METHODOLOGICAL NOTES

PACS numbers: **04.20.–q**, **04.40.–b**, **04.70.–s**

The sky of a universe as seen through a wormhole

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DOI: 10.3367/UFNe.0179.200908c.0861

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Abstract. **A method is discussed for calculating the gravitational lensing of light passing through a wormhole from one universe to another. Fundamental and characteristic features of radiation traveling through a wormhole are identified, allowing new methods for observing wormhole-specific characteristics and yielding techniques for determining the parameters of wormhole models.**

1. Introduction

Recently, there has been a substantial growth of interest in papers in relativistic astrophysics that discuss solutions with traversable wormholes (WHs). This interest is due to the design and construction of radio interferometers with high angular resolution, which in the future may allow distinguishing WHs from black holes.

In this paper, we consider a static and spherically symmetric WH solution that is only slightly different from the Reissner–Nordström extreme black hole solution (see [1, 2]). As was shown in [3, 4], by changing parameters in the equation of state of matter, the WH solution can be smoothly transformed into a solution with the horizon of the extreme Reissner–Nordström black hole, and the WH becomes nontraversable.

The exotic properties of WHs can be due to phantom matter that surrounds their throat and violates the zero-energy condition[1] (see, e.g., [3, 5]).

[1] This condition requires the inequality (NEC) $T_{ik}\zeta^i\zeta^k \geqslant 0$, where T_{ik} is the matter energy–momentum tensor and ζ^1 is the null 4-vector of a photon. The physical meaning of exotic phantom matter violating the NEC condition is in the possibility of finding a reference frame where the observed energy density is negative.

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Received 3 September 2008, revised 17 October 2008
Uspekhi Fizicheskikh Nauk **179** (8) 861 – 864 (2009)
DOI: 10.3367/UFNr.0179.200908c.0861
Translated by K A Postnov; edited by A M Semikhatov

The case of a small deviation of the WH from an electrically charged (or magnetically charged) extreme Reissner–Nordström black hole with charge q (the charge $q = m$ totally determines all the properties of this black hole[2]) can be of practical interest. In this case, the amount of phantom matter near the WH can be made arbitrarily small.

The unique properties of WHs can be manifested in the peculiarities of gravitational lensing (see, e.g., [6]). In this paper, we suggest a method for calculating those gravitational effects for different WH models that will in principle allow observing WHs and their properties in the future.

2. The Einstein equations

The metric tensor of a static and spherically symmetric WH can be written in the form

$$\begin{aligned} ds^2 &= \exp\left[2\phi(r)\right] dt^2 - \exp\left[\lambda(r)\right] dr^2 - r^2 \, d\Omega^2 \,, \\ d\Omega^2 &= d\theta^2 + \sin^2\theta \, d\varphi^2 \,. \end{aligned} \tag{1}$$

We consider matter with a linear equation of state in the case where the coupling of the energy density ε to the longitudinal pressure $p_\parallel$ (along the radius) and the transverse pressure $p_\perp$ (perpendicular to the radius) is determined by a constant coefficient $1 + \delta$:

$$1 + \delta = -\frac{p_\parallel}{\varepsilon} = \frac{p_\perp}{\varepsilon} \,. \tag{2}$$

For $\delta = 0$, the energy–momentum tensor determined by Eqns (1) and (2) corresponds to that of a static (and spherically symmetric) magnetic (or electric) field. Adding the δ term to the energy–momentum tensor leads to an addition to the radial magnetic field of phantom matter with $\varepsilon + p_\parallel < 0$ (without which the static existence of a WH is impossible).

We note that in contrast to a black hole, there is no real charge q in a WH: the monopole magnetic field lines in our universe enter the WH throat and then exit it in another universe to infinity.

[2] In this work, we use units such that $c = 1$ and $G = 1$.

