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Landau and modern physics*

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Contents

Abstract. This article describes the history of the creation and further development of Landau's famous works on phase transitions, diamagnetism of electron gas (Landau levels), and quantum transitions at a level crossing (the Landau–Zener phenomenon), and its role in modern physics.

1. Introduction

The fate of Landau's scientific legacy has been amazingly fortunate. It has been 46 years since he stopped doing science, but the number of citations per year of his papers is only increasing, rather than decreasing. It is rather difficult to give a quantitative estimate since, in many cases, when mentioning the Landau theory of phase transitions, the Ginzburg–Landau equation, the Landau–Lifshitz equation, Landau levels, the Landau criterion of superfluidity, the Landau theory of a Fermi liquid, etc., no citation to his original articles is given. Nevertheless, according to the science-bibliography website Scirus, the number of formal and informal citations of Landau's papers on phase transitions exceeds 30,000, of papers on a Fermi liquid is about 45,000, of Landau levels is 75,500, of the Landau–Lifshitz equation is 23,000, of Landau damping in plasma is 12,000. In the year 2000, at the anniversary session of the American Physical Society, there were six stands demonstrating the development of physics in the 20th century. At four of them, the name Landau was mentioned.

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The goal of these notes is to follow the development of Landau's ideas and their most important applications up to the present day. The breadth of Landau's contributions makes this task practically impossible for a single person. Originally, this article was planned as a review written by several authors, but unfortunately this idea did not work out. That is why the part of this review published here is not complete. Certainly, many wonderful papers by Landau are not mentioned here, such as those on astrophysics with predictions of neutron stars and a criterion of collapse, on the theory of elementary particles and field theory, on the theory of superfluidity, on the Fermi liquid theory, and many others. The material collected in these notes is related to three important and extremely popular articles by Landau: on phase transitions, on Landau levels, and on the Landau–Zener theory. Their further development and numerous applications clearly demonstrate that Landau's papers are an important part of modern physics and are at the center of its interests.

It is amazing how much one person was able to accomplish in the slightly more than thirty years given to him by his destiny for scientific work! A short acquaintance with Landau influenced my life very strongly. These notes are my grateful tribute to his blessed memory.

2. Phase transitions

2.1 Spontaneous symmetry breaking, the mesoscopic description, universality

In two articles [1, 2] published in the ominous year of 1937, Landau built a foundation for the general theory of phase

* This article is partially based on the text of my presentation given at the scientific-memorial session devoted to the 100th anniversary of the birth of L D Landau (1908–1968), held on 19–20 June 2008 in the Central House of Scientists, RAS, Moscow. The text of this article was created as a comment to the anniversary edition of the collected papers of L D Landau in 2008, but was not included in it for reasons independent of the author.

transitions with a change in symmetry. This theory has become the most fruitful in terms of its influence on the subsequent development of physics. The number of publications devoted to its development and experimental evidences amounts to the tens of thousands. Landau's theory of phase transitions is applied in crystallography, biology, high-energy physics and field theory, astrophysics and cosmology, not to mention statistical physics and many branches of condensed matter physics. Four Nobel Prizes have been awarded for papers devoted to the development of the ideas of this theory.

Landau was the first to introduce a concept of spontaneous symmetry breaking, which has spread widely in statistical physics and field theory. Often authors who write about the spontaneous symmetry breaking of a vacuum think that this idea is a folklore, but this is incorrect: this concept has a clearly defined authorship. Landau introduced it many years before it started to be used in field theory and high-energy physics, noting that symmetry breaking is equivalent to the emergence of a long-range order in many-particle systems.

Another important idea was first formulated in Landau's theory of phase transitions: the idea of a mesoscopic description of ordering media. Landau realized that when a system approaches a continuous phase transition or critical point, the correlation length grows and microscopic details of the system are no longer important. Only the initial symmetry and how it changes as a result of the transition are important. All significant events occur on the scale between the interparticle distance and the size of the whole system. The theory happens to be universal: phase transitions of a different nature, which have the same initial and final symmetries, are isomorphic. Landau suggested considering the amplitude of emerging (at the transition) irreducible representation of the initial symmetry group as a measure of symmetry breaking (an order parameter).

