem), the wave field in the isotropic medium (for $y \leqslant 0$) has the same form as in formula (4). However, the field inside the crystal (for $y \geqslant 0$) is now represented as the superposition of three rather than two components responsible for the bulk wave falling onto the interface and two reflected waves, namely, a bulk wave of the same branch as the falling wave and a bulk or nonuniform (localized) wave of the other branch. Later on, this expression will not be used, so we do not present it.

In both the cases (for polaritons and reflection), the stationary wave field at the interface should satisfy the ordinary electrodynamic continuity conditions for the tangential components of the electric and magnetic fields [2]:

$$\mathbf{E}_{\mathrm{t}}\Big|_{y=+0} = \mathbf{E}_{\mathrm{t}}\Big|_{y=-0}\,, \qquad \mathbf{H}_{\mathrm{t}}\Big|_{y=+0} = \mathbf{H}_{\mathrm{t}}\Big|_{y=-0}\,. \tag{5}$$

We define concrete characteristics of the wave field in the isotropic medium possessing the permittivity ε_m by using the concept of impedance (2). From the Maxwell equations we derive the expression for the wave localization parameter p_m in this medium:

$$p_{\mathrm{m}} = p_{\mathrm{m}}' - \mathrm{i}p_{\mathrm{m}}'' = \sqrt{\frac{\varepsilon_{\mathrm{m}}}{n^2} - 1} = \frac{R}{\zeta n}\,, \tag{6}$$

where we introduced the notation

$$R(\zeta n) \equiv \sqrt{1 - (\zeta n)^2}\,. \tag{7}$$

From the same equations it follows that the vector amplitudes of the wave field in the medium have the form

$$\begin{pmatrix} \mathbf{E}^0_{\mathrm{TM}} \\ \mathbf{H}^0_{\mathrm{TM}} \end{pmatrix} = \begin{pmatrix} \zeta(R, \zeta n, 0)^{\mathrm{T}} \\ (0,0,1)^{\mathrm{T}} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{E}^0_{\mathrm{TE}} \\ \mathbf{H}^0_{\mathrm{TE}} \end{pmatrix} = \begin{pmatrix} \zeta(0,0,-1)^{\mathrm{T}} \\ (R, \zeta n, 0)^{\mathrm{T}} \end{pmatrix}, \tag{8}$$

where the superscript T stands for transposition. We do not present the vector amplitudes in a crystal because, as will be shown, their explicit form is insignificant.

For polaritons, standard boundary conditions (5) in view of relationships (8) reduce to the system of four algebraic equations in the unknown coefficients C_α:

$$\begin{pmatrix} E^0_{\mathrm{I}x} & E^0_{\mathrm{II}x} & \zeta R & 0 \\ E^0_{\mathrm{I}z} & E^0_{\mathrm{II}z} & 0 & \zeta/R \\ H^0_{\mathrm{I}x} & H^0_{\mathrm{II}x} & 0 & -1 \\ H^0_{\mathrm{I}z} & H^0_{\mathrm{II}z} & 1 & 0 \end{pmatrix} \begin{pmatrix} C_{\mathrm{I}} \\ C_{\mathrm{II}} \\ C_{\mathrm{TM}} \\ C_{\mathrm{TE}} \end{pmatrix} = 0\,. \tag{9}$$

Here, for the sake of convenience some factors are included in the coefficients C_α.

3. Exact boundary condition at an arbitrary impedance value

We eliminate the quantities C_{TM} and C_{TE} related to an isotropic medium from Eqns (9) and reduce the system of four equations to that of two equations:

$$\left\{ \begin{pmatrix} E^0_{\mathrm{I}x} & E^0_{\mathrm{II}x} \\ E^0_{\mathrm{I}z} & E^0_{\mathrm{II}z} \end{pmatrix} + \zeta \begin{pmatrix} -RH^0_{\mathrm{I}z} & -RH^0_{\mathrm{II}z} \\ H^0_{\mathrm{I}x}/R & H^0_{\mathrm{II}x}/R \end{pmatrix} \right\} \begin{pmatrix} C_{\mathrm{I}} \\ C_{\mathrm{II}} \end{pmatrix} = 0\,. \tag{10}$$

