# Infrared-modified gravities and massive gravitons

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<u>Abstract.</u> We review some theoretical and phenomenological aspects of massive gravities in 4 dimensions. We start from the Fierz – Pauli theory with Lorentz-invariant mass terms and then proceed to Lorentz-violating masses. Unlike the former theory, some models with Lorentz violation have no pathologies in the spectrum in flat and nearly flat backgrounds and lead to an interesting phenomenology.

# 1. Introduction

Recently, there has been a revival of interest in attempts to construct models of gravity that deviate from general relativity at ultra-large distance and time scales, that is, models with infrared-modified gravity. The general approach is to view these models as possible low-energy limits of an unknown fundamental theory, and at the exploratory stage not to worry too much about issues like renormalizability, embedding into an ultraviolet-complete

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theory, etc. However, self-consistency problems do occur in low-energy theories, which severely limits the classes of acceptable models. But acceptable models can nevertheless be found, and their phenomenology turns out to be rather rich.

Besides pure curiosity, there were several original motivations for the recent increase in this activity. One of them is related to the cosmological constant problem and the observational evidence for the accelerated expansion of the Universe in the present epoch (for reviews from the theoretical standpoint, see, e.g., Refs [1–7]). This accelerated expansion may well be due to the cosmological constant (the vacuum energy density), a new weakly interacting field, or some other kind of dark energy, which, according to Ref. [8], contributes about 75% to the total energy density  $\rho_c$  in the present Universe. The problem is that the value  $\epsilon_A$  of the dark energy density is very small,

 $\epsilon_{\Lambda} \simeq 0.75 \rho_{\rm c} \simeq 4 \times 10^{-6} \ {\rm GeV} \ {\rm cm}^{-3} \, . \label{eq:electropy}$ 

This is many orders of magnitude smaller than the values that can be associated, on dimensional grounds, with the fundamental interactions—strong, electroweak, and gravitational:

$$\epsilon_A \sim 10^{-46} \epsilon_{\rm QCD} \sim 10^{-54} \epsilon_{\rm EW} \sim 10^{-123} \epsilon_{\rm grav}$$
.

In other words, the energy scale  $M_A = \epsilon_A^{1/4} \sim 10^{-3} \text{ eV}$  characteristic of dark energy<sup>1</sup> is much smaller than the energy scales of the known interactions,  $\Lambda_{\rm QCD} \simeq 200$  MeV,  $M_{\rm W} \simeq 80$  GeV, and  $M_{\rm Pl} \simeq 10^{19}$  GeV. The unnatural smallness of  $\epsilon_A$  (or  $M_A$ ) is precisely the cosmological constant problem.

In fact, there are two parts to this problem. One is that the contributions from the strong (quantum chromodynamics),

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electroweak, and gravitational sectors to the vacuum energy density should be of the order of  $\epsilon_{\rm QCD} \simeq \Lambda_{\rm QCD}^4$ ,  $\epsilon_{\rm EW} \sim M_{\rm W}^4$ , and  $\epsilon_{\rm grav} \sim M_{\rm Pl}^{4.2}$  Thus, the first part of the cosmological constant problem is to explain why  $\epsilon_A$  is almost zero. The second part is to understand why  $\epsilon_A$  is in fact nonzero, and what physics is behind the energy scale  $M_A$ .

Evidently, the first part of the dark energy problem may be solved by one mechanism or another that drives the cosmological constant to zero (for a review, see, e.g., Ref. [9]); such a mechanism would most probably operate at a cosmological epoch that preceded any known stage of cosmological evolution, but at which the state of the Universe was quite similar to the present one [10, 11].<sup>3</sup>

On the other hand, despite numerous attempts, no compelling idea has been put forward of how the value of  $M_A$  may be related to other known fundamental energy scales. One possible standpoint is that  $\epsilon_A$  is actually the cosmological constant (a time-independent quantity during the known history of our part of the Universe) and that its value is determined anthropically (for reviews, see, e.g., Refs [1, 12]): much larger values of  $|\epsilon_A|$  would be inconsistent with our existence. This standpoint implies that the Universe is much larger than its visible part and that  $\epsilon_A$  takes different values in different cosmologically large regions; we happen to have measured a small value of  $\epsilon_A$  merely because there is nobody in other places to measure (larger values of) the cosmological 'constant.'

Another option is that the accelerated expansion of the Universe is due to new low-energy (infrared) physics. Perhaps the best known examples are quintessence models (for reviews, see, e.g., Refs [2-6]), in which gravity is described by general relativity and the accelerated expansion is driven by (dark) energy of a new super-weakly interacting field (conventionally, but not necessarily, this field is a Lorentz scalar). The original idea of infrared-modified gravity is that, instead, gravitational laws are modified at cosmological distance and time scales, hopefully leading to the accelerated expansion without dark energy at all. This would certainly be an interesting alternative to dark energy, which might even be observationally testable.

Another original motivation for infrared-modified gravities came from theories with brane-worlds and extra dimensions of large or infinite size (for a review, see, e.g., Ref. [13]). In these theories, the ordinary matter is localized on a threedimensional hypersurface (brane) embedded in higherdimensional space. The idea [14-16] is that gravitons may propagate along 'our' brane for a finite (albeit long) time, after which they escape into extra dimensions. This would modify the brane-to-brane graviton propagator at large distances and time intervals, thus changing the gravitational interactions between particles on 'our' brane. If successful, models with this property would provide concrete and calculable examples of the infrared modification of gravity. Again, this idea is very hard to implement in a self-consistent way, and the models constructed so far have their intrinsic problems. In this regard, it is worth mentioning that there are claims of an exception: it has been argued [17] that the Dvali–Gabadadze–Porrati 'brane-induced gravity' model [18] (see Ref. [19] for a review) may be fully self-consistent, although at a first glance this model becomes strongly coupled at unacceptably large distances [20-22]. Interestingly, the DGP model has a self-accelerating branch of cosmological solutions [23-25], which, however, has phenomenologically unacceptable ghosts among perturbations of these solutions [20, 26].

Among other lines of thought, we mention theories attempting to incorporate MOND (modified Newtonian dynamics) [27-31], which modify gravity for explaining rotation curves of galaxies without dark matter, and RTG (relativistic theory of gravity) [32, 33], motivated by the desire to restore the full generality of energy and momentum conservation laws. It remains to be seen whether these theories can be made fully self-consistent and phenomenologically acceptable.

Recently, the massive graviton has been motivated from quite a different perspective [34]. Namely, there is a fairly widespread expectation that quantum chromodynamics (QCD) may be formulated in terms of a string theory of some sort. The known string theories, however, often have a massless spin-2 state in the spectrum, while QCD does not. The argument is that it is desirable to remove this state from the massless sector of string theory by giving it a mass. In terms of the effective four-dimensional low-energy theory, this task appears very similar to giving a mass to the graviton.

It is natural to expect that the infrared modification of gravity may be associated with the modification of the dispersion law  $\omega = \omega(\mathbf{p})$  of metric perturbations at low spatial momenta **p**, the simplest option being the graviton mass. In this review, we mostly discuss this type of theories, and stay in 4 dimensions. But we emphasize that theories of this type by no means exhaust all possible classes of theories with infrared-modified gravity. Other classes include scalartensor theories, in which the modification of gravity occurs due to the presence of extra field(s) (scalars), over and beyond the space-time metric, that are relevant in the infrared domain. There are examples of models belonging to this class that not only are phenomenologically acceptable but also lead to interesting cosmological dynamics, including the accelerated expansion of the Universe [35-37]. Another class of models involves condensates of vector and/or tensor fields (see, e.g., Refs [38, 39] and the references therein). The discussion of these and similar models is beyond the scope of this review.

As it often happens, irrespectively of the original motivations, theoretical developments lead to new insights. In the case of infrared-modified gravity and a modified graviton dispersion law, these are insights into self-consistency issues, on the one hand, and phenomenological implications, on the other. The reason behind self-consistency problems is the lack of manifest invariance under general coordinate transformations (or a nontrivial realization of these transformations). Indeed, unless extra fields are added to the gravitational sector of the theory, straightforward implementation of the requirement of this gauge symmetry leads in a unique way to general relativity (with the cosmological constant) plus possible higher-order terms irrelevant in the infrared domain. Once this gauge symmetry is broken, explicitly or spontaneously, gravity is infrared-modified, but new light degrees of freedom may appear among metric perturbations

<sup>&</sup>lt;sup>2</sup> For example, the QCD vacuum has a complex structure: there are quark and gluon condensates whose values are determined by highly complicated (and largely unknown) dynamics and depend on the QCD parameters ( $\Lambda_{\rm QGD}$  and quark masses) in a complicated way. The difference between the energy densities of this vacuum and the naive, perturbative one is certainly of the order of  $\epsilon_{\rm QCD}$ , and hence there is no reason whatsoever for the energy density of the physical vacuum to be 46 orders of magnitude smaller than  $\epsilon_{\rm QCD}$ .

<sup>&</sup>lt;sup>3</sup> A name suggested by G Ross for this scenario is 'deja vu Universe.'

in addition to spin-2 gravitons. These new degrees of freedom may be ghosts or tachyons, which is often unacceptable. Another dangerous possibility is that they may be strongly interacting at energy scales above a certain 'ultraviolet' scale  $\Lambda_{\rm UV}$ . This would mean that the theory of gravity becomes inapplicable at energies above  $\Lambda_{\rm UV}$ . If  $\Lambda_{\rm UV}$  is too low and if the new degrees of freedom do not effectively decouple, the theory cannot be considered phenomenologically acceptable. We see in what follows that problems of this sort are quite inherent in theories with manifest Lorentz invariance in the Minkowski background.

In four-dimensional models with infrared-modified gravity, avoiding the self-consistency problems is relatively easy if the Lorentz invariance is broken for excitations about a flat background. The main emphasis of this review is on models of this type [38-42]. Breaking of the Lorentz invariance is in fact quite natural in this context. Indeed, infrared modification of gravity may be considered an analog of the broken (Higgs) phase in gauge theories, gravity in a certain sense being the gauge theory of the Lorentz group. Gravity in the Higgs phase is thus naturally expected to be Lorentz violating. We discuss various aspects of theories of this type, including selfconsistency, technical naturalness, and phenomenology. The last point is quite interesting in a number of cases, because intuition gained in Lorentz-invariant field theories is often misleading when Lorentz invariance does not hold. At the end of this review, we return to the issue of the accelerated expansion of the Universe.

# 2. The Fierz – Pauli model

To better understand the problems arising in attempting to modify the gravitational interaction at large distances, it is instructive to first consider the *Lorentz-invariant* massive gravity. The Lorentz-invariant graviton mass term was proposed by Fierz and Pauli [43]; we refer to the corresponding model as the Fierz-Pauli model.

The theories belonging to the class we discuss in this section can be considered in various ways. A graviton mass term can simply be added to the Einstein-Hilbert action, as we do in Section 2.1. Equivalently, general relativity coupled to extra massless fields can be considered (see, e.g., Refs [44, 45] and the references therein). Once these fields acquire background values that depend on space-time coordinates, general covariance is broken, and the graviton acquires a mass in a manner reminiscent of the Higgs mechanism. Each of these approaches yields the same class of theories if the following requirements are satisfied: (i) Minkowski space-time is a legitimate background, i.e., the flat metric solves the field equations;<sup>4</sup> (ii) in the Minkowski background, there are no light fields except for metric perturbations; (iii) Lorentz invariance is unbroken in this background.

The issues arising in modified gravity theories are as follows: ghosts and the related instabilities; the absence of the zero-mass limit (the van Dam-Veltman-Zakharov discontinuity); strong coupling at the parametrically low ultraviolet (UV) energy scale; and the existence of a 'hidden' Boulware-Deser mode that is not seen in the analysis of perturbations about the Minkowski background but becomes propagating—and dangerous—once the background is curved, even slightly. In this section, we discuss these issues using the Fierz–Pauli model as an example.

# 2.1 Lorentz-invariant massive gravity in the Minkowski background

We consider general relativity plus the most general Lorentzinvariant graviton mass term added to the action. We parameterize the nearly Minkowski metric as

$$g_{\mu
u} = \eta_{\mu
u} + h_{\mu
u}$$

We sometimes need the expressions for  $g^{\mu\nu}$  and  $\sqrt{-g}$  up to the second order,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda} h^{\nu}_{\lambda} ,$$
  
$$\sqrt{-g} = 1 + \frac{1}{2} h^{\mu}_{\mu} + \frac{1}{8} h^{\mu}_{\mu} h^{\lambda}_{\lambda} - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} , \qquad (1)$$

where the indices are raised and lowered with the Minkowski metric  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . At the quadratic level about the Minkowski background, the general action for the Lorentz-invariant massive gravity is given by

$$S = M_{\rm Pl}^2 \int d^4x \left\{ L_{\rm EH}^{(2)}(h_{\mu\nu}) + \frac{\alpha}{4} h_{\mu\nu} h^{\mu\nu} + \frac{\beta}{4} (h_{\mu}^{\mu})^2 \right\},$$
(2)

where  $M_{\rm Pl}^2 = 1/(16\pi G)$ ,  $\alpha$  and  $\beta$  are arbitrary coefficients of dimension [mass squared], and  $L_{\rm EH}^{(2)}$  is the standard graviton kinetic term coming from the Einstein–Hilbert action. This term can be written as

$$L_{\rm EH}^{(2)} = \frac{1}{4} \left( \partial_{\lambda} h^{\mu\nu} \partial^{\lambda} h_{\mu\nu} - 2 \partial_{\mu} h^{\mu\nu} \partial_{\lambda} h^{\lambda}_{\nu} \right. \\ \left. + 2 \partial_{\mu} h^{\mu\nu} \partial_{\nu} h^{\lambda}_{\lambda} - \partial_{\mu} h^{\nu}_{\nu} \partial^{\mu} h^{\lambda}_{\lambda} \right).$$

$$(3)$$

When discussing metric perturbations, we use the convention that the Lagrangian and the action are related by

$$S = M_{\rm Pl}^2 \int \mathrm{d}^4 x L \,. \tag{4}$$

This simplifies the formulas; many of them do not then include  $M_{\rm Pl}$ .

In what follows, it is convenient to use both the above Lorentz-covariant form of the Lagrangian and the form corresponding to the (3+1) decomposition. The metric perturbations in the latter formalism are traditionally parameterized as follows [51]:

$$h_{00} = 2\varphi,$$
  

$$h_{0i} = S_i + \partial_i B,$$
  

$$h_{ij} = h_{ij}^{\text{TT}} - \partial_i F_j - \partial_j F_i - 2(\psi \delta_{ij} - \partial_i \partial_j E).$$
(5)

Here,  $h_{ii}^{\text{TT}}$  is a transverse traceless 3-tensor,

$$\hat{\mathbf{o}}_i h_{ij}^{\mathrm{TT}} = \mathbf{0} \,, \quad h_{ii}^{\mathrm{TT}} = \mathbf{0}$$

 $S_i$  and  $F_i$  are transverse 3-vectors,

$$\partial_i S_i = \partial_i F_i = 0$$

and the other variables are 3-scalars; hereafter, summation over spatial indices i, j = 1, 2, 3 is performed with the

 $<sup>^4</sup>$  It is worth noting that massive gravities with legitimate backgrounds other than the Minkowski one may have properties quite different from those exposed in this section. A well-studied example is massive gravity in the (anti-) de Sitter background [46–50].

Euclidean metric. Accordingly, the quadratic part of the Einstein-Hilbert term decomposes into tensor, vector, and scalar parts,

$$L_{\rm EH}^{(2)} = L_{\rm EH}^{\rm (T)} + L_{\rm EH}^{\rm (V)} + L_{\rm EH}^{\rm (S)} , \qquad (6)$$

where

$$L_{\rm EH}^{\rm (T)} = \frac{1}{4} \left( \partial_0 h_{ij}^{\rm TT} \, \partial_0 h_{ij}^{\rm TT} - \partial_k h_{ij}^{\rm TT} \, \partial_k h_{ij}^{\rm TT} \right), \tag{7}$$

$$L_{\rm EH}^{\rm (V)} = \frac{1}{2} \,\partial_k (S_i + \partial_0 F_i) \,\partial_k (S_i + \partial_0 F_i) \,, \tag{8}$$

$$L_{\rm EH}^{\rm (S)} = 2 \left[ \widehat{\partial}_k \psi \widehat{\partial}_k \psi - 3 \widehat{\partial}_0 \psi \widehat{\partial}_0 \psi + 2 \widehat{\partial}_k (\varphi - \widehat{\partial}_0 B + \widehat{\partial}_0^2 E) \widehat{\partial}_k \psi \right].$$

(9)

Likewise, the mass terms decompose as

$$L_{\rm m} = \frac{\alpha}{4} h_{ij}^{\rm (TT)} h_{ij}^{\rm (TT)} \tag{10}$$

$$+\frac{\alpha}{2}\left(\partial_{i}F_{j}\partial_{i}F_{j}-S_{i}S_{i}\right)$$
(11)

$$+ \left[ (\alpha + \beta) \varphi^{2} + 2\beta (3\psi - \Delta E) \varphi + (\alpha + \beta) (\Delta E)^{2} - 2(\alpha + 3\beta) \psi \Delta E + 3(\alpha + 3\beta) \psi^{2} + \frac{\alpha}{2} B\Delta B \right].$$
(12)

Hereafter, Lagrangians differing by a total derivative are not distinguished.

In general relativity, most of the fields entering the Lagrangian  $L_{\rm EH}$  do not propagate: the only propagating degrees of freedom are conveniently parameterized by  $h_{ij}^{\rm TT}$  and are transverse traceless gravitational waves. This feature is of course a consequence of the gauge invariance of general relativity, with gauge transformations, at the linearized level about the Minkowski background, having the form

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_{\mu}\zeta_{\nu}(x) + \partial_{\nu}\zeta_{\mu}(x),$$
 (13)

where  $\zeta_{\mu}(x)$  are arbitrary functions of coordinates.

Once the mass terms are added, gauge invariance is lost, and extra propagating degrees of freedom emerge. Indeed, the massless spin-2 graviton is to be expected to become a massive spin-2 particle. The graviton has two polarization states, but a massive particle has five, with helicities  $\pm 2$ ,  $\pm 1$ , and 0. In the (3 + 1)-language, these respectively correspond to two tensor modes, two vector modes, and one scalar mode. For general  $\alpha$ and  $\beta$ , however, there is one more scalar mode, which is necessarily a ghost. We consider this point, within the (3 + 1)formalism first.

In the tensor sector, the  $h_{ij}^{TT}$  remain two propagating degrees of freedom, whose mass is now given by

$$m_{\rm G}^2 = -\alpha \,. \tag{14}$$

To avoid tachyons, we consider the case  $\alpha < 0$  in what follows.<sup>5</sup> Because the tensor modes have healthy kinetic term (7), this sector is not problematic.

The vector sector contains a nondynamic field  $S_i$  that enters the action without time derivatives.

<sup>5</sup> We see shortly that  $\alpha > 0$  leads to an even more serious problem in the vector sector of the model.

We pause at this point to discuss, in general terms, two kinds of nondynamic fields. At the level of quadratic action, a nondynamic field may enter the action either linearly or quadratically. An example of the latter situation is given by the vector sector of massive gravity: there is a term  $S_i^2$  in Lagrangian (11), and also a term  $(\partial_k S_i)^2$  in (8). In this case, the nondynamic field can be integrated out: the field equation obtained by varying this field can be used to express this field through dynamic fields (those entering the action with time derivatives), and then nondynamic fields are eliminated by substituting the resulting expression into the action. The number of dynamic fields is not reduced in this way in general (there are important exceptions in gauge-invariant theories, which we encounter a number of times in this review).

Another possibility is that the action does not contain a term quadratic in a nondynamic field. This is the case, for instance, in the scalar sector of general relativity, whose action (9) is linear in the field  $\varphi$  (and *B*, which is also a nondynamic field, because it enters without time derivatives after integration by parts). Unlike in the quadratic case, the corresponding field equation is a constraint imposed on the dynamic fields, and the nondynamic field itself is a Lagrange multiplier. An important feature here is that the constraint reduces the number of dynamic fields, i.e., the number of the degrees of freedom.

This discussion is straightforwardly generalized to the case of several nondynamic fields: if the part of the action that is quadratic in these fields is nondegenerate, all these fields belong to the first category; otherwise, there are Lagrange multipliers, whose number equals the degree of degeneracy. After this general remark, we return to the vector sector and integrate over the field  $S_i$ . Its field equation is

$$\left(\Delta - m_{\rm G}^2\right)S_i = -\Delta \partial_0 F_i\,,$$

where  $\Delta$  is the 3-dimensional Laplacian. Expressing  $S_i$  through  $F_i$  from this equation and substituting it in the action, we obtain the action for the remaining field  $F_i$ . In a massless theory, this action is identically zero, and hence  $F_i$  does not have to satisfy any equation and is therefore arbitrary. This arbitrariness is of course a consequence of gauge freedom (13), in this case, with a transverse  $\zeta_i$ . With the mass terms added, the field  $F_i$  is dynamical. The Lagrangian for  $F_i$ , in the 3-dimensional momentum representation, is<sup>6</sup>

$$L_F = \frac{m_G^2}{2} \left[ \frac{\mathbf{p}^2}{\mathbf{p}^2 + m_G^2} \,\partial_0 F_i^*(\mathbf{p}) \,\partial_0 F_i(\mathbf{p}) - \mathbf{p}^2 F_i^*(\mathbf{p}) \,F_i(\mathbf{p}) \right].$$
(15)

To convert it into the standard form, we introduce the canonically normalized field

$$\mathcal{F}_i(\mathbf{p}) = M_{\rm Pl} \, m_{\rm G} \sqrt{\frac{\mathbf{p}^2}{\mathbf{p}^2 + m_{\rm G}^2}} F_i(\mathbf{p}) \tag{16}$$

and find that the linearized action is

$$S_{\mathcal{F}} = \int \mathrm{d}^3 p \, \mathrm{d}t \, \frac{1}{2} \left[ \partial_0 \mathcal{F}_i^* \partial_0 \mathcal{F}_i - (\mathbf{p}^2 + m_G^2) \, \mathcal{F}_i^* \mathcal{F}_i \right]$$

<sup>6</sup> Hereafter, **p** denotes a 3-momentum, while p is reserved for a 4-momentum.

Hence, the vector sector has two propagating degrees of freedom of mass  $m_G$  (we recall that the  $F_i$  are transverse, and hence only two components of  $F_i$  are independent). The number of propagating degrees of freedom in the tensor and vector sectors corresponds to the number of states of the massive graviton with the respective helicities  $\pm 2$  and  $\pm 1$ , in accordance with expectations.

We note here that for  $m_G^2 \equiv -\alpha < 0$ , Lagrangian (12) has a negative overall sign, and hence vector modes are ghosts in that case. This is even worse than the tachyon behavior of tensor modes.

We return to the theory with  $m_G^2 \equiv -\alpha > 0$ . It follows from (16) that the limit  $m_G \rightarrow 0$  is singular. Fluctuations in the canonically normalized field  $\mathcal{F}_i$  are finite in this limit, and therefore fluctuations in the vector part of the metric (the field  $F_i$ ) diverge as  $m_G^{-1}$ . At small but finite  $m_G$ , this implies that the quantum theory becomes strongly coupled at an ultraviolet energy scale  $\Lambda_{\rm UV}$  that is much lower than  $M_{\rm Pl}$ . In the vector sector, this scale is not unacceptably low, however. We discuss the strong coupling later, because it becomes a more serious problem not in the vector but in the scalar sector.

We now pass to the scalar sector and begin with the general case

$$\alpha \neq 0$$
,  $\alpha \neq -\beta$ ,  $\alpha \neq -2\beta$ .

After integrating by parts, we obtain the following form of the Lagrangian, including mass terms:

$$L^{(S)} = 2\left[-2\varphi\Delta\psi - 2\dot{\psi}\Delta B + 2\dot{\psi}\Delta\dot{E} - 3\dot{\psi}^{2} - \psi\Delta\psi + \frac{\alpha+\beta}{2}\varphi^{2} + \beta(3\psi-\Delta E)\varphi + \frac{\alpha+\beta}{2}(\Delta E)^{2} - (\alpha+3\beta)\psi\Delta E + 3\frac{\alpha+3\beta}{2}\psi^{2} + \frac{\alpha}{4}B\Delta B\right].$$
 (17)

In the case of general relativity ( $\alpha = \beta = 0$ ), the fields  $\varphi$  and *B* are Lagrange multipliers, resulting in the same constraint  $\psi = 0$ . Then the equation of motion obtained by varying  $\psi$  gives  $\varphi = \dot{B} - \ddot{E}$ ; varying *E* gives nothing new. There are no propagating degrees of freedom, and the fields *B* and *E* remain arbitrary. This is again due to gauge freedom (13), now with  $\zeta_0 \neq 0$  and  $\zeta_i = \partial_i \zeta_L$ .

In the massive case, the fields  $\varphi$  and *B* are no longer Lagrange multipliers, but are still nondynamical and can be integrated out. Integrating over *B*, we obtain an additional term in the Lagrangian,

$$L_B = -\frac{8}{\alpha} \dot{\psi} \Delta \dot{\psi} \,, \tag{18}$$

and integrating over  $\varphi$  gives another additional term

$$L_{\varphi} = -\frac{1}{\alpha + \beta} \left[ 2\Delta \psi - \beta (3\psi - \Delta E) \right]^2.$$
<sup>(19)</sup>

Then the Lagrangian for the remaining fields  $\psi$  and E becomes

$$L^{(S)} = L_B + L_{\varphi} + 2\left[2\dot{\psi}\Delta\dot{E} - 3\dot{\psi}^2 - \psi\Delta\psi + \frac{\alpha + \beta}{2}\left(\Delta E\right)^2 - (\alpha + 3\beta)\psi\Delta E + 3\frac{\alpha + 3\beta}{2}\psi^2\right].$$
 (20)

Both fields  $\psi$  and *E* are dynamical, and hence there are two propagating degrees of freedom in the scalar sector. Thus, there is an extra scalar mode in addition to the expected helicity-0 state of the massive graviton. This degree of freedom is actually a ghost (it has a negative sign of the kinetic term).