Quantitatively, Landau's theory was based on the self-consistent field approximation. Fluctuations were assumed to be negligibly small. Based on this approximation, Landau formulated group-theoretic rules allowing the classification of possible second-order phase transitions, under which the order parameter in the ordered phase continuously grows starting from zero, and the criterion for the transition to be always accompanied by jumps of some physical quantities (first-order phase transition). In the second of the cited papers, Landau showed that one-dimensional crystal order in two-dimensional and three-dimensional media is broken by fluctuations. A congenial statement was formulated by Peierls [3]: a long-range order in two-dimensional systems with spontaneously broken continuous symmetry is broken by fluctuations.

2.2 Classification of phase transitions. Order parameters

The development of Landau's ideas took place in several directions. One of them was the concrete implementation of Landau's general scheme for systems of the highest interest in physics. The first group-theoretic calculation of the phase transition between different crystal phases was performed by E M Lifshitz [4]. The literature on the reconstruction of crystals is widespread and partially connected with general crystallographic research. The conclusions of this work up to the middle of the 1990s were summarized in books by Izyumov and Syromyatnikov [5] and Tolédano and Dmitriev [6].

A review of phase transitions in crystals with initial colored (Shubnikov) group symmetry [7], which is physically interpreted as a magnetic crystallographic structure with discrete magnetic symmetry, is given in paper [8] based on Landau's theory. Andreev and Marchenko [9] have built a full classification of magnetic crystallographic structures with an initial magnetic symmetry group of rotation SO(3), which is typical for exchange interactions. The phenomenon of weak ferromagnetism in crystals was predicted by Dzyaloshinsky [10] based on investigations of group invariants using Landau's method, and was experimentally discovered by Borovik-Romanov [11]. A microscopic interpretation of weak ferromagnetism was given by Moriya [12].

Phase transitions in ferroelectrics were interpreted based on the Landau–Ginzburg–Devonshire theory [13], which had multiple experimental confirmations and technological applications [14].

In their famous study of 1950, Ginzburg and Landau gave a symmetry description of superconduction near the transition point [15]. They introduced a complex wave function of superconducting condensate as the order parameter. This work had a deep influence not only on the theory of superconductivity but also on many other areas of physics and mathematics. An analogous theory for superfluids was constructed by Ginzburg and Pitaevskii [16]. Gross [17] and Pitaevskii [18] have proposed a nonlinear theory of a weakly nonideal superfluid Bose gas, which is applicable at low temperatures and perfectly describes modern experiments with laser-cooled gases of alkali metal atoms. The Ginzburg–Landau theory and its application to the description of the vortex state in superconductors was the reason for awarding Abrikosov, Ginzburg, and Leggett the Nobel Prize in Physics 2003.

Based on the principles of Landau's theory, P-G de Gennes has built his theory of phase transitions from an isotropic liquid to a nematic liquid crystal, and from a nematic to a smectic liquid crystal (the so-called Landau–de Gennes theory) [19, 20]. This theory represents a significant part of a cycle of works on 'soft matter', for which de Gennes was awarded the Nobel Prize in Physics 1991. The universal use of liquid crystals in displays, monitors, and televisions has converted Landau's theory into an applied science.

Many first-order phase transitions, in particular, all transitions of liquid crystals from the isotropic to nematic phase, are close to second-order transitions: the jumps in volume and the specific heats of transition are small. The theory of weak melting–crystallization based on Landau's free energy was elaborated by Brazovskii [21]. Later on, it was used for the description of blue phases in liquid crystals.

2.3 Fluctuation theory

In the middle of the 1940s, there was a crisis in the theory of phase transitions, related to the great achievement of L Onsager — the exact solution of the two-dimensional Ising model [22]. This solution uncovered singularities of thermodynamic quantities, which were completely different from the predictions of Landau's theory. Precise measurements of the specific heat of liquid helium near the transition to the superfluid state (Buckingham and Fairbank [23]) and of argon near its critical point (Voronel' et al. [24]) have revealed a singular behavior which was not accounted for by the mean-field theory. This contradiction was explained by Levanyuk [25] and Ginzburg [26], who showed that the

fluctuations in the order parameter and entropy, even if they are weak when far away from the transition point, become strong in its vicinity. The range of temperatures within which the fluctuations are weak and the self-consistent field approximation is valid does not exist by far in every system susceptible to a phase transition [27]. In particular, it does not exist in the Ising model. However, there are reasons to believe that the behavior of systems in the region of strong fluctuations is universal in the above sense, and that it depends only on the initial symmetry and the method of its breaking. The hypothesis for universality in the fluctuation region was introduced by Vaks and Larkin [28] and was later proven by Kadanoff and Wegner [29].