Taking into account the matrix identity

$$\begin{pmatrix} -RH^0_{\mathrm{I}z} & -RH^0_{\mathrm{II}z} \\ H^0_{\mathrm{I}x}/R & H^0_{\mathrm{II}x}/R \end{pmatrix} = \begin{pmatrix} -H^0_{\mathrm{I}z} & -H^0_{\mathrm{II}z} \\ H^0_{\mathrm{I}x} & H^0_{\mathrm{II}x} \end{pmatrix} + (1-R)\begin{pmatrix} H^0_{\mathrm{I}z} & H^0_{\mathrm{II}z} \\ H^0_{\mathrm{I}x}/R & H^0_{\mathrm{II}x}/R \end{pmatrix} \tag{11}$$

and the explicit form of two-dimensional vectors $\mathbf{E}_{\mathrm{t}} = (E_{\mathrm{t}x}, E_{\mathrm{t}z})^{\mathrm{T}}$ and $\mathbf{H}_{\mathrm{t}} = (H_{\mathrm{t}x}, H_{\mathrm{t}z})^{\mathrm{T}}$ residing in the xz plane, namely

$$\mathbf{E}_{\mathrm{t}} = C_{\mathrm{I}}(E^0_{\mathrm{I}x}, E^0_{\mathrm{I}z})^{\mathrm{T}} + C_{\mathrm{II}}(E^0_{\mathrm{II}x}, E^0_{\mathrm{II}z})^{\mathrm{T}}\,, \tag{12}$$

$$\mathbf{H}_{\mathrm{t}} = C_{\mathrm{I}}(H^0_{\mathrm{I}x}, H^0_{\mathrm{I}z})^{\mathrm{T}} + C_{\mathrm{II}}(H^0_{\mathrm{II}x}, H^0_{\mathrm{II}z})^{\mathrm{T}}\,, \tag{13}$$

system (10) reduces to the following equation

$$\left\{ \mathbf{E}_{\mathrm{t}} + \zeta \mathbf{H}_{\mathrm{t}} \times \mathbf{n} + \zeta(1-R)\mathbf{N}\mathbf{H}_{\mathrm{t}} \right\}_{y=+0} = 0\,, \tag{14}$$

where the function $R(\zeta n)$ was defined in formula (7), and $\mathbf{N}(\zeta n)$ is the 2×2 matrix:

$$\mathbf{N}(\zeta n) = \begin{pmatrix} 0 & 1 \\ \dfrac{1}{R(\zeta n)} & 0 \end{pmatrix}. \tag{15}$$

Analysis of the reflection problem yields the same formula (14). The above-mentioned complication concerning making allowance for the additional partial wave does not affect the final result. Notice that equation (14) is equivalent to an initial set of conditions (5) and in this sense is exact. Impedance ζ in Eqn (14) is not assumed to be small and this expression only includes crystal fields (12) and (13). Thus, equation (14) is the natural generalization of Leontovich boundary condition (1).

4. Generalized impedance approximation

The dielectric properties of an isotropic medium possessing a large permittivity ε_{m} (i.e., a small impedance ζ) are similar to those of metals. In this case, function $R(\zeta n)$ in equation (14) [see formula (7)] can be expanded in powers of the small parameter $(\zeta n)^2$, holding an arbitrary number of terms and calculating the characteristics of the wave fields with any desired precision. This expansion comprises odd powers of the parameter ζ:

$$\mathbf{E}_{\mathrm{t}} + \zeta \mathbf{H}_{\mathrm{t}} \times \mathbf{n} + \frac{1}{2}\zeta^3 n^2 \left(\mathbf{N}_1 + \sum_{s=1}^{\infty} (\zeta n)^{2s} \frac{(2s-1)!!}{2^s (s+1)!} \mathbf{N}_{2s+1} \right) \mathbf{H}_{\mathrm{t}} = 0\,, \tag{16}$$

where the set of matrices $\mathbf{N}_m$ $(m = 2s+1)$ is defined by the expression

$$\mathbf{N}_m = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}. \tag{17}$$

