Physics – Uspekhi **52** (5) 487 – 493 (2009)

METHODOLOGICAL NOTES

PACS numbers: 03.50.De, **41.20.–q**, 41.60.Dk

Features of the transition radiation field

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DOI: 10.3367/UFNe.0179.200905c.0517

Contents

Abstract. The method of mirror images is used to calculate transition radiation (TR) at the plane interface between a vacuum and an ideally conducting medium. The total field is considered (that is, the TR field plus the fields of the uniformly moving charge and its mirror image) and its evolution in space and time is traced, considering fields as functions of space coordinates and time rather than represented by spectral components. Conditions are analyzed under which the separation and measurement of a TR field are possible.

1. Introduction

Transition radiation at a plane interface between two media was first discussed by V L Ginzburg and I M Frank [1] in 1946. They calculated the electromagnetic radiation generated by a particle traversing an interface. This problem has been studied in subsequent years in sufficient detail in a number of papers (see monographs [2–4]; many papers were also published later). The interest in transition radiation was stimulated by its important applications. It is used in high-energy physics for detecting charged particles, since the TR burst not only signals that a charged particle has passed but also allows the determination of its energy, the direction of its motion, and some other characteristics. Transition radiation can also be used to generate electromagnetic waves by high-current electron beams, to generate radiation in free electron lasers, and to control the beam of accelerated particles in accelerators.

Most of the published papers deal with the frequency spectrum and angular distribution of transition radiation. However, a different question can be posed in problems involving transition radiation: what is the behavior of an electromagnetic field at a fixed point in space at a given instant of time? This is a significant question because the electromagnetic field includes not only the emitted waves but also the bound fields (or fields accompanying moving charges, i.e., 'proper fields'). It is essential that two types of components need to be taken into account in real experiments: emitted and nonemitted fields (free and coupled fields). Of course, problems of these two types — determination of the spectral properties of radiation, and determination of its spatial and temporal characteristics — are interrelated. In fact, the latter problem has received much less attention so far than it deserves. This paper treats temporal and spatial properties of the total field in the simplest version of the problem covering transition radiation.

We consider the field arising when a charged particle is incident upon an ideally conducting flat surface. We assume that the charged particle moves in the medium with a constant velocity perpendicularly to the plane interface with an ideally conducting medium and crosses this interface. The evolution of the field in this problem demonstrates one of the examples of the transition process. It is transitory not in the sense that the particle crosses the interface but in the sense of the standard definition of a transition process as a phenomenon that accompanies the transition of a system from one stationary state to another. The first stationary state (the initial state) in the problem of a charge incident on an ideally conducting plane comprises the field of a uniformly moving charge. It is well known that a uniformly moving charge does not emit radiation and hence its approach to the surface produces no emission. After an impingement upon the conducting surface, the charge vanishes for the observer, so that the second stationary state (the final state) comprises the space free of charge. The transition from the initial to the final state is accompanied by emission of electromagnetic waves. The field that existed in the initial state is transformed into radiation and goes to infinity. The sections that follow trace the evolution of the field.

2. Formulation of the problem

We introduce in space a Cartesian coordinate system xyz. Let an ideally conducting surface lie in plane yz. We can also assume that the half-space $x < 0$ is occupied with an ideally conducting medium. Let us choose in half-space $x > 0$ an observation point P at a distance r_P from the origin at angle θ to the x-axis (Fig. 1). We can assume, without loss of

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Received 21 October 2008, revised 26 December 2008
Uspekhi Fizicheskikh Nauk **179** (5) 517 – 524 (2009)
DOI: 10.3367/UFNr.0179.200905c.0517
Translated by V I Kisin; edited by A Radzig

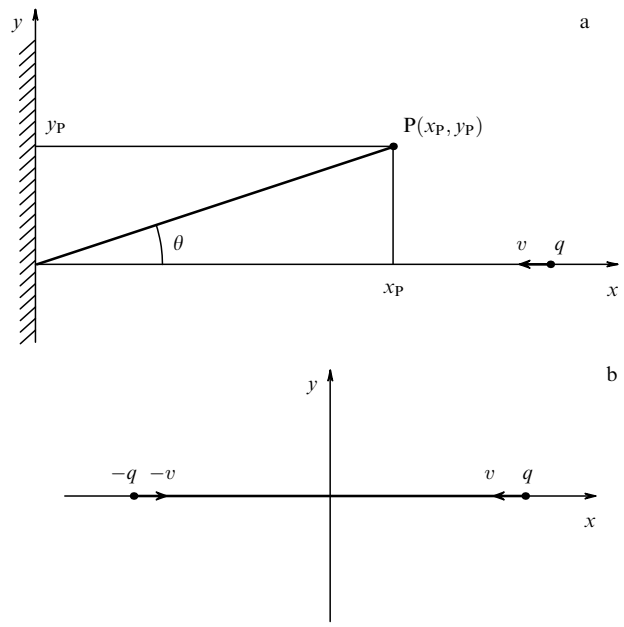

Figure 1. Geometry of the problem.

generality, that the point P lies in the plane xy. The coordinates of the observation point are $x_P = r_P \cos\theta$ and $y_P = r_P \sin\theta$. A point charged particle moves with the velocity v from the region $x > 0$ along the x-axis towards the conducting plane. The charge of the particle we denote by q. The particle impinges on the conducting surface $x = 0$ and then vanishes from observation. The problem is to find the electromagnetic field of a charge moving as described above.