Another hypothesis formulated in the middle of the 1960s was the scale invariance of critical fluctuations (Patashinskii and Pokrovskii [30, 31], Kadanoff [32], and in a narrow sense, only for the equation of state, by Widom [33], and by Domb and Hunter [34]). According to this hypothesis, when varying the linear scale the picture of fluctuations should not change if the units of measurement of the fluctuating fields were altered appropriately. Every fluctuating quantity, for example, an order parameter, entropy, temperature, etc. are characterized by their scaling dimensions (critical exponents). The hypothesis for scale invariance together with simple thermodynamic equations allowed finding a number of relations between the scaling dimensions, and therefore it reduced the number of independent dimensions, but did not solve the problem completely. In 1959, M E Fisher was one of the first who pointed out the significance of calculating the critical exponents, and introduced the index of anomalous dimension [35]. Powerful numerical methods of finding critical exponents using the first several series terms of the high-temperature perturbation theory allowed Domb et al. to find critical exponents of the three-dimensional Ising model with a precision of several percent [36]. However, a real theory did not yet exist.

In the late 1950s, Landau formulated the problem of the fluctuating theory of phase transitions as one of evaluating a partition function of the system whose Hamiltonian coincides with the free energy of the self-consistent field of the order parameter, which he introduced in 1937. Exactly on this way, K G Wilson found in 1972 a constructive solution to the problem of a phase transition [37–39], for which he was awarded the Nobel Prize in Physics 1982. To calculate a partition function, Wilson developed a new version of the method of renormalization group which was first introduced in quantum field theory by Gell-Mann and Low [40] and Stückelberg and Green [41]. An attempt to use the field-theory renormalization group in the theory of phase transitions was made by Di Castro and Jona-Lasinio [42], but it did not lead to quantitative results. In Wilson's method, the renormalization means a consecutive elimination of short-wave fluctuations, starting from the shortest waves and following to longer ones, and the calculation of the effective energy of the remaining long-wave fluctuations. This idea was put forward earlier by Kadanoff [32] but was accomplished only by Wilson. Another important idea of Wilson and Fisher [38], which allowed finding critical exponents analytically, is the expansion in the powers of a small parameter ε equal to a deviation of space dimensionality from the critical value 4, at which the critical exponents have the same values as in Landau's theory. The interaction between fluctuations leads only to logarithmic singularities which were found earlier in the work of Larkin and Khmel'nitskii [43], although in the real world this parameter is not small ($\varepsilon = 1$) and to prove this procedure in a strictly mathematical sense is difficult. Nevertheless, the calculation of critical exponents [44] employing the method of series summation in ε with the use of the asymptotic behavior of the high-order terms (found by L N Lipatov [45]) leads to results close to the best numerical calculations and to the data of the most precise experiments.

2.4 Dynamics of critical fluctuations

As the characteristic size of fluctuations grows, their motion gets slower. This is a so-called critical slowdown. The first quantitative theory of the critical slowdown was established by Landau and Khalatnikov [46] in 1954. It was based on a phenomenological assumption that the critical dynamics is of a purely relaxational character: the time derivative of the order parameter is proportional to the self-consistent field — that is, the derivative of thermodynamic potential with respect to the order parameter. The kinetic coefficient Γ relating these quantities was assumed to be independent of the distance from a transition point. Hence it followed that the relaxation time of the order parameter is proportional to the square of the correlation length. As in the static case, the fluctuations were considered to be negligibly small in comparison with the average values. A new effect on anomalous absorption of sound because of the excitation of critical fluctuations at a frequency on the order of the inverse relaxation time was predicted. This phenomenon was examined experimentally [47].