To see this, we concentrate on the terms with time derivatives. These come from the terms explicitly shown in (20) and from term (18). Therefore, for a given spatial momentum, the relevant part of the Lagrangian has the form

$$L_{\rm kin} = \frac{A}{2} \dot{\psi}^2 + B \dot{\psi} \dot{E}$$
$$= \frac{A}{2} \left( \dot{\psi} + \frac{B}{A} \dot{E} \right)^2 - \frac{B^2}{2A} \dot{E}^2 , \qquad (21)$$

where A and B are numerical coefficients (depending on the spatial momentum). It follows that irrespectively of the sign of A, one of the two degrees of freedom is a ghost. Of course, the theory is Lorentz invariant, and hence both modes have  $\omega^2 = \mathbf{p}^2$  at high spatial momenta, and the ghost exists at arbitrarily high spatial momenta  $\mathbf{p}$ .

It is clear from (19) that the case  $\alpha = -\beta$  is special.<sup>7</sup> This is precisely the Fierz–Pauli theory, where

$$\beta = -\alpha = m_{\rm G}^2$$
.

In this case, there is no quadratic term in  $\varphi$ , and therefore  $\varphi$  is a Lagrange multiplier. The corresponding constraint is

$$\Delta E = 3\psi - \frac{2}{\beta} \,\Delta\psi \,. \tag{22}$$

This constraint eliminates one degree of freedom out of two, and hence the only mode in the scalar sector is the helicity-0 state of a massive graviton with the normal (positive) sign of the kinetic term. Indeed, by inserting (22) in action<sup>8</sup> (17) and adding term (18), we find that the only remaining degree of freedom is  $\psi$ , and the Lagrangian has the kinetic term

$$L_{\mathrm{kin},\psi} = 6\dot{\psi}^2$$
.

In fact, the complete quadratic Lagrangian for  $\psi$  is

$$L_{\psi \,\mathrm{FP}} = 6 \left( \widehat{\partial}_{\mu} \psi \widehat{\partial}^{\mu} \psi - m_{\mathrm{G}}^{2} \psi^{2} \right), \qquad (23)$$

in full analogy, for example, with the Lagrangian for tensor modes  $h_{ii}^{TT}$ .

We finish this discussion with the following comment. Of course, in the case of the Minkowski background and Lorentz-invariant mass terms, the analysis is most straightforwardly performed in a Lorentz-covariant way. The (3 + 1)formalism used here certainly looks like an unnecessary

<sup>&</sup>lt;sup>7</sup> The cases  $\alpha = 0$  and  $\alpha = -2\beta$  are also special. For  $\alpha = -2\beta$ , the ghost has the same mass as the graviton, and because of this degeneracy, its wave function increases in time, i.e., behaves as  $t \exp(i\omega t)$ . For  $\alpha = 0$ , graviton mass (14) is zero, and it can be verified that the only degrees of freedom of the linearized theory are transverse traceless massless gravitons. Therefore, the theory with  $\alpha = 0$  cannot be considered a massive gravity, and we do not discuss this case further.

<sup>&</sup>lt;sup>8</sup> The question may arise as to whether this procedure is legitimate, since one of the field equations is apparently lost. The case is that in the original formulation, this would-be-lost equation is an equation determining the Lagrange multiplier  $\varphi$  in terms of  $\psi$ .

complication. Our analysis, however, does provide useful insights. First, it suggests that the problems of massive gravity are most severe in the scalar sector. Second, it shows that the Fierz-Pauli miracle occurs due to the special property of the field  $\varphi$ , which acts as the Lagrange multiplier and eliminates the undesirable degree of freedom. We see in Section 2.6 that this property is lost in curved backgrounds, and the extra degree of freedom—the Boulware-Deser mode—reappears in the spectrum. Finally, our discussion suggests a possibility that Lorentz-violating mass terms may give rise to a healthier theory, if they are chosen such that the unwanted degree of freedom is consistently eliminated. We discuss this possibility in Sections 3-5.

# 2.2 A ghost via the Stückelberg trick

A convenient way to isolate and study dangerous degrees of freedom is to use the Stückelberg formalism [44]. The idea is to enlarge the field content of massive gravity in such a way that the gauge invariance is restored, and then make a judicious choice of the gauge fixing condition. We use this trick in various sections of this review, and here we illustrate the Stückelberg approach by rederiving the ghost in the spectrum about the Minkowski background.

We again consider theory (2) with general mass terms. In the linearized theory, we introduce a new, Stückelberg field  $\xi_{\mu}$ and a new field  $\bar{h}_{\mu\nu}$  by writing

$$h_{\mu\nu} = h_{\mu\nu} + \partial_{\mu}\xi_{\nu}(x) + \partial_{\nu}\xi_{\mu}(x).$$
(24)

Then the linearized theory is invariant under the gauge transformations

$$\begin{split} \bar{h}_{\mu\nu}(x) &\to \bar{h}_{\mu\nu}(x) + \eth_{\mu}\zeta_{\nu}(x) + \eth_{\nu}\zeta_{\mu}(x) \ , \\ \\ \xi_{\mu} &\to \xi_{\mu} - \zeta_{\mu} \ . \end{split}$$

Importantly, because of the gauge invariance of general relativity, the Einstein – Hilbert part of the quadratic action, Eqn (3), is independent of  $\xi_{\mu}$ :

$$L_{\rm EH}^{(2)} = L_{\rm EH}^{(2)}(\bar{h}_{\mu\nu})$$

We note that we did not introduce new degrees of freedom: by imposing the gauge condition  $\xi_{\mu} = 0$ , we return to the original massive gravity. The trick is to impose a gauge condition on  $\bar{h}_{\mu\nu}$  instead, such that all independent components of  $\bar{h}_{\mu\nu}$ obtain nontrivial kinetic terms from the Einstein-Hilbert Lagrangian. This guarantees that the fields  $\bar{h}_{\mu\nu}$  and  $\xi_{\mu}$ decouple at high energies (for  $\alpha \neq -\beta$ ) and the properties of the dangerous modes can be derived from the Lagrangian involving the fields  $\xi_{\mu}$  only. We further comment on this procedure in what follows [see Eqns (33) and (34)].

We are interested in relatively high energies and spatial momenta,  $\omega^2$ ,  $\mathbf{p}^2 \ge |\alpha|$ ,  $|\beta|$ , and therefore keep the terms in the action that are of the highest order in derivatives. Because of the structure of (24), these terms come not only from the Einstein-Hilbert part of the action but also from the mass terms. This is a peculiarity of the Stückelberg formalism.

The choice of the gauge condition for  $\bar{h}_{\mu\nu}$  is not very important. The gauge  $\bar{h}_{00} = 0$ ,  $\bar{h}_{0i} = 0$ , of covariant gauges can be used.<sup>9</sup> In any case, the remaining components of  $\bar{h}_{\mu\nu}$ have nondegenerate terms with two time derivatives, which come from the Einstein–Hilbert action. For example, in the gauge  $\bar{h}_{00} = 0$ ,  $\bar{h}_{0i} = 0$ , i.e.,  $\bar{\varphi} = 0$ ,  $\bar{S}_i = 0$ , and  $\bar{B} = 0$ , the fields  $\bar{F}_i$ ,  $\bar{\psi}$ , and  $\bar{E}$  have nondegenerate terms with two time derivatives [see (8) and (9)]. We sometimes write, schematically,

$$L_{\rm EH}^{(2)} = (\partial \bar{h})^2$$

Kinetic terms for the field  $\xi_{\mu}$  come from the mass terms in action (2). They are

$$\frac{\alpha}{2} \left( \hat{\partial}_{\mu} \xi_{\nu} \right)^{2} + \left( \frac{\alpha}{2} + \beta \right) \left( \hat{\partial}_{\mu} \xi^{\mu} \right)^{2}.$$
(25)

The mass terms also induce a mixing between  $\xi_{\mu}$  and  $\bar{h}_{\mu\nu}$ , but as we discuss shortly, this mixing is unimportant for  $\alpha \neq -\beta$ at high momenta and frequencies,  $\mathbf{p}^2$ ,  $\omega^2 \ge |\alpha|, |\beta|$ . Once this mixing is neglected, the fields  $\bar{h}_{\mu\nu}$  and  $\xi_{\mu}$  decouple, as promised, and we can study the metric and Stückelberg sectors,  $\bar{h}_{\mu\nu}$  and  $\xi^{\mu}$ , independently.

In the metric sector, the kinetic part of the Lagrangian is just the gauge-fixed Einstein – Hilbert Lagrangian. Therefore, the only propagating modes in this sector are the  $\bar{h}_{ij}^{TT}$ . The other propagating modes belong to the Stückelberg sector. Once the mixing between  $\bar{h}_{\mu\nu}$  and  $\xi^{\mu}$  is taken into account, the propagating modes have nonvanishing contributions proportional to  $\bar{h}_{\mu\nu}$ , but this effect is small and can be neglected. To see explicitly how this works in the gauge  $\bar{h}_{00} = 0$ ,  $\bar{h}_{0i} = 0$ , we consider the vector sector in the (3+1)-decomposition language as an example. The full Lagrangian in this sector is

$$L^{(\mathbf{V})} = \frac{1}{2} \left( \partial_i \partial_0 \bar{F}_i \right)^2 + \frac{\alpha}{2} \left( \partial_i \bar{F}_j \right)^2 - \alpha \partial_i \bar{F}_j \partial_i \xi_j^{\mathrm{T}} - \frac{\alpha}{2} \left( \partial_\mu \xi_i^{\mathrm{T}} \right)^2,$$
(26)

where we set  $S_i = 0$  in accordance with our gauge choice and  $\xi_i^{T}$  is 3-dimensionally transverse,  $\partial_i \xi_i^{T} = 0$ . The first term in (26) is the Einstein–Hilbert term (8), the last term comes from (25), and the third term is precisely the one that mixes the metric and the Stückelberg field. The field equations are

$$\bar{F}_i + \alpha \xi_i^{\mathrm{T}} - \alpha \bar{F}_i = 0, \qquad (27)$$

$$\Box \xi_i^{\mathrm{T}} + \Delta \bar{F}_i = 0, \qquad (28)$$

were  $\Box$  is the standard d'Alembertian. For  $\omega^2$ ,  $\mathbf{p}^2 \gg m_G^2 \equiv -\alpha$ , these equations may be solved perturbatively in the small parameter  $\alpha$ . In the zeroth order, the first of these equations has no oscillating solutions, and hence  $F_i = 0$ , and the only propagating modes are  $\xi_i^T$ , as expected. These are helicity-1 states of the massive graviton in the Stückelberg picture. In the first order, it follows from (27) that

$$\bar{F}_i = \frac{\alpha}{2\omega^2} \, \xi_i^{\,\mathrm{T}} \,,$$

and therefore Eqn (28) becomes

$$\left(\omega^2 - \mathbf{p}^2\right)\xi_i^{\mathrm{T}} + \frac{\alpha \mathbf{p}^2}{\omega^2}\,\xi_i^{\mathrm{T}} = 0\,.$$

As promised, the second term here, which appears due to the mixing between  $\bar{F}_i$  and  $\xi_i^{T}$ , is negligible for  $\omega^2$ ,  $\mathbf{p}^2 \ge m_G^2$ .

The lesson from this exercise is twofold. First, it shows that neglecting the metric sector  $\bar{h}_{\mu\nu}$  is indeed legitimate

<sup>&</sup>lt;sup>9</sup> Unlike noncovariant gauges, covariant ones do not fix the gauge completely. There remain unphysical modes to worry about.

(except for helicity-2 states) as long as modes with  $\omega^2$ ,  $\mathbf{p}^2 \ge m_G^2$  are considered. Second, we see that the Stückelberg formalism is useless for studying modes with  $\omega^2 \le m_G^2$ ; restricting to only the Stückelberg sector may even result in losing some modes with  $\omega^2 \le m_G^2$ . Indeed, there is no guarantee that system of equations like (27), (28) does not have more slowly oscillating solutions than the equation  $\Box \xi_i^T = 0$ ; after all, Eqns (27) and (28) are both second order in time. In the theory considered in this section, the number of modes with high and low  $\omega$  is the same due to Lorentz invariance, but the last remark should be kept in mind when studying Lorentz-violating massive gravities.

We return to modes with  $\omega^2$ ,  $\mathbf{p}^2 \ge m_G^2$  and consider the Stückelberg sector. Expression (25) can be considered the most general Lorentz-invariant Lagrangian for the vector field  $\xi_{\mu}$ . It is well known that this Lagrangian has a ghost in the spectrum unless the two terms combine into the field strength tensor  $F_{\mu\nu}^2 = (\partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu})^2$ . Thus, we again see that the no-ghost situation is possible in the Fierz–Pauli case only.

Ghosts are unacceptable in a Lorentz-invariant theory. Therefore, we concentrate on the Fierz–Pauli theory in the rest of this section.

We show how the Stückelberg analysis works in the Fierz-Pauli case  $\beta = -\alpha = m_G^2$ . The relevant part of the mass term is then given by

$$L_{\rm m} = -\frac{m_{\rm G}^2}{2} \left( \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \right) \left( \partial^\mu \xi^\nu - \partial^\nu \xi^\mu \right) - m_{\rm G}^2 \left( \partial_\nu \xi^\mu \bar{h}_\mu^\nu - \partial_\mu \xi^\mu \bar{h}_\nu^\nu \right), \qquad (29)$$

where we omitted terms without derivatives but kept the kinetic mixing between  $\bar{h}_{\mu\nu}$  and  $\xi^{\mu}$ . The 4-dimensionally transverse part of  $\xi^{\mu}$ , which obeys  $\partial_{\mu}\xi^{\mu}_{tr} = 0$ , has a healthy kinetic term given by the first line in (29). On the other hand, the longitudinal part

$$\xi_{\mu} = \frac{1}{2} \, \partial_{\mu} \phi$$

has the kinetic term only due to mixing with the field  $\bar{h}_{\mu\nu}$ ; this is why the mixing term, which is subdominant for  $\alpha \neq -\beta$ , plays a key role now. Temporarily, we do not impose the gauge condition on  $\bar{h}_{\mu\nu}$ . Then the kinetic term for  $\bar{h}_{\mu\nu}$  and  $\phi$  is given by

$$L_{\rm EH}^{(2)}(\bar{h}_{\mu\nu}) - \frac{m_{\rm G}^2}{2} \left( \partial_{\mu} \partial_{\nu} \bar{h}^{\mu\nu} - \partial_{\mu} \partial^{\mu} \bar{h}^{\nu}_{\nu} \right) \phi \,. \tag{30}$$

It can be diagonalized [44] by noticing that the combination  $\partial_{\mu}\partial_{\nu}\bar{h}^{\mu\nu} - \partial_{\mu}\partial^{\mu}\bar{h}^{\nu}_{\nu}$  is proportional to the linearized Riemann curvature, and hence the second term in (30) has the structure  $m_G^2 R(\bar{h}_{\mu\nu}) \phi$ . Therefore, the kinetic term is diagonalized by a conformal transformation, which at the linearized level is given by

$$\bar{h}_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{m_{\rm G}^2}{2} \eta_{\mu\nu} \phi \,. \tag{31}$$

Then the kinetic term becomes

$$L_{\rm kin} = L_{\rm EH}^{(2)}(\hat{h}_{\mu\nu}) + \frac{3}{8} m_{\rm G}^4 \partial_{\mu} \phi \partial^{\mu} \phi \,. \tag{32}$$

Upon gauge fixing of  $h_{\mu\nu}$ , the longitudinal sector of the theory contains one degree of freedom  $\phi$  with a healthy kinetic term. In this way, we recover the absence of ghosts in the Fierz–Pauli theory in the Minkowski background.

We note that for general  $\alpha \neq -\beta$ , term (32) is subdominant compared to the term  $(\alpha + \beta)(\Box \phi)^2$  that arises from (26). Thus, mixing between the scalar parts of  $\bar{h}_{\mu\nu}$  and  $\xi^{\mu}$  is unimportant in this case, just as in the vector sector.

A general comment is in order. With gauge conditions imposed on  $\bar{h}_{\mu\nu}$ , the Stückelberg procedure may result in the occurrence of spurious solutions of the field equations. For example, in the gauge  $\bar{h}_{00} = 0$ ,  $\bar{h}_{0i} = 0$ , we have  $h_{00} = 2\partial_0\xi_0$ and  $h_{0i} = \partial_0\xi_i + \partial_i\xi_0$ . Varying the action of the original theory with respect to  $h_{\mu\nu}$ , we obtain the field equations

$$\frac{\delta S}{\delta h_{00}} = 0 , \quad \frac{\delta S}{\delta h_{0i}} = 0 , \quad \frac{\delta S}{\delta h_{ij}} = 0 .$$
(33)

The first two of these equations do not contain second time derivatives; these are constraints. On the other hand, substituting  $h_{00} = 2\partial_0\xi_0$  and  $h_{0i} = \partial_0\xi_i + \partial_i\xi_0$  in the action and then varying with respect to  $\xi_{\mu}$  and  $h_{ij}$ , we find

$$\partial_0 \left( \frac{\delta S}{\delta h_{00}} \right) + \partial_i \left( \frac{\delta S}{\delta h_{0i}} \right) = 0, \quad \partial_0 \left( \frac{\delta S}{\delta h_{0i}} \right) = 0, \quad \frac{\delta S}{\delta h_{ij}} = 0.$$
(34)

There are no constraints any longer; instead, all these equations are second order in time. Hence, system (34) has more solutions than (33). However, we are interested in propagating modes, i.e., solutions of the linearized field equations that have the form  $\exp(i\omega t - i\mathbf{p}\mathbf{x})$ . In this case, the left-hand sides of (33) oscillate unless they are identically zero, and hence the left-hand sides of (34) cannot vanish unless (33) are satisfied. System (34) has the same number of propagating modes as the original system (33). Also, energies and momenta of the solutions are the same in the original and Stückelberg formalisms: if a propagating mode is a ghost in the Stückelberg formalism, it is a ghost in the original theory. Indeed, in general there is a unique energy-momentum tensor (modulo terms that do not contribute to the total energy or momentum) that is conserved on solutions of the field equations. This last observation is also valid at the nonlinear level, and for adiabatically varying backgrounds.

#### 2.3 Van Dam-Veltman-Zakharov discontinuity

The Fierz-Pauli mass term changes the gravitational interaction both between two massive bodies and between a massive body and light. This interaction can be straightforwardly calculated in the weak-field approximation [52, 53]. The result is surprising: the prediction of light bending in the massive case is different from general relativity even in the limit of zero graviton mass. This is known as the van Dam – Veltman–Zakharov (vDVZ) discontinuity: the linearized Fierz–Pauli theory does not approach linearized general relativity as  $m_G \rightarrow 0$ . Taken at face value, this result would mean that the Fierz–Pauli gravity is ruled out, because the experimental measurement of light bending agrees with the general relativity (see, e.g., [54] and the references therein).

We consider this phenomenon in more detail. At the linearized level, the interaction between two sources of a gravitational field is given by

$$GT^{\mu\nu}P_{\mu\nu\lambda\rho}T^{\prime\lambda\rho}$$
,

where G is the gravitational coupling constant,  $P_{\mu\nu\lambda\rho}$  is the propagator of the gravitational field, and  $T^{\mu\nu}$  and  $T^{\prime\lambda\rho}$  are the energy-momentum tensors of the two sources. The point is

that the propagators are different in the massive and massless cases. Their structure in both cases is

$$P_{\mu
u\lambda
ho} \propto rac{\sum_i e^i_{\mu
u} e^i_{\lambda
ho}}{p^2 - m_{
m G}^2} \, ,$$

where  $e^{i}_{\mu\nu}$  are the graviton polarization tensors; in the massless case, the mass in the denominator is zero. Because the denominator is continuous in the zero-mass limit, the sum over the polarizations is responsible for the discontinuity.

In the massive case, there are 5 polarization tensors. The summation over these tensors gives

FP: 
$$P_{\mu\nu\lambda\rho} = \frac{1}{p^2 - m_G^2} \left\{ \frac{1}{2} \eta_{\mu\lambda} \eta_{\nu\rho} + \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\lambda} - \frac{1}{3} \eta_{\mu\nu} \eta_{\lambda\rho} + (p\text{-dependent terms}) \right\},$$
(35)

where the terms containing  $p_{\mu}$  are irrelevant because they do not contribute when contracted with the conserved energy– momentum tensors. In the massless case, there are only two polarizations. The propagator in this case takes the form

$$GR: P_{\mu\nu\lambda\rho} = \frac{1}{p^2} \left\{ \frac{1}{2} \eta_{\mu\lambda} \eta_{\nu\rho} + \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\lambda} - \frac{1}{2} \eta_{\mu\nu} \eta_{\lambda\rho} + (p\text{-dependent terms}) \right\}.$$
(36)

The difference between these two expressions is in the coefficient of the third term. This difference persists in the limit of zero mass; this is precisely the vDVZ discontinuity. We also note that the difference is in the part of the propagator that is coupled to the trace of the energy – momentum tensor.<sup>10</sup>

It is worth noting that the vDVZ discontinuity is specific to a spin-2 field. In the case of a vector field, the zero mass limit of the propagator coincides, modulo the longitudinal piece, with the propagator of the massless field, and hence the vDVZ discontinuity is absent.

In general, the coupling constants  $G_{GR}$  and  $G_{FP}$  in the massless and massive cases are different. The relation between them can be found by requiring that two nonrelativistic bodies interact with the same strength in the massive and massless theories. In the nonrelativistic limit, only the 00-component of the energy–momentum tensor contributes, and therefore we have

$$GR: G_{GR} T_{\mu\nu} P_{\mu\nu\lambda\rho} T'_{\lambda\rho} = \frac{1}{2} G_{GR} T_{00} T'_{00} \frac{1}{p^2},$$
  
FP:  $G_{FP} T_{\mu\nu} \tilde{P}_{\mu\nu\lambda\rho} T'_{\lambda\rho} = \frac{2}{3} G_{FP} T_{00} T'_{00} \frac{1}{p^2 - m_G^2}.$ 

<sup>10</sup> A question may arise whether the vDVZ discontinuity can be bypassed by abandoning the weak equivalence principle, i.e., by modifying the way in which gravity couples to matter. Indeed, in massive gravity, the consistency of the field equations does not require the covariant conservation of the source tensor (unlike in general relativity, where the gravitational part of the field equations — the Einstein tensor — obeys the Bianchi identity, and therefore the matter part — the energy – momentum tensor — must obey the covariant conservation law). The above analysis becomes inapplicable if instead of the coupling to the conserved energy – momentum tensor, the field  $h_{\mu\nu}$  couples to some tensor  $S^{\mu\nu}$  whose divergence is nonzero at finite  $m_G$  and vanishes in the massless limit only. This question has been studied in Ref. [55]; the result is that the vDVZ discontinuity cannot be bypassed in this way. In the zero-mass limit, this implies

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$$G_{\rm FP} = \frac{3}{4} G_{\rm GR} \equiv \frac{3}{4} G_{\rm Newton} . \tag{37}$$

We next consider the prediction for light bending in each case. The energy-momentum tensor of an electromagnetic wave is traceless. Hence, the third term in the propagator does not contribute, and we find the following expressions for the interaction strength:

$$GR: G_{GR} T_{00} T_{00}' \frac{1}{p^2},$$
  
FP:  $G_{FP} T_{00} T_{00}' \frac{1}{p^2 - m_G^2}.$ 

In view of Eqn (37), the light bending predicted in the massive theory in the limit of the vanishing graviton mass is 3/4 times that in the massless theory, general relativity.

Clearly, the discontinuity is related to the longitudinal polarizations of the graviton, i.e., to the Stückelberg field  $\xi_{\mu}$  discussed in Section 2.2. In what follows, we shed more light on the mechanism responsible for this phenomenon.

# 2.4 The Vainshtein radius

As we already noted, if the arguments in the previous section were strictly correct, they would imply that the graviton mass is exactly zero in the Lorentz-invariant theory. But these arguments have a loophole [56], because they rely on the linear approximation. In general relativity, this approximation is valid for distances much larger than the Schwarzschild radius of the source. Therefore, the gravitational bending of light that passes next to the surface of the Sun is described well in the linear regime.

The situation is different in the Fierz-Pauli gravity. It was argued in Ref. [56], by studying spherically symmetric classical solutions, that with a nonzero graviton mass, the linear approximation actually breaks down already at a distance much longer than the Schwarzschild radius, namely, at the distance called the Vainshtein radius,

$$r_{\rm V} = \left(\frac{M}{M_{\rm Pl}^2 m_{\rm G}^4}\right)^{1/5},\tag{38}$$

where *M* is the mass of the source. We note that the smaller the graviton mass, the larger is the distance where the nonlinear regime sets in. Taking the graviton mass to be of the order of the present Hubble parameter yields  $r_V \simeq 100$  kpc for the Sun, and therefore bodies orbiting the Sun, as well as light passing not far from the solar surface, feel the nonlinear gravitational interaction. The above argument for the incorrect bending of light in the massive theory is therefore not directly applicable. On the other hand, the nonlinearity of the Fierz-Pauli gravity in the entire solar system is a problem by itself.

The origin of the scale  $r_V$  is easy to understand by simple power counting [44]. We first recall how the Schwarzschild radius  $r_S = 2M/M_{Pl}^2$  appears as the expansion parameter in general relativity. Schematically, quadratic Einstein – Hilbert action (3) with a source term can be written as

$$\int d^4x \left[ M_{\rm Pl}^2 (\partial h)^2 + Th \right], \tag{39}$$

where h is the metric perturbation and T is the energymomentum tensor. The corresponding equations have the solution<sup>11</sup>

$$h = \frac{1}{\partial^2} \frac{T}{M_{\rm Pl}^2}$$

or, equivalently,

$$h = \frac{M}{M_{\rm Pl}^2 r} \,, \tag{40}$$

where *M* is the total mass of the source. This is the standard form of Newton's law. The nonlinear corrections to the action begin with terms of the type  $M_{Pl}^2 \int d^4x h(\partial h)^2$ . Requiring that these terms be small compared to quadratic contributions (39) yields the condition  $h \ll 1$ , i.e.,

$$\frac{M}{M_{\rm Pl}^2 r} \ll 1$$

for perturbation (40). This is precisely the condition  $r \ge r_S$  ensuring the validity of the linear approximation in general relativity.

