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Modern methods for the statistical description of dynamical stochastic systems

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1. Introduction

S M Rytov gave much attention to the development of functional methods of stochastic system analysis at All-Moscow Radiophysics Seminars he led. He dubbed them *radiomathematics*. I participated in these seminars from the end of the 1960s. S M Rytov frequently asked, me in particular, a question: "What are you studying?" I traditionally answered that solutions of stochastic equations (ordinary and partial differential, or integral) are functionals from random coefficients of these equations and that I am studying the dependence of the statistical characteristics of these solutions on various models and the statistical parameters of these coefficients. For about 30 years I considered this answer to be exhaustive, and only during the last 10–15 year did I realize all the topicality of the question "What are you studying?" and the total inadequacy of my usual answer. This is related to the fact that in recent years the attention of both theorists and experimenters has focused on the question of the links of dynamics pertaining to averaged characteristics of problem solution to the solution behavior in specific realizations. This is especially relevant to geophysical problems related to the atmosphere and ocean in which, by and large, the respective averaging ensemble is absent and experimenters as a rule deal with individual realizations. In this case, the results of statistical analysis frequently not only have nothing in common with the behavior of solutions in specific realizations but often simply contradict them. It is namely this that I would like to demonstrate in this report.

Three approaches are currently utilized in the analysis of a stochastic dynamical system.

The first approach is based on analyzing the Lyapunov stability of solutions to deterministic linear ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{x}(t) = A(t)\,\mathbf{x}(t)$$

and traditionally attracts the attention of many researchers. One analyzes here the upper bound of the problem solution

$$\lambda_{\mathbf{x}(t)} = \overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\ln\left|\mathbf{x}(t)\right|\,,$$

which is termed its characteristic exponent. When this approach is applied to stochastic dynamical systems, it is common that, to interpret and simplify the obtained results at the final stage, statistical analysis is invoked and statistical averages such, for example, as

$$\left\langle\lambda_{\mathbf{x}(t)}\right\rangle = \overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\left\langle\,\ln\left|\mathbf{x}(t)\right|\right\rangle\,,$$

are computed.

The drawbacks of this approach to stochastic dynamical systems are as follows:

(1) Such simplifying features of random parameters as stationarity in time, homogeneity, and isotropy in space are exploited only at the stage of final analysis.

(2) When passing to continual generalizations of ordinary differential equations (for example, in mechanics or the electrodynamics of continuous media), i.e., to partial differential equations (to fields), the analysis of Lyapunov stability is only possible through the series expansions of solutions in complete sets of orthogonal functions. If such a technique is applied to stochastic problems, a question emerges as to whether the operations of series expansion and statistical averaging are permutable. In particular, when statistical characteristics of random processes and fields are approximated by singular (generalized) functions (as, for example, in the approximation that fluctuations of system parameters are delta-correlated), these operations are not, as a rule, permutable.

The second approach is also traditional and relies on the analysis of moment and correlation functions of solutions to stochastic problems.

The drawback of this second approach is that commonly used methods of statistical averaging smooth the qualitative features of separate realizations and it is not uncommon for the obtained statistical characteristics to have nothing in common with the behavior of separate realizations.

In certain circumstances there exist, however, physical processes and phenomena occurring with the probability of one (i.e., happening in almost all realizations). They are called *coherent* (see monographs [1–4] and work [5] where this question is thoroughly discussed). To describe such phenomena, the third approach is applied. It is rooted in *the method of statistical topography* which studies, instead of moment functions, the statistical characteristics of some functionals describing precisely these coherent phenomena.

Below, we will illustrate these approaches as applied to simple physical problems.

2. Examples of dynamical systems

2.1 Diffusion of a passive inertialess admixture in a random velocity field

As the first example let us consider the relative diffusion of inertia-less particles in a random hydrodynamical flow with the velocity field $\mathbf{u}(\mathbf{r}, t)$ in the framework of the simplest kinematic equation for each particle:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) = \mathbf{u}\big(\mathbf{r}(t), t\big)\,, \quad \mathbf{r}(0) = \mathbf{r}_0\,.$$

Numerical modeling of this problem indicates that the dynamics of the system of particles are essentially dependent on whether the velocity field is solenoidal or divergent. Thus, Fig. 1a presents in a schematic way a fragment of the evolution exhibited by a system of particles (a two-dimensional case) for a particular realization of a solenoidal velocity field $\mathbf{u}(\mathbf{r})$ stationary in time. Nondimensional time here is related to statistical parameters of the field $\mathbf{u}(\mathbf{r})$. Initially, the particles were uniformly spread over the circle. In this case, they continue to fill the area confined by the deformed contour in a fairly uniform way. Only a strong contour irregularity of a fractal character develops.