The Landau–Khalatnikov theory has been generalized in various aspects. One of them is the transition to the region of large fluctuations. The simplest generalization was introduced by Pokrovskii and Khalatnikov [48], who assumed that the kinetic coefficient Γ remains a constant but the correlation radius and other physical quantities follow the laws of static scaling. A more detailed generalization was suggested by Ferrell et al. [49, 50], and in a more general and physically transparent form by Halperin and Hohenberg [51] (so-called dynamical scaling). They have noticed that in systems with a continuous order parameter, the dynamics of the ordered state is determined by the conservation laws which follow from the remaining continuous symmetry, and therefore in general is nondissipative. The long-wave excitations are propagating waves. Their velocity is determined by thermodynamics, e.g., by the Laplace equation for the speed of sound. Thus, the frequency of the wave in the critical region is determined by static scaling. The dynamics have a purely relaxational character above the critical point at any wavelength λ and below the critical point at wavelengths shorter than the correlation radius ξ (e.g., fluctuational thermal conductivity near the transition to the superfluid state). The hydrodynamic and relaxation frequencies coincide to an order of magnitude at $\lambda \sim \xi$. This relationship allows finding dynamic critical exponent z which determines the scaling dimension of frequency ω. For $\lambda \ll \xi$, we have $\omega \sim \lambda^{-z}$ and we can find the critical behavior of kinetic coefficients. This theory has found excellent agreement with experiments [1] [51].

The renormalization-group theory of dynamic critical phenomena has been developed in papers by Halperin, Hohenberg, Ma, and Siggia (see review [52]).

[1] A detailed description of experiments on dynamic scaling and their comparison to the theory was done in review [52].

3. Landau levels

3.1 Landau levels and the diamagnetism of metals

This is the commonly used name of the energy levels of a charged particle in constant uniform magnetic field. The solution to this very important question, which was as important in quantum mechanics as questions about the spectrum of the hydrogen atom and the quantum harmonic oscillator, was found by Landau [53] in 1930. Landau found that motion in the plane perpendicular to the magnetic field is quantized.[2] The energy levels are strongly degenerate in the quantum number corresponding to the center of the Larmor circle in classical mechanics. The number of states per unit area, belonging to a given Landau level, is equal to the ratio B/Φ_0 of a magnetic field induction to the magnetic flux quantum $\Phi_0 = h/2e$. In other words, each state carries one magnetic flux quantum. The energy of transverse motion of an electron on the nth Landau level is equal to

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega\,,$$

where $\omega = eB/mc$ is the cyclotron frequency. The solution of this relatively simple problem led Landau to the important conclusion that the quantization of the transverse motion is the reason for the diamagnetism of electrons in a metal, which could not be explained from the classical point of view, as was shown by Bohr [57] and van Leeuwen [58].

3.2 Magnetic oscillations: Shubnikov–de Haas and de Haas–van Alphen effects

At the end of his article [53] Landau noted that the quantization of energy levels leads to oscillations of diamagnetic susceptibility in a changing field. However, he believed that examining these oscillations experimentally would be hard because of the nonuniformity of the magnetic field. Most probably he did not know about the Shubnikov–de Haas experiment [59], which was also performed in 1930. In 1937, Landau learned from D Shoenberg about the de Haas–van Alphen effect [60]: oscillations of magnetic susceptibility vs. magnetic field. Landau immediately made a preliminary calculation of the oscillations and handed it to Shoenberg, but was not able to publish it because of his arrest. In 1939, Shoenberg published this calculation [61], obviously under Landau's[3] name, as complementary material to his experimental work [63]. Landau made the calculation for an extremely simplified model of free electron gas. A more realistic approach, which accounted for the band structure of the energy spectrum and the anisotropy of metals, was introduced independently by I M Lifshits [64] and Onsager [65]. They showed that quasiclassical orbits of electrons in momentum space are the cross-sections of the isoenergetic surface by planes perpendicular to the magnetic field. The orbits of electrons in the configuration space differ from orbits in momentum space by a rotation and scale factor, which is chosen in such a way that a magnetic flux through the orbit is equal to an integer number of magnetic flux quanta. Using this, Lifshits and Kosevich [66] elaborated the theory of the de Haas–van Alphen effect for an arbitrary dispersion of electrons. Lifshits and Pogorelov [67] found a solution to the inverse problem on reconstruction of a Fermi surface using experimental data on the de Haas–van Alphen effect. Numerous measurements of this effect, as well as of the Shubnikov–de Haas effect, allowed creating an atlas of Fermi surfaces [68].