The moving charge excites on the conducting surface induced charges and currents, which become sources of the electromagnetic field. These induced currents and charges can be determined from boundary conditions on the ideally conducting surface. An alternative method of satisfying the boundary conditions consists in introducing additional auxiliary charges which are known as electrical images of the original charge q.

Consider now two pointlike charges in free space, which are moving towards each other along the x-axis with equal velocities v. The first charge q travels from $+\infty$ in the negative direction of the x-axis, and the second charge, $-q$, moves from $-\infty$ in the positive direction of the x-axis. We refer to the charge $-q$ as the image-charge. The law of motion of the first charge will be defined by the equation $x = -vt$. The position of the second charge will be defined by the equation $x = vt$. The charges take up thus positions always at points symmetrical relative to the plane $x = 0$. Obviously, at the instant of time $t = 0$ these charges meet and cancel each other out.

It is not difficult to demonstrate that with this arrangement of charges the tangential components of the electric field in the plane $x = 0$ vanish; in other words, those conditions hold true which would hold on an ideally conducting plane boundary. The magnetic fields in the plane $x = 0$ possess only tangential components and add up (are doubled). Hence, the ideally conducting plane $x = 0$ creates in the half-space $x > 0$ the same field as two charges: the charge q and the image-charge $-q$ moving symmetrically with respect to the former charge.

We considered the case in which the charge velocities are directed perpendicularly to the specular interface. This case was first discussed for the problem of transition radiation by V L Ginzburg and I M Frank [1]. In fact, an even more general statement holds true. If for any law describing the motion of the original charge the image-charge moves along a symmetrical curve (i.e., the trajectories of the original charge and the electrical image of the charge are mirror-symmetrical), then the same conditions hold on the symmetry plane as on a boundary with an ideally conducting surface.

The incident charge and the image-charge move towards each other and meet at the instant of time $t = 0$ at the point $x = 0$ on the ideally conducting plane. We can assume that each of the charges was moving up to the point $x = 0$ with constant velocity and stopped there. The problem of field determination is thereby reduced to the problem of radiation emission of a uniformly moving charged particle in response to instantaneous stopping. We consider this problem in Section 3.

3. Field created by an instantaneously stopped charge

The field set up in space by instantaneously stopping a charge can be found in the same way in which E Purcell calculated the field arising from instantaneous starting [5] (see also Ref. [6]). Let us recall how Purcell treated the field arising from instantaneous starting of a charge.

Let a charged particle be at the origin of a Cartesian coordinate system (Fig. 2a). Until the instant of time $t = 0$ the particle is at rest, and at the instant $t = 0$ begins to move in the positive direction of the x-axis with the velocity v. We need to find the field of the charged particle for this law of motion.

The assumption of an instantaneous start is to a certain extent an idealization. In realistic conditions, a finite change in velocity requires a finite time. However, if we consider radiation of a sufficiently low frequency, the assumption of an instantaneous velocity jump is justified.

Let us surround the starting point with a sphere of radius $r = ct$. Inside this sphere, the solution of Maxwell's equations gives the field of a charge moving uniformly with the velocity v:

$$E_x^q = q(1-\beta^2)\,\frac{x - vt}{\left[(1-\beta^2)(y^2+z^2) + (x-vt)^2\right]^{3/2}}\,, \quad (1)$$

$$E_y^q = q(1-\beta^2)\,\frac{y}{\left[(1-\beta^2)(y^2+z^2) + (x-vt)^2\right]^{3/2}}\,, \quad (2)$$

where $\beta = v/c$ is the reduced velocity of the charge.

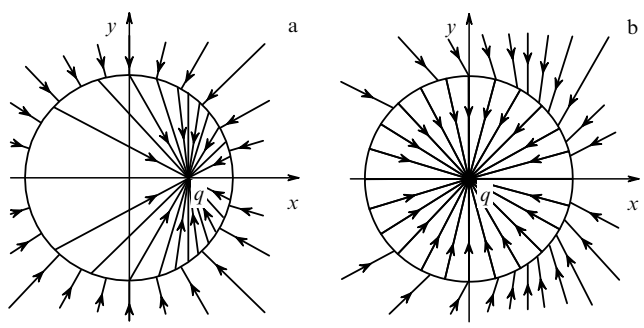

Figure 2. Spatial distribution of an electric field in the case of (a) instantaneous starting, and (b) instantaneous stopping of the electric charge.

Outside of the sphere with radius $r = ct$, the field coincides with the Coulomb field of the charge at rest at the origin of coordinates. The sphere of radius $r = ct$ passes, as it expands, through the observation point P. Hence, however large is the distance from the observer to the starting point, the Coulomb field of the charge at rest at the instant $t = r/c$ is replaced with the field of a uniformly moving charge residing at the point $x = vt$. This structure of the field is imposed by the retardation effect which is caused by the finite value of the speed of light. If the observer is very far from the starting point, the field at the observation point remains for a long time that of a charge at rest, even though this charge has already been in uniform motion for quite some time. In the case of stopping, the field of the charge offers similar features: if the observer is located far from the point where the charge stops, then the field at the observation point continues to be that of a moving charge for a long time after the charge has stopped.