For the potential velocity field $\mathbf{u}(\mathbf{r})$, however, particles uniformly spread over a square at the initial instant of time form cluster areas as they evolve with time. Figure 1b presents a fragment of such an evolution, obtained through numerical simulation. We emphasize once again that formation of *clusters* in this case is a purely kinematic effect. Apparently, after averaging over an ensemble of realizations of the random velocity field this feature of the dynamics of particles will disappear.

Consider the joint dynamics of two particles. In this case, the probability density for the distance between the particles (provided the initial distance between them is small) is log-normal and moment functions of the distance (for example, in the two-dimensional case) grow with time exponentially:

$$\big\langle l^n(t)\big\rangle = l_0 \exp\left\{\frac{1}{8}\Big[2(D^{\mathrm{s}} - D^{\mathrm{p}})\,n + 3D^{\mathrm{p}}\,n^2\Big]\right\},$$

where D^{s} and D^{p} pertain to the solenoidal and potential components of the spectral function of field $\mathbf{u}(\mathbf{r}, t)$.

There also exists a deterministic function called *the curve of typical realization* (CTR), which describes the main tendency of temporal behavior exhibited by the random process $l(t)$. For the problem considered here, this function, similarly, turns out to be an exponential function of time:

$$l^*(t) = l_0 \exp\left\{\frac{1}{4}\big(D^{\mathrm{s}} - D^{\mathrm{p}}\big)\,t\right\},$$

and it is related to *the Lyapunov exponent*.

The CTR is essentially dependent on the sign of the difference $D^{\mathrm{s}} - D^{\mathrm{p}}$. In particular, for a solenoidal velocity field ($D^{\mathrm{p}} = 0$) we have an exponentially growing typical realization. In the other limit, for a potential velocity field ($D^{\mathrm{s}} = 0$), the typical realization is an exponentially decaying curve, i.e., particles would tend to coalesce. Consequently, *clusters* should form, i.e., zones of particle centering located in regions largely devoid of particles, which agrees with the results of numerical simulations. Thus, the inequality $D^{\mathrm{s}} < D^{\mathrm{p}}$ should hold for particle clustering in this problem.

The exponential growth of moments arises from overshoots of the process $l(t)$ relative to the curve of typical realization $l^*(t)$ both toward large and small values of l. It is a purely statistical effect caused by averaging over the entire ensemble of realizations.

Thus, we arrive at an apparent contradiction between the character of behavior exhibited by statistical characteristics of the process $l(t)$ and its behavior in concrete realizations. Let us formulate two clarifying remarks.

Remark 1. *The curve of typical realization* (*CTR*)

The statistical characteristics of a random process $z(t)$ are described by the probability density $P(t; z) = \big\langle \delta\big(z(t) - z\big)\big\rangle$ and integral distribution function

$$F(t; z) = \mathrm{Prob}\big(z(t) < z\big) = \big\langle \theta\big(z(t) - z\big)\big\rangle$$
$$= \int_{-\infty}^{z} \mathrm{d}z'\, P(t; z')\,,$$

where $\delta(z)$ is the Dirac delta function, and $\theta(z)$ is the Heaviside function equal to 1 for $z > 0$, and to 0 for $z < 0$.

The curve of typical realization for the random process $z(t)$ is referred to as a deterministic curve $z^*(t)$ which is *the*