Equation (16) is qualitatively distinct from the Leontovich condition (1): in the matrix relationship between vectors $\mathbf{E}_{\mathrm{t}}$ and $\mathbf{H}_{\mathrm{t}}$, the refraction factor n arises in an explicit form. In the reflection problem, the factor n is uniquely related to the angle of wave incidence that is assumed to be prescribed. In the case of a polariton, n is the unknown parameter that is determined from the existence condition for a nontrivial solution to the homogeneous system (10) and, of course, depends on ζ itself. One might think that an uncertainty arises in the structure of expansion (16) with respect to ζ. However, one should keep in mind that the refraction factor $n = c/v$ essentially presents the dimensionless 'slowness' of the polariton and should be of the order of unity, sufficiently weakly depending on ζ for $|\zeta| \ll 1$. For example, in Section 5.1 we will show [see formula (29)] that for an isotropic dielectric with the permittivity ε the estimate $n \approx \sqrt{\varepsilon}(1 - \zeta^2\varepsilon/2)$ is valid.

In view of the given considerations, from expansion (16) it follows that the discrepancy between Leontovich approximation (1) and the exact boundary condition starts from the term $\sim \zeta^3$; hence, in the framework of this approach the quadratic corrections $(\sim \zeta^2)$ to the wave fields are correct. It is interesting that Leontovich himself paradoxically underestimated the accuracy of the approximation introduced. In Ref. [1] he compared the reflection coefficient of the electromagnetic wave, calculated in approximation (1), with its exact expression that follows from the Fresnel formula, and it appeared to him that they differ by terms of the order of ζ^2. Meanwhile, a more thorough consideration gives a difference that is of the next order: $\sim \zeta^3$ (see Section 5.2). This is a rather good accuracy even in the visible wavelength range. For example, for aluminium [6] the parameter $|\zeta| \approx 0.25$ corresponds to the vacuum wavelength $\lambda_0 \approx 0.45$ μm, which is the boundary value between violet and blue light. In this case, $|\zeta|^3 \approx 0.016$ and moving to a longer wavelength range substantially improves the accuracy (see Table 1).

Equation (16) makes it possible to intentionally improve the accuracy of the description by breaking the series at will and in this sense it is the *generalized impedance approximation*.

Table 1. Values of the components of the surface impedance $\zeta = \zeta' + i\zeta''$ in the visible and near-IR spectral ranges for aluminium at room temperature (using data taken from Ref. [6]).

λ_0, µm*	Aluminium	
	ζ'	$-\zeta''$
0.40	0.0229	0.267
0.45	0.0246	0.244
0.50	0.0234	0.215
0.55	0.0236	0.197
0.60	0.0253	0.178
0.65	0.0268	0.163
0.70	0.0296	0.150
0.75	0.0316	0.142
0.80	0.0353	0.136
0.85	0.0373	0.135
0.90	0.0331	0.135
0.95	0.0259	0.133
1.00	0.0199	0.129
1.10	0.0121	0.118
1.20	0.0092	0.108

* λ_0 is the vacuum wavelength.

Surely, one should judge the quality of approximation by comparing the solutions obtained with the corresponding exact (or more accurate) solutions, rather than by estimating the accuracy of writing the boundary condition. With this aim, we will perform below such an analysis using the example of two simple problems with known exact solutions.

5. Wave fields at the interface between an isotropic dielectric and an isotropic metal

In order to estimate the accuracy of describing particular wave characteristics in the framework of initial (1) and generalized (16) impedance approximations, we will consider below two different stationary wave fields with the TM-polarization, corresponding to a surface polariton and reflection problem, near the interface between two isotropic media: a dielectric with the permittivity ε and a metal with the surface impedance ζ. As was mentioned, exact solutions are known for both these cases [2].