Figure 2a plots the pattern of the lines of force inside and outside the sphere with radius $r = ct$. We see that the inner and outer lines of force do not join — that is, they suffer discontinuity on the surface of the sphere. However, such a discontinuity is in contradiction with the physical conditions of the problem. The lines of force can originate and terminate only on charges. Hence, a discontinuity of the lines of force on a spherical surface would mean that the surface of the sphere is charged. However, we consider the charge q located at the point $x = vt$, and no other charges can appear. This implies that the lines of force must be continuous — that is, each line of force inside the sphere must continue as a line outside of the sphere. This transfer occurs through lines of force located on the surface of the sphere of radius $r = ct$. The radiation emitted in the problem in question is determined precisely by these lines of force lying on the surface of the expanding sphere.

Indeed, a sphere with radius $r = ct$ expands at the speed of light. The lines of force lying on the sphere are perpendicular to the direction of propagation. The field described by the lines of force lying on the sphere thus meets all the conditions that must be met by the electromagnetic wave. It should be noted that in the problem we are discussing the field inside the sphere contains no radiation but is the field of a uniformly moving charge. There is no radiation field outside the sphere either — this is the field of a charge at rest. The radiation field is nonzero only on the sphere with radius $r = ct$.

The above structure of the radiation field is also implied by the relation holding for the field of the moving charge written in terms of the Liénard–Wiechert potentials. The field of the moving charge has the form [5]

$$\mathbf{E} = q\,\frac{1-\beta^2}{(R-\mathbf{R}\mathbf{v}/c)^3}\left(\mathbf{R}-\frac{\mathbf{v}}{c}\,R\right) + \frac{q}{c^2(R-\mathbf{R}\mathbf{v}/c)^3}\left[\mathbf{R}\left[\left(\mathbf{R}-\frac{\mathbf{v}}{c}\,R\right)\dot{\mathbf{v}}\right]\right], \tag{3}$$

where $\mathbf{R}$ is the distance between the observation point and the moving charge, $\mathbf{v}$ is the velocity of the charge, and $\dot{\mathbf{v}}$ is its acceleration. All quantities on the right-hand side of formula (3) are taken at the instant of time $t' = t - R/c$. The term containing $\dot{\mathbf{v}}$ determines the radiation field of the moving charge. If we consider the instantaneous stopping of a charge moving at a velocity $\mathbf{v}$, then $\dot{\mathbf{v}} = -\mathbf{v}\delta(t)$. Therefore, the radiation field in the case of instantaneous stopping of a charge is proportional to the factor $\delta(t - R/c)$, i.e., it constitutes a spherical wave whose field differs from zero only on the spherical surface of radius $R = ct$.

There is a one-to-one correspondence in the problem at hand between the lines of force inside and outside the sphere. If we consider, to be specific, the problem of charge instantaneous stopping, the line of force inside the sphere that forms an angle θ with the direction of motion continues as the line of force outside the sphere at an angle θ' to the same direction, so that

$$\tan\theta' = \gamma\tan\theta\,, \tag{4}$$

where $\gamma = \left[1-(v/c)^2\right]^{-1/2}$ is the so-called Lorentz factor (reduced energy of the particle). These two lines are joined by a segment of a line of force lying on the sphere and therefore are different parts of the same line of force.

The strength of the radiation field can be determined in the following manner. We introduce a spherical system of coordinates with the origin at the starting point of the charge and the axis pointing along the vector of the charge's velocity. Consider an area of the spherical surface, which is resided at an angle to the velocity of the charge. We wish to calculate the electric field flux through this area. Obviously, this flux equals the difference between the fluxes produced by the inner and outer fields. It is easy to find the external field flux. Namely, the flux of the external field across an area on the spherical surface, corresponding to the solid angle $\mathrm{d}\Omega$, equals $q\,\mathrm{d}\Omega$. We now calculate the flux of the internal field. For this purpose, we set $x = r\cos\theta$, $y = r\sin\theta$ in formulas (1) and (2). This yields expressions for the fields E_x and E_y on the sphere:

$$E_x^q\big|_{r=ct} = \frac{q(1-\beta^2)}{t^2}\,\frac{c\cos\theta - v}{\left[(1-\beta^2)c^2\sin^2\theta + (c\cos\theta - v)^2\right]^{3/2}}\,, \tag{5}$$

$$E_y^q\big|_{r=ct} = \frac{q(1-\beta^2)}{t^2}\,\frac{c\sin\theta}{\left[(1-\beta^2)c^2\sin^2\theta + (c\cos\theta - v)^2\right]^{3/2}}\,, \tag{6}$$

where $\beta = v/c$.

The flux of the field $\mathbf{E}$ across an element $\mathrm{d}S$ of the surface of the sphere equals $\mathbf{E}\mathbf{n}\,\mathrm{d}S$, i.e., $(n_xE_x + n_yE_y)\,\mathrm{d}S$, where $\mathbf{n} = (n_x, n_y)$ is a normal to the surface of the sphere. Taking into account that $n_x = \cos\theta$, $n_y = \sin\theta$, we obtain from formulas (5) and (6) an expression for the flux P in terms of the surface element that corresponds to the element $\mathrm{d}\Omega$ of the solid angle:

$$\mathrm{d}P = E_x\cos\theta + E_y\sin\theta = q(1-\beta^2)\,\frac{1}{(1-\cos\theta)^2}\,\mathrm{d}\Omega\,. \tag{7}$$