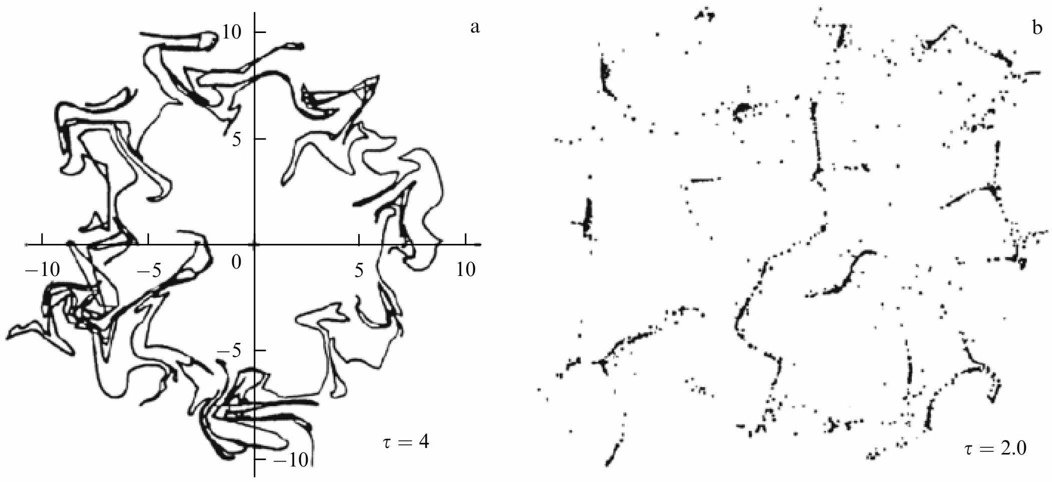

Figure 1. Results of modeling diffusion of a system of particles in solenoidal (a) and potential (b) random velocity fields.

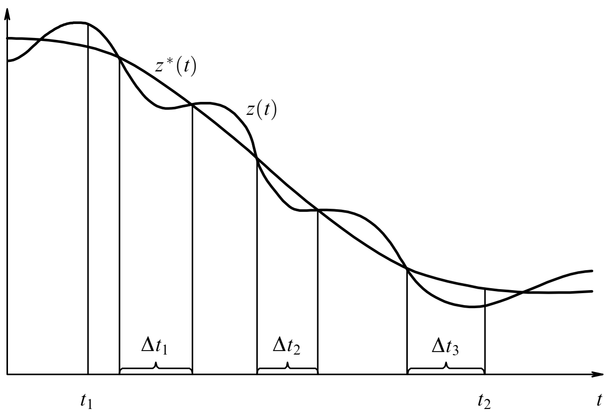

Figure 2. On the definition of the curve of typical realization for a random process.

median of the integral distribution function and is defined as a solution of the algebraic equation

$$F\big(t; z^*(t)\big) = \frac{1}{2}\,.$$

The motivation behind it is the property of a median that for any time interval (t_1, t_2) the random process $z(t)$ evolves as if it twists around the curve $z^*(t)$ in such a way that the mean time during which the inequality $z(t) > z^*(t)$ holds, coincides with the mean time during which the opposite inequality, $z(t) < z^*(t)$, is observed (Fig. 2), namely

$$\big\langle T_{z(t)>z^*(t)}\big\rangle = \big\langle T_{z(t)<z^*(t)}\big\rangle = \frac{1}{2}\,(t_2 - t_1)\,.$$

The curve of typical realization, although obtained with the help of contemporaneous probability density, is defined, nevertheless, for any time $t \in (0, \infty)$.

For a Gaussian random process $z(t)$, the CTR coincides with the mean value of the process, namely $z^*(t) = \big\langle z(t)\big\rangle$.

Remark 2. *Log-normal random process*
Define the log-normal random process by means of a stochastic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y(t; \alpha) = \big[-\alpha + z(t)\big]\, y(t; \alpha)\,, \qquad y(0; \alpha) = 1\,,$$

where $z(t)$ is the Gaussian process with parameters $\big\langle z(t)\big\rangle = 0$ and $\big\langle z(t)\, z(t')\big\rangle = 2D\delta(t - t')$. Its contemporaneous probability density is described by the Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial y}\, y\right) P(t; y, \alpha) = D\, \frac{\partial}{\partial y}\, y\, \frac{\partial}{\partial y}\, yP(t; y, \alpha)\,,$$

$$P(0; y, \alpha) = \delta(y - 1)\,.$$

The characteristic feature of the solution to this equation is the emergence of a long flattened *tail* for $Dt \gg 1$ implying the increased role of large overshoots of process $y(t; \alpha)$ in forming contemporaneous statistics. As a consequence of this, its moment functions

$$\big\langle y^n(t; \alpha)\big\rangle = \exp\left[n\left(n - \frac{\alpha}{D}\right) Dt\right],$$

$$\left\langle \frac{1}{y^n(t; \alpha)}\right\rangle = \exp\left[n\left(n + \frac{\alpha}{D}\right) Dt\right], \quad n = 1, 2, \ldots$$

grow exponentially with time for $n > \alpha/D$.