In the case of the Fierz-Pauli massive gravity, this condition has to be satisfied as well. But there is a stronger constraint. In the Stückelberg language of Section 2.2, this constraint comes from the analysis of the field  $\xi_{\mu}$ . Of particular importance is its scalar part,  $\xi_{\mu} = \partial_{\mu}\phi$ . It follows from the discussion at the end of Section 2.2 that the action for the fields  $\hat{h}_{\mu\nu}$  and  $\phi$  in the presence of a conserved source  $T_{\mu\nu}$  schematically has the form

$$\int d^4x \Big[ M_{\rm Pl}^2 (\hat{\partial}\hat{h})^2 + M_{\rm Pl}^2 m_{\rm G}^4 (\hat{\partial}\phi)^2 + T\hat{h} + m_{\rm G}^2 T\phi + \dots \Big],$$
(41)

where we again omitted numerical coefficients and did not explicitly write the mass terms for  $\hat{h}_{\mu\nu}$  and  $\phi$ . The kinetic term here is the same as in (32), and the source term is obtained from the standard expression  $h^{\mu\nu}T_{\mu\nu}$  using (24) and (31), together with the linearized conservation law  $\partial_{\mu}T^{\mu\nu} = 0$ . We note that after the diagonalization of the kinetic term via (31), matter becomes directly coupled to the field  $\phi$ . Solving the equations of motion, we find that at distances much shorter than  $m_{\rm G}^{-1}$ , the gravitational potential  $\hat{h}$  is given by Eqn (40) and that  $m_{\rm G}^2\phi$  is of the same order,

$$m_{\rm G}^2\phi = \frac{M}{M_{\rm Pl}^2r} \,.$$

This formula implies that  $\phi$  itself is large and singular in the limit  $m_{\rm G} \rightarrow 0$  [cf. Section 2.1, Eqn (16)]. This is the origin of the nonlinearity at long distances from the source.

Indeed, a nonlinear generalization of the Fierz–Pauli mass term would contain higher powers of the perturbation  $h_{\mu\nu}$ . The lowest term of this type is just  $h^3$ . This term gives rise to the nonlinear contribution to the action of the form

$$\int d^4x \, M_{\rm Pl}^2 m_{\rm G}^2 (\hat{\sigma}^2 \phi)^3 \,. \tag{42}$$

Another source of terms of the same order is the nonlinearity of gauge transformations in general relativity. In general, a coordinate transformation  $x^{\mu} \rightarrow x^{\mu} + \zeta^{\mu}$  corresponds to the following gauge transformation of the metric:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = g_{\mu\nu}(x+\zeta) + \partial_{\mu}\zeta^{\lambda}g_{\nu\lambda}(x+\zeta) + \partial_{\nu}\zeta^{\lambda}g_{\mu\lambda}(x+\zeta) + \partial_{\mu}\zeta^{\lambda}\partial_{\nu}\zeta^{\rho}g_{\lambda\rho}(x+\zeta).$$
(43)

With  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , it follows in the quadratic order (in both  $h_{\mu\nu}$  and  $\zeta^{\mu}$ ) that

$$egin{aligned} &h'_{\mu
u} = h_{\mu
u} + \partial_{\mu}\zeta_{
u} + \partial_{
u}\zeta^{\lambda} \; \partial_{
u}\zeta_{\lambda} + \partial_{\mu}\zeta^{\lambda} \; h_{
u\lambda} + \partial_{
u}\zeta^{\lambda}h_{\mu\lambda} \,, \end{aligned}$$

where the indices are still raised and lowered with the Minkowski metric. Accordingly, in this order, the change of variables in (24) has the form

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} + \partial_{\mu}\xi^{\lambda}\partial_{\nu}\xi_{\lambda} + \partial_{\mu}\xi^{\lambda}\bar{h}_{\nu\lambda} + \partial_{\nu}\xi^{\lambda}\bar{h}_{\mu\lambda}.$$
(44)

The field  $\xi^{\mu}$  still does not enter the Einstein–Hilbert part of the action, while the mass term receives the contribution whose schematic form is

$$\int \mathrm{d}^4 x M_{\rm Pl}^2 \, m_{\rm G}^2 (\partial \xi)^3 \,, \tag{45}$$

that is, the contribution of form (42) with  $\xi_{\mu} = \partial_{\mu}\phi$ .

The linearized theory is valid when contribution (42) is smaller than the quadratic term. This requirement leads to the condition

$$\frac{M}{M_{\rm Pl}^2 m_{\rm G}^4 r^5} \ll 1$$

which is equivalent to  $r \ll r_V$  with  $r_V$  given by Eqn (38).

The situation may be improved by tuning the explicit  $h^3$  terms in the nonlinearly generalized Fierz – Pauli action such that the leading correction  $(\partial^2 \phi)^3$  be absent. In this way, the onset of the nonlinear regime may be pushed to smaller scales, namely, to the distance

$$r_* = \left(\frac{M}{M_{\rm Pl}^2 m_{\rm G}^2}\right)^{1/3}.$$
 (46)

It can be shown that the situation cannot be improved further [44]. For a graviton mass of the order of the present Hubble parameter,<sup>12</sup> the nonlinear regime occurs at distances below  $r_* \sim 10$  pc from the Sun, which still covers the whole Solar System.

We discuss how the analysis presented here is related to the study of the vDVZ discontinuity. It is clear from (41) that the gravitational field  $\bar{h}_{\mu\nu}$  coupled to matter is a mixture of the two fields  $\hat{h}_{\mu\nu}$  and  $m_G^2 \eta_{\mu\nu} \phi$ . The field  $\hat{h}_{\mu\nu}$  has the same kinetic term  $L_{\rm EH}(\hat{h}_{\mu\nu})$  as the linearized gravitational field in general relativity, while  $m_G^2 \phi$  has the kinetic term of a gravi-scalar. Both fields interact with matter at roughly the same strength. In the massless limit, keeping only the part  $\hat{h}_{\mu\nu}$ , yields the

<sup>&</sup>lt;sup>11</sup> Throughout this review, when presenting power-counting arguments, we ignore numerical factors and signs.

<sup>&</sup>lt;sup>12</sup> Hereafter, we assume that greater values of  $m_{\rm G}$  would be inconsistent with cosmology. Even without this assumption, the requirement that Newton's law remains valid at the scale of galaxy clusters would give  $m_{\rm G} \lesssim 10 \ {\rm Mpc}^{-1}$ . The estimates here and in the rest of this section would change by about two orders of magnitude, with no change of the conclusions.

To summarize, distances below which massive gravity is in the nonlinear regime are not less than given by (46) and hence are very large. One may hope that the nonlinear interactions would modify the theory so as to make the massless limit smooth [56]. This indeed happens in some cases, an example being the DGP model [22], where the nonlinear interactions mainly affect the gravi-scalar sector and essentially decouple it from other modes in the small-mass limit, eliminating the extra contribution to the propagator responsible for the vDVZ discontinuity. This mechanism, however, does not work in the case of the Fierz-Pauli theory [57]. Thus, the Fierz-Pauli gravity in the Minkowski background is already problematic at the classical level: it most likely contradicts precision tests of general relativity. It becomes even more problematic at the quantum level, as we discuss below.

#### 2.5 Strong coupling

At the quantum level, the above nonlinearity problem manifests itself as a strong coupling at the energy scale that is much lower than the naive expectation.

Both massless and massive gravities should be treated as low-energy effective theories valid at energies (more precisely, momentum transfers) below some ultraviolet (UV) scale  $\Lambda_{UV}$ . Above this scale, these theories are meant to be extended to some 'fundamental' theories (UV completions) with better UV behavior. The situation here is analogous to the theory of the massive self-interacting vector field, whose possible UV completion is a non-Abelian gauge theory with the Higgs mechanism. In the case of general relativity, the UV completion is most likely the string/M-theory; whether there exists a UV completion of massive gravity is not known (in fact, the discussion in this section suggests that the Lorentz-invariant massive gravity does not have a UV completion at all).

The effective theories can be trusted only at distances  $r \ge \Lambda_{UV}^{-1}$ ; at shorter distances, the putative UV-complete theory has to be considered, which most likely has quite different properties. Experimentally, Newtonian gravity has been tested down to submillimeter distances [58 61], and hence an effective low-energy theory must be valid down to those distances. This implies

$$\Lambda_{\rm UV}^{-1} \lesssim 10^{-2} \text{ cm}, \text{ or } \Lambda_{\rm UV} \gtrsim 10^{-3} \text{ eV}.$$

On the other hand, an upper limit on the UV energy scale  $\Lambda_{UV}$  can be obtained within the low-energy effective theory itself. The tree-level scattering amplitudes calculated in the low-energy theory grow with the center-of-mass energy E and eventually, at some energy E, become large and hit the unitarity limit. Above this energy  $E \sim \Lambda$ , the effective theory becomes strongly coupled, and hence cannot be trusted. The entire framework makes sense if the UV-completion scale is below the strong-coupling scale,

$$\Lambda_{\rm UV} \lesssim \Lambda \,. \tag{47}$$

The well-known illustration is again a non-Abelian gauge theory with the Higgs mechanism. The strong-coupling scale of its effective low-energy theory — the theory of the massive self-interacting vector field — is

$$\Lambda = \frac{m_{\rm V}}{g} \,, \tag{48}$$

where  $m_V$  is the vector boson mass and g is the gauge coupling. This low-energy theory is extended to its UV completion—the gauge theory in the Higgs phase—at the scale  $\Lambda_{\rm UV} = m_{\rm H}$ , where  $m_{\rm H}$  is the Higgs boson mass: at that scale, new degrees of freedom, the Higgs bosons, show up. Because  $\Lambda = m_{\rm V}/g = v$ , where v is the Higgs vacuum expectation value, and  $m_{\rm H} = \sqrt{\lambda}v$ , where  $\lambda < 1$  is the Higgs self-coupling, inequality (47) indeed holds in this example.

In general relativity, the strong-coupling scale is  $\Lambda = M_{\rm Pl}$ , and therefore its UV completion may occur well above accessible energies.<sup>13</sup> In massive gravity, the strong-coupling scale  $\Lambda$ , and hence the UV-completion scale  $\Lambda_{\rm UV}$ , are certainly much below  $M_{\rm Pl}$ . Naively, the strong-coupling scale would be estimated as

1.10

$$A \sim (M_{\rm Pl} m_{\rm G})^{1/2}$$
 (49)

This is a direct analog of (48). Indeed, we consider the transverse component of the Stückelberg field  $\xi^{\mu}$ , which obeys  $\partial_{\mu}\xi^{\mu}_{tr} = 0$ . The kinetic term in the Lagrangian for this component comes from the Fierz–Pauli mass term and schematically has the form

$$L_{\rm tr}^{(2)} = M_{\rm Pl}^2 \, m_{\rm G}^2 (\partial \xi_{\rm tr})^2 \,. \tag{50}$$

The terms that are cubic in  $h_{\mu\nu}$  and come from a nonlinear generalization of the Fierz–Pauli term, as well as from the nonlinear change of variables in (44), give rise to the interaction terms schematically written in (45). Had the form (50) been common to the kinetic terms for both transverse and longitudinal components of  $\xi^{\mu}$ , we would have introduced the canonically normalized field  $\chi^{\mu} = M_{\rm Pl}m_{\rm G}\xi^{\mu}$  and found that it enters the kinetic term with the unit coefficient, while the interaction Lagrangian is

$$L_{\rm int} = \frac{1}{M_{\rm Pl}m_{\rm G}} \left(\partial\chi\right)^3. \tag{51}$$

This theory would indeed have the strong coupling scale (49), because it is this parameter that suppresses higher-order operator (51). The analysis of other higher-order operators,  $(\partial\xi)^4$ , etc., would lead to the same conclusion. The same argument applied to theories of self-interacting vector fields leads to estimate (48), hence the analogy between (48) and (49).

Scale (49) is actually quite interesting phenomenologically. For the graviton mass of the order of the present Hubble parameter, the corresponding distance is

$$(M_{\rm Pl} m_{\rm G})^{-1/2} \simeq 0.05 \,\,{\rm mm}\,.$$
 (52)

If this were the true scale of the UV completion, we would expect novel phenomena in the gravitational sector at submillimeter distances. In the Fierz-Pauli theory, however,

<sup>&</sup>lt;sup>13</sup> This is not a necessity: the inequality in (47) may be strong. In this regard, an interesting possibility is offered by TeV-scale gravities, where  $\Lambda_{\rm UV}$  is of the order of a few TeV, and new phenomena occur at collider energies. This possibility is reviewed, e.g., in [13, 62–64].

the strong-coupling scale is actually much lower than estimate (49), and the corresponding distance is much longer than (52).

The problem occurs in the longitudinal sector, where  $\xi_{\mu} = \partial_{\mu}\phi$ . We recall Eqns (41) and (42). According to (41), the canonically normalized field is  $\chi = M_{\rm Pl} m_{\rm G}^2 \phi$ , and (42) implies that its self-interaction has the form

$$L_{\rm int} = \frac{1}{M_{\rm Pl}m_{\rm G}^4} \left(\partial^2 \chi\right)^3$$

The scale that suppresses this higher-order operator is now

$$\Lambda = (M_{\rm Pl} m_{\rm G}^4)^{1/5} \,. \tag{53}$$

This is the actual strong-coupling scale in the general Fierz– Pauli theory. For a graviton mass of the order of the present Hubble parameter, this scale is of the order  $10^{-21}$  eV  $\sim (10^{16} \text{ cm})^{-1}$ , which is clearly too low.

In the perturbative framework, the origin of this strong coupling is the growth of the propagator and the wave functions of the longitudinal components of a massive graviton with energy, much like the case of a massive non-Abelian vector field. We consider the four-graviton scattering amplitude represented by the diagrams in Fig. 1. The external lines of the diagrams for longitudinal gravitons behave as  $E^2/m_{\rm G}^2$ . The 4-vertex gives the factor  $E^2/M_{\rm Pl}^2$ . Therefore, the first diagram gives the contribution of the order of

$$\frac{E^{10}}{M_{\rm Pl}^2 m_{\rm G}^8}$$

The second diagram is of the same order because two leading contributions to the propagator cancel in the on-shell amplitude [65]. Thus, the scattering amplitude indeed becomes large at energies of the order of (53). This has been checked by an explicit calculation of the amplitude [65].

The strong-coupling scale can be pushed to higher energies by a judicious choice of the interaction terms. Indeed, a nonlinear extension of the Fierz-Pauli theory can be chosen such that the cubic terms  $(\partial^2 \phi)^3$  vanish. Then the fourth-order terms are

$$L_{\rm int} = M_{\rm Pl}^2 m_{\rm G}^2 (\partial^2 \phi)^4 = \frac{1}{M_{\rm Pl}^2 m_{\rm G}^6} (\partial^2 \chi)^4,$$

and the strong-coupling scale is

$$\Lambda = (M_{\rm Pl} m_{\rm G}^2)^{1/3} \,. \tag{54}$$

This is the best that can achieved [44], because there are not only self-interactions of the field  $\phi$  (these can be



Figure 1. Four-graviton scattering in the first two orders of the perturbation theory (diagrams *a* and *b*, respectively). Only the *a*-channel diagram is shown in the second order, panel *b*.

canceled by an appropriate choice of higher-order terms in  $h_{\mu\nu}$ ) but also interactions between the longitudinal component of the Stückelberg field,  $\xi_{\mu} = \partial_{\mu}\phi$ , and the transverse component  $\xi_{tr}^{\mu}$ .

For the graviton mass  $m_{\rm G}$  of the order of the present Hubble parameter, it follows from (54) that  $\Lambda \sim 3 \times 10^{-13} \text{ eV} \sim (10^8 \text{ cm})^{-1}$ . This is also unacceptably low.<sup>14</sup> We conclude that the Fierz–Pauli theory suffers from a severe strong-coupling problem.

# 2.6 The Fierz-Pauli theory in curved backgrounds: The Boulware-Deser mode

**2.6.1 The cosmological background.** If the background is not exactly Minkowskian, the Fierz–Pauli cancellation no longer works, and the ghost or tachyon mode reappears in the spectrum. This mode exists for arbitrarily high spatial momenta, and hence it is unacceptable phenomenologically. This phenomenon is known as the Boulware–Deser instability [66]. Importantly, it occurs irrespectively of the way the Fierz–Pauli theory is generalized to a curved space–time [57].

To see the appearance of the Boulware–Deser mode explicitly, we consider an example of the cosmological background. As noted above, the scalar sector is the most problematic; indeed, as we find shortly, the Boulware–Deser mode emerges precisely in this sector. We therefore concentrate on the scalar sector in what follows.

We begin with general relativity. Let the metric with perturbations have the form

$$ds^{2} = a^{2}(\eta)(\eta_{\mu\nu} + h_{\mu\nu}) dx^{\mu} dx^{\nu}, \qquad (55)$$

where  $\eta$  is the conformal time. We note that we have changed the definition of  $h_{\mu\nu}$  here; in Sections 2.1–2.5,  $h_{\mu\nu}$  denoted the deviation of  $g_{\mu\nu}$  from the background metric, while here this deviation is equal to  $a^2h_{\mu\nu}$ . In what follows, we raise and lower indices by the Minkowski metric, and hence  $h_{\mu}^{\nu} = \eta^{\nu\lambda}h_{\mu\lambda}$  by definition. Another convention is that summation over spatial indices is performed using  $\delta_{ij}$ , and we never use covariant derivatives in explicit formulas. Hence, the dependence on the scale factor is always explicit.

Linearized gauge transformations (13) are generalized using (43). According to our conventions, we define  $\zeta_{\mu} = \eta_{\mu\nu} \zeta^{\nu}$  and write the gauge transformations in this background as follows:

— spatial,  $\zeta_i = -\partial_i \zeta_L$ :

$$\delta B = \zeta_{\rm L}', \quad \delta E = \zeta_{\rm L} \,, \tag{56}$$

— temporal,  $\zeta_0$ :

$$\delta B = \zeta_0 \,, \quad \delta \varphi = \zeta'_0 + \mathcal{H}\zeta_0 \,, \quad \delta \psi = \mathcal{H}\zeta_0 \,, \tag{57}$$

where we use notation (5) and specify the scalar sector. Hereafter, the prime denotes  $\partial/\partial \eta$  and

$$\mathcal{H} = \frac{a'}{a} \, .$$

To consider an expanding universe, we introduce a positive cosmological constant, the corresponding term in

<sup>&</sup>lt;sup>14</sup> This discussion refers to a flat background. It may in principle happen that effects due to curvature push the strong-coupling scale to higher values, as occurs, for instance, in the DGP model [22]. This does not happen in the Fierz-Pauli case [57].

the action being

$$S_A = -6H_0^2 M_{\rm Pl}^2 \int \mathrm{d}x \sqrt{-g} \,.$$

Here, the constant  $H_0^2$  is, by virtue of the Einstein equations, the Hubble parameter of the de Sitter space in the theory without a graviton mass and without matter. The background equations are

$$\mathcal{H}^2 = H_0^2 a^2 , 2\mathcal{H}' + \mathcal{H}^2 = 3H_0^2 a^2 ,$$
 (58)

and their solution, the de Sitter space-time scale factor, is

$$a = -\frac{1}{H_0\eta}$$

The quadratic part of the cosmological constant term is

$$S_{A}^{(2)} = 2H_{0}^{2}M_{\text{Pl}}^{2} \int d^{3}x \, d\eta \, a^{4} \left[\frac{3}{2} \, \varphi^{2} - 9\varphi\psi + 3\varphi\Delta E -\frac{9}{2} \, \psi^{2} + 3\psi\Delta E + \frac{3}{2} \, (\Delta E)^{2} - \frac{3}{2} \, (\partial_{i}B)^{2}\right],$$
(59)

where  $\Delta = \partial_i \partial_i$ .

By appearance, the part of the action in (59) resembles the graviton mass term, because it contains no derivatives of the fields. This term does not have the Fierz – Pauli structure, and both the  $\varphi^2$  term and  $(\partial_i B)^2$  term are present. Hence, unlike in the Minkowski background, none of these fields is a Lagrange multiplier, and it may seem that the theory has two dynamic scalars,  $\psi$  and *E*. This is not the case in general relativity: the two modes remaining after integrating over  $\varphi$  and *B* are puregauge modes.

Indeed, we consider the Einstein-Hilbert and cosmological terms together. Off shell (that is, for an arbitrary background), the quadratic part is

$$S_{\text{EH}+A}^{(2)} = 2M_{\text{Pl}}^{2} \int d^{3}x \, d\eta \, a^{2} \bigg\{ -2\varphi \Delta \psi - 2\psi' \Delta B + 2\psi' \Delta E' + 3\psi \psi'' - \psi \Delta \psi + \mathcal{H}(2\varphi \Delta B - 2\varphi \Delta E' + 6\phi \psi') + \left( -\frac{9}{2} \mathcal{H}^{2} + \frac{3}{2} H_{0}^{2} a^{2} \right) \varphi^{2} + \left( -\frac{9}{2} \mathcal{H}^{2} - \frac{9}{2} H_{0}^{2} a^{2} \right) \psi^{2} + \left[ (\mathcal{H}^{2} - H_{0}^{2} a^{2}) \left( 9\varphi \psi - 3\varphi \Delta E + \frac{3}{2} (\partial_{i} B)^{2} \right) + (2\mathcal{H}' + \mathcal{H}^{2} - 3H_{0}^{2} a^{2}) \left( -\psi \Delta E - \frac{1}{2} (\Delta E)^{2} \right) \bigg] \bigg\}.$$
(60)

Because of the background equations of motion, the last two lines in this expression vanish, and the action simplifies to

$$S_{\text{EH}+A}^{(2)} = 2M_{\text{Pl}}^2 \int d^4x a^2 \left\{ -2\varphi\Delta\psi - 2\psi'\Delta B + 2\psi'\Delta E' + 3\psi\psi'' - \psi\Delta\psi + \mathcal{H}(2\varphi\Delta B - 2\varphi\Delta E' + 6\varphi\psi') + \left[\mathcal{H}^2(-3\varphi^2 - 9\psi^2)\right] \right\}.$$
(61)

As expected, *B* and  $\varphi$  are nondynamic fields with a nondegenerate quadratic term. Their equations of motion give

$$\varphi = \frac{1}{\mathcal{H}}\psi', \quad B = \frac{1}{\mathcal{H}}\psi + E'.$$
(62)

The miracle is that after these expressions are substituted in action (62), integration by parts yields a vanishing quadratic Lagrangian,  $L_{\text{EH}+A}^{(2)}(\psi, E) = 0$ , where, again, the equations for background were used. Therefore,  $\psi$  and E are arbitrary functions of  $x^{\mu}$ , while  $\varphi$  and B are related to them via (62). These configurations are pure gauges of form (56) and (57). To verify this, we again use equations for the background (in particular,  $\mathcal{H}' = \mathcal{H}^2$ ).

This miracle of course happens because of gauge invariance. Once gauge invariance is broken explicitly by the graviton mass terms, miracles do not happen, and the Boulware–Deser mode appears.

We introduce the mass term generalized to a curved space-time,

$$S_{\mathrm{m}}=S_{\mathrm{m}}(g_{\mu\nu},\,\eta_{\mu\nu})\,.$$

There is much arbitrariness at this stage: general covariance is explicitly broken, and  $S_m$  may contain various combinations of  $g_{\mu\nu}\eta^{\mu\nu}$ ,  $g^{\mu\nu}\eta_{\mu\nu}$ ,  $\sqrt{g}$ , etc. The discussion that follows is not sensitive to the particular form of the mass term; it is only assumed to be independent of the derivatives of the metric, to become the Fierz – Pauli term in the Minkowski limit, and, for simplicity, to be proportional to a single mass parameter  $m_G^2$ . To illustrate the general analysis, we use the simplest generalization of the Fierz – Pauli mass term,

$$S_{\rm m} = \frac{1}{2} M_{\rm Pl}^2 \int d^4 x \left\{ -\frac{m_{\rm G}^2}{2} \eta^{\mu\lambda} \eta^{\nu\rho} (g_{\mu\nu} - \eta_{\mu\nu}) \right. \\ \left. \times \left( g_{\lambda\rho} - \eta_{\lambda\rho} \right) + \frac{m_{\rm G}^2}{2} \left[ \eta^{\mu\nu} (g_{\mu\nu} - \eta_{\mu\nu}) \right]^2 \right\}.$$
(63)

We stress that this form is used for illustration purposes only.

There is a coordinate frame where  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ , and we assume that the background space-time is homogeneous and isotropic in this frame. Then the metric has the general Friedman-Robertson-Walker form (plus perturbations)

$$ds^{2} = a^{2}(t) [n^{2}(t) (1 + h_{00}) dt^{2} + 2n(t) h_{0i} dt dx^{j} + (-\delta_{ij} + h_{ij}) dx^{i} dx^{j}].$$

In this frame, the background is characterized by two metric functions, a(t) and n(t). It is still convenient to work with the conformal time  $\eta$ , that is, to perform the change of variables  $d\eta = n(t) dt$ . In other words, we work in the conformal frame where the background metric has form (55). Consistency of the field equations implies an equation relating  $n(\eta)$  and  $a(\eta)$ , which generically has the form n' = f(n, a) a' (see the Appendix). We note, however, that its solution is not unique: at a given instant of time, n and a can be chosen arbitrarily.

Once the mass term is added, the quadratic part of the action in the cosmological background has a very general structure,

$$S_{\rm EH+A+m_{G}}^{(2)} = 2M_{\rm Pl}^{2} \int d^{3}x \, d\eta a^{2} \left\{ -2\varphi \Delta \psi - 2\psi' \Delta B + 2\psi' \Delta E' + 3\psi \psi'' - \psi \Delta \psi + \mathcal{H}(2\varphi \Delta B - 2\varphi \Delta E' + 6\varphi \psi') + \left[ \frac{m_{\varphi}^{2}}{2} \, \varphi^{2} - \frac{m_{B}^{2}}{2} \, (\widehat{o}_{i}B)^{2} + \frac{m_{E}^{2}}{2} \, (\Delta E)^{2} + \frac{m_{\psi}^{2}}{2} \, \psi^{2} + \mu_{1}\varphi \psi + \mu_{2}\varphi \Delta E + \mu_{3}\psi \Delta E \right] \right\}.$$

$$(64)$$

The terms in the last two lines have a three-fold origin. First, there are terms that do not vanish in the limit  $m_G^2 \rightarrow 0$ ; these are the terms in the last line of Eqn (64). Second, the background equations no longer coincide with Eqn (61), and hence the last two lines in (58) do not vanish. Finally, there are contributions due to the mass term  $S_m$  itself. We give a more detailed treatment of the last two contributions in the Appendix.