Notice that relation (7) does not include the radius of the sphere, i.e., the electric field flux is only determined by the element $\mathrm{d}\Omega$ of the solid angle. This is perfectly understandable because the electric field strength decreases with increasing radius r as r^{-2}, while the area of the surface element corresponding to the solid angle $\mathrm{d}\Omega$ is proportional to r^2. This implies, among other things, that the field of a uniformly moving charge cannot be the radiation field. Indeed, it is typical for the radiation field that the electromagnetic energy flux into the solid angle $\mathrm{d}\Omega$ is independent of the radius r. However, the electromagnetic energy flux represents in fact a bilinear combination of the electric and magnetic fields (the Poynting vector $\mathbf{P} = 4\pi c\mathbf{E}\times\mathbf{H}$) and, consequently, in this case the square of the field strength (not its first power) must decrease with increasing r as r^{-2}, and the radiation field itself

will decrease with distance as r^{-1}. It is not difficult to show that the electric field flux across the entire surface of the sphere of radius $r = ct$ enclosing the charge equals, by virtue of Gauss's theorem, $4\pi q$. This means that both internal and external fields are depicted by the same number of lines of force.

The difference between the inner and outer field fluxes across a portion of the spherical surface gives us the increment of the number of lines of force on a given portion of the sphere. We denote the strength of the radiation field on the sphere by E_θ^{rad}. As follows from the symmetry of the problem, the radiation field depends only on θ. Then, the condition of conservation of the number of lines of force (or an equivalent condition $\operatorname{div}\mathbf{E} = 0$) leads to the expression

$$\frac{\mathrm{d}E_\theta^{\text{rad}}}{\mathrm{d}\theta} = \frac{q}{r}\left[\frac{1-\beta^2}{(1-\beta\cos\theta)^2} - 1\right]\sin\theta\,. \tag{8}$$

Expression (8) can be regarded as a differential equation for the tangential component E_θ^{rad} of electric field, which differs from zero on the expanding sphere. We can, however, point to certain characteristic features of the field E_θ^{rad} even before solving equation (8). The lines of force of this field lie on the surface of the sphere whose center resides at the point from which the charge started. This sphere expands at the speed of light — that is, the charge is always inside the sphere. An important feature of the field E_θ^{rad} is that as the sphere radius increases, the strength of this field decreases in proportion to r^{-1}. Consequently, the field E_θ^{rad} possesses the property of a radiation field. It propagates at the speed of light and is perpendicular to the direction of propagation at every point.

In order to avoid misinterpretation, note that the field strength of a uniformly moving charge described by expressions (5) and (6) also has a tangential component on the sphere; we shall not give this expression here. However, in contrast to the radiation field E_θ^{rad}, the tangential component of the field defined by Eqns (5) and (6) decreases as r^{-2}.

The solution of differential equation (8), satisfying the condition $E_\theta^{\text{rad}} = 0$ for $\theta = 0$, is written in the form

$$E_\theta^{\text{rad}} = \frac{q}{r}\,\delta(r-ct)\,\frac{\beta\sin\theta}{1-\beta\cos\theta}\,, \tag{9}$$

where the delta function of the argument $r - ct$ takes into account the fact that the field E_θ^{rad} is nonzero only on a sphere of radius $r = ct$, which expands at the speed of light.

Hence, we see that the radiation field in the approximation of charge instantaneous starting is formed at the moment of starting and afterwards the wave packet of the radiation field propagates as described by law (8), which gives an expression for the unique nonzero component of the electric radiation field in the case of instantaneous starting of a charge.

Consider now the radiation field that arises in the case of a charge stopping instantaneously. Assume here that a charge q was moving uniformly in the positive direction of the x-axis with a velocity v and at the instant of time $t = 0$ stopped at a point $x = 0$. In this case, the solution of Maxwell's equation inside the sphere of radius $r = ct$ with the center at the point $x = 0$ gives the Coulomb field of the charge at rest at the origin of coordinates. The field outside the sphere with radius $r = ct$ is the field of the uniformly moving charge whose velocity is v (because the signal that the charge had stopped has not yet reached points located outside the light sphere). It is readily seen that the difference between the field fluxes inside and outside the sphere is equal in absolute value to, but has the opposite sign of, the quantity that we calculated for the case of an instantaneous start of a charge. This implies that the radiation field E_θ in the case of charge stopping is of equal magnitude and of opposite sign to the field (9) that we had for instantaneous starting.

Figure 2b depicts the electric field of a charged particle that moved along the x-axis at a constant velocity v and stopped abruptly at the point $x = 0$. If in the case of starting outside a sphere of radius $r = ct$ the field is that of a uniformly moving charge, and the field inside the sphere is that of a charge at rest at the point $x = 0$, then, in the case of a charge stopping, these fields are permuted: the field inside the sphere is that of a charge at rest, and the field outside the sphere is that of a uniformly moving charge.

4. Transition radiation field

As we mentioned in Section 2, the transition radiation field in the case of normal incidence of the charge on the ideally conducting surface constitutes the sum of the radiation fields produced by the charge and by its electrical image, the two moving towards each other and meeting at a point on the surface. It is not difficult to find from the analysis given above the geometry of the transition radiation field, both for a charge incident on a conducting surface and for a charge escaping from it.