5.1 Quasibulk polaritons

In the problem on TM-polariton propagation along the interface between two isotropic media (see Fig. 1b), the depth profile of the electromagnetic field in the dielectric is described by the expression

$$\begin{pmatrix} \mathbf{E}(y) \\ \mathbf{H}(y) \end{pmatrix} = C \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \end{pmatrix} \exp\left(\mathrm{i}p' - p''\right) ky \,. \quad (18)$$

For the parameters involved in Eqn (18), by full analogy with Eqns (6) and (8), we obtain

$$\begin{pmatrix} \mathbf{E}^0 \\ \mathbf{H}^0 \end{pmatrix} = \begin{pmatrix} n\,(-p, 1, 0)^{\mathrm{T}} \\ \varepsilon\,(0, 0, 1)^{\mathrm{T}} \end{pmatrix}, \quad (19)$$

$$p = p' + \mathrm{i}p'' = \sqrt{\frac{\varepsilon}{n^2} - 1} \,. \quad (20)$$

Leontovich boundary condition (1) in this case takes especially simple form

$$E_x^0 - \zeta H_z^0 = 0\,, \quad \text{i.e.,} \quad pn + \zeta\varepsilon = 0\,. \quad (21)$$

Taking into account formulas (20) and (21), we arrive at the following expression for the refraction parameter

$$n = \sqrt{\varepsilon}\left(1 - \frac{1}{2}\,\zeta^2\varepsilon\right). \quad (22)$$

In view of the above consideration, notice that in this expression the term quadratic in ζ is correct.

In the next approximation (16), retaining only one term, $\sim \zeta^3$, in addition to relationship (1), we obtain the expression for the refraction parameter

$$n = \sqrt{\varepsilon}\left(1 - \frac{1}{2}\,\zeta^2\varepsilon + \frac{3}{8}\,\zeta^4\varepsilon^2\right). \quad (23)$$

The additional term in formula (23) characterizes the relative accuracy of expression (22):

$$\frac{\Delta n}{n} \approx \frac{3}{8}\,\zeta^4\varepsilon^2\,. \quad (24)$$

We now estimate error (24) in the short-wavelength region of the visible range ($\lambda_0 \approx 0.45$ µm), assuming for the estimate $|\zeta| \approx 0.25$ and $\varepsilon = 2.25$. It turns out that even in this spectral range the Leontovich approximation provides rather high precision in determining the refraction parameter: $\Delta n/n \sim 0.7\%$. Actually, this is the estimate of the upper error boundary for the whole visible range — the accuracy increases at longer wavelengths.

Of course, the expression (23) for n is far more accurate. The estimate of the error in this case can be obtained in a manner quite similar to that used in deriving formula (24): n is calculated in the approximation that makes allowance for the additional term $\sim \zeta^5$ in expansion (16), then expression (23) is subtracted from the result and the difference is divided by $n \approx \sqrt{\varepsilon}$. Omitting details, we obtain the error for approximate expression (23):

$$\frac{\Delta n}{n} \approx -\frac{5}{16}\,\zeta^6\varepsilon^3\,, \quad (25)$$

which at the same parameters yields the relative error of order 0.1%.

It is interesting that the insertion of approximate formulae (22) and (23) within the limits of their accuracy into exact equation (20) yields the same expression for the localization parameter:

$$p = p' + \mathrm{i}p'' = -(\zeta' + \mathrm{i}\zeta'')\sqrt{\varepsilon}\,, \quad (26)$$

in which the first omitted terms are on the order of ζ^3 and ζ^5, respectively. Such a coincidence is explained by the fact that formula (26) is exact [2]. Since $\zeta'' < 0$ [see relationship (2)], the localization condition for the wave field, $p'' = -\zeta''\sqrt{\varepsilon} > 0$, is fulfilled in this case.

It should be noted that in this case the existence of polariton localization itself is a consequence of nonzero impedance for the metal under consideration, $\zeta \neq 0$. At $\zeta = 0$, when the metal is an ideal conductor, from formula (26) we obtain $p = 0$, so that the polariton features a bulk character and is not damped when propagating, and its wave field does not penetrate into the metal. At small imaginary parts of impedance ζ'', the value of p'' is also small, and the wave field is characterized by weak localization, and the polariton becomes quasibulky.