There are generically no specific relations between the terms in the last two lines of (64). The fields  $\varphi$  and *B* are still nondynamical, but integrating over them no longer gives zero action for  $\psi$  and *E*. Instead, the action contains terms with two time derivatives of  $\psi$  and *E*, some of which are explicit in (64) and some emerge after  $\varphi$  and *B* are integrated out (we note that the terms in (64) proportional to  $\varphi$  and *B* do not contain second time derivatives of  $\psi$  and *E*, and hence higher time derivatives of these fields do not appear). Both  $\psi$  and *E* are dynamic fields, and hence the scalar sector has two propagating modes. This is in contrast to the theory in the Minkowski background, with a single propagating mode in the scalar sector. The extra mode is precisely the Boulware – Deser degree of freedom.

We now specify the near-Minkowski background,

$$\mathcal{H}^2 \ll m_{\rm G}^2$$
,  $|a-1| \ll 1$ ,  $|n-1| \ll 1$ .

In this limit,  $\mathcal{H}$  coincides with the standard Hubble parameter. We are interested in relatively high momenta,  $\mathbf{p}^2 \ge m_G^2$ . A detailed analysis reveals the following features. First, the properties of the scalar perturbations are different in the two ranges of momenta:

1) 
$$\mathbf{p}^2 \ll \frac{m_{\rm G}^4}{\mathcal{H}^2}$$
,  
2)  $\mathbf{p}^2 \gg \frac{m_{\rm G}^4}{\mathcal{H}^2}$ . (65)

Hence, the high-momentum limit and the Minkowski limit do not commute. We discuss range 1) in the Appendix and here we briefly summarize the results. There are indeed two propagating modes. One of them is the Fierz-Pauli mode, whose dispersion relation remains  $\omega^2 = \mathbf{p}^2$ , up to small corrections. The second mode is a ghost or tachyon-ghost (tachyon and ghost at the same time). Being a ghost means that the energy is unbounded from below; if the mode is simultaneously a tachyon, it exponentially increases in time.

We now consider range 2), i.e., the high-momentum limit. To integrate over nondynamic fields, we solve equations obtained by varying the action with respect to  $\varphi$ and *B*. These equations are written explicitly in the Appendix [see Eqns (167) and (168)]. We need the expression for  $\varphi$  in the leading order in derivatives and the expression for *B* in both the leading and subleading order; the reason is that there are cancellations. The corresponding expressions are

$$\varphi = \frac{1}{\mathcal{H}} \psi',$$
  
$$B = \frac{1}{\mathcal{H}} \psi + E' - \frac{m_{\varphi}^2}{2\mathcal{H}^2} \frac{1}{\Delta} \psi' - \frac{3}{\Delta} \psi' - \frac{\mu_2}{2\mathcal{H}} E.$$

Substituting these in action (64) and integrating by parts, we arrive at the action for dynamic fields,

$$S_{\rm EH+A+m_G}^{(2)} = 2M_{\rm Pl}^2 \int d^3x \, d\eta a^2 \left\{ -\left[1 - \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{m_B^2}{2\mathcal{H}^2}\right] \partial_i \psi \, \partial_i \psi \right. \\ \left. + \left(3 + \frac{m_{\varphi}^2}{2\mathcal{H}^2}\right) (\psi')^2 + \frac{\mu_2 - m_B^2}{\mathcal{H}} \, \psi' \Delta E \right. \\ \left. - \frac{m_B^2}{2} \, \partial_i E' \, \partial_i E' + \frac{m_E^2}{2} \left(\Delta E\right)^2 \right\}.$$
(66)

We note that the terms with the highest derivatives,  $\psi' \Delta E'$ , have canceled.

We now see explicitly that there are two propagating modes. We also see that their action (66) is singular in the Minkowski limit.<sup>15</sup> Indeed, comparing (64) with the Lagrangian in Minkowski space-time [Eqn (12) with  $\alpha = -\beta = -m_G^2$ ] shows that in the Minkowski limit,

$$m_B^2 \to -\frac{m_G^2}{2} , \quad \mu_2 \to -m_G^2 ,$$
 (67)

and  $m_E^2$  and  $m_{\varphi}^2$  tend to zero. Thus, the first and the third terms in (66) have coefficients that diverge in the Minkowski limit, in which  $\mathcal{H} \to 0$ . Furthermore, for  $\mathcal{H}^2 \ll m_G^2$ , the first term in (66) has an overall positive sign (because of the first relation in (66)), which corresponds to negative energy. This energy is unbounded from below, and hence there is a ghost or a tachyon in the spectrum. We show in the Appendix that in model (67), one of the modes still has the dispersion relation

$$\omega^2 = \mathbf{p}^2$$
,

while the other mode is tachyonic or nontachyonic depending on the relation between a - 1 and n - 1.

**2.6.2 The Stückelberg treatment.** A lesson from the above analysis is that once the gauge invariance is dropped, there are two scalar propagating modes in curved backgrounds, irrespective of how close these backgrounds are to Minkowski space-time. The Minkowski limit is singular, and for a nearly Minkowski space-time, one of these modes is necessarily pathological. Another lesson is that the explicit analysis of this Boulware-Deser mode is rather cumbersome. At the same time, the Boulware-Deser phenomenon is relatively straightforward to see in the Stückelberg formalism [41, 44, 57, 67].

We consider backgrounds that only slightly differ from Minkowski space – time. In this case, the perturbation theory in  $h_{\mu\nu}$  is adequate. The quadratic Lagrangian has been discussed in previous sections. Generically, in the cubic order, we have the following contributions to the Lagrangian:

$$m_{\rm G}^2 \left[ \lambda_1 (h_{\mu}^{\mu})^3 + \lambda_2 h_{\mu}^{\mu} h_{\nu\lambda} h^{\nu\lambda} + \lambda_3 h_{\mu\nu} h_{\lambda}^{\nu} h^{\lambda\mu} \right], \tag{68}$$

with  $\lambda_{1,2,3}$  of the order of unity. For a nontrivial background, the field  $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$  has a nonzero background part  $h_{\mu\nu}^{(c)}$ . To perform the Stückelberg analysis, we change the variables

<sup>&</sup>lt;sup>15</sup> According to the above discussion, this is the limit in which  $\mathbf{p}^2$  is taken to infinity, and only then the background metric tends to the Minkowski metric.

in the way dictated by (43),

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x+\xi) + \partial_{\mu}\xi^{\lambda} \bar{g}_{\nu\lambda}(x+\xi) + \partial_{\nu}\xi^{\lambda} \bar{g}_{\mu\lambda}(x+\xi) + \partial_{\mu}\xi^{\lambda} \partial_{\nu}\xi^{\rho} \bar{g}_{\lambda\rho}(x+\xi) , \qquad (69)$$

where

$$\begin{split} \bar{g}_{\mu\nu}(x+\xi) &= \eta_{\mu\nu} + h_{\mu\nu}^{(c)}(x+\xi) + h_{\mu\nu}(x+\xi) \\ &= \eta_{\mu\nu} + h_{\mu\nu}^{(c)}(x) + \bar{h}_{\mu\nu}(x) \\ &+ \partial_{\lambda} h_{\mu\nu}^{(c)}(x) \, \xi^{\lambda} + \partial_{\lambda} \bar{h}_{\mu\nu}(x) \, \xi^{\lambda} + \dots \, . \end{split}$$

Here,  $\bar{h}_{\mu\nu}$  and  $\xi^{\mu}$  are perturbations, and  $\bar{h}_{\mu\nu}(x)$  is meant to be gauge fixed. As before, the Einstein–Hilbert action does not contain the field  $\xi^{\mu}$ . Concentrating on the longitudinal Stückelberg field  $\xi_{\mu} = \partial_{\mu}\phi$  and inserting decomposition (69) into both the quadratic Fierz–Pauli term and cubic term (68), we obtain the quadratic action for  $\phi$  as

$$m_{\rm G}^2 \left[ \frac{3}{8} m_{\rm G}^2 \partial_\mu \phi \, \partial^\mu \phi + \tilde{\lambda}_1 h_{\mu\nu}^{\rm (c)} \, \partial^\mu \, \partial^\nu \phi \Box \phi + \tilde{\lambda}_2 h_{\mu}^{\rm (c)\,\mu} (\Box \phi)^2 \right],$$
(70)

where we keep Fierz–Pauli contribution (32) that is independent of  $h_{\mu\nu}^{(c)}$ , as well as the part that is proportional to the background  $h_{\mu\nu}^{(c)}$  and has the largest number of derivatives. Omitting other terms is legitimate for studying slowly varying backgrounds and perturbations whose momenta obey  $\omega^2$ ,  $\mathbf{p}^2 \ge m_G^2$ .

One point to note is that any configuration obeying  $\Box \phi = 0$  solves the field equation following from (70). This explains why we have always found a mode with the dispersion relation  $\omega^2 = \mathbf{p}^2$  when studying the theory in the cosmological background. More important is the fact that Lagrangian (70) is of the fourth order in the derivatives of  $\phi$ , and therefore there is a ghost in the spectrum. To see this explicitly and to estimate the mass of the ghost, we consider a simplified version of (70),<sup>16</sup>

$$m_{\rm G}^2 \left[ m_{\rm G}^2 \partial_\mu \phi \partial^\mu \phi + \tilde{\lambda} h^{\rm (c)} (\Box \phi)^2 \right].$$

This Lagrangian is equivalent to

$$\begin{split} m_{\rm G}^2 \left[ m_{\rm G}^2 \partial_\mu \phi \partial^\mu \phi + 2\tilde{\lambda} h^{\rm (c)} \partial_\mu \chi \partial^\mu \phi - \tilde{\lambda} h^{\rm (c)} \chi^2 \right] \\ &= m_{\rm G}^2 \left[ m_{\rm G}^2 \left( \partial_\mu \phi + \frac{\lambda}{m_{\rm G}} h^{\rm (c)} \partial_\mu \chi \right)^2 \right. \\ &- \left( \frac{\tilde{\lambda} h^{\rm (c)}}{m_{\rm G}} \right)^2 (\partial_\mu \chi)^2 - \tilde{\lambda} h^{\rm (c)} \chi^2 \right], \end{split}$$

where  $\chi$  is a new field. The first term in the last expression corresponds to the modified Fierz-Pauli mode  $[\phi + (\lambda/m_G) h^{(c)}\chi]$ , and the second term is the kinetic term for the Boulware-Deser mode  $\chi$ . This term has a negative sign, and hence the Boulware-Deser mode is a ghost (depending on the sign of  $\lambda h^{(c)}$ , it may be a tachyon-ghost at sufficiently low momenta). The local value of its mass squared is of the order of

$$m_{\rm BD}^2 \simeq \frac{m_{\rm G}^2}{h^{\rm (c)}} \,. \tag{71}$$

This explains why the high-momentum limit and the Minkowski limit do not commute, as we have seen explicitly when studying the theory in the cosmological background.

The mass in (71) diverges as the background approaches the Minkowski limit. This property might raise the hope that the Boulware–Deser instability is not very dangerous. If the mass of the ghost turns out to be larger than the UV scale  $\Lambda_{\rm UV}$ , the above analysis becomes unreliable, since an unknown UV completion of the theory must be considered at energies exceeding  $\Lambda_{\rm UV}$ . This observation, however, does not save the Fierz–Pauli theory [57]. Indeed, away from an astrophysical source of mass M, we have

$$h^{(\mathrm{c})} \simeq rac{M}{M_{\mathrm{Pl}}^2 r}$$

and hence

$$m_{\rm BD}^2 \simeq \frac{r}{r_*^3} \,,$$
 (72)

where  $r_*$  is the radius given by (46). Hence, the Boulware– Deser instability definitely occurs in the interval  $(r_*\Lambda_{\rm UV})^2 r_* > r > r_*$ , in which  $m_{\rm BD} < \Lambda_{\rm UV}$  and at the same time the linear approximation is valid. This interval is not empty unless  $\Lambda_{\rm UV}^{-1} \gtrsim r_*$ . We recall that the value of  $r_*$  for the Sun is of the order of 10 pc. Hence, the Fierz–Pauli theory with  $\Lambda_{\rm UV}^{-1} \gtrsim r_*$  is unreliable in the Solar System; if the graviton had a Lorentz-invariant mass, we would either encounter rapid instabilities or have to deal with an unknown UV completion instead of the effective Fierz–Pauli theory.

To conclude, Lorentz-invariant massive gravities in four dimensions are full of pathologies. One way towards bypassing these pathologies is to drop the Lorentz invariance.

# 3. Lorentz-violating theories: generalities

#### 3.1 Lorentz-violating mass terms

In this and the following sections, we study a class of theories with Lorentz-violating mass terms. We assume that Minkowski space – time is a solution of the corresponding field equations and that the Euclidean symmetry of 3-dimensional space is not explicitly broken in the perturbation theory related to this background. Then the quadratic action for perturbations on the Minkowski background is

$$S^{(2)} = S_{\rm EH}^{(2)} + S_{\rm m} \,, \tag{73}$$

where  $S_{\text{EH}}^{(2)}$  is the quadratic part of the Einstein – Hilbert term, explicitly given by (3), and  $S_{\text{m}}$  is the graviton mass term. The Lagrangian of the latter is

$$L_{\rm m} = \frac{1}{4} \left[ m_0^2 h_{00} h_{00} + 2m_1^2 h_{0i} h_{0i} - m_2^2 h_{ij} h_{ij} \right. \\ \left. + m_3^2 h_{ii} h_{jj} - 2m_4^2 h_{00} h_{ii} \right].$$
(74)

Here, as before,  $h_{\mu\nu}$  are perturbations of the Minkowski metric. The Fierz-Pauli Lagrangian is obtained when all

<sup>&</sup>lt;sup>16</sup> The argument below is straightforwardly generalized to the case in (70). Using the spatial Fourier representation, we write the Lagrangian in the form  $A\ddot{\phi}^2 + B\dot{\phi}^2 + C\phi^2$ , where coefficients depend on **p**. Second-order time derivatives are eliminated by introducing a new field  $\chi$ ; the resulting structure of the kinetic terms is  $B\dot{\phi}^2 + 2A\dot{\chi}\dot{\phi}$ . This structure implies that there is a ghost [cf. (21)].

masses in Eqn (74), except  $m_0$ , are taken to be equal,

FP: 
$$m_0^2 = 0$$
,  $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_G^2$ 

This property explains the conventions used in Eqn (74). In what follows, we let *m* denote the overall scale of the masses  $m_0, \ldots, m_4$ .

We again use the (3+1)-decomposition in (5). This formalism is particularly appropriate here because it fully respects the 3-dimensional Euclidean invariance, the only symmetry that is not explicitly broken by the general mass term. The Lagrangian in the tensor sector is the sum of kinetic term (7) and the mass term

$$L_{\rm m,\,T} = -\frac{m_2^2}{4} \, h_{ij}^{\rm TT} \, h_{ij}^{\rm TT}$$

Hence, there are two propagating tensor modes with the relativistic dispersion relation

 $\omega^2 = \mathbf{p}^2 + m_{\rm G}^2 \,,$ 

where

$$m_{\rm G} = m_2 \tag{75}$$

is the mass of tensor gravitons. The requirement that these modes not be tachyonic gives

 $m_2^2 \ge 0$ .

We assume in what follows that this is the case.

In the vector sector, the quadratic Lagrangian is the sum of the Einstein–Hilbert part in (8) and the mass term

$$L_{\rm m, V} = \frac{m_1^2}{2} S_i S_i - \frac{m_2^2}{2} \partial_i F_j \partial_i F_j \,.$$

A novelty here, with respect to the Fierz–Pauli case, occurs at the special value  $m_1 = 0$ . In this case, the field  $S_i$  is a Lagrange multiplier, leading to the constraint F = 0. Hence, there are no propagating modes in the vector sector, unlike in the Fierz–Pauli theory,

$$m_1 = 0$$
: no propagating vector modes. (76)

For  $m_1 \neq 0$ , the analysis of the vector modes parallels that in Section 2.1. For  $m_1^2 > 0$ , the vector sector contains two normal propagating modes. The canonically normalized propagating field is now

$$\mathcal{F}_i(\mathbf{p}) = M_{\rm Pl} m_1 \sqrt{\frac{\mathbf{p}^2}{\mathbf{p}^2 + m_1^2}} F_i(\mathbf{p})$$

with the dispersion relation

$$\omega^2 = \frac{m_2^2}{m_1^2} \left( \mathbf{p}^2 + m_1^2 \right).$$

In the cases  $m_1^2 < 0$  and  $m_2 \neq 0$ , the modes are ghosts or tachyons at high spatial momenta, and we therefore impose the restriction

We now turn to the scalar sector. The full quadratic Lagrangian is

$$L_{\rm S}^{(2)} = 2 \left[ \partial_k \psi \partial_k \psi - 3 \partial_0 \psi \partial_0 \psi + 2 (\partial_k \varphi \partial_k \psi + \partial_k B \partial_0 \partial_k \psi + \partial_0 \Delta E \partial_0 \psi) \right] + \left[ \frac{m_0^2}{2} \varphi^2 + \frac{m_1^2}{4} (\partial_i B)^2 + \frac{3(3m_3^2 - m_2^2)}{2} \psi^2 - (3m_3^2 - m_2^2) \psi \Delta E + \frac{1}{2} (m_3^2 - m_2^2) (\Delta E)^2 + m_4^2 \varphi (3\psi - \Delta E) \right].$$
(77)

For general masses, there are two propagating modes, one of which is a ghost. Indeed, for  $m_1 \neq 0$ , integrating over the field *B* results in the following contribution to the Lagrangian [cf. (18)]:

$$L_B = \frac{8}{m_1^2} \dot{\psi} \Delta \dot{\psi} \,. \tag{78}$$

The field  $\varphi$  can also be integrated out, and the corresponding contribution to the Lagrangian of the dynamic fields  $\psi$  and *E* does not contain time derivatives. Hence, the terms with time derivatives in the resulting Lagrangian for  $\psi$  and *E* again have structure (21), implying that there is a ghost. Generally, the ghost exists at all spatial momenta and frequencies, and the observations to be made in Section 3.6 do not help. The ghost mode must be eliminated.

#### 3.2 Eliminating the second scalar mode

While the theory is not healthy in general, the ghost mode does not exist at special values of masses. This is the case, in particular, if either  $\varphi$  or *B* or both remain the Lagrange multiplier(s). The point is that the corresponding constraint suppresses the second mode in the scalar sector, while the remaining mode, if it exists, may well be normal. The two choices of the mass pattern that do the job are  $m_0 = 0$  and  $m_1 = 0$ . We discuss them in turn.

**3.2.1**  $m_0 = 0$ . In the case  $m_0 = 0$ , the field  $\varphi$  is a Lagrange multiplier, leading to the constraint

$$2\Delta\psi = m_4^2 (3\psi - \Delta E) \,. \tag{79}$$

Assuming that  $m_1 \neq 0$  and  $m_4 \neq 0$ , we integrate over the field *B* with result (78) and express  $\Delta E$  in terms of  $\psi$  using constraint (79). Then  $\psi$  is the only remaining dynamic field. The terms in its Lagrangian that are relevant at high momenta and frequencies,  $\omega^2$ ,  $\mathbf{p}^2 \gg m^2$ , are

$$L_{\psi} = 4 \left[ 2 \left( \frac{1}{m_4^2} - \frac{1}{m_1^2} \right) \partial_0 \partial_i \psi \, \partial_0 \partial_i \psi - \frac{m_2^2 - m_3^2}{m_4^4} \, (\Delta \psi)^2 \right] + \dots,$$

where the omitted terms have at most two derivatives. This Lagrangian is healthy at  $\omega^2$ ,  $\mathbf{p}^2 \ge m^2$  if

$$m_1^2 > m_4^2 > 0$$
,  $m_2^2 > m_3^2$ .

We see below, however, that this case is problematic.

Within the class of theories with  $m_0 = 0$ , there are subclasses in which more conditions are imposed on the masses. As an example, it already follows from the above analysis that the case  $m_4 = 0$ , the case  $m_4 = m_1$ , and the case

 $m_1^2 \ge 0$ .

 $m_2 = m_3$  are all special. A detailed study of these 'boundaries' is given in Ref. [40].

**3.2.2**  $m_1 = 0$ . For  $m_1 = 0$ , the field *B* is a Lagrange multiplier. The corresponding constraint is  $\dot{\psi} = 0$ , implying  $\psi = 0$  for propagating modes. Inserting  $\psi = 0$  into the action, we find that there remain no terms with time derivatives, and hence there are no propagating modes in the scalar sector. The vector sector has the same property [see (76)]. Thus, the only propagating modes in the theory with  $m_1 = 0$  are tensor gravitons with mass (75). We discuss this theory in detail in Section 5.

**3.2.3**  $m_2 = m_3$ ,  $m_4 = 0$ . Inspecting Lagrangian (77) reveals one more special case,  $m_2 = m_3$  and simultaneously  $m_4 = 0$ . It is now the field *E* rather than the nondynamic fields  $\varphi$  and *B* that plays a special role. The field  $\Delta E$  enters the Lagrangian linearly, and the corresponding field equation is

$$2\tilde{\psi} + (3m_3^2 - m_2^2)\,\psi = 0\,.$$

Thus, there are no high-frequency modes of  $\psi$ , irrespective of spatial momenta. If we are interested in high-frequency modes only, we must set  $\psi = 0$ , which leads to a Lagrangian without time derivatives. Hence, there are no propagating modes of high frequencies in this case.

We thus see that there are special cases where Lorentzviolating massive gravity does not contain ghosts in the linearized theory on the Minkowski background. Further analysis of numerous issues raised in Section 2 is conveniently performed in the Stückelberg formalism.

#### 3.3 Symmetries vs fine tuning

As we saw in Section 2.6, eliminating the second scalar mode in the Minkowski background is by itself insufficient for making the theory healthy. In the Lorentz-invariant theory, the absence of the second mode in the Minkowski background is due to the fine-tuning relation  $\alpha = -\beta$  imposed on the mass term in Lagrangian (2). This fine tuning is, however, destroyed in curved backgrounds, and the second, Boulware-Deser, mode reappears. Likewise, similar finetuning relations are problematic in Lorentz-violating theories. This can be seen explicitly in the theory with  $m_0 = 0$ and no other relations between the masses. It is convenient to use the Stückelberg formalism and proceed in analogy to Section 2.6.2. In the quadratic order, the Stückelberg part of metric (69) that contains the derivatives of  $\xi$  is

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} + \partial_{\mu}\xi^{\lambda}\partial_{\nu}\xi_{\lambda} .$$
 (80)

We concentrate on the terms involving the field  $\xi_0$ . In the Minkowski background, these come from the second and fifth terms in Lagrangian (74) and are given by

$$L_{\rm m} = \frac{1}{2} m_1^2 (\partial_0 \xi_i + \partial_i \xi_0) (\partial_0 \xi_i + \partial_i \xi_0) - m_4^2 \partial_0 \xi_0 \partial_i \xi_i + \dots ,$$

where omitted terms contain  $\xi_i$  only. Upon integrating the second term by parts, we see that  $\xi_0$  is not a dynamic field, and hence there is at most one propagating degree of freedom in the scalar sector, the longitudinal part of  $\xi_i$ . This is in accordance with the discussion in Section 3.2.1.

Once the background is slightly different from the Minkowski one,  $g_{\mu\nu}^{(c)} = \eta_{\mu\nu} + h_{\mu\nu}^{(c)}$ , the last property is lost. Indeed, due to the quadratic term in (80), the mass terms

themselves include the combination

$$-\frac{1}{2} m_4^2 h_{00} h_{ii}^{(c)} = -\frac{1}{2} m_4^2 (\partial_0 \xi_0)^2 h_{ii}^{(c)} + \dots$$

The field  $\xi_0$  becomes dynamical, the second mode reappears, and in some backgrounds (with the appropriate sign of  $h_{ii}^{(c)}$ ), this mode is a ghost.

There is an elegant way out of this fine-tuning problem, however [40]. Relations between the masses, instead of being results of fine tuning, may be consequences of unbroken gauge symmetries, which are parts of the gauge symmetry of general relativity. These residual gauge symmetries may then be expected to protect the theory from becoming pathological when it is extended to curved backgrounds and/or generalized to include possible UV effects (these are to be discussed in what follows). In several cases, this approach leads to healthy infrared-modified gravities.

Various residual gauge symmetries can be imagined [40]. In this review, we discuss only a few of them, which either are known to give rise to interesting theories or serve as examples of the failure of this approach. The first unbroken symmetry we consider is

$$x^i \to x^i + \zeta^i(x^i, t) \,. \tag{81}$$

This symmetry implies that all masses except  $m_0$  vanish; this is the symmetry of the ghost condensate theory [38].

The second symmetry to be discussed is

$$t \to t + \zeta^0(x^i, t) \,. \tag{82}$$

This symmetry leads to the constraint  $m_0 = m_1 = m_4 = 0$ . We see in Section 3.5 that the corresponding theory has problems with the stability against UV effects.

The third symmetry is

$$x^{i} \to x^{i} + \zeta^{i}(t) . \tag{83}$$

This symmetry is sufficient to ensure that  $m_1 = 0$ , while other masses are unconstrained. We found in Section 3.2.2 that the linearized theory in the Minkowski background is free of pathologies in this case. We see in what follows that the corresponding theory [40–42] is healthy both in the nearly Minkowski and in the general cosmological backgrounds. It is UV-stable as well. In fact, as we discuss in Section 5, this theory is quite interesting from the phenomenological standpoint.

#### 3.4 Lorentz-violating scalars

A convenient way to analyze the behavior of an infraredmodified gravity of the type we discuss in this review, and also to promote the perturbation theory on Minkowski background to a full low-energy effective theory, is to start with a generally covariant theory with additional scalar fields  $\phi^{\alpha}$ ,  $\alpha = 0, 1, 2, 3$ , which we call Goldstone fields. Breaking of the Lorentz invariance occurs when these fields acquire background values that depend on space-time coordinates. In this approach, the Lorentz invariance is broken *spontaneously*, because the original action of the theory is Lorentz invariant, but the background is not.