3.3 Quantum Hall effect

With the creation of two-dimensional electron systems, inversion layers in silicon, and GaAs – AlGaAs heterojunctions, objects appeared for which the theory of magnetic Landau levels became fully applicable. In 1980, a completely unexpected effect was discovered in these systems, which is known as the quantum Hall effect (QHE) [69]. It turned out that at a relatively low temperature and a high mobility of carriers the Hall conductivity $\sigma_{\rm H}$, i.e., the ratio of the current flowing in the direction perpendicular to the electric and magnetic fields to the magnitude of the magnetic field is quantized. In addition, the Hall conductivity quantum is equal to e^2/h with a very high precision of $\sim 10^{-7}$. More precisely, this means that when a magnetic field B varies then $\sigma_{\rm H}$, which is on average proportional to $1/B$, becomes a step function of $1/B$ with the values of $\nu e^2/h$ on its steps (ν is an integer number). Klaus von Klitzing was awarded the Nobel Prize in Physics 1985 for the discovery of the integer Hall effect. The reason for the stepwise dependence of $\sigma_{\rm H}$ on $1/B$ is the localization of all states of a given Landau level by impurities, with the exception of a single edge state which corresponds to the drift of an electron along the boundary of the sample in the direction given by the magnetic field and the force holding the electron inside the sample. When all remaining states on the Landau level are occupied, the scattering on impurities for the edge state is not effective, since it is impossible to change the direction of rotation, and unoccupied states on the next Landau level are separated by an energy gap. The drift of the edge state gives the same effect as the motion of all Larmor circles (electrons) of the filled Landau level. Indeed, integer values of $\sigma_{\rm H}$ in units of e^2/h correspond to the drift of all electrons with the flux density $\nu B/\Phi_0$ per unit area with the typical drift velocity cE/B.

In 1983, Tsui, Störmer and Gossard [70] discovered the fractional quantum Hall effect: the Hall conductivity is quantized in fractions with an odd denominator, for example, $1/3$, $2/3$, $4/3$, $2/5$, $3/5$ in the same units. This constitutes a more complicated phenomenon based on the interaction between Landau electrons. Laughlin [71] has explained this phenomenon using as an example the filling factor $1/3$, when there are 3 magnetic flux quanta per electron. In this case, electrons can more successfully avoid each other at small distances than at fractional concentrations with large denominators, and thus the Coulomb repulsion energy is reduced. This is the reason for the appearance of a gap in the spectrum, but the rest is similar to the integer QHE. The excitations in this strongly interacting incompressible electron liquid carry as before only one flux quantum, and therefore they have a fractional charge of $1/3$. For the

[2] Landau had predecessors: in 1928, I I Rabi solved the problem of Dirac's electron in an external magnetic field [54], and in the same year V A Fock solved the problem of a harmonic oscillator in a magnetic field [55]. In 1930, independently of Landau, the solution of the Schrödinger equation for a nonrelativistic charged particle in a magnetic field was found by Frenkel' and Bronshtein [56]. However, in none of these papers, the number of states on a magnetic level was found, and no linkage between quantization and diamagnetism and oscillations was established. This seemingly was the reason why the levels got Landau's name.

[3] A friend and coauthor of Landau, Rudolf Peierls, participated in writing this Appendix (see Shoenberg's book [61] and the historical review by M I Kaganov [62]).

discovery and explanation of the fractional quantum Hall effect Tsui, Störmer, and Laughlin were awarded the Nobel Prize in Physics 1998.

Recently, K S Novoselov and A K Geim with colleagues have learned how to produce isolated crystalline sheets of graphite called graphene [72]. This material has opened new horizons in the study and use of Landau levels. Graphene is an ideal semimetal whose Fermi level resides at the boundary between two bands touching each other. Near the intersection point, the electron dispersion possesses an ultrarelativistic form: $\varepsilon = vp$ with the velocity $v \sim 10^8$ cm s^{-1}, which plays the role of the speed of light. The Landau energy levels of the ultrarelativistic electrons are equal to

$$\varepsilon_n = v\sqrt{2nm\hbar\omega}\,.$$

For small n, these levels are on the order of the geometric mean of the atomic and cyclotron frequencies. For fields of several Tesla, this energy falls in the range of several hundred kelvins. This means that the QHE and oscillatory effects can be observed at room temperature, whereas in usual metals these effects are observed only at helium temperatures. Indeed, the QHE at room temperature has been examined experimentally [73].