In introducing a nonzero impedance, both the parameter n and speed v of polariton propagation become complex values. This is related to dissipation of the wave field penetrating into the metal, which, in turn, limits the mean free path L of polariton in its propagation along the metal boundary. Principally, the concept of the polariton itself retains a physical meaning until its mean free path is substantially longer than the wavelength λ in dielectric: $L \gg \lambda$. The polariton mean free path L can be estimated from the relationship $L \operatorname{Im} k = 1$, where $k = \omega n/c$ is the wavenumber. In view of $\omega/c = 2\pi/\lambda\sqrt{\varepsilon}$, we obtain

$$L = \frac{\lambda}{2\pi \operatorname{Im}(n/\sqrt{\varepsilon})} . \tag{27}$$

By substituting the parameter n from expression (22) into formula (27), in the Leontovich approximation we find

$$L = -\frac{\lambda}{2\pi\varepsilon\zeta'\zeta''} . \tag{28}$$

Taking into account relationship (23), this estimate has a low accuracy characterized by the relative error $\sim \zeta^2$ (or more precisely $\sim \zeta''^2$); however, this is just the case where a high accuracy is not necessary. The estimates based on formula (28) with the parameters ζ' and ζ'' for aluminium, taken from Table 1, and with the typical value of $\varepsilon = 2.25$ show that, with increasing the wavelength λ_0 in the interval 0.4–1.2 μm, the ratio L/λ rises approximately from 12 to 70, i.e., the criterion $L/\lambda \gg 1$ holds perfectly in the IR range, and rather well in the whole visible range.

We note that actually all the analytical estimates of this section can be directly obtained from the exact solution for the problem on propagation of TM-polarization waves localized near the interface between two isotropic media with two different permittivities (see Ref. [2]). If one medium is metal, then the corresponding permittivity should be considered a complex parameter with a large absolute value and negative real part. In this case, the exact expression for n in terms of the impedance concept, which is valid for all values of impedance ζ, is written out in the form

$$n = \sqrt{\frac{\varepsilon}{1 + \zeta^2\varepsilon}} . \tag{29}$$

Hence, the Leontovich approximation can be considered quite reasonable, as applied to the description of the polariton. It gives exact expression (26) for the localization parameter p, and formula (22) for the refraction parameter n, which is characterized by relative error (24) not exceeding 1% for Al in the visible wavelength range.

5.2 Reflection problem

The total wave field in the dielectric in this case comprises separate fields of two volume components, namely, the wave falling to the interface and the wave reflected from it:

$$\begin{pmatrix} \mathbf{E}(y) \\ \mathbf{H}(y) \end{pmatrix} = C^{\mathrm{i}} \begin{pmatrix} \mathbf{E}^{0\mathrm{i}} \\ \mathbf{H}^{0\mathrm{i}} \end{pmatrix} \exp(-\mathrm{i}pky) + C^{\mathrm{r}} \begin{pmatrix} \mathbf{E}^{0\mathrm{r}} \\ \mathbf{H}^{0\mathrm{r}} \end{pmatrix} \exp(\mathrm{i}pky) , \quad y \geqslant 0 . \tag{30}$$

Here, the following notation was introduced:

$$\begin{pmatrix} \mathbf{E}^{0\mathrm{i,r}} \\ \mathbf{H}^{0\mathrm{i,r}} \end{pmatrix} = \begin{pmatrix} n(\pm p, 1, 0)^{\mathrm{T}} \\ \varepsilon(0, 0, 1)^{\mathrm{T}} \end{pmatrix} , \tag{31}$$

$$p = \cot\theta , \qquad n = \sqrt{\varepsilon}\sin\theta , \tag{32}$$

where θ is the angle between the wave vector of the incident wave and the y-axis (see Fig. 1c).

Leontovich boundary condition (1) in this case yields the relationship

$$(E_x^{0\mathrm{i}} - \zeta H_z^{0\mathrm{i}})C^{\mathrm{i}} + (E_x^{0\mathrm{r}} - \zeta H_z^{0\mathrm{r}})C^{\mathrm{r}} = 0 , \tag{33}$$

from which we obtain the reflection coefficient

$$r = \frac{C^{\mathrm{r}}}{C^{\mathrm{i}}} = \frac{1 - \chi}{1 + \chi} . \tag{34}$$

Here, we introduced notation $\chi = \zeta\sqrt{\varepsilon}/\cos\theta$. Expression (34) is valid in the case where the parameter χ can be assumed to be small. In this case, the angle of incidence θ *a fortiori* cannot be close to 90° which corresponds to grazing incidence.