Figure 3a is a diagram of the transition radiation field created when a charge is escaped normally to the ideally conducting surface. The charge leaves the surface at the point $x = 0$. The field inside the sphere of radius $r = ct$ ($t = 0$ is the moment of ejection) is equal to the sum of the fields created by the charge and by its electrical image that fly away at a velocity v in opposite directions; it is described by the following expressions

$$E_x^q = q(1-\beta^2)\left\{\frac{r_{\text{P}}\cos\theta - vt}{\left[(1-\beta^2)r_{\text{P}}^2\sin^2\theta + (r_{\text{P}}\cos\theta - vt)^2\right]^{3/2}} + \frac{r_{\text{P}}\cos\theta + vt}{\left[(1-\beta^2)r_{\text{P}}^2\sin^2\theta + (r_{\text{P}}\cos\theta + vt)^2\right]^{3/2}}\right\}, \tag{10}$$

$$E_y^q = q(1-\beta^2)\left\{\frac{r_{\text{P}}\sin\theta}{\left[(1-\beta^2)r_{\text{P}}^2\sin^2\theta + (r_{\text{P}}\cos\theta - vt)^2\right]^{3/2}} + \frac{r_{\text{P}}\sin\theta}{\left[(1-\beta^2)r_{\text{P}}^2\sin^2\theta + (r_{\text{P}}\cos\theta + vt)^2\right]^{3/2}}\right\}. \tag{11}$$

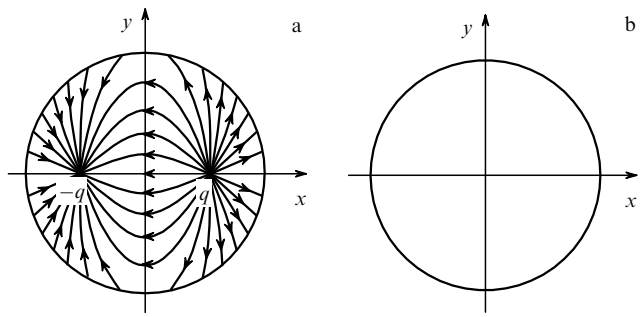

Figure 3. The spatial distribution of an electric field in the cases of instantaneous (a) starting, and (b) stopping of two oppositely charged particles. The particles start from (figure a) and stop at (figure b) the same point.

The transition radiation field resides on the surface of a sphere of radius $r = ct$. It is readily seen that the transition radiation field in this case consists of radiation fields appeared in starting the charge and its electrical image. The charge escapes to the region $x > 0$, while the image-charge is to be found in the region $x < 0$. Let us consider the radiation field in that half-space where the charge is moving. The radiation field created in instantaneous starting of a given charge and an image-charge is described by the expressions*

$$E_x^{\text{rad}} = \frac{q}{r_{\text{P}}}\,\delta(r_{\text{P}} - ct)\left(\frac{\beta\sin^2\theta}{1-\beta\cos\theta} + \frac{\beta\sin^2\theta}{1+\beta\cos\theta}\right) = \frac{2q}{r_{\text{P}}}\,\delta(r_{\text{P}} - ct)\,\frac{\beta\sin^2\theta}{1-\beta^2\cos^2\theta}\,, \tag{12}$$

$$E_y^{\text{rad}} = \frac{q}{r_{\text{P}}}\,\delta(r_{\text{P}} - ct)\left(\frac{\beta\sin\theta\cos\theta}{1-\beta\cos\theta} + \frac{\beta\sin\theta\cos\theta}{1+\beta\cos\theta}\right) = \frac{2q}{r_{\text{P}}}\,\delta(r_{\text{P}} - ct)\,\frac{\beta\sin\theta\cos\theta}{1-\beta^2\cos^2\theta}\,. \tag{13}$$

Figure 3b depicts the pattern of the field arising when a charged particle is incident on an ideally conducting surface. The field inside a sphere with a center at the point of incidence is absent. As the charge and its electrical image meet, the field becomes zero in the entire space. Therefore, once the incident charge reaches the ideally conducting medium, the field in the space $x > 0$ drops to zero everywhere except on the spherical envelope of radius $r = ct$, which expands away from the transition point at the speed of light. The lines of force of the transition radiation field lie on the surface of the sphere. In the case in question, the transition radiation constitutes the sum of radiation fields arising from the charge and its electrical image stopping. We remind the reader that we consider the field in that half-space in which the particle moves. The radiation that gets into this half-space ($x > 0$) is that emitted by the charge and propagating backward, and that emitted by the image-charge and propagating forward with respect to the direction of motion.

Figure 3 presents field patterns after the charge escaped from the space $x < 0$ (Fig. 3a) and after its 'annihilation' with the electrical image on the $x = 0$ interface (Fig. 3b). The instants of time corresponding to 'fly-in' ('annihilation') and 'fly-out' were denoted here as $t = 0$. Until this instant (if what we mean is the incidence of the charge on the boundary with an ideal conductor), the half-space $x > 0$ contains the fields of the uniformly moving charge and its electrical image. Until the instant $t = 0$, the field in the half-space $x > 0$ is absent if we consider the charge escaping, while in the case of a charge incident on the $x = 0$ interface the field becomes similar to the field of a dipole whose moment tends linearly to zero.