For example, in Minkowski space – time, the background fields are

$$\bar{\phi}^0 = a\Lambda^2 t \,,$$
  
$$\bar{\phi}^i = b\Lambda^2 x^i \,, \tag{84}$$

where  $\Lambda$  is a parameter with the dimension of mass, and a and b are coefficients of the order of unity. In our convention, the fields  $\phi^{\alpha}$  have the dimension of mass. Background fields (84) are solutions of the equations of motion if the Lagrangian contains their derivatives only. This property automatically implies that the Lagrangian is invariant under the shift symmetry  $\phi^{\alpha}(x) \rightarrow \phi^{\alpha}(x) + \lambda^{\alpha}$  with constant  $\lambda^{\alpha}$ . This means that the translational symmetry of (3 + 1)-dimensional space – time is unbroken by background (84) because the translation can be compensated by shifting the fields  $\phi^{\alpha}$ . Likewise, to preserve the spatial rotation symmetry, we require that the Lagrangian be invariant under SO(3) rotations of the fields,  $\phi^i \rightarrow A_j^i \phi^j$ . Thus, we are led to consider theories whose actions, at the one-derivative level, have the general form

$$S = S_{\rm EH} + S_{\phi} \,, \tag{85}$$

where  $S_{\rm EH}$  is the Einstein – Hilbert action and

$$S_{\phi} = \int \mathrm{d}^4 x \sqrt{-g} \,\Lambda^4 F(X, V^i, Y^{ij}, \mathcal{Q}) \,, \tag{86}$$

with

$$X = \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 ,$$
  

$$V^i = \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^i ,$$
  

$$Y^{ij} = \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j ,$$
  

$$Q = \frac{1}{\Lambda^8} \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ijk} \partial_\mu \phi^0 \partial_\nu \phi^i \partial_\lambda \phi^j \partial_\rho \phi^k .$$
(87)

Internal indices *i*, *j*, *k* are to be contracted in action (86) with either  $\delta_{ij}$  or  $\epsilon_{ijk}$ . Hereafter, to simplify power counting, we do *not* use convention (4) when writing the Lagrangian for the Goldstone fields. The combination *Q* is in fact not independent (apart from possible subtleties related to the presence of the  $\epsilon$  symbol): its square can be expressed in terms of *X*,  $V^i$ , and  $Y^{ij}$ . In what follows, we therefore consider only functions *F* depending on the first three combinations.

The energy-momentum tensor of configuration (84) vanishes in Minkowski space-time, and hence Minkowski space-time is a legitimate background, if a and b are such that

$$-\frac{1}{2}F + a^{2}\frac{\partial F}{\partial X} = 0,$$
  

$$\frac{1}{2}F\delta_{ij} + b^{2}\frac{\partial F}{\partial Y^{ij}} = 0,$$
  

$$\frac{\partial F}{\partial V^{i}} = 0$$
(88)

with  $X = a^2$ ,  $Y^{ij} = -b^2 \delta^{ij}$ , and  $V^i = 0$ . In what follows, we often set a = b = 1 by field redefinition.

The theory with action (86) is to be considered an effective field theory valid at low energies only.<sup>17</sup> The UV cutoff  $\Lambda_{UV}$  in

this theory must be somewhat below  $\Lambda$  (cf. Section 2.5). Indeed, expanding the fields on background (84),

$$\phi^{\alpha} = \bar{\phi}^{\alpha} + \pi^{\alpha} \,,$$

we obtain the following structure of the Lagrangian for perturbations:

$$L_{\pi} = \left(\Im\pi\right)^{2} + \frac{1}{\Lambda} \left(\Im\pi\right)^{3} + \dots, \qquad (89)$$

implying that  $\Lambda_{UV} \leq \Lambda$ . In this regard, an important point is the UV stability of the theory [40]. In low-energy effective theories, there is no reason to think that the low-energy Lagrangian contains terms with first derivatives only. Therefore, the effects of higher-derivative terms such as  $\Lambda^{-2}g^{\mu\nu}g^{\lambda\rho}\partial_{\mu}\partial_{\nu}\phi^{\alpha}\partial_{\lambda}\partial_{\rho}\phi^{\alpha}$  must be taken into account. Naively, these terms are suppressed below the cutoff scale, i.e., for  $\mathbf{p}^2, \omega^2 \leq \Lambda^2$ . But if the kinetic terms in (89) have a special structure, the higher-derivative terms may become important. We encounter examples of this sort in what follows.

Turning on gravity, still in Minkowski space – time and in background (84), we observe that the gauge transformation  $x^{\mu} \rightarrow x^{\mu} + \zeta^{\mu}(x)$  corresponds to the following transformation of the fields  $\pi^{\alpha}$ :

$$\pi^{\alpha}(x) \to \pi^{\alpha}(x) + \Lambda^2 \zeta^{\alpha}(x)$$
.

Hence,

$$\xi^{\alpha} = \Lambda^{-2} \pi^{\alpha} \tag{90}$$

are the Stückelberg fields of the previous sections. In the unitary gauge  $\pi^{\alpha} = 0$ , we have  $X = 1 - h_{00}$ ,  $V^{ij} = -1 - h_{ij}$ , etc., and hence the part of the action quadratic in the  $h_{\mu\nu}$  contains mass term (74), the scale of graviton masses being

$$m = \frac{\Lambda^2}{M_{\rm Pl}} \,, \tag{91}$$

which agrees with (49). Hence, the class of theories (85) indeed has all the expected properties of Lorentz-violating massive gravity. A convenient feature of this construction is that the behavior of the theory for  $\mathbf{p}^2, \omega^2 \ge m_G^2$  in or near the Minkowski background can be analyzed by studying the Goldstone sector only. Also, the theory away from the Minkowski background is well defined.

Needless to say, for the general Lagrange function F, the theory is pathological. As we discussed in Section 3.3, it may not be pathological if a part of the gauge symmetry of general relativity remains unbroken. In that case, Goldstone action (86) does not have the generic form. For example, the residual gauge invariance  $t \rightarrow t + \zeta^0(x^i, t)$  [see (82)] in the Goldstone language implies that the Lagrange function F is invariant under the change of variables

$$\phi^0 \to \phi^0 + \Xi^0(\phi^i, \phi^0) \tag{92}$$

with an arbitrary function  $\Xi^0(\phi^i, \phi^0)$ . Indeed, only in this case is background (84) invariant under the gauge transformation  $t \to t + \zeta^0(x^i, t)$  accompanied by a field redefinition. It is in this way that the Lagrangian, and hence graviton mass terms, become constrained by the requirement of residual gauge invariance in the Goldstone framework.

<sup>&</sup>lt;sup>17</sup> What is exactly meant by low energies becomes clear after the effects of higher-order terms are understood: these are energies and/or momenta at which higher-order terms increase sharply. Because the actions we discuss are Lorentz invariant, the way the UV cutoff  $\Lambda$  enters the action is dictated by the Lorentz invariance. On the other hand, the energy value at which the low-energy theory ceases to work may be different from the spatial momentum value, due to a spontaneous Lorentz violation by background scalar fields.

#### 3.5 An example of a UV unstable theory

To illustrate the problem with the UV stability that may be encountered in an otherwise healthy theory, we consider the model with residual gauge symmetry (82), implying the constraint on the Lorentz-violating graviton masses  $m_0 = m_1 = m_4 = 0$ . In the Goldstone language, this symmetry translates into field transformation (92). This can be a symmetry of Goldstone action (86) only if the field  $\phi^0$  is absent altogether. Hence, the Goldstone sector of the theory has three fields  $\phi^i$  and at the one-derivative level, the action is

$$S_{\phi} = \int \mathrm{d}^4 x \sqrt{-g} \, \Lambda^4 F(Y^{ij}) \,,$$

where  $Y^{ij}(\phi^i)$  is given by (87). As pointed out in Section 3.3, the general Lagrangian for the theory of Goldstone fields, viewed as a low-energy effective theory, contains higher-order terms, e.g.,

$$\Delta F = \frac{1}{\Lambda^4} g^{\mu\nu} g^{\lambda\rho} \partial_\mu \partial_\nu \phi^i \partial_\lambda \partial_\rho \phi^i \,. \tag{93}$$

We see in what follows that in the model discussed here, these terms are important and, in fact, lead to pathologies in the spectrum.

We consider this theory in the Minkowski background, temporarily discarding higher-order terms. The Lorentz invariance is broken by the background

$$\bar{\phi}^i = \Lambda^2 x^i \,, \tag{94}$$

which satisfies the field equations for the Goldstone fields. Expanding the fields near this background,  $\phi^i = \bar{\phi}^i + \pi^i$ , we obtain the quadratic Lagrangian for the Stückelberg fields  $\pi^i$ . This Lagrangian involves the first and second derivatives of the Lagrange function *F* evaluated at  $Y^{ij} = Y^{ij}(\bar{\phi}^i) = -\delta^{ij}$ , which we parameterize as

$$\begin{aligned} \frac{\partial F}{\partial Y^{ij}} \left( \bar{\phi} \right) &= F_1 \delta_{ij} \,, \\ \frac{\partial^2 F}{\partial Y^{ij} \partial Y^{kl}} \left( \bar{\phi} \right) &= F_{21} \delta_{ij} \delta_{kl} + F_{22} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \,. \end{aligned}$$

The quadratic Lagrangian is

$$L_{\pi} = F_1 \partial_{\mu} \pi^i \partial^{\mu} \pi^i + 2F_{21} \partial_i \pi^i \partial_j \pi^j + 2F_{22} (\partial_i \pi^j \partial_i \pi^j + \partial_i \pi^j \partial_j \pi^i) .$$
(95)

At first sight, this Lagrangian describes three scalar fields with a healthy kinetic term. But this is inconsistent with the absence of propagating modes in the vector sector for  $m_1 = 0$ [see (3.1)]. The resolution of this discrepancy is related to the requirement that the energy-momentum tensor of background field configuration (94) vanish, and hence the Minkowski metric be a solution of the complete set of field equations. The corresponding conditions are read off from Eqns (88), which in the absence of the combinations X and V<sup>i</sup> yield

$$F = 0$$
,  $\frac{\partial F}{\partial Y^{ij}} = 0$  at  $\phi^i = \overline{\phi}^i$ 

Hence,  $F_1 = 0$  in (95), and therefore the one-derivative action actually corresponds to a theory with no propagating modes: at this level, all Stückelberg fields enter the action without time derivatives, and none of them is a dynamic field.

Once the higher-order terms are added, the situation changes. Terms like (101) contain time derivatives, and there is no symmetry that would forbid them. In terms of the fields  $\pi^i$ , these contributions have the structure

$$\Delta L_{\pi} = \frac{1}{\Lambda^2} \left[ \left( \hat{\partial}_0^2 \pi^i \right)^2 - \left( \hat{\partial}_0 \hat{\partial}_i \pi^j \right)^2 + \dots \right].$$
 (96)

These contributions dominate at high *frequencies*, precisely because Lagrangian (95) does not contain time derivatives, i.e., precisely because the fields  $\pi^i$  are not dynamical at the one-derivative level. With the higher-order terms included, the fields  $\pi^i$  become propagating, and their dispersion relation is

$$\omega^4 = \operatorname{const} \mathbf{p}^2 \Lambda^2$$
.

This means that at least one of the modes for each  $\pi^i$  is tachyonic, and the corresponding 'frequency' is high even at moderate spatial momenta (being, nevertheless, smaller than the cut-off scale  $\Lambda_{\rm UV}$ ). The model is therefore unacceptable.

Hence, fields that are nondynamical in the Minkowski background and at the level of the one-derivative Lagrangian are potentially dangerous. They may become propagating in curved backgrounds and/or due to higher-order terms in the Lagrangian. We refer to the first possibility as the Boulware–Deser instability, while the second is called the UV sensitivity [40].

To conclude the discussion of the model studied here, we note that in the language of metric perturbations, the UV sensitivity is the sensitivity to derivative terms in the Lagrangian for  $h_{\mu\nu}$ . For example, the first, most unwelcome contribution to (96), in terms of metric perturbations, corresponds to the term

$$\Delta S_{h} = \int d^{4}x \Lambda^{2} \left(\frac{1}{2} \,\partial_{i}h_{00} - \partial_{0}h_{0i}\right)^{2}$$
$$= M_{\rm Pl}^{2} \int d^{4}x m_{\rm G}^{2} \,\frac{1}{\Lambda^{2}} \left(\frac{1}{2} \,\partial_{i}h_{00} - \partial_{0}h_{0i}\right)^{2}$$

where we recall relation (90), which implies the correspondence  $h \simeq \Lambda^{-2} \partial \pi$ , and use (91). This term is invariant under residual gauge transformations (82) and is suppressed by the anticipated UV scale (49) as compared to the graviton mass terms, and therefore there is no reason for it to be absent. Thus, we have found that the theory with two nonvanishing graviton masses  $m_2$  and  $m_3$  has tachyons in the spectrum, once generic one-derivative terms in  $h_{\mu\nu}$  consistent with symmetry (82) are added.

#### 3.6 Not-so-dangerous instabilities

To conclude this section, we digress to a more phenomenological discussion of instabilities in Lorentz-violating theories. In these theories, tachyons and/or ghosts are allowed if they exist at low frequencies (particle energies) only. In viable theories, the frequency cutoff  $\Lambda_{tc}$  for tachyons can be somewhat higher than the present Hubble scale  $H_0$ ; for ghosts, the cutoff  $\Lambda_{gh}$  can be many orders of magnitude higher than  $H_0$ . We discuss this in some detail, assuming that ghosts and tachyons are coupled to the ordinary matter only gravitationally.

We consider tachyons first, and suppose, as an example, that the dispersion relation is

$$\omega^2 = -\mathbf{p}^2, \qquad (97)$$

for  $|\mathbf{p}| \ll \Lambda_{tc}$ , and the frequency is normal,  $\omega^2 > 0$  for  $|\mathbf{p}| > \Lambda_{tc}$  (an example of such a dispersion law is  $\omega^2 = -\mathbf{p}^2 + \Lambda_{tc}^{-2}\mathbf{p}^4$ ). Then, in an expanding universe,  $\omega$  scales for  $|\mathbf{p}| \ll \Lambda_{tc}$  as

$$\left|\omega(t)\right| = \frac{\Omega}{a(t)} ,$$

where  $\Omega$  is a constant conformal frequency. There is a characteristic time instant  $t_A$  in the history of the Universe at which

$$H(t_A) = A_{\rm tc}$$

Before that instant, would-be tachyonic modes with  $\omega(t) \leq \Lambda_{tc}$  are over-damped and do not develop, and hence an exponential growth of any mode is possible only after  $t_A$ . The largest growth factor corresponds to modes that become tachyonic just at the instant  $t_A$ , i.e., the modes with

$$\omega(t_A) \equiv \frac{\Omega}{a(t_A)} \simeq \Lambda_{\rm tc} \,.$$

Indeed, modes of higher conformal frequencies still oscillate at  $t = t_A$ , while modes of lower conformal frequencies still do not develop at  $t \sim t_A$ . At present, the largest growth factor for the field amplitude is

$$\exp\left(\int_{t_A}^{t_0}\frac{a(t_A)}{a(t)}\,\Lambda_{\rm tc}\,{\rm d}t\right),\,$$

where  $t_0$  denotes the present time. The inhomogeneities in the tachyon field produce gravitational potentials comparable to those of ordinary matter with the energy density perturbations

$$\delta 
ho \simeq \Lambda_{\rm tc}^4 \exp\left(2\int_{t_A}^{t_0} \frac{a(t_A)}{a(t)} \Lambda_{\rm tc} \,\mathrm{d}t\right),$$

where we estimated the preexponential factor on dimensional grounds and neglected the energy redshift in writing this factor. The bound on  $A_{\rm tc}$  comes from the requirement that this inhomogeneous energy density not exceed the observationally allowed value, e.g.,  $10^{-4}\rho_{\rm c}$  (the exact number is unimportant here). Approximating the cosmological expansion by  $a \propto t^{2/3}$  (which corresponds to the Universe dominated by nonrelativistic matter), we find

$$\delta\rho \approx \Lambda_{\rm tc}^4 \exp\left(3t_0^{1/3}t_A^{2/3}\Lambda_{\rm tc}\right) = \Lambda_{\rm tc}^4 \exp\left[4\left(\frac{\Lambda_{\rm tc}}{H_0}\right)^{1/3}\right]$$

Requiring that  $\delta \rho \lesssim 10^{-4} \rho_c \sim 10^{-4} M_{\rm Pl}^2 H_0^2$ , we have

$$\frac{A_{\rm tc}}{H_0} \lesssim \frac{1}{64} \left[ \ln \left( 10^{-4} \, \frac{M_{\rm Pl}^2}{H_0^2} \right) \right]^3 \sim 3 \times 10^5 \,.$$

We conclude that the frequency cutoff for tachyons with dispersion relation (97) must be of the order of  $\Lambda_{\rm tc} \sim 10^5 H_0$  or lower.

The bound on  $\Lambda_{tc}$  rather strongly depends on the form of the dispersion relation for tachyons. In any case, it is somewhat higher, but not very much higher, than  $H_0$ .

We now turn to ghosts. The instability in this case is due to pair creation of ghosts and usual particles from the vacuum,



Figure 2. The decay of the vacuum into two ghosts and two photons via creation of a virtual graviton.

the process allowed by the energy–momentum conservation due to the negative energy of ghost particles. The strongest bound [68] on the frequency cutoff  $\Lambda_{\rm gh}$  comes from the process

vacuum 
$$\rightarrow \phi + \phi + \gamma + \gamma$$
, (98)

where  $\phi$  and  $\gamma$  respectively denote a ghost and a photon. We assume that ghosts experience gravitational interactions only. Then this process is described by Fig. 2 and its rate per unit volume is estimated on dimensional grounds as<sup>18</sup>

$$\Gamma \simeq rac{\Lambda_{
m gh}^8}{M_{
m Pl}^4}$$

We note that all particles in (98) are on-shell, and hence  $\Lambda_{\rm gh}$  is the cutoff of energy, not the usual UV cutoff of momentum transfer. In Lorentz-invariant theories,  $\Lambda_{\rm gh} = \infty$  and the rate is infinite. This corresponds to an infinite volume of the Lorentz group. In other words, in Lorentz-invariant theories, process (98) with certain momenta of outgoing particles has its boosted counterparts, and the phase space is therefore infinite. This is not the case in Lorentz-violating theories. There, photons created in process (98) have energies  $E_{\gamma} \leq \Lambda_{\rm gh}$ , and their number density in the present Universe, and hence the flux near the Earth, is of the order of

$$F\simeq \Gamma t_0$$
.

The flux per energy interval is

$$\frac{\mathrm{d}F}{\mathrm{d}E_{\gamma}}\left(E_{\gamma}\sim\Lambda_{\mathrm{gh}}\right)\simeq\frac{\Lambda_{\mathrm{gh}}^{7}}{M_{\mathrm{Pl}}^{4}}t_{0}$$

This flux has to be smaller than the EGRET differential flux,

$$\frac{\mathrm{d}F}{\mathrm{d}E_{\gamma}} = 7 \times 10^{-9} \left(\frac{E_{\gamma}}{450 \text{ MeV}}\right)^{-2.1} \text{ cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1} \text{ MeV}^{-1}.$$

This requirement gives [68]

$$\Lambda_{\rm gh} \lesssim 3 {\rm ~MeV}$$

Hence, the frequency cutoff in Lorentz-violating theories with ghosts may be relatively high.

<sup>18</sup> We assume here that the 3-momentum cutoff is also of the order of  $\Lambda_{\rm gh}$ .

# 4. Ghost condensate: modification of gravity without graviton mass

We now consider an example of a UV-stable theory, the 'ghost condensate' model [38]. The questions discussed above—the absence of the extra scalar mode near Minkowski space-time, protection by a residual symmetry against its reappearance in curved backgrounds and due to higher-derivative corrections—play a key role in the construction of this model. Although the graviton remains massless in the ghost condensate model, the simplicity of this model makes it a good introduction to more complicated models of massive gravity.

As we discussed in Section 3.2, a convenient way to modify the gravitational interaction in the infrared range is to introduce additional scalar fields. In the simplest case, this is just a single scalar field  $\phi$  (cf. Section 3.4) with the action reminiscent of Eqn (86),

$$S_{\phi} = \Lambda^4 \int \mathrm{d}^4 x \sqrt{-g} F(X) \,, \tag{99}$$

with X given by the first equation in (87),  $X = g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi / \Lambda^4$ . The field equation derived from this action is

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left[ F_X(X) \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi \right] = 0, \qquad (100)$$

where  $F_X = \partial F/\partial X$ . In Minkowski space-time and with gravity switched off, this equation has a solution linearly growing in time ('ghost condensate'),

$$\phi_{\rm v} = \alpha \Lambda^2 t \,, \tag{101}$$

where  $\alpha$  is an arbitrary constant. This background obviously breaks the Lorentz symmetry. Time translations are also broken, but the diagonal combination of time translations and shifts of  $\phi$  by a constant remain a symmetry, and hence the energy is conserved. In the 'unitary' gauge  $\phi = \phi_v$ , the variable X reduces to  $X = \alpha^2 g^{00}$  and action (99) becomes a function of  $g^{00}$ . This action is invariant under space-timedependent transformations of spatial coordinates

$$x^{i} \to \tilde{x}^{i} = \tilde{x}^{i}(x^{i}, t), \qquad (102)$$

as discussed at the end of Section 3.3 [see Eqn (81)]. This symmetry plays an important role in the construction of the ghost condensate model.

Once the back reaction of ghost condensate on gravitational background is switched on, the parameter  $\alpha$  is no longer arbitrary. For the Minkowski space to be a solution of the Einstein equations for  $\phi = \phi_v$ , the energy-momentum tensor of  $\phi_v$  must vanish,

$$T_{\mu\nu} = \left[ 2\partial_{\mu}\phi \partial_{\nu}\phi F_X - \eta_{\mu\nu}F \right]_{\phi=\phi_{\nu}} = 0.$$

This equation leads to the two conditions [cf. (88)]

$$2\alpha^2 F_X(\alpha^2) - F(\alpha^2) = 0,$$
  
 $F(\alpha^2) = 0.$ 
(103)

The second of these two conditions is the usual tuning of the cosmological constant to zero. When this condition is

satisfied, the first of Eqns (103) implies that  $F_X(\alpha^2) = 0$ . We assume in what follows that extrema of F(X) occur at  $X \neq 0$ . Thus,  $\alpha$  is nonzero, and the field can be redefined such that conditions (103) are satisfied for  $\alpha = 1$ .

In the expanding universe, the ghost condensate is automatically driven to the point  $F_X = 0$ . This follows from field equation (100). Indeed, this equation can be regarded as the covariant conservation equation for the current

$$J^{\mu} = F_X(X) g^{\mu\nu} \partial_{\nu} \phi \,.$$

In the cosmological setting, the field  $\phi$  is consistently taken to depend on time only, and hence the only nonvanishing component of this current is the density  $J^0$ . Its covariant conservation implies that, like other densities, it decays in time,

$$J^0 \propto \frac{1}{a^3} \,,$$

which means that  $F_X(X)$  becomes negligibly small at late times.

In the unitary gauge and at the quadratic level in metric perturbations  $h_{\mu\nu}$ , action (99) in the Minkowski background becomes

$$S_{\phi}^{(2)} = \Lambda^4 \, \frac{F_{XX}}{2} \int \mathrm{d}^4 x h_{00}^2 = \frac{1}{2} \, M_{\rm Pl}^2 m_0^2 \int \mathrm{d}^4 x h_{00}^2 \,, \qquad (104)$$

where  $F_{XX} = [d^2 F/dX^2](\alpha = 1)$  is a constant. With the Einstein–Hilbert part of the action added, this yields Eqn (74) with all masses equal to zero except  $m_0$ . Thus, at the level of the two-derivative action, there are no propagating degrees of freedom (cf. Section 3.2).

The same can be seen in the Stückelberg language by replacing  $h_{00} \rightarrow 2(M_{\rm Pl}m_0)^{-1}\partial_0\pi$  in Eqn (104). The resulting action for the Stückelberg field  $\pi$ ,

$$2\int \mathrm{d}^4 x (\hat{\mathrm{o}}_0 \pi)^2 \,, \tag{105}$$

has no gradient term and describes a mode with the dispersion relation

$$\omega^2 = 0. \tag{106}$$

If the action of the ghost condensate model contained contribution (99) only, its effect would simply be a (partial) gauge fixing of general relativity, and therefore in the sector with the initial condition X = 1, the theory would describe the Einstein gravity in a particular gauge. This situation is specific to the ghost condensate model; we see in the next section that in the general case, modifications to gravity already arise from the first term in the derivative expansion of the action.

The degeneracy of action (105) (the absence of spatial gradient terms) signals that the nonpropagating Stückelberg mode can become propagating once higher-derivative corrections are added. These corrections are routine in the effective low-energy theories, but they play a crucial role here. Symmetry (81) restricts the general form of these corrections. In the Goldstone language, the next-order contribution to the action contains higher derivatives acting on the Goldstone field  $\phi$ , such as  $\partial^2 \phi$  and  $\partial_\mu \partial_\nu \phi$ , suppressed by powers of  $\Lambda$ . When expanded in the quadratic order in  $\pi$ ,

these terms modify action (105) to

$$2\int d^{4}x \left\{ (\partial_{0}\pi)^{2} + \frac{c_{0}}{\Lambda^{2}} (\partial_{0}^{2}\pi)^{2} + \frac{c_{1}}{\Lambda^{2}} \partial_{0}^{2}\pi \partial_{i}^{2}\pi + \frac{c_{2}}{\Lambda^{2}} (\partial_{i}^{2}\pi)^{2} + \dots \right\},$$

where  $c_i$  are numerical coefficients roughly of the order of unity. The dispersion relation for the Stückelberg mode becomes

$$\omega^{2} = \frac{c_{0}}{\Lambda^{2}} \,\omega^{4} + \frac{c_{1}}{\Lambda^{2}} \,\omega^{2} \mathbf{p}^{2} + \frac{c_{2}}{\Lambda^{2}} \,\mathbf{p}^{4} \,. \tag{107}$$

There are two solutions of this equation. The first one is

$$\omega^2 = \frac{\Lambda^2}{c_0} + O(\mathbf{p}^2) \,.$$

This solution is irrelevant because it falls outside the region of validity of the low-energy effective theory, which is  $\omega \ll \Lambda$ . The second solution represents a modification of the dispersion relation  $\omega^2 = 0$ , which becomes

$$\omega^2 = \frac{c_2}{\Lambda^2} \mathbf{p}^4 + O\left(\frac{\mathbf{p}^6}{\Lambda^4}\right). \tag{108}$$

This solution describes a slowly propagating mode [38] that is nontachyonic if  $c_2 > 0$ . According to (105), this mode is not a ghost for  $F_{XX}(\alpha = 1) > 1$ . Hence, the theory is healthy at high spatial momenta.