In the next approximation, by retaining the term $\sim \zeta^3$ in expansion (16) and neglecting higher-order terms, instead of expression (34) we have

$$r = \frac{1 - \chi + (1/8)\chi^3 \sin^2 2\theta}{1 + \chi - (1/8)\chi^3 \sin^2 2\theta} . \tag{35}$$

It is easy to verify that the relative difference between the second and first approximations is determined by the quantity

$$\frac{\Delta r}{r} = \frac{1}{4}\chi^3 \sin^2 2\theta . \tag{36}$$

Worthy of note, however, is the fact that the expansion parameter χ in formulas (34), (35) at an ordinary choice of the parameters $|\zeta| = 0.25$ and $\varepsilon = 2.25$ is not small even at angles θ far from normal incidence. Hence, in a short-wavelength part of the visible range the error $\Delta r/r$ in approximation (1) is not so small as in the polariton theory despite its smallness compared to the above-mentioned Leontovich pessimistic estimates. At $\theta = 45°$, the estimates yield $|\Delta r/r| \sim 4\%$.

Similarly to obtaining estimate (25) one can show that formula (35) is characterized by the relative error

$$\frac{\Delta r}{r} = \frac{1}{64}\chi^5 \sin^4 2\theta . \tag{37}$$

In this case, error (37) is two orders of magnitude less, $|\Delta r/r| \sim 0.06\,\%$, at the same values of the parameters ($|\zeta| = 0.25$, $\varepsilon = 2.25$, $\theta = 45°$).

Relative errors (36) and (37) are shown in Fig. 2 versus the wavelength and angle of incidence. The curves are plotted by using the data for aluminium taken from Table 1. As one can see in Fig. 2a, the accuracy of the Leontovich approximation noticeably rises at longer wavelengths. Nevertheless, in the short-wavelength part of the visible range the error is maintained at an admissible level, but only for the angles of incidence far from the singular value $\theta = 90°$. In the next approximation, the accuracy is much better. From Fig. 2b one can see that in this case, even for the greatest considered angle of incidence $\theta = 80°$, the error in the short-wavelength part of visible range is less than 1.5%.

Obviously, expansions (34) and (35) can also be derived from the known exact Fresnel formula [1, 2], which in our notations takes the form

$$r = \frac{1 - \chi\sqrt{1 - (1/4)\chi^2 \sin^2 2\theta}}{1 + \chi\sqrt{1 - (1/4)\chi^2 \sin^2 2\theta}} , \tag{38}$$

where the parameter χ can already take an arbitrary value.

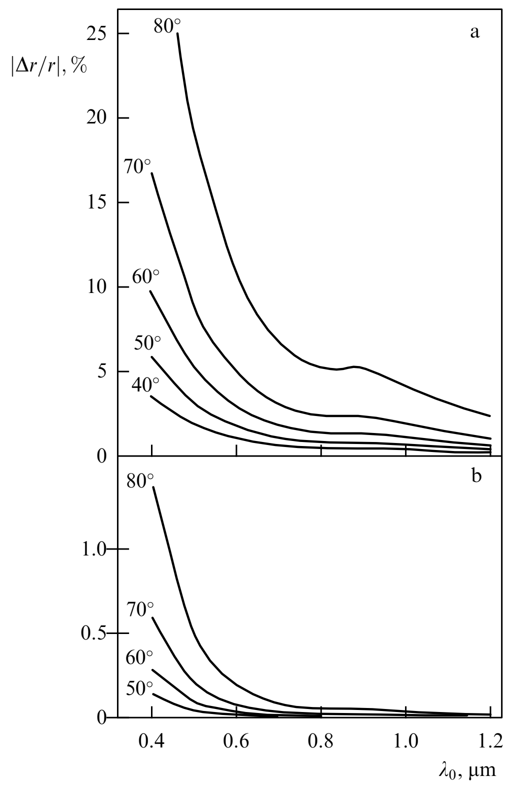

Figure 2. Relative error in calculations of the reflection coefficient r versus wavelength λ_0 at various angles of incidence: in the Leontovich approximation (a), where $|\Delta r/r| \sim \zeta^3$, and in the next approximation (b), where $|\Delta r/r| \sim \zeta^5$. Values of the angles of incidence θ are given alongside of each curve.