We introduce the notation

$$x = \frac{r}{q}\,, \qquad \exp(-\lambda) \equiv 1 - \frac{y(x)}{x}\,,$$
$$\xi(x) \equiv 8\pi\varepsilon q^2\,, \qquad z(x) \equiv \left(1 - \frac{1}{x}\right) \tag{3}$$

and let the prime denote the derivative with respect to x. The Einstein equations corresponding to this metric have the form[3]

$$8\pi\varepsilon r^2 = \xi x^2 = -\exp(-\lambda)(1 - x\lambda') + 1\,, \tag{4}$$

$$8\pi p_{\parallel} r^2 = -(1+\delta)\xi x^2 = \exp(-\lambda)(1 + 2x\phi') - 1\,, \tag{5}$$

$$8\pi p_{\perp} r^2 = (1+\delta)\xi x^2 = \exp(-\lambda) \times \left(x^2\phi'' + x^2\phi'^2 - x\frac{\lambda'}{2} - x^2\phi'\frac{\lambda'}{2} + x\phi'\right). \tag{6}$$

The Bianchi identities ($T^k_{l;k} \equiv 0$) are contained in the Einstein equations and have the form

$$(1+\delta)(\ln\xi)' + 4(1+\delta)(\ln x)' + \delta\phi' = 0\,. \tag{7}$$

By defining the WH throat radius[4] r_0 (or x_0) as

$$x_0 \equiv y(x_0)\,, \tag{8}$$

we rewrite Eqns (4), (5), and (7) in the following conventional form:

$$y(x) = x_0 + \int_{x_0}^{x} \xi x^2\,\mathrm{d}x\,, \tag{9}$$

$$\left[\ln(\xi x^4)\right]' = \frac{\delta}{2(1+\delta)}\,\frac{\xi x(1+\delta) - y/x^2}{1 - y/x}\,, \tag{10}$$

$$\exp\left[\phi(x)\right] = (\xi x^4)^{-(1+\delta)/\delta}\,. \tag{11}$$

For $\delta = 0$, this solution takes the form of the extreme Reissner–Nordström solution:

$$y_{[\delta=0]}(x) = 2 - \frac{1}{x}\,, \qquad \xi_{[\delta=0]}(x) = \frac{1}{x^4}\,,$$
$$\exp\left[\phi_{[\delta=0]}(x)\right] = 1 - \frac{1}{x}\,. \tag{12}$$

The problem for the Einstein equations set up above is to obtain their analytic solution in the first approximation in the small parameter δ, with the subsequent application of this solution to the calculation of gravitational lensing on a WH.

In the linear approximation in δ, Eqn (10) can be rewritten as

$$\frac{\partial\ln(\xi x^4)}{\partial x} = -\delta\,\frac{\partial\ln z}{\partial x}\,. \tag{13}$$

From here, using that ξ must coincide with $\xi_{[\delta=0]}$ at infinity, we obtain the approximate solution

$$\xi x^4 = z^{-\delta}\,, \qquad y(x) = x_0 + \frac{z^{1-\delta} - z_0^{1-\delta}}{1-\delta} \qquad \left(z_0 \equiv 1 - \frac{1}{x_0}\right),$$
$$\exp\phi = z^{1+\delta}\,. \tag{14}$$

The relation of the throat radius x_0 to δ is determined by the coincidence of the asymptotic forms of $\exp(-\lambda)$ and $\exp(2\phi)$ as $x \to \infty$:

$$x_0 + \frac{1 - z_0^{1-\delta}}{1-\delta} \to 2(1+\delta)\,. \tag{15}$$

The throat radius can be found from this transcendental equation. A detailed analysis of this equation yields the asymptotic condition

$$\lim_{\delta\to 0}\delta = (x_0 - 1)^2\,, \tag{16}$$

where the parameter δ is determined by the matter equation of state in the WH.

For $\delta > 0$, the obtained solution no longer corresponds to a black hole because the metric coefficient $\exp(2\phi)$ does not vanish anywhere (since $x_0 > 1$), and hence there is no horizon. Therefore, this solution corresponds to a WH with the throat at $x = x_0$.