When deriving formulas (12) and (13) it was assumed that the surface on which a charge is incident is ideally conducting. In reality, metal surfaces can be regarded as ideally conducting only in the range of optical and radio frequencies. At shorter wavelengths (e.g., in the range of soft X-rays), the reflection coefficient gets smaller and tends to zero with increasing frequency. This circumstance can be taken into account approximately in the following way. We rewrite the expression for the delta function $\delta(r_{\text{P}} - ct)$ as

$$\delta(r_{\text{P}} - ct) = \frac{1}{c}\,\delta\left(t - \frac{r_{\text{P}}}{c}\right). \tag{14}$$

Expanding the delta function into the Fourier integral we obtain

$$\delta\left(t - \frac{r_{\text{P}}}{c}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[\mathrm{i}\omega\left(t - \frac{r_{\text{P}}}{c}\right)\right]\mathrm{d}\omega\,. \tag{15}$$

The integral on the right-hand side of formula (15) is taken over all frequencies from $-\infty$ to ∞. In reality, though, we need to take into account only those frequencies for which the boundary can be regarded as ideally conducting. Assume now that the plane boundary can be treated as ideally conducting at frequencies up to ω_{o}. We choose the frequency ω_{o} in such a way that lower-frequency waves undergo specular reflection, while waves with higher frequencies pass across the interface without reflection. Obviously, there is no such abrupt boundary in real systems and the frequency ω_{o} can be estimated only within an order of magnitude. The above arguments imply that in expansion (15) we need to take into account frequencies in the interval $|\omega| < \omega_{\text{o}}$. Then, the integral on the right-hand side of formula (15) takes the form

$$\delta_{\omega_{\text{o}}}\left(t - \frac{r_{\text{P}}}{c}\right) = \frac{1}{2\pi}\int_{-\omega_{\text{o}}}^{\omega_{\text{o}}} \exp\left[\mathrm{i}\omega\left(t - \frac{r_{\text{P}}}{c}\right)\right]\mathrm{d}\omega = \frac{1}{\pi}\,\frac{\sin\omega_{\text{o}}(t - r_{\text{P}}/c)}{t - r_{\text{P}}/c}\,. \tag{16}$$

Replacing $\delta(r - ct)$ in formulas (12) and (13) by the expression $(1/c)\delta_{\omega_{\text{o}}}(t - r_{\text{P}}/c)$, we find

$$E_x = \frac{q}{c\pi r_{\text{P}}}\,\frac{\sin\omega_{\text{o}}(t - r_{\text{P}}/c)}{t - r_{\text{P}}/c}\,\frac{\beta\sin^2\theta}{1-\beta^2\cos^2\theta}\,, \tag{17}$$

$$E_y = \frac{q}{c\pi r_{\text{P}}}\,\frac{\sin\omega_{\text{o}}(t - r_{\text{P}}/c)}{t - r_{\text{P}}/c}\,\frac{\beta\sin\theta\cos\theta}{1-\beta^2\cos^2\theta}\,. \tag{18}$$

Let us choose in the half-space $x > 0$ an observation point P with coordinates x_{P} and y_{P} in such a way that the radius vector connecting this point to the origin of coordinates is at an angle θ to the x-axis (Fig. 1a). We will assume that the observation point P lies sufficiently close to the interface and to the line of motion of the charge. Here we use 'sufficiently close' in the sense that the field of the uniformly moving charge and the transition radiation field at the observation point are of comparable magnitudes.

Furthermore, let us consider how the field at P changes with time. If we consider the field generated by the particle that is escaped from a metallic surface, then the observation point is first reached by the radiation field and then by the eigenfields of the escaped charge and its electrical image. Figure 4a shows electric field strength as a function of time, $E(t)$, at different distances from the observation point to the charge's trajectory. The curves were plotted for an electron at an energy $\gamma = 10$, which escaped at right angles to the surface of the metal. The distance from the planar metal surface to the measurement plane was chosen to be $x_{\text{P}} = 5$ cm, and the observation angle took the values $\theta = 1/2\gamma$, $1/\gamma$, and $3/2\gamma$ (solid, dashed, and dotted curves, respectively).

* Compared to the Russian original of this paper, the following two formulas have been changed by the author in English proofreading. (*Editor's note.*)

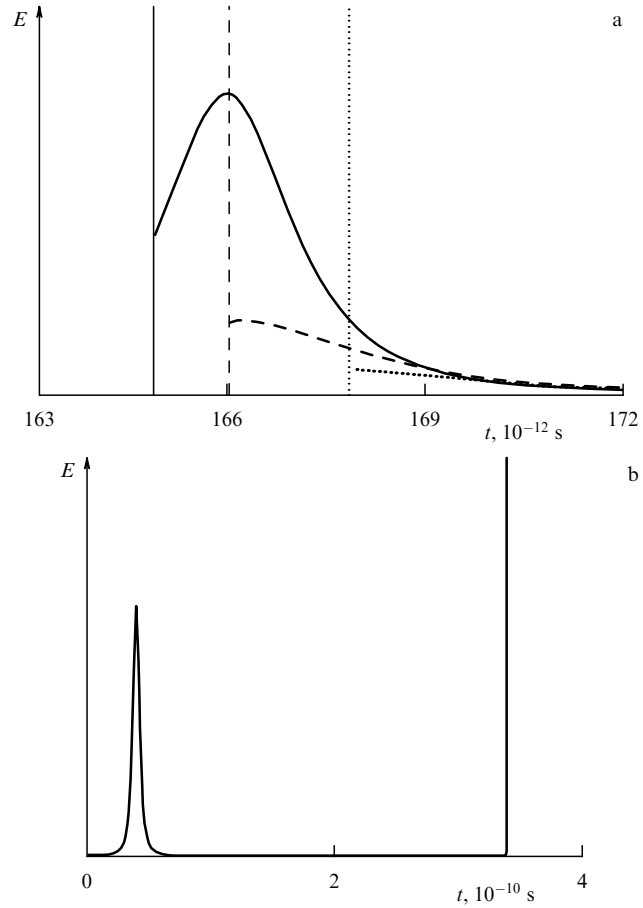

Figure 4. Electric field strength as a function of time after a charge (a) escapes from a conducting plane, and (b) hits the conducting plane; $\gamma = 10$, and $x_{\mathrm{P}} = 5$ cm. The solid line in figure (a) plots $E(t)$ for the observation angle $\theta = 1/2\gamma$, the dashed curve for $\theta = 1/\gamma$, and the dotted curve for $\theta = 3/2\gamma$.