For a log-normal process one finds $\langle \ln y(t)\rangle = -\alpha t$ and, consequently, the parameter $-\alpha = (1/t)\langle \ln y(t)\rangle$ is the *Lyapunov characteristic exponent*, while the CTR of the process $y(t; \alpha)$ turns out to be a curve exponentially decaying with time:

$$y^*(t) = \exp\big(\big\langle \ln y(t)\big\rangle\big) = \exp(-\alpha t)\,.$$

Consider now a continual generalization to the problem of the diffusion of an inertialess passive admixture. In this case, the admixture density field $\rho(\mathbf{r}, t)$ is described by the continuity equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\, \mathbf{u}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t) = 0\,, \quad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,. \tag{1}$$

The total admixture mass is preserved as the admixture evolves with time, namely

$$M = M(t) = \int \mathrm{d}\mathbf{r}\, \rho(\mathbf{r}, t) = \int \mathrm{d}\mathbf{r}\, \rho_0(\mathbf{r}) = \text{const}\,.$$

To describe the local behavior of admixture field realizations in space in a random velocity field $\mathbf{u}(\mathbf{r}, t)$, one needs the probability distribution for the admixture density. Based on stochastic equation (1), we derive an equation for the probability density of the admixture density (concentration) field:

$$\left(\frac{\partial}{\partial t} - D_0 \Delta\right) P(\mathbf{r}, t; \rho) = D_\rho\, \frac{\partial^2}{\partial \rho^2}\, \rho^2 P(\mathbf{r}, t; \rho)\,,$$

where the diffusion coefficient in the ρ-space, $D_\rho = D^{\mathrm{p}}$, is only related to the potential component of field $\mathbf{u}(\mathbf{r}, t)$. The solution to this equation takes the form

$$P(\mathbf{r}, t; \rho) = \frac{1}{2\rho\sqrt{\pi D t}} \exp\left(D_0 t\, \frac{\partial^2}{\partial \mathbf{r}^2}\right) \times \exp \frac{\ln^2\big[\rho \exp(\alpha t)/\rho_0(\mathbf{r})\big]}{4Dt}\,. \tag{2}$$

If the initial admixture density is everywhere uniform, $\rho_0(\mathbf{r}) = \rho_0 = \text{const}$, the probability distribution of density is independent of $\mathbf{r}$ and can be described by the equation

$$\frac{\partial}{\partial t}\, P(t; \rho) = D_\rho\, \frac{\partial^2}{\partial \rho^2}\, \rho^2 P(t; \rho)\,. \tag{3}$$

From equation (3) it follows, in particular, that the probability distribution is log-normal and moment functions of the density field, beginning from the second one, exponentially grow with time $\tau = D_\rho t$:

$$\big\langle \rho^n(\mathbf{r}, t)\big\rangle = \rho_0^n \exp\big[n(n-1)\,\tau\big]\,.$$

From the viewpoint of single-point characteristics of density field $\rho(\mathbf{r}, t)$, the problem in this case is statistically equivalent to a random process $\rho(t)$, whose probability density obeys Fokker–Planck equation (3), whereas the CTR exponentially decays with time at any fixed point in space:

$$\rho^*(t) = \rho_0 \exp(-\tau)\,.$$

This gives evidence of the presence of a clustering behavior for fluctuations of medium density in arbitrary divergent flows.

The probability distribution (2) also enables learning about some characteristic features of the spatio-temporal structure of the density field realizations.