Mode (108) modifies the gravitational interaction (in particular, the Newtonian potential) at distances longer than  $r_c = 1/m_0$ . On the other hand, the time scale at which these modifications build up is parametrically larger,  $t_c = \Lambda/m_0^2$  [40]. The reason is again that the modification of gravity only occurs in the next-to-leading order in the derivative expansion. As the mass  $m_0$  tends to zero, the scales  $r_c$  and  $t_c$  tend to infinity, and the modifications are smoothly switched off. In this sense, the van Dam–Veltman–Zakharov phenomenon is absent in the ghost condensate model. We note, however, that according to (104), the mass  $m_0$  is related to the UV scale as

$$m_0^2 M_{\rm Pl}^2 = F_{XX} \Lambda^4$$
.

Hence, the limit of vanishing mass corresponds to the limit  $\Lambda \rightarrow 0$  (at fixed  $M_{\rm Pl}$ ), and therefore the validity region of the low-energy effective theory shrinks to zero in this limit.

The ghost condensate model does not exhibit the Boulware-Deser instability either. In contrast to the example considered in Section 3.5, the only scalar field  $\pi$  present in the theory has, in a flat background, dispersion relation (108), which is a consequence of residual symmetry (102). In a slightly curved background, this dispersion relation may acquire additional terms with small coefficients controlled by the background curvature. The appearance of new contributions to the dispersion relation-for instance, a term proportional to  $\mathbf{p}^2$  with a negative coefficient — may cause a tachyonic instability at low spatial momenta. This is precisely what happens in some cosmological backgrounds [69] (although this instability is not particularly dangerous). On the other hand, the terms induced by a slightly curved background cannot change the sign of the leading  $\omega^2$  term, and hence a propagating mode does not become a ghost. The situation is therefore different from the case of the Boulware – Deser mode of Section 2.6, where one of the scalar modes is necessarily a ghost in the curved background.

The ghost condensate model and its modifications have unusual properties. Some of these properties are potentially interesting from the standpoint of phenomenology and cosmology, while others serve as examples of novel phenomena that may emerge once the Lorentz invariance is broken. We briefly describe some of them.

Due to the Lorentz violation and mixing of the slowly propagating field  $\pi$  with metric perturbations, gravitational fields of moving sources are different from gravitational fields of sources that are at rest with respect to the ghost condensate. In particular, there is a memory effect: moving bodies leave 'star tracks' in the ghost condensate [70, 71].

The ghost condensate itself may be regarded as matter with rather unusual properties. In particular, lumps of this matter can in principle antigravitate [38]. More generic is the property that the presence of the ghost condensate in space leads to an instability of the Jeans type with the time scale that is parametrically large compared to ordinary fluids of the same energy density [38]. This property is again related to the presence of the slowly propagating mode  $\pi$ .

The nonlinear dynamics of the ghost condensate are also quite rich. An evolving ghost condensate tends to form caustics [72], much in common with caustics in some other scalar theories [73]. Away from the caustics, the ghost condensate dynamics are the same as the dynamics of a fluid with the equation of state  $p \propto \rho^2$ . Another possible effect is the nonperturbative instability of background (101), leading to the formation of microscopic negative-energy 'holes' [74].

Lorentz violation makes the physics of black holes considerably different from that in general relativity. The least dramatic effect is the accretion of the ghost condensate onto black holes [75, 76]. More exotic are the possibilities that black hole systems may violate the second law of thermo-dynamics [77], signals may escape from black holes [78], and black holes may have hair [79].

A cosmologically interesting class of models is obtained by adding a potential term to the action, such that instead of (99), the action is chosen as

$$S_{\phi} = \Lambda^4 \int \mathrm{d}^4 x \sqrt{-g} \left[ F(X) - V(\phi) \right].$$

Then both the kinetic term F(X) and the potential term  $V(\phi)$ contribute to the energy-momentum tensor. The field  $\phi$  is still growing, albeit not quite according to (101). This may be used for constructing models of inflation with the ghost condensate serving as the inflaton [80] and models for dark energy driving the present accelerated expansion of the Universe [81, 82]. Interestingly, the field  $\phi$  grows even if the potential increases as  $\phi$  increases; in this case, the field  $\phi$  rolls up the potential. This gives rise to phantom behavior [69, 83] in which the energy density grows in time, and the equation of state is  $p = w\rho$  with w < -1 (and w depends on time in general). This is one of a few examples of phantom matter without UV pathologies: in most other cases, a phantom equation of state is obtained in theories with unacceptable tachyons and/or ghosts in the UV range (see, however, Refs [84, 85]). If phantom behavior occurs at the inflationary stage of the cosmological evolution, the consequence is the blue-tilted spectrum of primordial tensor perturbations (as opposed to the red-tilted spectrum predicted by theories where the inflaton is an ordinary scalar field). The dark energy driving the present accelerated expansion may also have a phantom equation of state, the feature potentially detectable by future observations of SNe1a (see, e.g., Ref. [86]). Perhaps the most striking possibility is that a phantom may give rise to bouncing cosmology: in general relativity, the relation  $p < -\rho$  implies that

$$\dot{H} > 0$$

where *H* is the Hubble parameter, and hence the transition from a contracting to an expanding universe (from H < 0 to H > 0) becomes possible. Indeed, solutions of this sort have been found [69, 87] and explored [88, 88] in ghost condensate models with suitable potentials  $V(\phi)$ . The bounce in these models occurs in a controllable and self-consistent way.

# 5. The minimal model of a massive graviton

# 5.1 The linearized theory

An interesting theory [40–42], without obvious pathologies and with massive gravitons, is obtained by considering the case of residual gauge symmetry (83), which leads to the condition  $m_1 = 0$ . This symmetry,  $x^i \rightarrow x^i + \zeta^i(t)$ , translates into the symmetry of the Goldstone Lagrangian

$$\phi^i \to \phi^i + \Xi^i(\phi^0) \,, \tag{109}$$

with three arbitrary functions  $\Xi^i$ . At the one-derivative level, there are two combinations of the Goldstone fields that respect this symmetry,

$$\begin{split} X &= \frac{1}{\Lambda^4} g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 \,, \\ W^{ij} &= \frac{1}{\Lambda^4} \left( g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^i \, \frac{g^{\lambda\rho} \partial_\lambda \phi^0 \partial_\rho \phi^j}{\Lambda^4 X} \right) \\ &= Y^{ij} - \frac{V^i V^j}{X} \,, \end{split}$$

where  $Y^{ij}$  and  $V^i$  are defined in (87). Hence, at this level, the Goldstone action is

$$S_{\phi} = \int \mathrm{d}^4 x \sqrt{-g} F(X, W^{ij}) \,, \tag{110}$$

where the indices *i*, *j* are contracted using  $\delta_{ij}$ .

We first discuss this theory at the linearized level on the Minkowski background. The background Goldstone fields are given by (84). By field redefinitions, we set a = b = 1 and write the background fields simply as

$$\bar{\phi}^0 = \Lambda^2 t \,, \tag{111}$$

$$\bar{\phi}^i = \Lambda^2 x^i \,. \tag{112}$$

It follows from (88) that the energy-momentum of this configuration vanishes if

$$-\frac{1}{2}F + \frac{\partial F}{\partial X} = 0, \qquad (113)$$
$$\frac{1}{2}F\delta_{ij} + \frac{\partial F}{\partial W^{ij}} = 0$$

with X = 1 and  $W^{ij} = -\delta^{ij}$ . Switching metric perturbations on and using the unitary gauge  $\phi^{\alpha} = \bar{\phi}^{\alpha}$ , we find that the theory on the Minkowski background is gravity with Lorentz-violating mass terms (74), with the only constraint

 $m_1 = 0$ .

The other mass parameters are independent of each other, and are expressed through *F* and its first and second derivatives at X = 1,  $W^{ij} = -\delta^{ij}$ . Hence, at the level of the one-derivative Goldstone action, neither the vector nor the scalar sector contains propagating modes, as we discussed in Section 3.2.2, and tensor gravitons (two degrees of freedom) have the mass  $m_G = m_2$ . In this sense, the model can be regarded as the minimal model of a massive graviton.

It is instructive to switch off gravity and consider the Goldstone sector of this theory on the Minkowski background but away from point (113). The quadratic Lagrangian for the perturbations  $\pi^{\alpha} = \phi^{\alpha} - \bar{\phi}^{\alpha}$  is obtained from (110) and has the general form

$$L_{\pi} = \frac{a}{2} (\dot{\pi}^{0})^{2} - \frac{b}{2} (\partial_{i}\pi^{0})^{2} + c\dot{\pi}^{0}\partial_{i}\pi^{i} + \frac{d_{1}}{2} (\partial_{i}\pi^{i})^{2} + \frac{d_{2}}{2} (\partial_{i}\pi^{j})^{2}, \qquad (114)$$

where  $b = 2(\partial F/\partial X)(\bar{\phi})$ , and the constants *a*, *c*, *d*<sub>1</sub>, and *d*<sub>2</sub> contain second derivatives of *F* at  $\phi = \bar{\phi}$ . It is easy to see that the fields  $\pi^i$  are nondynamical. Their equations of motion in the vector sector give  $\pi^{Ti} = 0$ , where  $\pi^{Ti}$  is the transverse part,  $\partial_i \pi^{Ti} = 0$ . Hence, there are no nontrivial modes in the vector sector even for a general linearly increasing background. The equation of motion for the longitudinal part of  $\pi^i$  gives

$$\pi^i = {
m const}\, {\partial_i\over\Delta}\, {\dot\pi}^0\,.$$

Substituting this expression in the equation of motion for  $\pi^0$  gives

$$\tilde{a}\ddot{\pi}^0 - b\Delta\pi^0 = 0\,,$$

where  $\tilde{a}$  is a combination of the constants  $a, c, d_1$ , and  $d_2$ . For a general linearly increasing Goldstone background, the dispersion relation is  $\omega^2 = \text{const } \mathbf{p}^2$ . It follows that with a suitable choice of parameters, this mode is neither tachyon nor ghost [40, 41].

At the point

$$\frac{\partial F}{\partial X} = 0, \qquad (115)$$

the dispersion relation is  $\omega^2 = 0$  at the level of the onederivative action. In fact, this special point is basically coincident with Minkowski point (113), because we neglect gravity here and cannot therefore discriminate between different values of *F* at  $\phi = \overline{\phi}$ . Overall, the situation in the scalar and vector sectors is very similar to that in the ghost condensate theory.

The absence of propagating modes associated with the fields  $\pi^i$  (rather than  $\pi^0$ ) is by no means an accident. Given background (111), symmetry (109) implies that the theory is invariant under the infinitesimal transformations

$$\pi^i \to \pi^i + \Xi^i(t)$$

This means that at the one-derivative level, the Lagrangian does not contain time derivatives of the fields  $\pi^i$ , and hence these fields are not dynamical. This is, of course, explicit in (114). Thus, the dispersion relation  $\mathbf{p}^2 = 0$ , characteristic of nonpropagating modes, is protected in this model by symmetry (109).

The last observation is useful in discussing the UV sensitivity issue in this model. Under the assumption that higher-derivative terms respect symmetry (109), these terms cannot contain  $\partial_0^2 \phi^i$  (in the reference frame where the background  $\bar{\phi}^0$  has form (111)), and in terms of the perturbations  $\pi^{\alpha}$ , they are quadratic combinations of

$$\begin{split} \Lambda^{-1}\partial_0\partial_j\pi^i \,, \quad \Lambda^{-1}\partial_j\partial_k\pi^i \,, \quad \Lambda^{-1}\partial_0^2\pi^0 \\ \Lambda^{-1}\partial_0\partial_j\pi^0 \,, \quad \Lambda^{-1}\partial_j\partial_k\pi^0 \,. \end{split}$$

Once these terms are added to Lagrangian (114), the fields  $\pi^i$  formally become dynamical, but it is straightforward to see that the equation for the corresponding dispersion relation has the form

$$\mathbf{p}^2 \left[ \omega^2 - \operatorname{const} \Lambda^2 + O(\mathbf{p}^2) \right] = 0.$$

Hence, the would-be new propagating modes have the dispersion relation

$$\omega^2 = \operatorname{const} \Lambda^2 + O(\mathbf{p}^2)$$
.

Because the frequencies are of the order of the UV cutoff, these modes are actually absent in the low-energy theory. In this sense, the theory is UV stable: upon switching on higher-derivative terms, only the mode with the dispersion relation  $\mathbf{p}^2 = 0$  remains.

At this point, it is worth discussing the physical interpretation of modes with the dispersion relation  $\mathbf{p}^2 = 0$ . They can be considered the degrees of freedom with infinite propagation velocity (unlike the ghost condensate mode, which has zero velocity at the one-derivative level and acquires a small velocity due to higher-derivative terms). Physically, they describe sound waves propagating through the rigid coordinate frame selected in space by the functions  $\phi^i$ . The rigidity of this frame is ensured by symmetry (83) and the SO(3) symmetry of the Goldstone action, which allow moving and rotating this frame only as a whole. Infinitely fast propagating modes do not imply the causality violation in the absence of Lorentz invariance, but allow instantaneous transfer of information. This leads to a number of unusual effects related to black hole physics [77, 79]. A detailed discussion of the properties of these modes in a simplified QED model can be found in Refs [90, 91].

The higher-order terms are also important for the remaining dynamic field  $\pi^0$  if the background satisfies Eqn (115). In that case, the one-derivative dispersion relation  $\omega^2 = 0$  is transformed into

$$\omega^2 = \operatorname{const} \frac{\mathbf{p}^4}{\Lambda^2} \,. \tag{116}$$

Therefore, the spectrum of the low-energy effective theory is the same as in the ghost condensate case, except that tensor gravitons are massive in the model discussed here.

Symmetry (109) protects the theory from the Boulware– Deser instability as well. In a nearly Minkowski space–time, and for a background nearly the same as in (111) and (112), a reference frame can be chosen such that the background  $\bar{\phi}^0$  has precisely form (111). In that frame, the above analysis retains its validity: the fields  $\pi^i$  are nondynamical in the lowenergy effective theory, at least for  $\omega^2$ ,  $\mathbf{p}^2 \ge m_G^2$ , and one dynamic mode associated with the field  $\pi^0$  remains. Its dispersion relation coincides with (116), modulo corrections proportional to the deviation of the background from Minkowski space.

To conclude this part, we digress to mention that a theory that shares a number of properties inherent in the model considered here was obtained in [92] in quite a different context of bi-gravity theories [93]. It contains two symmetric tensor fields,  $g_{1\mu\nu}$  and  $g_{2\mu\nu}$ , with their own Einstein-Hilbert actions, and involves a nonderivative coupling between them,

$$S_{\rm int} = \int d^4 x (g_1 g_2)^{1/4} V(g_1_{\mu\nu}, g_2_{\lambda\rho}) \,. \tag{117}$$

The entire theory is taken to be invariant under space-time diffeomorphisms. With fine tuning that sets the cosmological constant equal to zero, this theory allows a Lorentz-violating solution, for which the two metrics are flat but not proportional to each other [92]; in a certain reference frame,  $g_{1\mu\nu} =$ diag (1, -1, -1, -1) and  $g_{2\mu\nu} = a^2 \operatorname{diag} (c^2, -1, -1, -1)$ . Mixing term (117) gives rise to Lorentz-violating mass terms in the action for perturbations about this background. In the tensor sector of perturbations, there are two transverse traceless gravitons, one massless and one with a nonvanishing mass, which at high enough momenta propagate with different velocities  $v_1 = 1$  and  $v_2 = c$ , and oscillate from one to the other. Interestingly, the diffeomorphism invariance of the original theory imposes a number of constraints on the mass terms, one of which is analogous to the constraint  $m_1 = 0$ defining the model we discuss in this section. As a result, there are no propagating modes other than the two transverse traceless gravitons, unless derivative terms are added to action (117). At the linearized level, the gravitational potential between massive bodies in this bi-metric theory on a Lorentz-violating background generally has form (132), but the linearly increasing part can be eliminated by imposing a certain dilatation symmetry. All these features are direct counterparts of the properties of the model we consider in this section.

#### 5.2 Phenomenology

By analogy to conventional field theory, it may be expected that a nonzero graviton mass leads to an exponential suppression of the gravitational potential at distances greater than the inverse graviton mass. That mass would then be constrained by the experimental data. This *is not* the case in the model described by action (110), the reason being the violation of Lorentz invariance. We see below that the gravitational potential remains unchanged at the linear level, at least in some region of the parameter space. In this region, the behavior of the model is similar to general relativity in many respects and may be phenomenologically acceptable. At the same time, there may exist a number of interesting and potentially detectable effects, the nonzero graviton mass being one of them.

**5.2.1 Newton's law.** Newton's law emerges from general relativity in the linear approximation. To derive its analog in the model described by action (110), it is instructive to return

to the unitary gauge, where the perturbations of the Goldstone fields are absent and the only perturbations are those of the metric. This simplifies the comparison with general relativity. As in Section 2.1, it is convenient to decompose the metric perturbations in accordance with Eqn (1). The quadratic part of the action is then given by

$$L^{(2)} = L_{\rm EH}^{(2)} + L_{\rm m} + L_{\rm s} \,, \tag{118}$$

where  $L_{\rm EH}^{(2)}$ ,  $L_{\rm m}$ , and  $L_{\rm s}$  respectively come from the Einstein – Hilbert, mass, and source terms. The Einstein – Hilbert term is given by Eqn (6), and the mass and source terms are

$$L_{\rm m} = M_{\rm Pl}^2 \left\{ -\frac{1}{4} m_2^2 (h_{ij}^{\rm TT})^2 - \frac{1}{2} m_2^2 (\partial_i F_j)^2 + m_0^2 \varphi^2 + (m_3^2 - m_2^2) (\Delta E)^2 - 2(3m_3^2 - m_2^2) \psi \Delta E + 3(3m_3^2 - m_2^2) \psi^2 + 2m_4^2 \Delta E - 6m_4^2 \varphi \psi \right\},$$
(119)

$$L_{s} = -T_{00} \left( \varphi + \partial_{0} B - \partial_{0}^{2} E \right) - T_{ii} \psi + \left( S_{i} + \partial_{0} F_{i} \right) T_{0i} + \frac{1}{2} h_{ij} T_{ij} .$$
(120)

The notation for the masses is the same as in (74); the masses  $m_i^2$  are combinations of the first and second derivatives of the function F, the parameter  $\Lambda$ , and the Planck mass. As discussed above, their overall scale is  $m \sim \Lambda^2/M_{\rm Pl}$ . The source term contains an external energy – momentum tensor  $T_{\mu\nu}$ , which we assume to be conserved. All combinations coupled to the components of  $T_{\mu\nu}$  are gauge invariant. The one multiplying  $T_{00}$ ,

$$\Phi \equiv \varphi + \partial_0 B - \partial_0^2 E \,,$$

plays the role of the Newtonian potential in the nonrelativistic limit of general relativity.

In the tensor sector, only the transverse traceless perturbations  $h_{ij}^{\text{TT}}$  are present (two degrees of freedom). Their field equation is that of a massive field with the mass  $m_{\text{G}} = m_2$ . We note that the massive tensor field does not necessarily have five polarizations in a Lorentz-violating theory. Examples of this phenomenon have already been discussed in the previous sections.

In the vector sector, the field equations are

$$-\Delta(S_i + \widehat{o}_0 F_i) = -T_{0i}, \qquad (121)$$

$$\partial_0 \Delta (S_i + \partial_0 F_i) + m_2^2 \Delta F_i = \partial_0 T_{0i} \,. \tag{122}$$

Taking the time derivative of Eqn (121) and adding it to Eqn (122) gives

 $F_i = 0$ 

if  $m_2^2 \neq 0$ . Thus, the vector sector of the model behaves the same as in the Einstein theory in the gauge  $F_i = 0$ . There are no propagating vector perturbations and the interaction of sources is not modified in the vector sector unless nonlinear effects or higher-derivative terms are taken into account.

The interaction potential between static sources (the Newtonian potential) is determined by the scalar sector of

the model. The field equations for scalar perturbations are

$$2\Delta\psi + m_0^2\varphi + m_4^2\Delta E - 3m_4^2\psi = \frac{T_{00}}{2M_{\rm Pl}^2}, \qquad (123)$$

$$2\Delta \Phi - 2\Delta \psi + 6\partial_0^2 \psi - (3m_3^2 - m_2^2) \Delta E + 3(3m_3^2 - m_2^2) \psi - 3m_4^2 \varphi = \frac{T_{ii}}{2M_{\rm Pl}^2}, \qquad (124)$$

$$-2\Delta \hat{\sigma}_{0}^{2}\psi + (m_{3}^{2} - m_{2}^{2})\Delta^{2}E - (3m_{3}^{2} - m_{2}^{2})\Delta\psi + m_{4}^{2}\Delta\varphi = -\frac{\hat{\sigma}_{0}^{2}T_{00}}{2M_{\rm Pl}^{2}}, \qquad (125)$$

$$2\Delta \partial_0 \psi = \frac{\partial_0 T_{00}}{2M_{\rm Pl}^2} \,. \tag{126}$$

Equation (126) implies

$$\psi = \frac{1}{\Delta} \frac{T_{00}}{4M_{\rm Pl}^2} + \psi_0(x^i) , \qquad (127)$$

where  $\psi_0(x^i)$  is an arbitrary time-independent function. Equations (123) and (125) imply that

$$\varphi = \frac{2m_2^2 m_4^2}{\mathfrak{M}} \psi + \frac{2(m_3^2 - m_2^2)}{\mathfrak{M}} \Delta \psi_0 , \qquad (128)$$

$$\Delta E = \left(3 - \frac{2m_0^2 m_2^2}{\mathfrak{M}}\right) \psi - \frac{2m_4^2}{\mathfrak{M}} \,\Delta \psi_0 \,, \tag{129}$$

where

$$\mathfrak{M} = m_4^4 - m_0^2 (m_3^2 - m_2^2) \,.$$

Finally, substituting Eqns (127)-(129) in Eqn (124), we find the gauge-invariant potential

$$\Phi = \frac{1}{\Delta} \frac{T_{00} + T_{ii}}{4M_{\rm Pl}^2} - 3 \frac{\partial_0^2}{\Delta^2} \frac{T_{00}}{4M_{\rm Pl}^2} + \left(3 - \frac{2m_0^2m_2^2}{\mathfrak{M}}\right) \\
\times \frac{m_2^2}{\Delta} \left(\frac{1}{\Delta} \frac{T_{00}}{4M_{\rm Pl}^2} + \psi_0\right) + \left(1 - \frac{2m_2^2m_4^2}{\mathfrak{M}}\right)\psi_0. \quad (130)$$

The first two terms in the right-hand side of Eqn (130) are the standard contributions in the Einstein theory, the first one becoming the Newtonian potential in the nonrelativistic limit. Thus, except for the  $\psi_0$ -dependent terms, the gauge-invariant potentials  $\Phi$  and  $\psi$  differ from their analogs in the Einstein theory  $\Phi_E$  and  $\psi_E$  by the mass-dependent third term in the right-hand side of Eqn (130),

$$\psi = \psi_{\rm E} ,$$
  

$$\Phi = \Phi_{\rm E} + \left(3 - \frac{2m_0^2 m_2^2}{\mathfrak{M}}\right) \frac{m_2^2}{\Delta^2} \frac{T_{00}}{4M_{\rm Pl}^2} .$$
(131)

The second term in Eqn (131) vanishes if all masses uniformly tend to zero, and hence, in the massless limit, both potentials  $\psi$  and  $\Phi$  become the same as in general relativity. This means the absence of the vDVZ discontinuity in the model.

For a static source, Eqn (131) leads to a modification of the Newtonian potential of a point mass M, which in coordinate space takes the form

$$\Phi = G_{\rm N} M \left( -\frac{1}{r} + \mu^2 r \right), \tag{132}$$

where

$$\mu^{2} = -\frac{1}{2} m_{2}^{2} \left( 3 - \frac{2m_{0}^{2}m_{2}^{2}}{\mathfrak{M}} \right).$$
(133)

Because the potential is increasing, the perturbation theory breaks down at distances  $r \gtrsim 1/(G_N M \mu^2)$ . This would be unacceptable for relatively large graviton masses. But the modification of the potential is absent in the case  $3\mathfrak{M} = 2m_0^2 m_2^2$  (and  $\mathfrak{M} \neq 0$ ). We see in what follows that this condition can be ensured by a particular dilatation symmetry,<sup>19</sup> which is automatically enforced at the cosmological attractor, i.e., at late times of the cosmological evolution.

The freedom in choosing the time-independent function  $\psi_0(x)$  that enters the above gravitational potentials corresponds to the presence of the scalar mode with the dispersion relation  $\omega^2 = 0$ . As discussed in Section 5.1, this mode is an analog of the ghost condensate mode and becomes dynamical with the higher-derivative terms in the action taken into account, acquiring the dispersion relation  $\omega^2 \propto \mathbf{p}^4$ . The value of  $\psi_0$  is determined by the initial conditions. In the linear regime, a nonzero value of  $\psi_0$  would mean the presence of an incoming 'ghost condensate wave.' Hence, for the purpose of finding the potential between sources, the physical choice is  $\psi_0(x^i) = 0$ . We note, however, that this choice is not so evident in the cosmological context.