Strictly speaking, the series expansion in δ is not fully correct, as can be seen by direct substitution of Eqns (14) in (5) and (6). However, a WH solution can be found in the opposite way: the metric components $\exp(2\phi)$ and $\exp\lambda$ can be taken from Eqns (14), and expressions for $p_{\parallel}$ and $p_{\perp}$ can be found from Eqns (5) and (6). Such an approach is as correct as the conventional solution of the Einstein equations.

3. Light passing through the throat

Let another universe contain $N \gg 1$ stars of the same luminosity, and all these stars be uniformly distributed over the sky.

In our universe, an observer looking through a WH throat at the stars in another universe sees their nonuniform distribution in the throat. This is because the WH throat refracts and distorts the light from these stars. Clearly, these distortions looks centrally symmetric relative to the WH throat center.

Let the observer now look at only part of these stars through a thin ring centered at the throat center; the ring radius is h and its thickness is $\mathrm{d}h$. Hence, he observes the solid angle $\mathrm{d}\Omega$ of the sky of another universe with $\mathrm{d}\Omega = 2\pi|\sin\theta|\,\mathrm{d}\theta$. Here, $\theta(h)$ is the deviation angle of light rays passing through the WH throat.[5] Because the total solid angle is 4π, the observer sees $\mathrm{d}N = N\,\mathrm{d}\Omega/(4\pi)$ stars.[6] Here, the visible density of stars (per unit area of the ring $\mathrm{d}S = 2\pi h\,\mathrm{d}h$) is $J = \mathrm{d}N/\mathrm{d}S$. We thus obtain

$$J(h) = \frac{N|\sin\theta|}{4\pi h}\,\frac{\mathrm{d}\theta}{\mathrm{d}h}\,. \tag{17}$$

[3] The derivation of these equations can be found, e.g., in [7, problem 5k, § 100].

[4] The WH throat is (by definition) the minimum possible radius r squared standing before the angular element $\mathrm{d}\Omega^2$ in the metric.

[5] By definition, direct light ray propagation is through the WH throat center.

[6] Because the light deviation angle θ can be greater than π, it follows that the total solid angle is also greater than 4π, but this replacement reduces to another constant (instead of 4π) and does not affect the final result.

In [3], $\theta(h)$ was obtained as

$$\theta(h) = 2\int_{x_0}^{\infty} \frac{h/q}{x^2\sqrt{(1-y/x)\left(\exp(-2\phi) - h^2/(qx)^2\right)}}\,\mathrm{d}x\,. \tag{18}$$

It then follows that

$$\frac{\mathrm{d}\theta}{\mathrm{d}h} = \frac{2}{q}\int_{x_0}^{\infty} \frac{\exp(-2\phi)}{x^2\sqrt{1-y/x}\,\left[\exp(-2\phi) - h^2/(qx)^2\right]^{3/2}}\,\mathrm{d}x\,. \tag{19}$$

Using formulas (14), (17), (18), and (19) allows finding the expression for the visible density of stars $J(h)$ in the wormhole.

We note that the maximum possible target parameter $h = h_{\max}$ for which one can observe stars from another universe can be derived from Eqn (18). This parameter is restricted by the zero of the second factor in the radicand in Eqn (18):

$$\exp(-2\phi) - \frac{h^2}{(qx)^2} \geqslant 0 \Rightarrow \frac{h}{q} \leqslant x\exp(-\phi)\,. \tag{20}$$

The maximum possible value of h therefore corresponds to the minimal possible values of $x\exp(-\phi)$. Making trivial calculations and using Eqn (14), we obtain

$$\frac{h_{\max}}{q} = 4\times 2^{\delta} \approx 4 \tag{21}$$

as $\delta \to 0$.

The integral in Eqn (19) diverges, but the total deviation angle of a photon $\theta_{\max}$ in Eqn (18) is finite. The divergence in Eqn (19) at $h = h_{\max}$ is due to the mathematically infinite density of stars in the elementary solid angle for $h = h_{\max}$. This mathematical singularity does not relate to physical processes and can be removed by carrying out observations of this region of the WH with any physical detector that always has some finite angular resolution, which determines the observed density of stars in the elementary solid angle [instead of (17)] near $h = h_{\max}$.