4. Landau–Zener transitions

4.1 History and formulation of the results

In 1932, Landau formulated and solved one of the most important dynamic problems of quantum mechanics [74]. Independently from him in the same year, although a bit later, this problem was also solved by three outstanding theorists: C Zener [75], E G G Stückelberg [76], and E Majorana [77]. In the literature it is known as the Landau–Zener problem. Below we formulate the problem and its solution in modern terms and definitions. The problem is as follows. The Hamiltonian of a system H depends on the number of parameters $\mathbf{R}$. The discrete energy levels $E_n(\mathbf{R})$ and vectors of the state (wave functions) $\left|\Psi_n(\mathbf{R})\right\rangle$ corresponding to them continuously depend on these parameters. If the parameters $\mathbf{R}$ vary with time t, the energy is not conserved and also depends on time. However, if the parameters vary slowly, the vector of state is equal to $\left|\Psi_n\big(\mathbf{R}(t)\big)\right\rangle$ in the leading approximation, i.e., the system follows the continuous change of parameters with time without making a transition to other states. This is the so-called adiabatic approximation. It is not valid if some of the levels cross. Near the crossing point of the levels there are intensive transitions between the corresponding states. In the Landau–Zener theory it is assumed that only two levels cross, and all other levels are far away. In this approximation, the problem reduces to finding the states of a two-level system, i.e., to the problem of spin-1/2 motion in a time-dependent magnetic field. Using the fact that the time interval for the transitions is narrow, one can assume the time dependence of the distance between the approaching levels to be linear. Therefore, one can represent the transition frequency Ω as a linear function of time: $\Omega(t) = \dot{\Omega}t$. The matrix element Δ of the transition between the two initial (so-called diabatic) states is considered to be independent of time. In this approximation, the spin Hamiltonian takes the simple form

$$H = \hbar\dot{\Omega}t\sigma_z + \Delta\sigma_x\,,$$

where σ_x and σ_z are the Pauli matrices. The time-dependent Schrödinger equation is reduced to an exactly solvable equation for a parabolic cylinder. This is exactly how Zener solved this problem. Instead of this, Landau went around the crossing point in the complex time plane, which allowed him to avoid using the functions of the parabolic cylinder. The transition matrix calculated by Landau and Zener has the diagonal matrix element (amplitude of survival)

$$\alpha = \exp\left(-\pi\gamma\right),$$

and the off-diagonal element

$$\beta = -\sqrt{2\pi}\,\exp\left(-\frac{\pi\gamma}{2} + \frac{\mathrm{i}\pi}{4}\right)\left[\sqrt{\gamma}\,\Gamma(-\mathrm{i}\gamma)\right]^{-1}$$

(calculated by Zener), where $\gamma = \Delta^2/\left(\hbar^2|\dot{\Omega}|\right)$ is the dimensionless Landau–Zener parameter. The large values of γ correspond to the adiabatic regime when the system resides on one of the two adiabatic levels which are separated by the energy gap Δ, and performs the transition from one diabatic level to the other. The small values of γ correspond to the fast (anti-adiabatic) regime and to small transition probabilities. The Landau–Zener formula gives values of transition amplitudes beyond the limits of applicability of perturbation theory. The transition takes time $\tau_{\mathrm{LZ}} = \Delta/\left(\hbar|\dot{\Omega}|\right)$ (the Landau–Zener time).

4.2 Applications

The generality of the formulation has allowed applying the Landau–Zener theory to a wide variety of phenomena. Landau had in mind the description of a certain class of molecular reactions called predissociation. Later on, the Landau–Zener theory was applied widely in chemistry, chemical physics, and biochemistry. In particular, it is applied for the description of such an important process as the transfer of charge along with its energy [78]. Landau–Zener transitions play a significant role in photosynthesis [79].