**5.2.2 Cosmological solutions.** At the time of writing this review, only spatially flat cosmological solutions are known in model (110). The flat cosmological ansatz is<sup>20</sup>

$$ds^{2} = dt^{2} - a^{2}(t) dx_{i}^{2},$$
  

$$\phi^{0} = \phi(t), \quad \phi^{i} = A^{2}x^{i}.$$
(134)

For this ansatz,  $W^{ij} = -a^{-2}\delta^{ij}$ , and hence the function *F* in Eqn (110) depends only on *X* and *a*, F = F(X, a). The Einstein equations reduce to the Friedman equation,

$$\left(\frac{\dot{a}}{a}\right)^{2} = \frac{1}{6M_{\text{Pl}}^{2}} \left\{ \rho_{\text{m}} + 2\Lambda^{4}XF_{X} - \Lambda^{4}F \right\}$$
$$\equiv \frac{1}{6M_{\text{Pl}}^{2}} \left\{ \rho_{\text{m}} + \rho_{1} + \rho_{2} \right\}, \qquad (135)$$

where  $\rho_{\rm m}$  is the energy density not including Goldstone fields. The field equation for  $\phi^0$  is

$$\partial_t \left( a^3 \sqrt{X} F_X \right) = 0. \tag{136}$$

The field equations for  $\phi^i$  are satisfied automatically. In principle, it is straightforward to solve this system of equations for any given function F(X, a). Upon integration, Eqn (136) gives an algebraic equation that determines X as a function of the scale factor a. This makes Eqn (135) a closed equation for the scale factor a(t).

From the standpoint of cosmological applications, of particular interest are solutions where the scale factor a(t) tends to infinity at late times. Because the graviton masses are linear combinations of the function F(X, a) and its derivatives, the question arises as to whether they remain finite or tend to zero in this limit and whether the effective-theory description remains valid. Indeed, Eqn (136) implies that either X or  $F_X$  tends to zero at late times, which suggests that the graviton masses might tend to zero as well.

We consider a particular class of functions F such that X(a) found from Eqn (136) asymptotically tends to some power of a at large a. This is not a very restrictive assumption: for instance, it is satisfied by any algebraic function F(X, a). Then there exists a real constant  $\gamma$  such that the combination  $X^{\gamma}/a^2$  tends to a nonzero value as  $a \to \infty$ . Equation (136) implies that  $XF_X = \text{const} \sqrt{X}/a^3$ ; this determines the dependence of the energy component  $\rho_1$  on the scale factor,

$$\rho_1 = \operatorname{const} \frac{1}{a^{3-1/\gamma}} \,. \tag{137}$$

This relation generalizes the behavior found in the ghost condensate model, where the energy density of the ghost condensate scales as  $1/a^3$  [38] (this behavior is recovered from Eqn (137) as  $\gamma \to \infty$ ).

For  $\gamma > 1/3$ , the energy density  $\rho_1$  behaves like the dark energy component with negative pressure. Its equation of state varies between that of cold dark matter, w = 0 (for  $\gamma = +\infty$ ), and that of the cosmological constant, w = -1 (for  $\gamma = 1/3$ ). For  $0 < \gamma < 1/3$ , the term  $\rho_1$  increases with *a*. This corresponds to the energy density component with a highly negative equation of state, w < -1. Without fine tuning, this contribution cannot be canceled by the term  $\rho_2$ , and hence the Hubble rate diverges as  $a \to \infty$ , leading to the breakdown of the low-energy effective theory and rapid instabilities [94]. In what follows, we assume that  $\gamma$  does not belong to this range. For  $\gamma < 0$ , the energy density  $\rho_1$  corresponds to a fluid with positive pressure.

To see that the graviton masses remain finite and the effective field theory description is valid in the limit  $a \to \infty$ , it is convenient to replace X by the new variable  $Z = X^{\gamma}/a^2$ . The function F(X, a) becomes a function of Z and a,  $\tilde{F}(Z, a) = F(Z^{1/\gamma}a^{2/\gamma}, a)$ . We note that it satisfies the relation  $\gamma Z \tilde{F}_Z = X F_X$ , where  $\tilde{F}_Z = \partial \tilde{F}/\partial Z$ . With this notation, Eqn (136) becomes

$$\gamma a^{3-(1/\gamma)} Z^{1-(1/2\gamma)} \tilde{F}_Z(Z,a) = A,$$
 (138)

where A is an integration constant. This equation determines Z as a function of a. By construction, this dependence is such that  $Z(a \rightarrow \infty) = Z_0$ , where  $Z_0$  is some constant.

If we assume further that the function F(Z, a) is regular as  $a \to \infty$ , then at late times we have

$$F(X,a) = \tilde{F}(Z,a) \to F_0(Z).$$
(139)

In terms of the original variables, this means that in the limit  $a \to \infty$ , the function  $F(X, W^{ij})$  depends only on the combination  $X^{\gamma}W^{ij}$ . This corresponds to the following dilatation symmetry of the Goldstone action:

$$\begin{split} \phi_0 &\to \lambda \phi_0 \,, \\ \phi_i &\to \lambda^{-\gamma} \phi_i \,, \end{split} \tag{140}$$

<sup>&</sup>lt;sup>19</sup> Like other symmetries discussed in Section 3.3.3, this dilatation symmetry, Eqn (140), may be regarded as an unbroken part of the diffeomorphism invariance.

<sup>&</sup>lt;sup>20</sup> In principle, a more general time-dependent ansatz for the scalar fields can be written, namely,  $\phi^i = \Lambda^2 C(t) x^i$ , where C(t) is an arbitrary function of time. For models respecting symmetry (140), which we mainly consider in what follows, this ansatz leads to the same cosmological evolution as ansatz (134) (see Ref. [41] for the details).

which is equivalent, in the unitary gauge, to the unbroken part of the diffeomorphism invariance,  $t \to \lambda t, x^i \to \lambda^{-\gamma} x^i$ . In this case, we have

$$\rho_2 = -\Lambda^4 F_0(Z_0) \,,$$

which behaves like a cosmological constant (assuming  $F_0(Z_0) \neq 0$ ). Likewise, as  $a \to \infty$ , the graviton masses become functions of  $Z_0$  and remain finite in general.

The models obeying Eqn (139) have an interesting feature, which is a consequence of symmetry (140). It is straightforward to verity that Eqn (140) implies the following relations among the graviton masses in Minkowski space:

$$m_0^2 = -3\gamma m_4^2$$
,  $\gamma (m_2^2 - 3m_3^2) = m_4^2$ . (141)

These relations ensure that the parameter  $\mu^2$  defined by Eqn (133) is zero, i.e., the correction to the Newtonian potential (the last term in Eqn (131)) vanishes. Thus, at late times, apart from the effects of the higher-derivative terms, the only modification of gravity at the linear level is the nonzero mass of the two polarizations of the graviton.

A particularly simple case occurs if the function F depends only on the combination

$$Z^{ij} = X^{\gamma} W^{ij} \,. \tag{142}$$

If  $\gamma > 1/3$  or  $\gamma < 0$ , the evolution drives the system to the point  $\tilde{F}_Z = 0$ , in full similarity with the ghost condensate model. In the case  $0 < \gamma < 1/3$  and regular  $\tilde{F}$ ,  $Z^{ij}$  diverges at large *a*. This breaks the validity of the low-energy effective theory.

**5.2.3 Massive gravitons.** We consider the properties of massive gravitons in more detail; specifically, we consider the experimental constraints on the graviton mass and the possibility of graviton creation in the early Universe. For simplicity, in this section, we limit ourselves to the model with the action

$$S_{\rm G} = \Lambda^4 \int \mathrm{d}^4 x \sqrt{-g} F(Z^{ij}) \,, \tag{143}$$

where the  $Z^{ij}$  are given by Eqn (142). In this model, there are no corrections to the Newtonian potential at the linear level, and therefore the tests of (linearized) gravity based on the Solar System and Cavendish-type experiments [95] are automatically satisfied. The constraints on the graviton mass come from the emission and/or propagation of gravitational waves.

Observations of the slowdown of the orbital motion in binary pulsar systems [96] are considered an indirect proof of the existence of gravitational waves. The agreement of these observations with general relativity implies that the graviton mass cannot be larger than the characteristic frequency of the emitted gravitational waves. This frequency is set by the period of the orbital motion, which is of the order of 10 hours, implying the following limit on the graviton mass:

$$\frac{m_{\rm G}}{2\pi} \equiv v_{\rm G} \lesssim 3 \times 10^{-5} \,\mathrm{Hz} \approx (10^{15} \,\mathrm{cm})^{-1} \sim (70 \,\mathrm{a.u.})^{-1}. \,(144)$$

Thus, the maximum allowed graviton mass is comparable to the inverse size of the Solar System, which is a very large mass (short distance) by cosmological standards. Gravitons of such a mass can serve as dark matter candidates if they can be produced in the early Universe in sufficient numbers. Indeed, if the graviton mass is large enough,  $(mv)^{-1} \leq 1 \text{ kpc} \sim 3 \times 10^{21} \text{ cm}$ , where  $v \sim 10^{-3}$  is the typical velocity in the halo, then massive gravitons may cluster in galaxies and account for the dark matter in galactic halos.

It is straightforward to estimate the cosmological abundance of relic massive gravitons. The massive gravitons are described by the transverse traceless perturbation of the metric,  $h_{ij}^{\text{TT}}$ . The quadratic action for  $h_{ij}$  (we omit the superscript in what follows) in the expanding Universe has the form

$$M_{\rm Pl}^2 \int d^3k \, d\eta a^2(\eta) \left[ (h_{ij}')^2 - (\partial_k h_{ij})^2 - m_{\rm G}^2 a^2(\eta) \, h_{ij}^2 \right], \, (145)$$

where  $\eta$  is conformal time and the prime denotes  $\partial/\partial \eta$ . Equation (145) has the form of the action of a minimally coupled massive scalar field. Therefore, similarly to scalar bosons, massive gravitons are produced efficiently during inflation (cf. Ref. [97]).

To be specific, we consider a scenario where the Hubble parameter  $H_i$  is constant during inflation. This scenario may be realized, for instance, in hybrid models of inflation [98]. First, it must be verified that the phenomenologically relevant values of parameters correspond to the regime below the cutoff scale of the effective theory, i.e.,  $H_i \leq \Lambda$ . For the energy scale of inflation  $E_i \sim \sqrt{H_i M_{\text{Pl}}}$ , this implies that

$$E_{\rm i} < m_{\rm G}^{1/4} M_{\rm Pl}^{3/4} \approx 10^7 \,{\rm GeV} \,(m_{\rm G} \times 10^{15} \,{\rm cm})^{1/4} \,.$$
 (146)

This value is high enough to be consistent with everything else in cosmology (in particular, to allow successful baryogenesis), even for graviton masses of the order of the current Hubble scale.

We next consider the production of massive gravitons. With the above scenario of inflation assumed, the perturbation spectrum for massive gravitons is that for the minimally coupled massive scalar field in the de Sitter space [99],

$$\langle h_{ij}^2 \rangle \simeq \frac{1}{4\pi^2} \left(\frac{H_i}{M_{\rm Pl}}\right)^2 \int \frac{\mathrm{d}k}{k} \left(\frac{k}{H_{\rm i}}\right)^{2m_{\rm G}^2/(3H^2)}.$$
 (147)

Importantly, for long enough inflation, the present physical momenta of most of the gravitons are smaller than the present Hubble scale.

Metric fluctuations remain frozen until the Hubble parameter becomes smaller than the graviton mass, and afterwards they start to oscillate with the amplitude decreasing as  $a^{-3/2}$ . The energy density in massive gravitons at the beginning of oscillations is of the order of

$$\rho_* \sim M_{\rm Pl}^2 m_{\rm G}^2 \langle h_{ij}^2 \rangle \simeq \frac{3 H_i^4}{8 \pi^2} \ , \label{eq:rho}$$

where we neglected a prefactor, which is roughly of the order of unity. Today, the fraction of the energy density in massive gravitons is

$$\Omega_{\rm g} = \frac{\rho_*}{z_*^3 \rho_{\rm c}} = \frac{\rho_*}{z_{\rm e}^3 \rho_{\rm c}} \left(\frac{H_{\rm e}}{H_*}\right)^{3/2},\tag{148}$$

where  $z_*$  is the redshift at the start of oscillations,  $H_* \sim m_{\rm G}$  is the Hubble parameter at that time,  $H_{\rm e} \approx 0.4 \times 10^{-12} \, {\rm s}^{-1}$  is

the Hubble parameter at the matter/radiation equality, and  $z_e \approx 3200$  is the corresponding redshift. Combining all factors, we obtain

$$\Omega_{\rm g} \sim 3 \times 10^3 (m_{\rm G} \times 10^{15} \,{\rm cm})^{1/2} \left(\frac{H_{\rm i}}{\Lambda}\right)^4.$$
 (149)

This estimate assumes that the number of e-foldings during inflation is large,  $\ln N_e > H^2/m_G^2$ , which is quite natural in the inflation model considered here.

According to Eqn (149), massive gravitons are produced efficiently enough to comprise all of the cold dark matter if the value of the Hubble parameter during inflation is about one order of magnitude below the scale  $\Lambda$ . Interestingly, it follows that  $\Omega_{\rm g} \sim 1$  when the initial energy density in the metric perturbations is close to the cutoff scale,  $\rho_*^{1/4} \sim \Lambda$ . This suggests that other mechanisms of production unrelated to inflation may naturally lead to the same result,  $\Omega_{\rm g} \sim 1$ .

If massive gravitons have been produced in substantial amounts during the evolution of the Universe, they can be observed by gravitational wave detectors. At distances shorter than the wavelength, the effect of a transverse traceless gravitational wave on test massive particles in the Newtonian approximation is described by the acceleration  $\dot{h}_{ij}x^j/2$  (see, e.g., Ref. [96] for a review). The same is true for massive gravitational waves, the only difference being that the wavelengths are longer in the nonrelativistic case, and therefore the Newtonian description works for a larger range of distances. Thus, nonrelativistic waves of the same frequency.

To estimate the amplitude of the gravitational waves, we assume that they comprise all of the dark matter in the halo of our Galaxy. The energy density in nonrelativistic gravitational waves is of the order of  $M_{\rm Pl}^2 m_G^2 h_{ij}^2$ . Equating this to the local halo density  $\rho_0 \sim 0.3$  GeV cm<sup>-3</sup>, we obtain

$$h_{ij} \sim 10^{-10} \left( \frac{3 \times 10^{-5} \,\mathrm{Hz}}{v_{\mathrm{G}}} \right).$$
 (150)

At frequencies  $10^{-6} - 10^{-5}$ Hz, this value is many orders of magnitude above the expected sensitivity of the LISA detector [101]. Thus, LISA may observe massive gravitational waves even if their abundance is much lower than that required to play the role of dark matter. We note that in the nearby frequency range  $10^{-9} - 10^{-7}$ Hz, there is a restrictive bound [102] at the level  $\Omega_g < 10^{-9}$  on the stochastic background of gravitational waves, coming from the timing of millisecond pulsars [103]. Hence, it is possible that the model can be tested by the re-analysis of the already existing data on pulsar timing. This re-analysis would have to take into account that, unlike the usual gravitational waves, relic massive gravitons produce a monochromatic line at the frequency equal to the graviton mass. Such a narrow line with the relative width  $\Delta v/v \sim 10^{-6}$  is a distinctive signature of the model.

Another possible signature is the time delay of a gravitational wave signal compared to electromagnetic radiation. In terms of the wave frequency f and the distance D to the source, the time delay is given by

$$\Delta t = \frac{D}{2} \left(\frac{m_{\rm G}}{2\pi f}\right)^2$$

(assuming that  $f \ge m_G$ ). As an example, we consider gravitational waves emitted during the merger of two massive black

holes — one of the promising processes from the standpoint of gravitational wave detection. The frequency of these waves is of the order of the gravitational radius of the resulting black hole,

$$f \sim R_{\rm S}^{-1} = \frac{M_{\rm Pl}^2}{2M} \,,$$

where *M* is the black hole mass. Thus, for  $m_{\rm G} \sim 10^{-15} \,{\rm cm}^{-1}$ , the time delay is

$$\Delta t \sim \frac{D}{2} \left(\frac{Mm_{\rm G}}{\pi M_{\rm Pl}^2}\right)^2 \sim 5 \times 10^{-6} \left(\frac{D}{\rm Mpc}\right) \left(\frac{M}{M_{\odot}}\right)^2 [\rm s].$$

This is probably too small to be detected for solar-mass black holes, but may be detectable for heavier ones.

5.2.4 Refined cosmological tests: growth of perturbations. Given that some models of massive gravity pass the most obvious experimental tests, the question arises whether they may provide a viable alternative to general relativity in describing subtler effects. One of these effects is the structure formation. In the standard cosmology based on general relativity, the formation of the observed structure in the Universe is explained by the growth of primordial perturbations, mostly during the matter-dominated stage (see, e.g., [104, 105] and the references therein). The conventional theory is in good agreement with observations if the dark matter component has the right properties [8, 106, 107]. It is not obvious that general relativity can be modified without spoiling this agreement. We demonstrate in this section that the massive gravity model described by action (143) is an example of such a modification, i.e., this model successfully passes the structure formation test even though the graviton mass is very large by cosmological standards. This again illustrates the fact that in a Lorentz-violating theory, the mass of a transverse traceless graviton has very little to do with the properties of 3-dimensionally scalar modes.

Perturbations relevant for structure formation are 3-dimensional scalars. In massive gravity, the scalar sector contains additional scalar fields that may alter the growth rate and make the model incompatible with observations. Without gauge fixing, the scalar sector contains metric perturbations  $\varphi$  (not to be confused with the Goldstone fields  $\phi^0$ ,  $\phi^i$ ), *B*,  $\psi$ , and *E* defined in accordance with Eqns (5) and (55), perturbations of the Goldstone fields  $\pi_0$  and  $\pi_L$  (the longitudinal part of  $\pi_i$ ), and perturbations of ordinary matter. In total, there are 9 scalar perturbations, one of which can form 7 gauge-invariant combinations whose dynamics are responsible for the structure formation. The complete set of equations that govern the behavior of these perturbations can be found in Ref. [108].

The system of equations for perturbations can be reduced to two equations for the gauge-invariant gravitational potentials  $\Phi$  and  $\Psi$ . In general relativity, they satisfy the relation  $\Phi - \Psi = 0$ . In massive gravity, this relation changes to

$$\Phi - \Psi = \vartheta(x^i) \, a^{1/\gamma - 1} \,, \tag{151}$$

where  $\vartheta(x^i)$  is an arbitrary function of spatial coordinates, which arises as an integration constant. The origin of this constant is the presence of a mode with the dispersion relation  $\omega^2 = 0$ . We have already encountered the appearance of such a constant in Section 5.2.1. The second equation is a closed equation for  $\Psi$ ,

$$\frac{\partial^2 \Psi}{\partial a^2} + \frac{1}{a} \left( 4 + 3c_s^2 + \frac{H'}{H^2} \right) \frac{\partial \Psi}{\partial a} + \frac{1}{a^2} \left[ \left( 1 + 3c_s^2 \right) + 2\frac{H'}{H^2} - \frac{c_s^2 \Delta}{H^2} \right] \Psi = \left[ \frac{\gamma c_s^2 \Delta}{H^2} - \left( 3c_s^2 + \frac{1}{\gamma} + 2\frac{H'}{H^2} \right) \right] \vartheta a^{1/\gamma - 3} .$$
(152)

In terms of the solutions of this equation, the density contrast is expressed as

$$\delta_{\rho} = \frac{2M_{\rm Pl}^2}{\rho_{\rm m}} \left( \gamma \Delta - 3H^2 \right) a^{1/\gamma - 3} \vartheta - \frac{2M_{\rm Pl}^2}{a^2 \rho_{\rm m}} \left[ 3H^2 \left( 1 + a \frac{\partial}{\partial a} \right) - \Delta \right] \Psi, \qquad (153)$$

where  $\vartheta$  is the same time-independent function of the spatial coordinates as in Eqn (151).

The standard cosmological perturbations are recovered by setting the graviton masses to zero,  $m_i^2 = 0$ . In this case,  $\Phi - \Psi = 0$ , i.e.,  $\vartheta(x^i) = 0$ . Then the equations for perturbations become identical to those in the Einstein theory. We note that the function  $\vartheta$  is determined by the initial conditions. Setting  $\vartheta = 0$  would eliminate the  $\vartheta$ -dependent terms in Eqns (152) and (153) and bring these equations to the conventional form, even in the case  $m_G^2 \neq 0$ . Hence, there always exist initial conditions such that model (143) exhibits the standard rate of perturbation growth and is therefore compatible with observations. Furthermore, at some values of the parameter  $\gamma$ , the part of the perturbations that is proportional to  $\vartheta(x^i)$  grows more slowly than the conventional part and is therefore subdominant, such that the agreement with observations is achieved for any function  $\vartheta(x^i)$  unless it is too large.

In the case of matter perturbations in a matter-dominated universe, Eqn (152) reduces to the equation

$$\frac{\partial^2 \Psi}{\partial a^2} + \frac{7}{2a} \frac{\partial \Psi}{\partial a} + \left(\frac{1}{\gamma} - 1\right) a^{1/\gamma - 3} \vartheta = 0,$$

which differs from the standard case by the presence of the inhomogeneous term proportional to  $\vartheta$ . The solution of this equation is given by

$$\Psi = -\frac{2\gamma}{2+3\gamma} a^{1/\gamma-1} \vartheta(x^i) + a^{-5/2} c_1(x^i) + c_2(x^i) ,$$

where  $c_i(x^i)$  are integration constants. Substituting this solution in Eqn (153), we find the density contrast

$$\delta_{\rho} = \left(\frac{2M_{\text{Pl}}^2 a}{\rho_0} \Delta + 3\right) \frac{c_1(x^i)}{a^{5/2}} + 2\left(\frac{aM_{\text{Pl}}^2}{\rho_0} \Delta - 1\right) c_2(x^i) + \frac{6\gamma}{2+3\gamma} a^{1/\gamma-1} \left(\frac{a\gamma M_{\text{Pl}}^2}{\rho_0} \Delta - 1\right) \vartheta(x^i), \qquad (154)$$

where  $\rho_0$  is the present energy density of matter. The first two terms in this equation are precisely the ones that appear in the standard Einstein theory, the second term describing the linear growth of the perturbations,  $\delta_\rho \propto a$ . The difference from the conventional case is in the third term in the righthand side of Eqn (154). The perturbations corresponding to this term increase proportionally to  $a^{1/\gamma}$ . For  $\gamma > 1$  or  $\gamma < 0$ , these 'anomalous' perturbations increase more slowly than the standard ones.

At the epoch of radiation domination, the situation is similar. For a relativistic fluid, we have  $c_s^2 = w = 1/3$ , and hence Eqn (152) becomes

$$\frac{\partial^2 \Psi}{\partial a^2} + \frac{4}{a} \frac{\partial \Psi}{\partial a} - \frac{M_{\rm Pl}^2 \Delta}{\rho_{\rm r}} \Psi + \left(\frac{1}{\gamma} - 1 - \frac{a^2 \gamma M_{\rm Pl}^2 \Delta}{\rho_{\rm r}}\right) a^{1/\gamma - 3} \vartheta = 0,$$
(155)

where  $\rho_r$  is the present energy density of radiation. For a generic value of  $\gamma$ , the solution of this equation is cumbersome. For simplicity, we concentrate on the modes that are shorter than the horizon size,  $k^2/a^2 \gg H^2$ . The density contrast calculated in accordance with Eqn (153) has the standard oscillating piece and an extra part proportional to  $\vartheta$ ,

$$\delta_{\rho} \sim c_{1}(x^{i}) \sin y + c_{2}(x^{i}) \cos y + 2\gamma \left(\frac{\rho_{r}}{k^{2} M_{Pl}^{2}}\right)^{(1/\gamma - 1)/2} \\ \times \left[ -y^{1 + 1/\gamma} + \int_{0}^{y} \mathrm{d}x \, x^{1 + 1/\gamma} \sin(y - x) \right] \vartheta \,, \tag{156}$$

where  $y = \eta k/\sqrt{3}$  is proportional to the scale factor and  $c_i(x^i)$  are two integration constants. It follows from this expression that for  $-1 \le \gamma < 0$ , the  $\vartheta$ -dependent contribution to the density contrast decays in time, and hence only the standard contribution remains. Thus, in this range of  $\gamma$ , the perturbations behave just as predicted by general relativity in both the matter and radiation-dominated epochs.

Another interesting case is  $\gamma = 1$ . This case is special because the *a*-dependence of the last term in Eqn (155) disappears at  $\gamma = 1$ . In fact, it can be shown in this case that the dependence on  $\vartheta$  cancels in the density contrast, and hence only the standard part of perturbations remains.

At other values of  $\gamma$ ,  $\vartheta$ -dependent contributions to the perturbations increase in the radiation-dominated Universe. Whether a model of this sort is compatible with observations depends on the unknown function ('integration constant')  $\vartheta(x^i)$ . It is worth noting that this function may become slowly varying in time when higher-derivative corrections to action (143) are taken into account. It remains to be understood whether these corrections can drive  $\vartheta(x^i)$  to zero during inflation, in which case the dependence on the initial value of  $\vartheta(x^i)$  is eliminated and the model is compatible with observations at any value of the parameter  $\gamma$ .

**5.2.5** Nonlinear solutions: black holes. The approach based on Goldstone fields with action (86) (compared, e.g., to the Fierz–Pauli model) is fully nonlinear. We have already used this fact in Section 5.2.2, where we derived cosmological solutions in massive gravity. Another interesting question related to nonlinear gravitational dynamics is the existence and properties of black holes. Rapid progress in observational techniques will allow a quantitative study of astrophysical black holes in the near future, including mapping the metric near the black hole horizon [109–114]. It is therefore important to understand what kind of deviations from general relativity are possible, at least in principle.

Black hole properties are universal in general relativity in the sense that the black hole metric is uniquely characterized by the black hole mass and angular momentum. This property is related to the causal structure of the black hole space-time and is a consequence of the 'no-hair' theorems [115–118]. Thus, properties of black holes are extremely 'resistant' to modifications of gravity theories. For example, they remain unchanged in scalar – tensor theories [115, 116, 119. 120]. Constructing an alternative model of a black hole is therefore a challenging problem.

We see in this section that black hole properties in massive gravity do differ from those in general relativity. In other words, black holes do have 'hair' in massive gravity. The origin of this hair lies in the instantaneous interaction present in Lorentz-violating massive gravity. Their existence is thus related to the mode with the dispersion relation  $\mathbf{p}^2 = 0$ , which is in turn a consequence of symmetry (83), as was noted in Section 5.2. For simplicity, we again limit our discussion to the particular class of models with action (143). Because the presence of the instantaneous interaction is a generic property of Lorentz-violating massive gravities (in particular, models with action (110)), we expect our conclusions to apply to a wider class of models than considered in this section.