The initial jump in the electric field strength, described by the delta function in expression (9), corresponds to the instant of arrival of the transition radiation field at the point of observation. At subsequent instants of time, the evolution is determined by the fields of the charges moving away — the given charge and image-charge. The figure shows that if the angle at which observation is conducted is increased, it results in changing not only the amplitude of the field but also in changing the shape of the pulse of the eigenfields of the charges. As the observation angle increases, the asymmetry of the pulse shape grows. As follows from the figure, the eigenfields of particles will, at the observation point P located at an angle $\theta = 1/\gamma$ to the direction of motion of the particle, form only one-half of the bell-shaped pulse generated by a single particle flying past the observation point in free space. It is clear that increasing the observation angle θ results in a steep drop in the amplitude and length of the pulse of a particle's eigenfield.

If we consider the field that the particle generates when it is incident on the ideally conducting surface (Fig. 1a, charge q), the first to be registered is the eigenfield of the particle, and then the radiation field. Figure 4b plots the electric field strength E as a function of time. The maximum pulse amplitude corresponds to the instant of time t_0: the time of flight of the charge across a point x_{P} located at the shortest distance from the observation point P. As the charge moves on, it reaches the conducting surface and generates a pulse of transition radiation. This pulse arrives at the observation point at the instant $t = t_0 + \Delta t$, where $\Delta t = x_{\mathrm{P}}/v + (x_{\mathrm{P}}^2 + y_{\mathrm{P}}^2)^{1/2}/c$ is the delay time. The delay time is summed over the time of motion of the particle from point x_{P} to the conducting surface, x_{P}/v, and the time of propagation of the pulse of transition radiation from point $x = 0$ to the observation point, $(x_{\mathrm{P}}^2 + y_{\mathrm{P}}^2)^{1/2}/c$.

It should be noted that in this case the field in the space $x > 0$ consists, after the particle has reached the conducting surface, only of the transition radiation field. The transition radiation field itself is in fact a spherical wave of the form (12) and (13); note also that these relations are valid for any value of r_{P}. The spherical wave of transition radiation is formed precisely at the instant ot time the particle impinges on the surface.

The total field in the problem at hand is the sum of three components: the field of a uniformly moving charge, the field of its electrical image, and the transition radiation field. These fields depend differently on the coordinates. The eigenfield of the charge and the field of its electrical image decrease with distance as $1/r^2$. The transition radiation field decreases as $1/r$. Hence, if r exceeds a certain threshold level, the transition radiation field becomes stronger than the eigenfield of the charge.

Let us evaluate the distance from the interface at which the transition radiation field becomes equal to the eigenfield. We compare the components E_y of the electric field. The transition radiation field is given by formula (18), and the eigenfield of the charge by formula (11). We set these fields equal to calculate the distance to the transition point where these fields become equal. We will consider the fields at a distance $r = ct$ from the transition point, so that the radius vector to the observation point is oriented at an angle θ to the direction of motion of the charge. Under these conditions, the eigenfield at the observation point takes the form

$$E_y^q = \frac{q(1-\beta^2)}{c^2 t^2} \frac{\sin\theta}{(1-\beta\cos\theta)^3}\,, \tag{19}$$

and the transition radiation field is written out as

$$E_y^{\mathrm{tr}} = \frac{2q}{\lambda_{\mathrm{o}} ct} \frac{\beta\sin\theta\cos\theta}{1-\beta^2\cos^2\theta}\,. \tag{20}$$

Here, $\lambda_{\mathrm{o}} = 2\pi c/\omega_{\mathrm{o}}$ is the wavelength corresponding to the maximum frequency ω_{o} at which specular reflection still takes place. By setting the fields equal to each other, we obtain the expression

$$\frac{1-\beta^2}{ct} \frac{1}{(1-\beta\cos\theta)^2} = \frac{2\beta}{\lambda_{\mathrm{o}}} \frac{\cos\theta}{1+\beta\cos\theta}\,, \tag{21}$$

from which one can determine the distance $r = ct$ from the transition point. Equality (21) yields the expression for r:

$$r = \lambda_{\mathrm{o}} \frac{1-\beta^2}{2\beta\cos\theta} \frac{1+\beta\cos\theta}{(1-\beta\cos\theta)^2}\,. \tag{22}$$

Formula (22) demonstrates that as the angle θ increases, the distance r at which the eigenfield and the transition radiation field equal each other diminishes. If the measurement is conducted at an angle $\theta = 1/\gamma$ at which the transition radiation has the maximum amplitude, then at large values of γ we can assume that $1 - \beta\cos\theta \simeq 1/\gamma^2$. Formula (22) then

yields the following relation

$$r \simeq \lambda_o \gamma^2 . \tag{23}$$

The quantity on the right-hand side of Eqn (23) represents the path on which radiation forms at the frequency ω_o, i.e., at the maximum frequency in the spectrum of the transition radiation. Rough estimates for metals show that the frequency ω_o is of order 10^{16}. The corresponding wavelength is $\lambda_o \simeq 2 \times 10^{-5}$ cm.