The Landau–Zener theory has also found multiple and important applications in physics. At the earlier stage it was applied to the theory of atomic and molecular collisions accompanied by transitions between electron levels, in particular, charge-exchange collisions. The theory of these processes and their role in plasma physics was introduced by B M Smirnov, E E Nikitin, and others and described in detail in books by Smirnov [80], where one can also find experimental data. An alternative approach to this problem was described in the review by Solov'ev [81].

A P Kazantsev and colleagues showed [82, 83] that the Landau–Zener transitions of atoms in a standing light field lead to the appearance of a band structure in a spectrum of atoms which interact resonantly with this field. If a slow field amplitude depends periodically on time, the atomic states are described by Floquet–Bloch wave functions and quasienergy [84]. Similar phenomena also occur in highly excited (Rydberg) atoms under the influence of a strong microwave field [85].

It is interesting to note that the number of citations of Landau's above-mentioned paper has increased significantly in the last few years. This is related to the appearance of new experimental objects whose dynamics is determined by the Landau–Zener processes. Here, one should first mention qubits, which are the elementary units of a quantum

computer. They operate with quantum states instead of binary numbers. Qubits have already been created experimentally. One type of qubit consists of two small superconductors connected via a Josephson junction [86]. Here, the role of diabatic states is played by the states of a Cooper pair in one of the two superconductors. Another realization of the qubit [87] is a so-called quantum dot, a tiny (several nanometers in size) piece of a semiconductor from which, with the use of a gate, almost all but a few electrons are pushed out. The diabatic states in this system are the spin states of an electron in an external magnetic field. In these and other qubits, the controlled superpositions of two quantum states are obtained by means of the Landau–Zener process.

In the last 10–15 years, molecular magnets have been studied intensively. This is a family of molecules containing 100–200 atoms in which a ferromagnetic core (atoms of iron, cobalt, or manganese) is held together by organic bridges. The most popular are the molecules abbreviated as Mn_{12} and Fe_8. The former was investigated by an experimental group at the City College of New York under the guidance of M Sarachik, while the latter was studied by the collaboration of French experimentalists from Grenoble and Italian experimentalists from Florence (B Barbara, W Wernsdorfer, D Gatteshi, R Sessoli). Theoretical papers by E Chudnovsky, A Garg, N Prokofiev, and J Villain played an important role. The results of this research are presented in the book [88]. Both Mn_{12} and Fe_8 molecules have spin $S = 10$ and really represent nanometer-sized magnets. Experimentalists deal with molecular single crystals but the ferromagnetic cores of their molecules are so far away from each other that their interaction is negligibly small. Therefore, the magnetic properties of these systems are those of a single molecule. When measuring magnetization as a function of the magnetic field, the hysteresis curves were obtained in both substances, which testified to a new phenomenon called molecular hysteresis. At a temperature below 0.5 K, the hysteresis curves stop changing, which proves the quantum character of the hysteresis. In the hysteresis curves one can see steps at particular values of the magnetic field. These steps were attributed to Landau–Zener transitions appearing at the crossing of terms with large spins, which are initially split by the anisotropy of the molecular field and are controlled by a time-dependent magnetic field. Thus, Landau–Zener transitions are the real reason for the quantum hysteresis.

Several years ago experimentalists working with alkali metal atoms cooled by a laser light to temperatures on the order of $10^{-8} - 10^{-7}$ K learned how to obtain diatomic molecules at the so-called Feshbach resonance [89]. The atoms were placed in an external magnetic field, and by changing it they passed that magnitude at which the Zeeman energy of a pair of atoms becomes equal to the binding energy of a molecule. This is accompanied by the Landau–Zener process, and the atoms turn into molecules. In many theoretical papers, authors have attempted to apply directly the Landau–Zener theory to this problem [90]. However, the experiments were performed with degenerate Bose or Fermi gases. The identification of pairs of atoms in this gas is impossible because identical particles are quantum-mechanically indistinguishable. This problem should be solved by the methods of many-particle theory (see the next section).