The most straightforward approach to the problem would be to try to find black hole solutions explicitly and to see if they differ from general relativity black holes. But this appears to be a prohibitively difficult task. The problem may be simplified by addressing a slightly different question: does massive gravity have a black hole solution with exactly the same metric as in general relativity? Answering this question requires finding a configuration of the Goldstone fields such that for the given black hole metric, all equations of motion (the Einstein equations and the equations of motion of the scalar fields) are satisfied. If this is possible, then the solution with the given metric exists. Alternatively, if this is not possible, black hole solutions are modified in massive gravity.

For the black hole metric to be a solution of the Einstein equations, the energy–momentum tensor of the Goldstone fields  $\phi^0$  and  $\phi^i$  must vanish in the exterior of the black hole,

$$0 = T_{\mu\nu} = -g_{\mu\nu}F + 2\frac{\delta F}{\delta W^{ij}} \left\{ \left( \gamma \frac{W^{ij}}{X} + \frac{V^i V^j}{X^2} \right) \partial_\mu \phi^0 \partial_\nu \phi^0 + X^\gamma \partial_\mu \phi^i \partial_\nu \phi^j - \frac{V^i}{X} \left( \partial_\mu \phi^0 \partial_\nu \phi^j + \partial_\nu \phi^0 \partial_\mu \phi^j \right) \right\}, \quad (157)$$

where  $g_{\mu\nu}$  is the black hole metric. It is clear from Eqn (157) that with a possible exception of some very special functions *F*, the energy–momentum tensor does not vanish, because the vanishing would require 10 equations to be satisfied for 4 unknowns.

We now recall that our model is constructed such that the energy – momentum tensor of the Goldstone fields vanishes in Minkowski space. This is achieved by choosing the vacuum solutions for the Goldstone fields, Eqns (111) and (112), such that Eqns (113) are satisfied. For model (143), these equations give

$$F = 0, \qquad \frac{\delta F}{\delta W^{ij}} = 0$$

in the Minkowski vacuum, where

$$W^{ij} = -\delta^{ij}.\tag{158}$$

Thus, we can make  $T_{\mu\nu}$  vanish if we find a configuration of the Goldstone fields such that Eqns (158) are satisfied *in the background metric of the black hole.* 

There are fewer equations in (158) than in (157), but they are still too many: system (158) contains 6 differential

equations for only 4 unknown functions  $\phi^0$  and  $\phi^i$ . Consequently, if there are no degeneracies, these equations cannot be satisfied and we expect that the Goldstone fields cannot be adjusted such that their energy–momentum tensor is zero.

An equivalent form of Eqn (158) can be obtained by passing to the unitary gauge. In this gauge, Eqn (158) becomes

$$(g^{00})^{\gamma}g_{ii}^{-1} = -\delta^{ij}.$$
(159)

In geometrical terms, solving Eqn (159) is equivalent to finding, for a given metric, the coordinate frame in which the constant-time slices are conformally flat. This reformulation of Eqn (158) is particularly convenient.

In the case of the Schwarzschild black hole, there actually *exists* a solution of Eqns (158). Equivalently, there exists a coordinate frame in which the spatial part of the metric is conformally flat, the so-called Gullstrand – Painlevé frame. In this frame, the black hole metric has the form

$$ds^{2} = d\tau^{2} - \left(dx^{i} - \frac{R_{S}^{1/2}}{r^{3/2}} x^{i} d\tau\right)^{2},$$

where  $R_{\rm S}$  is the Schwarzschild radius of the black hole and  $r = \sqrt{x_i^2}$ , and the scalar field configuration that solves Eqns (158) is simply

$$\phi^0 = \Lambda^2 \tau, \qquad \phi^i = \Lambda^2 x^i. \tag{160}$$

Transforming back to the Schwarzschild coordinates, we find

$$\phi^{0} = \Lambda^{2} \left[ t + 2\sqrt{rR_{\rm S}} + R_{\rm S} \ln \left( \frac{\sqrt{r} - \sqrt{R_{\rm S}}}{\sqrt{r} + \sqrt{R_{\rm S}}} \right) \right].$$

with the  $\phi^i$  still given by Eqns (160). Thus, Schwarzschild black holes are solutions of massive gravity as well.

The situation is different in the case of a rotating black hole: the above miracle does not happen and, as expected, Eqns (158) and (159) do not have solutions. In fact, conformally flat spatial slicings are an important ingredient in the numerical simulations of black hole mergers, and their existence for various solutions of the Einstein equations has therefore been extensively studied [121, 122]. In particular, it was proved that a conformally flat slicing of the Kerr metric is impossible due to the existence of a nontrivial invariant of the quadrupole origin [122],

$$\Upsilon = -112\pi J^2 \,. \tag{161}$$

Moreover, the results in Ref. [122] imply that not only the Kerr metric but also an arbitrary axisymmetric vacuum solution of the Einstein equations with nonzero angular momentum has a nonvanishing value of  $\Upsilon$  and, consequently, does not allow conformally flat spatial slicings. Therefore, there are no configurations of the Goldstone fields such that their energy–momentum tensor is zero in the background of the Kerr or any other metric with nonzero angular momentum. Consequently, rotating black holes in massive gravity have to be different from those in the Einstein theory.

The fact that rotating black holes are modified in the presence of Goldstone fields, as compared to their general relativity counterparts, is in accordance with the expectation that black holes may have 'hair' in massive gravity. The existence of hair can be demonstrated explicitly in a simplified model of Lorentz-violating electrodynamics with the action [79]

$$S = S_{\rm EH} + \int d^4x \sqrt{-g} \left\{ F(X) - \frac{1}{4} F_{\mu\nu}^2 + m^2 G^{\mu\nu} A_{\mu} A_{\nu} \right\},$$
(162)

where  $S_{\rm EH}$  is the Einstein-Hilbert action, X is given by Eqn (87), and  $G^{\mu\nu}$  is the 'effective metric'

$$G^{\mu\nu} = g^{\mu\nu} - \frac{\partial^{\mu}\phi^0 \partial^{\nu}\phi^0}{X}$$

This model is analogous to massive gravity in that it allows instantaneous interactions [90] that are responsible for the presence of black hole hair. Moreover, it can be shown [79] that the standard charged rotating black holes are not solutions in this model, in full similarity with rotating black holes not being solutions in massive gravity.

To demonstrate the existence of electromagnetic hair in model (162), it must be shown that there exist nontrivial static finite-energy solutions for the linearized perturbations of the electromagnetic field in the background of the Schwarzschild black hole. In the sector with the angular momentum l = 1, the vector field can be parameterized by 4 real functions of a radial variable  $\rho$  (see Ref. [79] for explicit expressions). The equation for the perturbations of the vector field translates into a coupled system of ordinary differential equations for the radial functions. It must be shown that there exists a solution of this system that is regular both at infinity and at the black hole horizon  $\rho \to -\infty$ .

The existence of a regular solution can be demonstrated by counting the decreasing and increasing modes in the asymptotic regions. Here, we only present the results; details can be found in Ref. [79]. It can be shown that one of the four radial functions decouples and the corresponding equation does not have regular solutions. The equations for the remaining three radial functions can be rewritten as a single fourth-order equation. Therefore, an arbitrary solution is parameterized by four real parameters, one of which is the overall normalization. At the infinity, there are two decreasing and two increasing solutions. Requiring the general solution to decrease at the infinity fixes two of these three parameters. At the horizon, there are one singular and three regular solutions. The remaining free parameter can therefore be used to eliminate the singular part and obtain the solution regular everywhere — the "dipole hair."

Overall, the following picture emerges. Black holes in massive gravity have no reason to be universal. In particular, the metric of a rotating black hole can (and must, according to the direct analysis) be different from that in the Einstein theory. The differences between different possible metrics — black hole hair – depend on the collapse history. It has been argued [79] that these differences, as well as the deviations from the standard metric, are of the order of unity only at distances much larger than the inverse graviton mass  $m^{-1}$ , and are likely to be suppressed by the factor  $\sim (ml)^2$  at distances  $l \ll m^{-1}$ , unless the parameters of the model are tuned. Given the existing constraint on the graviton mass in (132), the effects of black hole hair in the simplest models are observable only for the largest black holes, with masses  $10^9 M_{\odot}$ .

# 6. Conclusion

To summarize, Lorentz-invariant massive gravity in 4 dimensions has severe self-consistency problems. It has either ghosts in the perturbation spectrum about Minkowski space or an unacceptably low UV energy scale at which strong coupling sets in, plus the Boulware – Deser ghost mode away from the Minkowski background. Because of the Lorentz invariance, the pathological ghost modes exist at arbitrarily high spatial momenta, and therefore the vacuum in this theory is catastrophically unstable. Presently, no way of fixing these problems is known, and it appears rather unlikely that this theory can be made healthy and phenomenologically acceptable.

Infrared-modified gravities may be less problematic in theories with extra spatial dimensions and brane worlds. Among the most widely discussed models of this sort is the DGP model, whose normal (as opposed to self-accelerated) branch does not have ghosts in the spectrum and may or may not have an acceptably high UV strong-coupling scale.

In this review, we followed another route and discussed Lorentz-violating theories. Among those, we concentrated on a subclass of theories that have only the metric as a dynamical field in the unitary gauge and which have Minkowski space as a solution of the field equations. Under these conditions, the lowest-order terms in the action in the Minkowski background are mass terms for metric perturbations. Hence, the emphasis in this review was on Lorentz-violating massive gravities. There is a plethora of other possibilities, some of which are reviewed, e.g., in Refs [124, 125].

Once the spectrum of a theory is not Lorentz invariant, ghosts and, to lesser extent, tachyons become phenomenologically acceptable if they exist only at sufficiently low spatial momenta and energies and only weakly (e.g., gravitationally) interact with matter. Furthermore, some Lorentz-violating massive gravities do not have obvious pathologies at all, and are phenomenologically acceptable even for the relatively high energy scale of Lorentz violation. Unlike the Fierz-Pauli theory that has a vDVZ discontinuity, these theories are smooth, perturbative deformations of general relativity at the classical level, while their UV strong-coupling scale at the quantum level is not dangerously low. The most appealing among these theories are the ones where some part of the diffeomorphism invariance of general relativity is unbroken, the feature that ensures the stability of these theories against deformation of the background and/or generation of higherorder terms in the action.

A general problem we must mention in this regard is the UV completion of these theories. Unlike general relativity, which is believed to be an effective low-energy theory descending from string theory, massive gravities do not have obvious string-theory completions. We believe this issue is worth investigating in the future.

Massive gravities of the sort we discuss in this review are conveniently analyzed by using the Stückelberg–Goldstone formalism. This formalism involves scalar fields whose background values roll along either time-like or space-like directions, or both. The advantage is that the full general covariance is restored, and hence at energies and momenta exceeding the graviton mass scale, the new modes, over and beyond the gravitons of general relativity, are perturbations of these scalar fields, which effectively decouple from gravity (except for the Fierz–Pauli case). In this way, the spectrum of the theory is studied rather straightforwardly. Furthermore, the Goldstone action can be regarded as a nonlinear generalization of the graviton mass terms, and the nonlinear properties of the resulting theory, such as cosmology and black holes, can then be studied.

Rolling scalar fields are interesting in many respects, even though their perturbations may be gauged away such that the theory involves only the metric in the unitary gauge. In the cosmological context, rolling scalar fields are capable of giving rise to the late-time accelerated expansion of the Universe, with a nontrivial equation of state of the effective dark energy. It is worth noting that other theories with IRmodified gravity are often unable to do that. For example, in theories with vector field condensates, the condensates may tend to constant values at late times. Then there is a general argument showing that the late-time evolution of the Universe is basically the same as in general relativity,<sup>21</sup> possibly with the cosmological constant [126, 127]. The argument is as follows. A spatially flat, homogeneous, and isotropic metric has the general form

$$ds^{2} = N^{2}(t) dt^{2} - a^{2}(t) \delta_{ij} dx^{i} dx^{j}.$$

This form is symmetric under time reparameterizations and space dilations,

$$t \to t'(t), \quad x^i \to \lambda x^i,$$
 (163)

with an arbitrary function t'(t) and an arbitrary constant  $\lambda$ . With the matter fields fixed at their vacuum values (in the locally Minkowski frame), the only dynamic variables are N(t) and a(t), and the action for these variables should respect symmetries (163). The only action that is local in time, is consistent with these symmetries, and has no more than two time derivatives is

$$S(N,a) = \tilde{M}_{\rm Pl}^2 \int \mathrm{d}t \, \frac{1}{N} \left(\frac{\dot{a}}{a}\right)^2 - \tilde{A} \int N \, \mathrm{d}t \,,$$

where  $M_{Pl}$  and A need not coincide with the genuine Planck mass and cosmological constant. This action has precisely the same form as the action of general relativity with the cosmological constant, specified to homogeneous and isotropic space. No matter what condensates are present in the Universe, its evolution proceeds according to the Friedman equation, possibly with modified forms of Newton's constant and the cosmological constant,<sup>22</sup> if the condensates are independent of the space-time point (in the locally Minkowski frame) and are consistent with the homogeneity and isotropy of space.

Despite apparent generality, this argument does not apply to the rolling scalar fields just because their background values depend on a space-time point. We gave an explicit example of a nontrivial late-time cosmological evolution in Section 5.2.2. More possibilities emerge if a scalar potential for the rolling field(s) is added, as was discussed at the end of Section 4.

Lorentz-violating massive gravities have a number of other interesting features. Massive gravitons are candidates for dark matter particles; in this case, dark-matter detection is a job for future (and maybe even present) gravitational wave searches. Unlike in Lorentz-invariant theories, black holes are expected to have rich properties. On the phenomenological side, this opens up the possibility of searching for Lorentz violation by measuring the metrics of black holes in the vicinity of their horizons. From the theoretical standpoint, Lorentz-violating massive gravities may be used to gain better insight into both the classical and quantum aspects of black hole physics. The studies of these fascinating issues have started only recently, and rapid progress in this area can be expected.

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# 7. Appendix

In this appendix, we give details of the treatment of small perturbations about the cosmological background in the Lorentz-invariant massive gravity. We use the setup and notation of Section 2.6.1. To give explicit examples, we use the theory with the particular form of the mass term given by (63). With the cosmological constant term included, the background equations in conformal time are, in general,

$$\mathcal{H}^{2} = H_{0}^{2} a^{2} + \epsilon_{0} ,$$

$$2\mathcal{H}' + \mathcal{H}^{2} = 3H_{0}^{2} a^{2} + \epsilon_{s} ,$$
(164)

where  $\epsilon_0(a, n)$  and  $\epsilon_s(a, n)$  come from the mass term. As noted in Section 2.6.1, the consistency of these equations implies an equation relating  $n(\eta)$  and  $a(\eta)$ , which generically has the form n' = f(n, a) a', but this is irrelevant for the discussion here because a and n can take arbitrary values at a given moment of time. We are interested in nearly Minkowski backgrounds, for which  $|a - 1| \leq 1$ ,  $|n - 1| \leq 1$ . Because the mass term is quadratic in metric perturbations of the Minkowski background, the source functions  $\epsilon_0$  and  $\epsilon_s$ vanish for a = 1, n = 1, and near these values, we have

$$\epsilon_0, \ \ \epsilon_s = O(a-1) + O(n-1).$$

For example, in case (63), the source functions are

$$\epsilon_{0} = -\frac{1}{2} m_{G}^{2} (a^{2} - 1) n,$$
  

$$\epsilon_{s} = -\frac{m_{G}^{2}}{2n} \left[ 2(a^{2} - 1) + (a^{2}n^{2} - 1) \right].$$
(165)

We begin with the range of momenta 1) in (65). In this range, the parameters  $\mathcal{H}^2$ , |a-1|, and |n-1| are the smallest parameters in the problem; we formally take them to be of the same order. We use the fact that the Minkowski values for the parameters entering (64) are the ones in the Fierz-Pauli theory, and hence

$$m_{\psi}^{2} = 3m_{G}^{2} + O(\mathcal{H}^{2}), \quad m_{B}^{2} = -\frac{1}{2}m_{G}^{2} + O(\mathcal{H}^{2}),$$
  

$$\mu_{1} = 3m_{G}^{2} + O(\mathcal{H}^{2}), \quad \mu_{2} = -m_{G}^{2} + O(\mathcal{H}^{2}),$$
  

$$\mu_{3} = -2m_{G}^{2} + O(\mathcal{H}^{2}), \quad (166)$$

<sup>&</sup>lt;sup>21</sup> The condensates may still evolve in the present cosmological epoch, leading to dark energy with a nontrivial equation of state. This possibility has been explored in the literature.

<sup>&</sup>lt;sup>22</sup> In fact, the possibility that the 'cosmological' form of Newton's constant and the 'Newton's law' form of Newton's constant may be different is of phenomenological interest [128] (see also [129]).

while the rest of the parameters are  $O(\mathcal{H}^2)$ , meaning that they vanish in the Minkowski limit. To find the number of dynamic modes and obtain their dispersion relations, we write the system of linear equations for perturbations and calculate its determinant to the order  $\mathcal{H}^2$ . The equations are

$$\varphi: \quad m_{\varphi}^{2}\varphi + 2\mathcal{H}\Delta B + (-2\Delta\psi + 6\mathcal{H}\psi' + \mu_{1}\psi) + (-2\mathcal{H}\Delta E' + \mu_{2}\Delta E) = 0, \quad (167)$$

$$B/\Delta: \quad 2\mathcal{H}\varphi + m_B^2 B - 2\psi' = 0, \qquad (168)$$

$$\begin{split} \psi : & (-2\Delta\varphi - 6\mathcal{H}\varphi' - 6q\varphi + \mu_1\varphi) + (2\Delta B' + 4\mathcal{H}\Delta B) \\ & + (6\psi'' - 2\Delta\psi + 4\mathcal{H}\psi' + 2q\psi + m_{\psi}^2\psi) \\ & + (-2\Delta E'' - 4\mathcal{H}\Delta E' + \mu_3\Delta E) = 0 \,, \end{split}$$

$$\begin{split} E/\Delta: \quad (2\mathcal{H}\varphi'+2q\varphi+\mu_2\varphi)+(-2\psi''-4\mathcal{H}\psi'+\mu_3\psi) \\ &+m_E^2\Delta E=0\,, \end{split}$$

where  $q = \mathcal{H}' + 2\mathcal{H}^2 = O(\mathcal{H}^2)$ . We now calculate the determinant of this system to find the number of modes and their dispersion relations. After passing to Fourier space and using (166), we find

$$\frac{1}{\Delta} \operatorname{Det} = m_{G}^{6} (3\omega^{2} - 3\mathbf{p}^{2} - 3m_{G}^{2}) + (12m_{G}^{2}\mathcal{H}^{2} + 2m_{G}^{2}m_{\phi}^{2})\omega^{4} + O(m_{G}^{2}\mathcal{H}^{2}\omega^{2}\mathbf{p}^{2}) + O(m_{G}^{2}\mathcal{H}^{2}\mathbf{p}^{4}).$$
(169)

Here, we assume that  $\mathbf{p}^2 \ge m_G^2$ ,  $\omega^2 \ge m_G^2$ , and keep only those new terms (with respect to Minkowski space) that are proportional to the highest power of  $\omega$ . We note that cancelations have occurred: in particular, the terms of the order  $\mathcal{H}^2 \mathbf{p}^2 \omega^4$  have canceled. These cancelations are a remnant of the gauge invariance: the terms of the order  $\mathcal{H}^2 \mathbf{p}^2 \omega^4$  would be independent of the graviton mass, and would therefore remain in de Sitter space for massless gravitons, which would be inconsistent with gauge invariance.

Because the determinant is of the fourth order in  $\omega$ , there are two modes. One of them has the dispersion relation of (the longitudinal component of) the massive graviton. This result is valid in the range of momenta 1) only; indeed, the terms neglected in (169) are large at high momenta. We discuss the high-momentum limit later on. In the range of momenta under discussion here, the second, Boulware–Deser mode has  $\omega^2 \ge \mathbf{p}^2$ , and its frequency is given by

$$\omega^2 = -\frac{3m_G^4}{12\mathcal{H}^2 + 2m_{\varphi}^2}.$$
 (170)

The discussion here is valid for an arbitrary mass term, not necessarily of form (63). For any mass term having the Fierz-Pauli form in the Minkowski background, we have  $m_{\varphi}^2 \sim \mathcal{H}^2$ , and therefore there is no smooth Minkowski-space limit for the frequency of the Boulware-Deser mode. This mode can be both tachyonic and nontachyonic; in example (63), this depends on the sign of a - 1. Indeed, in this example,  $m_{\varphi}^2 = -6\mathcal{H}^2 - 3\epsilon_0$ , and the frequency is there-

fore given by

S

$$\omega^2 = \frac{m_0^4}{2\epsilon_0}$$

and its sign is opposite to the sign of a - 1 [see (165)].

To see whether the Boulware – Deser mode is a ghost, we use the fact that its frequency (170) is independent of spatial momenta if they belong to region 1). Thus, to obtain the action for this mode, we can omit terms with the Laplacian in Eqns (167) and (168), except for the terms containing  $\Delta E$  (here, we treat  $\Delta E$  as a field, on equal footing with  $\psi$ ). We thus obtain

$$\begin{split} \varphi &= \frac{1}{m_{\varphi}^2} \left[ 2\mathcal{H}(\Delta E - 3\psi)' + m_{\rm G}^2(\Delta E - 3\psi) \right], \\ B &= \frac{1}{m_B^2} \left( 2\psi' - 2\mathcal{H}\varphi \right), \end{split}$$

where we used the leading-order expressions  $\mu_1 = 3m_G^2$  and  $\mu_2 = -m_G^2$ . Substituting these expressions in action (64), integrating by parts, and again omitting terms with the spatial Laplacian and terms suppressed by the ratio  $\mathcal{H}^2/m_G^2$ , we find the action for the dynamic fields  $\psi$  and  $\Delta E$ :

$$\begin{split} S^{(2)}_{\mathrm{EH}+A+m_{\mathrm{G}}} &= 2M_{\mathrm{Pl}} \int \mathrm{d}^{3}x \,\mathrm{d}\eta a^{2} \\ &\times \left[ -\frac{6\mathcal{H}^{2}+m_{\varphi}^{2}}{3m_{\varphi}^{2}} \left(\Delta E'-3\psi'\right)^{2} -\frac{m_{\mathrm{G}}^{4}}{2m_{\varphi}^{2}} \right. \\ &\times \left(\Delta E-3\psi\right)^{2} + \frac{1}{3} \left(\Delta E'\right)^{2} \right]. \end{split}$$

This expression is again valid for any mass term, not necessarily (63). Here, we neglected the terms  $m_{\psi}^2 \psi^2$  and  $\mu_3 \psi \Delta E$  because their contributions to the action for the Boulware – Deser mode are suppressed by  $\mathcal{H}^2/m_G^2$ . It is clear from this action that the Boulware – Deser field with dispersion relation (170) is  $\Delta E - 3\psi$ , and that when this field is not a tachyon, it is a ghost. Indeed, this field is not a tachyon for  $6\mathcal{H}^2 + m_{\phi}^2 < 0$ , which also implies  $m_{\phi}^2 < 0$ , and the kinetic term for this field is therefore negative. We also note that the action for this field is singular in the Minkowski limit, i.e., in the limit  $\mathcal{H} \to 0$ , in which  $m_{\phi}^2 \to 0$  as well.

We repeat that this analysis is valid for  $\mathbf{p}^2 \ll m_G^4/\mathcal{H}^2$  only. Passing to the high-momentum limit using expression (169) would be incorrect, partly because the terms not explicitly written in (169) are important at high momenta, partly because the terms of higher orders in  $\mathcal{H}$  are potentially higher order in  $\mathbf{p}^2$ .

We now proceed to high-momentum limit 2) in (65). The analysis leading to action (66) applies to any mass term. This action has the general form

$$S_{\rm EH+A+m_G}^{(2)} = 2M_{\rm Pl} \int d^4x \, a^2 \left\{ A\psi \Delta \psi + C(\psi')^2 + B\psi' \Delta E + \frac{m_B^2}{2} E' \Delta E' + \frac{m_E^2}{2} \left( \Delta E \right)^2 \right\}.$$

To proceed further, we consider the mass term in (63). After straightforward calculation, we then find

$$\begin{split} m_{\varphi}^{2} &= -6\mathcal{H}^{2} - 3\epsilon_{0} , \quad m_{B}^{2} = -3\epsilon_{0} - \frac{1}{2} m_{G}^{2} a^{2} n , \\ \mu_{2} &= -3\epsilon_{0} - m_{G}^{2} a^{2} n , \quad 1 - \frac{\mathcal{H}'}{\mathcal{H}^{2}} = \frac{3\epsilon_{0} - \epsilon_{s}}{2\mathcal{H}^{2}} , \quad m_{E}^{2} = -\epsilon_{s} . \end{split}$$

Hence,

$$\begin{split} A &\equiv 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{m_B^2}{2\mathcal{H}^2} = \frac{1}{4\mathcal{H}^2} \left( -2\epsilon_{\rm s} - m_{\rm G}^2 a^2 n \right), \\ B &\equiv \frac{\mu_2 - m_B^2}{\mathcal{H}} = -\frac{m_{\rm G}^2 a^2 n}{2\mathcal{H}}, \\ C &\equiv 3 + \frac{m_{\phi}^2}{2\mathcal{H}^2} = -\frac{3\epsilon_0}{2\mathcal{H}^2}. \end{split}$$

The dispersion relations are obtained by solving the equations of motion for  $\psi$  and *E*. There are two modes, one with

$$\omega_1^2 = \mathbf{p}^2$$

and the other with

$$\omega_2^2 = \frac{\epsilon_{\rm s}}{3\epsilon_0} \, \mathbf{p}^2 \, .$$

These expressions are valid for all a and n, not necessarily close to 1.

One of these modes is a tachyon or a ghost. Indeed, the energy positivity requires

$$A > 0$$
,  $C > 0$ ,  $m_B^2 < 0$ ,  $m_E^2 > 0$ .

Now,

$$\omega_2^2 = 2\mathcal{H}^2 \, \frac{A}{m_B^2} \,,$$

and therefore these requirements imply that  $\omega_2^2 < 0$ , that is, a tachyon. We note that the second mode is superluminal in certain backgrounds.

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