Remark 3. *Statistical topography of a random density field*
To be more illustrative, we limit ourselves here to the case of two dimensions as well. In statistical topography, important knowledge of the spatial behavior of realizations is provided by the analysis of isolines defined as

$$\rho(\mathbf{r},t) = \rho = \text{const}\,.$$

In particular, the mean values of such functionals of the density field as the total area where $\rho(\mathbf{r},t) > \rho$:

$$S(t,\rho) = \int \mathrm{d}\mathbf{r}\,\theta\big(\rho(\mathbf{r},t)-\rho\big) = \int \mathrm{d}\mathbf{r} \int_{\rho}^{\infty} \mathrm{d}\widetilde{\rho}\,\delta\big(\rho(\mathbf{r},t)-\widetilde{\rho}\big)\,,$$

and the total mass of the admixture comprised within this area:

$$M(t,\rho) = \int \mathrm{d}\mathbf{r}\,\rho(\mathbf{r},t)\,\theta\big(\rho(\mathbf{r},t)-\rho\big)$$
$$= \int \mathrm{d}\mathbf{r} \int_{\rho}^{\infty} \mathrm{d}\widetilde{\rho}\,\widetilde{\rho}\delta\big(\rho(\mathbf{r},t)-\widetilde{\rho}\big)\,,$$

are defined by a single-point probability density and are expressed as

$$\big\langle S(t,\rho)\big\rangle = \int_{\rho}^{\infty} \mathrm{d}\widetilde{\rho} \int \mathrm{d}\mathbf{r}\,P(\mathbf{r},t;\widetilde{\rho})\,,$$
$$\big\langle M(t,\rho)\big\rangle = \int_{\rho}^{\infty} \mathrm{d}\widetilde{\rho}\,\widetilde{\rho} \int \mathrm{d}\mathbf{r}\,P(\mathbf{r},t;\widetilde{\rho})\,.$$

Hence, it is seen, in particular, that for $\tau \gg 1$ the mean area of the regions, where the density is in excess of a given level ρ, decays with time according to the law

$$\big\langle S(t,\rho)\big\rangle \approx \frac{1}{\sqrt{\pi\rho\tau}} \exp\left(-\frac{\tau}{4}\right) \int \mathrm{d}\mathbf{r}\,\sqrt{\rho_0(\mathbf{r})}\,,$$

while the mean mass of the admixture inside them, namely

$$\big\langle M(t,\rho)\big\rangle \approx M - \sqrt{\frac{\rho}{\pi\tau}} \exp\left(-\frac{\tau}{4}\right) \int \mathrm{d}\mathbf{r}\,\sqrt{\rho_0(\mathbf{r})}$$

tends monotonically to the total mass. This once again confirms the conclusion drawn earlier that admixture particles tend to coalesce with time in clusters — the compact regions of augmented density surrounded by rarefied regions.

It should be noted that for a spatially homogeneous field $\rho(\mathbf{r},t)$ these expressions can be simplified, yielding for specific quantities per unit area the following expressions

$$\big\langle s(t,\rho)\big\rangle = \int_{\rho}^{\infty} \mathrm{d}\widetilde{\rho}\,P(t;\widetilde{\rho})\,,\quad \big\langle m(t,\rho)\big\rangle = \int_{\rho}^{\infty} \mathrm{d}\widetilde{\rho}\,\widetilde{\rho}\,P(t;\widetilde{\rho})$$

linked with the solution to equation (3).

2.2 Waves in a randomly inhomogeneous medium

As the second example, let us consider the problem of wave propagation in random media.

We begin with a one-dimensional problem which corresponds to waves in layered media.

Let a layer of a chaotically inhomogeneous medium occupy the space $L_0 < x < L$, and a plane wave $u_0(x) = \exp\big[-\mathrm{i}k(x-L)\big]$ be incident on it from the region $x > L$. Due to the presence of inhomogeneities, there appears a wave reflected from the layer with the reflection coefficient $R_L = u(L) - 1$, and a wave leaving the layer with the transmission coefficient $T_L = u(L_0)$. Inside the layer, the wave field satisfies the boundary value problem:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,u(x) + k^2[1+\varepsilon(x)]\,u(x) = 0\,,$$
$$u(L) + \frac{\mathrm{i}}{k}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\bigg|_{x=L} = 2\,,\qquad u(L_0) - \frac{\mathrm{i}}{k}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\bigg|_{x=L_0} = 0\,,$$

where the function $\varepsilon(x)$, which we regard as a random one, describes the inhomogeneities of the medium.

Under the assumption that the statistical characteristics of function $\varepsilon(x)$ are known, the statistical problem amounts to searching for the statistical characteristics of the wave field intensity $I(x) = |u(x)|^2$ inside the inhomogeneous medium and at its boundaries.