4.3 Development of the theory

A M Dykhne has proposed a modification of the theory for the case where the crossing of the diabatic electron terms takes place not on the real axis but somewhere else in the complex plane of time [91]. This result was discussed with Landau and was approved by him. It is known in the literature as the Dykhne–Landau formula.[4]

At the crossing of electron terms of a molecule one has to deal not with a single level but with the bands consisting of vibrational and rotational sublevels. The problem of the crossing of a band of parallel levels with a single level was solved by Demkov and Osherov [92]. Some particular cases of crossings of many levels have been found by Demkov and Ostrovsky [93]. The transition probabilities in all these solutions are given by a product of the probabilities of the consecutive Landau–Zener transitions. Carroll and Hioe [94] have found a solution to the problem of the motion of an arbitrary spin in a magnetic field, when one of its components (z) depends linearly on time, and another one (x) is a constant. In this case, $2S + 1$ diabatic levels cross simultaneously. The exact solution is possible due to SO(3) symmetry and is given by the matrix of the Landau–Zener transition in terms of Jacobi polynomials. The most general construction, which allows finding a solution to a wide class of problems with the crossing of many diabatic levels, was introduced by N A Sinitsyn [95]. All exact solutions found earlier have entered into this class as particular cases. As for the general case of the crossing of many diabatic levels in different but sufficiently close points at different angles, only the amplitude of survival at the level with maximal or minimal slope is known. It is equal to the product of the Landau–Zener amplitudes of survival for the points of consecutive crossing with other levels. This result was initially formulated as a hypothesis by Brundobler and Elser [96], and later was proven analytically by Shytov [97], who generalized Landau's method, and algebraically by Volkov and Ostrovsky [98] and Dobrescu and Sinitsyn [99].

The application of the Landau–Zener theory to qubits and molecular magnets has raised the question of the role of noise. Noise is responsible for the decoherence and errors in quantum calculations. The thermal noise in molecular magnets leads to a significant narrowing of the molecular hysteresis loop even at a temperature of 1 K. There are longitudinal noise, which distorts adiabatic levels but does not lead to transitions, and transverse noise, which causes spontaneous transitions. Purely longitudinal noise has been studied in a number of papers. The most detailed analysis was given by Ao and Rammer [100], who considered a number of limiting cases (in this article one can find a rather extensive bibliography). Unfortunately, we cannot describe the results obtained in Ref. [100] using simple physical patterns. The situation is reversed in the case of transverse noise. The pioneering work about fast transverse noise was introduced by Kayanuma [101], but this paper is not transparent in terms of physics, and a spectral composition of noise in it is taken rather artificially. A significant simplification was reached in the paper by Pokrovsky and Sinitsyn [102]. They showed that noise-induced transitions accumulate during a time which is

[4] I participated in a discussion where Landau found a mistake in Dykhne's initial solution. The same day Dykhne and Landau independently found the correct method. Landau refused to be a coauthor of the study. Later on, Landau's approach was published in the new edition of *Quantum Mechanics: Non-Relativistic Theory* (Moscow: Nauka, 1963) which was edited by him but published after his car accident, without citing Dykhne. According to Pitaevskii's recollection, Landau told him that there was a mistake in Dykhne's formula. However, by checking this formula one can show that the results of both authors coincide apart of notations.

much longer than the Landau–Zener time. Noise transitions and Landau–Zener transitions turn out to be separated in time, which allows solving this problem exactly. The noise transitions, at a given instant of time, are caused by the spectral component of the noise which is at resonance with the current frequency of the two-level system. This allows using this frequency as the noise analyzer. A strong classical transverse noise leads to the equal occupation of two levels for any value of γ. This conclusion becomes invalid in the case of fast quantum noise, which was analyzed in paper [103]. Strong quantum noise brings a two-level system into equilibrium with noise or to a steady state if the noise is not thermal. The interaction of transverse and longitudinal noise leads to a change in the transition matrix element Δ. If the noise is due to phonons, this renormalization results in the isotopic effect.

The Landau–Zener transitions in degenerate atomic Fermi gas with the creation of diatomic molecules have been described in papers [104–106]. In the first of them, a perturbation theory for the calculation of the number of created molecules has been developed up to the second order in the parameter $\delta = g^2 n/\left(\hbar^2|\dot{\Omega}|\right)$ (g denotes the atom–molecule coupling constant, and n is the gas density). It was shown that Fermi repulsion reduces the probability of molecule production in comparison with the same process in a gas of independent atomic pairs. The consideration in paper [105] is given beyond the perturbation theory at the expense of a significant simplification of the model. A more general analysis [106] confirms the reduction of molecule production due to Fermi repulsion beyond the perturbation theory.

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