The distortion of light rays passing through the WH throat leads not only to a redistribution of the star density but also to a change in their visible brightness as follows: upon increasing the target parameter h, the brightness of a star changes. This is because increasing the radius of the ring h through which light passes changes the solid angle element into which this light is scattered. The corresponding change in the star brightness is proportional to $\kappa = \mathrm{d}S/\mathrm{d}\Omega$. The total brightness of all stars visible from the unit area of such a ring is therefore given by $\kappa\,\mathrm{d}N/\mathrm{d}S$.

As $N \to \infty$, one can no longer see individual images of stars — they become smeared out due to the restricted angular resolution of the detector. It therefore follows from the equality $\kappa\,\mathrm{d}N/\mathrm{d}S = N/(4\pi)$ that the mean brightness is independent of h, i.e., the brightness is uniform. At small values of N, individual stars can be discerned in the WH, and hence there is no uniform brightness.

Thus, we find that as $N \to \infty$, the visible brightness of the part of the WH inside its throat is independent of the target parameter, and the wormhole looks like a uniform spot at all wavelengths, independently of the WH model. This result is universal: it is independent of the choice of the specific WH model but depends only on the positive total WH mass.

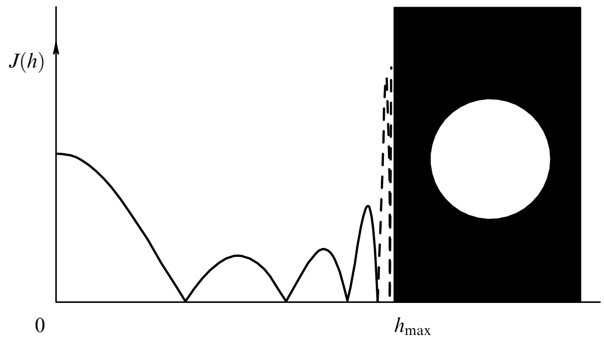

Figure 1. The plot of $J(h)$ for $h \in (0, h_{\max})$ and $\delta = 0.001$ (left). The sky of another universe as seen through a WH throat as $N \to \infty$ (to the right).

In this connection, we note that an erroneous conclusion was made in [8] regarding the observation of a nonuniform light distribution from another universe through a WH throat. In that paper, an incorrect physical interpretation was given to correct mathematical expressions.

4. Conclusion

Despite the obtained result on a uniform light distribution inside a WH throat for any WH model, we note that in the real universe, the number of visible stars is finite (although very large). This implies that if the angular resolution of the detector in our universe is sufficiently high, the varying star density in the throat $J(h)$ can be measured. The plot for $\delta = 0.001$ is shown in the left side of Fig. 1. The sharp minima in this plot correspond to zeros of the sine function in Eqn (17). For sufficiently large values of the target parameter $h \sim h_{\max}$, the light rays deviate by large angles ($\theta > \pi$), and hence sharp dips appear close to the points $\theta = \pi n$. However, the observed brightness of stars tends to infinity in the vicinity of such 'dips' (lensing), which ultimately provides a uniform (on average) light flux through the WH throat as shown in the right side of Fig. 1.

The positions of these dips depend on δ, and therefore measuring them allows determining the parameters of the equation of state of WH matter and other characteristics of the WH model (similarly to inferring information from the light spectra).

Thus, we suggested a method of calculating the effects of gravitational lensing for light passing through wormholes, as well as a way to observe distinct features of wormholes.

In conclusion, I thank V Strokov, N S Kardashev, and I D Novikov for their fruitful discussions of the paper and the useful remarks.

The work was supported by the RFBR grants 07-02-01128-a, 08-02-00090-a, Science Schools grants NSh-626.2008.2, NSh-2469.2008.2, and the RAS program "The Origin and Evolution of Stars and Galaxies 2008".

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