A statistical analysis of the solution to this problem indicates that for a sufficiently thick layer, namely, $D(L-L_0) \gg 1$ [where the quantity D is related to statistical characteristics of $\varepsilon(x)$], $|T_L| \to 0$ with probability one and, consequently, $|R_L| \to 1$, i.e., the half-space ($L_0 \to -\infty$) of the randomly inhomogeneous medium totally reflects the incident wave. Thus, a *dynamical localization of the wave field* in this layer occurs.

However, the mean value of wave field intensity is constant in the half-space of the random medium, while higher moments normalized to their values at the layer boundary are described by the expression

$$\big\langle I^n(L-x)\big\rangle = \exp\big[Dn(n-1)(L-x)\big]\,,$$

i.e., the intensity of the wave field has a log-normal probability distribution, and moment functions grow exponentially along the direction deep into the medium.

In this case, the CTR for the wave intensity in the medium is described by an exponentially decaying function

$$I^*(x) = 2\exp\big[-D(L-x)\big]$$

and coincides with the Lyapunov exponent; the quantity $l_{\text{loc}} = 1/D$, dubbed the *localization length*, sets the spatial scale for the decay of the wave field intensity in separate realizations.

Thus, it becomes apparent that the statistics form through large overshoots relative to the typical realization curve. Figure 3 shows two realizations of wave field intensity in a sufficiently thick layer, obtained through numerical simulations. It apparently illustrates the tendency of fast exponential decay (with large overshoots toward both ever larger intensity and zero).

Consider now wave propagation in a randomly inhomogeneous three-dimensional medium based on the scalar parabolic equation

$$\frac{\partial}{\partial x}\,U(x,\mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}}U(x,\mathbf{R}) + \frac{\mathrm{i}k}{2}\,\varepsilon(x,\mathbf{R})\,U(x,\mathbf{R})\,,$$
$$U(0,\mathbf{R}) = U_0(\mathbf{R})\,. \tag{4}$$

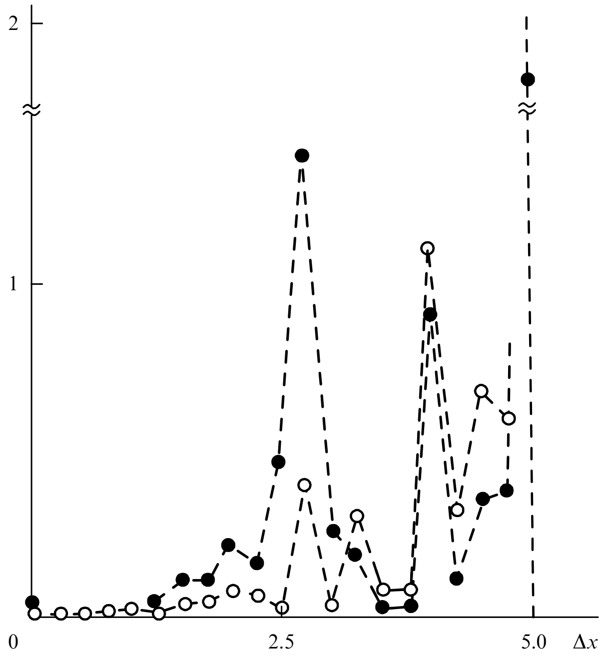

Figure 3. Numerical simulation of dynamic localization for two realizations of medium inhomogeneity.

Here, x is the coordinate in the direction of wave propagation, $\mathbf{R}$ are the coordinates in the transverse plane, and $\varepsilon(x, \mathbf{R})$ is the deviation of permittivity from unity.

On introducing the amplitude and phase of the wave field as

$$U(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\{\mathrm{i}S(x, \mathbf{R})\},$$

the transfer equation can be written for the intensity of the wave field $I(x, \mathbf{R}) = |U(x, \mathbf{R})|^2$ in the form

$$\frac{\partial}{\partial x} I(x, \mathbf{R}) + \frac{1}{k} \nabla_{\mathbf{R}} \{\nabla_{\mathbf{R}} S(x, \mathbf{R}) I(x, \mathbf{R})\} = 0,$$

$$I(0, \mathbf{R}) = I_0(\mathbf{R}). \qquad (5)$$

Hence, it follows that in the general case of an arbitrary incident beam the wave power in the plane $x = \mathrm{const}$ is preserved:

$$E_0 = \int I(x, \mathbf{R}) \, \mathrm{d}\mathbf{R} = \int I_0(\mathbf{R}) \, \mathrm{d}\mathbf{R}.$$