We assumed in our arguments that the recording instrument has constant sensitivity in the entire spectral range of transition radiation. In other words, we supposed that the recording instrument can measure field strength changes over sufficiently short intervals of time (in this case, over intervals on the order of $\Delta t \simeq 1/\omega_o$). If, however, the instrument possesses sufficient sensitivity only in a limited spectral interval, it is necessary to compare the spectral characteristics of fields in the same limited frequency range.

In reality, electromagnetic fields are usually measured in a limited spectral interval. Therefore, instead of dealing with field strengths as functions of time, we switch to Fourier transforms. Components of Fourier transforms of the transition radiation fields, $E_x^{\rm rad}(\omega)$ and $E_y^{\rm rad}(\omega)$, are written in the form

$$E_x^{\rm rad}(\omega) = \frac{q}{\pi c r_{\rm P}} \frac{\beta \sin^2\theta}{1-\beta^2\cos^2\theta} \exp\left(\mathrm{i}\,\frac{\omega}{c}\,r_{\rm P}\right), \tag{24}$$

$$E_y^{\rm rad}(\omega) = \frac{q}{\pi c r_{\rm P}} \frac{\beta \sin\theta\cos\theta}{1-\beta^2\cos^2\theta} \exp\left(\mathrm{i}\,\frac{\omega}{c}\,r_{\rm P}\right). \tag{25}$$

As we expect for radiation fields, the transition radiation fields vary in inverse proportion to radius $r_{\rm P}$.

Let us now consider the Fourier components of the eigenfield of the charge. From expression (10) it follows that the component E_x changes sign at the instant of time at which the charge is at the point $x = x_{\rm P}$, $y = 0$. The integral of this component over time is zero. The pulse of the field E_x at $t \approx x_{\rm P}/v$ is nearly sinusoidal at a frequency $\omega \approx \gamma v/y_{\rm P}$, so that its spectrum consists of a narrow band of frequencies in the vicinity of the value $\omega = \gamma v/y_{\rm P}$. The component E_y of the charge eigenfield is a bell-shaped pulse with an amplitude $E_y \approx q\gamma/y_{\rm P}^2$ and the characteristic width $\tau \approx y_{\rm P}/\gamma v$. The spectrum of the pulse contains all frequencies up to $\omega \approx \gamma v/y_{\rm P}$.

The eigenfield of the charge differs essentially from the radiation field. The radiation field is expanded over waves which propagate in all possible directions. In contrast to this, the field of a uniformly moving charge contains waves whose direction of propagation coincides with that of the charge velocity. The expansion of the eigenfield into the Fourier integral gives the following expression for the components $E_x^q(\omega)$ and $E_y^q(\omega)$:

$$E_x^q(\omega) = -\mathrm{i}\,\frac{q\omega}{\pi v^2\gamma^2}\,K_0\left(\frac{\omega r\sin\theta}{\gamma v}\right)\exp\left(\mathrm{i}\,\frac{\omega}{v}\,x_{\rm P}\right), \tag{26}$$

$$E_y^q(\omega) = \frac{q\omega}{\pi v^2\gamma^2}\,K_1\left(\frac{\omega r\sin\theta}{\gamma v}\right)\exp\left(\mathrm{i}\,\frac{\omega}{v}\,x_{\rm P}\right), \tag{27}$$

where K_0 and K_1 are the modified Bessel functions of imaginary argument (Mcdonald's functions), and $r = (y_{\rm P}^2 + z_{\rm P}^2)^{1/2}$ is the distance from the observation point P to the line of motion of the charge (i.e., from the x-axis).

The waves of the eigenfield propagate along the x-axis. The field strength rapidly decreases away from the trajectory of the charge motion (i.e., with increasing r). Formulas (26) and (27) describe the field of a single uniformly moving charge in free space. The reader will recall that the total field in the problem of transition radiation is the sum of the fields produced by the incident charge and that of the image-charge.

When comparing transition radiation fields [formulas (24) and (25)] with the eigenfields of charges in motion [formulas (26) and (27)], it should be remembered that the Fourier components of the fields differ both in amplitude and in phase. This fact should be taken into account.

Notice that the distance from the transition point over which the wave zone of the transition radiation is formed was evaluated in Ref. [7]. Verzilov [7] assumed that a finite-sized emitting zone is formed around the transition point in the plane of the interface. In this case, one can determine the distance $r_{\rm P}$ from the transition point over which the transition radiation field has the form of an expanding spherical wave, i.e., is a function of $r_{\rm P}$, namely, $\exp(\mathrm{i}kr_{\rm P})/r_{\rm P}$.

5. Conclusion

The above analysis implies that transition radiation is formed as a spherical wave expanding right from the transition point. In this sense, transition radiation exists as a spherical wave at any distance, no matter how small, from the transition point. The wave zone arises where the transition radiation field becomes stronger than the total accompanying field of the charge and its electrical image. One needs to remember, though, that this conclusion holds true in the case of an ideally conducting plane boundary.

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