6. Conclusions

One of the main results of these notes is obtaining exact formula (14) for electromagnetic fields in a dielectric at the boundary with a metal at an arbitrary impedance of the latter. The generalized form of the impedance approximation (16) that follows from equation (14) gives the possibility of controlling the accuracy of calculated wave characteristics for electromagnetic fields by retaining the desired number of terms in expansion (16) in powers of impedance ζ. If necessary, one can utilize the exact boundary condition (14) expressed in the same terms of the surface impedance. Since expansion (16) comprises odd powers of impedance ζ, by cutting off the series on the term of order ζ^m ($m = 1, 3, 5, \ldots$), we have definitely correct summands of order ζ^{m+1} in calculating the parameters of wave fields. The dielectric adjacent to an isotropic metal may be an isotropic body or anisotropic crystal.

The impedance approximation formulated by Leontovich ($m = 1$) differs from the exact boundary condition by terms on the order of $\sim \zeta^3$. This is just the order of magnitude for the error in describing the coefficient of electromagnetic wave reflection from the boundary between an isotropic dielectric and a metal. At the same time, the Leontovich approximation yields the higher accuracy $\sim \zeta^4$ in describing the propagation velocity for the polariton wave field, and the calculated polariton localization parameter near the interface coincides with the exact relationship altogether.

Notice that the accuracy in the description of reflection in this approximation is worse not only because of the lower degree of the parameter ζ that defines the error. As we have seen, the small parameter in this case is $\zeta/\cos\theta$ rather than ζ, which eliminates the possibility of describing grazing incidence near $\theta \approx 90°$. This is why, as one can see from Fig. 2a, at large angles of incidence and short wavelengths corresponding to insufficiently small ζ (i.e., for $\cos\theta \leqslant \zeta$) the Leontovich approximation fails. However, as follows from Fig. 2a, far from the singular angle $\theta \approx 90°$ the Leontovich approximation remains admissible over all the visible range. To enhance the accuracy of the description near the grazing incidence it suffices to fall outside the limits of the Leontovich approximation by adding to the term that is linear in impedance the next expansion terms of our generalized impedance approximation (16). Figure 2b demonstrates that just the first such term reduces the calculation error by 1–2 orders in magnitude.

In view of the very small penetration depth d_{m} of the electromagnetic field in metal, the metal half-space in our consideration (and in practice) can be replaced by a thin metal deposition (metallization) of the dielectric surface without affecting the results of the consideration. It is important that the thickness of the deposition be noticeably greater than the characteristic depth d_{m}. On the basis of the criterion $kp''_{\mathrm{m}} d_{\mathrm{m}} = 1$ and taking into account the fact that $p_{\mathrm{m}} = 1/\zeta n$ and $k = 2\pi n/\lambda_0$, we obtain the following estimate for the depth d_{m}:

$$d_{\mathrm{m}} = \frac{\lambda_0 |\zeta|^2}{2\pi|\zeta''|} \,. \tag{39}$$

For aluminium at $\lambda_0 = 0.6$ μm, formula (39) yields $d_{\mathrm{m}} \approx 0.018$ μm.

In Section 5, the test analysis of the discussed impedance approximations was performed for an example of two exactly solvable problems concerning the surface polariton and reflection of a plane wave from the boundary of the isotropic dielectric–metal structure. Meanwhile, similar unsolved problems on a metallized crystal boundary promise a richer physical picture for both the polariton and reflection cases to judge from the available material [7, 8] obtained in zero approximation $\zeta = 0$ when the metal is an ideal conductor. The analysis performed in this work constitutes the basis for a more general consideration of these problems as well.

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