Equation (5) shares its form with Eqn (1). It therefore can be treated as the transfer equation for a conservative admixture in a potential velocity field. As a consequence, realizations of the intensity field have a cluster character, whereas this clustering manifests itself through *caustic structures*. By way of example, Fig. 4 displays photos of a cross section of a laser beam propagating in a turbulent medium in a laboratory setup, for various fluctuations of permittivity. The appearance of the caustic structure of the wave field is vividly seen.

Let us introduce the amplitude and phase of the wave field and the complex phase of the wave:

$$U(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\big(\mathrm{i}S(x, \mathbf{R})\big) = \exp\big(\phi(x, \mathbf{R})\big),$$

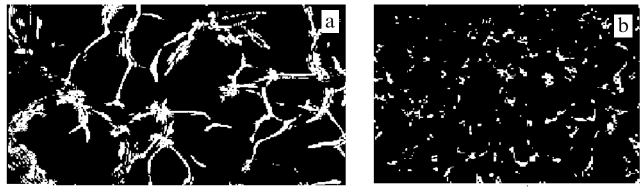

Figure 4. Cross section of a laser beam propagating in a turbulent medium in laboratory conditions (a) in the region of strong focusing, and (b) in the region of strong (saturated) fluctuations.

where

$$\phi(x, \mathbf{R}) = \chi(x, \mathbf{R}) + \mathrm{i}S(x, \mathbf{R}).$$

$\chi(x, \mathbf{R}) = \ln A(x, \mathbf{R})$ is the wave amplitude level, and $S(x, \mathbf{R})$ is the wave phase fluctuations relative to the phase kx of the incident wave. Proceeding from parabolic equation (4), one can obtain, for the complex phase, a nonlinear equation of the so-called Rytov *method of smooth perturbations* (MSP):

$$\frac{\partial}{\partial x} \phi(x, \mathbf{R}) = \frac{\mathrm{i}}{2k} \Delta_{\mathbf{R}} \phi(x, \mathbf{R}) + \frac{\mathrm{i}}{2k} \big[\nabla_{\mathbf{R}} \phi(x, \mathbf{R})\big]^2 + \mathrm{i} \frac{k}{2} \varepsilon(x, \mathbf{R}).$$

For the case of a plane incident wave, which will only be considered further, it can be assumed that $U_0(\mathbf{R}) = 1$ without loss of generality and, consequently, that $\phi(0, \mathbf{R}) = 0$. In this case, the random field $\phi(x, \mathbf{R})$ is statistically homogeneous in the plane $\mathbf{R}$ and all its single-point statistical characteristics are independent of the parameter $\mathbf{R}$.

Remark 4. *The Rytov smooth perturbation method*

The method of smooth perturbations was proposed by S M Rytov when analyzing the problem of light diffraction by ultrasonic waves in 1938. A M Obukhov applied this method in 1953 to treat the diffraction effects accompanying wave propagation in random media in the framework of perturbation theory. Earlier, analogous studies were carried out in the approximation of geometrical optics (acoustics). This technique has not lost its relevance even now providing the basic mathematical apparatus for various technical applications.

In the first order of the MSP, the statistical properties of amplitude fluctuations are characterized by the variance of amplitude level, i.e., by the parameter $\sigma_0^2(x) = \langle \chi_0^2(x, \mathbf{R}) \rangle$, in which case $\langle \chi_0(x, \mathbf{R}) \rangle = -\sigma_0^2(x)$. Regarding the variance of wave intensity, which is called the *scintillation index*, it is written down in the first approximation as

$$\beta_0(x) = \langle I^2(x, \mathbf{R}) \rangle - 1 = \big\langle \exp\big[4\chi_0(x, \mathbf{R})\big] \big\rangle - 1 \approx 4\sigma_0^2(x).$$

In this case, the intensity of the wave field is a log-normal random field and all statistical moments of the wave field intensity grow with an increase in the parameter $\beta_0(x)$, i.e., with the distance travelled by the wave. Now, a statistically equivalent random process $I(x)$ can be considered, for which the CTR of wave field intensity decays exponentially with distance:

$$I^*(x) = \exp\left(-\frac{1}{2} \beta_0(x)\right),$$

at any fixed point $\mathbf{R}$ in space. This is indicative of the emergence of a cluster (caustic) structure in the intensity field. The formation of statistics (for instance, of moment functions $\langle I^n(x, \mathbf{R})\rangle$) proceeds through large overshoots of the process $I(x)$ with respect to this curve.

The description of intensity fluctuations, obtained in the first order of the MSP, is valid for $\beta_0(x) \leqslant 1$. As the parameter $\beta_0(x)$ increases further, this approximation becomes violated and the nonlinear character of the equation for the complex phase of the wave field has to be taken into account. This range of fluctuations, called *the region of strong focusing*, is very difficult for analytical research. For even larger values of parameter $\beta_0(x)$, the statistical characteristics of intensity reach saturation, in which case $\beta(x) \to 1$ as $\beta_0(x) \to \infty$. This region of the parameter $\beta_0(x)$ variations is called *the region of strong intensity fluctuations*.

In this region, the statistical characteristics of the wave field cease to depend on the distance and one has

$$\left\langle I^n(x, \mathbf{R})\right\rangle = n!\,, \qquad P(x, I) = \exp(-I)\,.$$

In this case, the mean specific area of regions within which $I(x, \mathbf{R}) > I$ and the mean specific power concentrated in these regions are constant and do not describe the behavior of the wave field intensity in separate realizations. Likewise, passage to a statistically equivalent random process is not informative in this case since its curve of typical realization assumes a constant value. An understanding of the wave field structure in specific realizations can only be gained in this case from the analysis of such quantities as the specific mean length of contours and mean specific number of wave field intensity contours. These quantities continue to grow with the parameter $\beta_0(x)$, implying that the splitting of contours takes place (see Fig. 4).

3. Conclusions

In closing, I would like to reiterate once more the main point of this talk. The approach to analysis of stochastic dynamical problems rooted in the ideas of stochastic topography, which enables, given the one-point statistical characteristics of processes and fields, determining quantitative and qualitative characteristics of behavior of their particular realizations for all times (in the entire space), has emerged as a result of discussions with experimenters who largely deal with separate realizations. For a comprehensive description of stochastic dynamical systems, it is insufficient to formulate a basic equations with respective boundary and initial conditions. It is necessary first and foremost to understand which coherent phenomena (occurring with the probability of unity, i.e., in almost all realizations of their solutions) are contained in these systems, and proceed with a statistical analysis in a related way.

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Development of the radiative transfer theory as applied to instrumental imaging in turbid media

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1. Introduction

This talk presents the basic elements of instrumental imaging theory in media with strongly anisotropic scattering and a technique devised to compute images of diffusively reflecting objects, accounting for the effects of light absorption and multiple scattering. It discusses peculiarities of different variants of the radiative transfer equation in the small-angle approximation, used in the imaging theory and optical coherence 'tomography' (OCT) of turbid media. A new method of computing the temporal moments of a pulsed light beam transmitted through a layer of a turbid medium is described. The results of theoretical and experimental studies of shadow noises in OCT images of turbid media with fluctuating optical parameters are outlined.

By scattering light a turbid medium limits the visibility range of objects located within it and becomes visible itself. Therefore, the development of the methods and theory of instrumental imaging in turbid media was directed toward solving two interconnected tasks — the removal of the adverse influence of the medium on the visibility of objects, and the remote sensing of inherent optical properties of the medium itself.

The Koshmider equation [1] expresses the fundamental result of the imaging theory by relating the image contrast of a black object (observed in the sky background near the horizon) to the light attenuation coefficient in the atmosphere. The relationships for estimating the contrast of the image and visibility range of underwater objects under natural illumination were obtained in a now classical work by Duntley [2]. In this case, it was assumed that the angular size of the observed object is small, so that its apparent radiance is attenuated by the medium according to Buger's law. The need in a more universal imaging theory emerged in connection with the development of laser methods of underwater vision.

Pioneering works in this area were performed under the supervision of A V Gaponov-Grekhov in the Radiophysical Research Institute (NIRFI in *Russ. abbr.*) (Gor'ky) in the 1960s. They have led to the design of the first prototype of a laser-pulse system of underwater imaging with the help of