Physics – Uspekhi **51** (7) 723 – 725 (2008)

METHODOLOGICAL NOTES PACS numbers: **01.55.+b**, *06.20.fa*, 89.75.Da

Demonstrativeness of using energy rather than mass as the unit of measure for a number of problems in physics, mechanics, and geophysics

G S Golitsyn

DOI: 10.1070/PU2008v051n07ABEH006537

Abstract. Changing from the mass – length – time to the energy – length – time system of units is suggested as a means by which a number of problems in physics, mechanics, and geophysics can be more easily and conveniently solved using similarity analysis and dimensional methods. Eight examples are presented, with the derivations of the Stefan – Boltzmann radiation law, total kinetic energy of a hurricane, cosmic ray energy spectrum, etc.

Excellent books exist with descriptions of the substantiation of the similarity and dimensional methods and with practical formulas for using them [1, 2]. If a complete set of equations and initial or boundary conditions for them are not available, the parameters defining the sought-for quantity can be chosen by applying physical (or intuitive) arguments. This quantity possessing some dimension is sought as a monomial from a product of dimensions of the defining parameters, each of which is raised to some power. Equating power exponents on the left and on the right, we obtain a set of linear algebraic equations. The solution of this set specifies the monomial that determines the sought-for quantity in the case where the number of independent measurement units coincides with the number of parameters defining the problem. If the number of these parameters is greater than the number of measurement units, similarity parameters arise and this typically dictates the range of applicability of the results obtained.

It is standard practice to choose for measurement units the units of mass M, length L, and time T. Then, for example, the dimension of energy is $[E] = ML^2T^{-2}$, density $[\rho] = ML^{-3}$, pressure $[p] = ML^{-1}T^{-2}$, energy flux density $[q] = MT^{-3}$, etc. As our first example, we shall consider the most frequently used method of dimensions, as applied to the problem of a high-power explosion.

Example 1. It is assumed that at the initial instant of time an explosion in a medium with density ρ releases energy E. We need to find the radius and velocity of the generated shockwave as a function of time t. The size of the explosive device is assumed small in comparison with the shockwave radius (to be found) — that is, times close to the initial instant are not considered. The solution is meaningful as long as the pressure jump in the wave remains large in comparison with the ambient pressure which is considered negligible here.

We will limit the consideration to searching for the wave radius; in compliance with standard formulas, we seek it in the form $r = E^a\rho^b t^c$, or in terms of dimensionality as

$$L = (ML^2T^{-2})^a\,(ML^{-3})^b\,T^c\,.$$

Setting the dimensions equal on both sides, we obtain $a + b = 0$ with the dimension of mass, $2a - 3b = 1$ with the dimension of length, and $-2a + c = 0$ with the dimension of time. This linear system of algebraic equations has a unique solution, as its determinant equals $-5 \neq 0$, and takes the form $a = 1/5 = -b$, $c = 2/5$, so that

$$r(t) = \left(\frac{Et^2}{\rho}\right)^{1/5}. \qquad (1)$$

At the same time, however, classics showed [1, 2] that the system of units can be chosen arbitrarily. It is convenient to make a choice in such a way that measurement units enter a minimum possible number of key parameters. This is clarified by the example of choosing the following system of units: time T, energy E, length L — the TEL system of units. Then, the dimension of energy is $[E] = E$, density $[\rho] = ET^2L^{-5}$, radius $[r] = L$, and time $[t] = T$. Now we can write out expression (1) right away by looking at the dimension of density. Indeed, we need to exclude the dimensions of energy and time in the expression for a shockwave radius, which is done by dividing energy by density, multiplying the quotient by t^2, and taking the fifth root of the radicand.

Example 2. Finding the expression for the constant in the Stefan – Boltzmann radiation law in terms of the Planck constant $\hbar$ and speed of light c. Temperature Θ will be measured in energy units $[\Theta] = E$. The dimensionality of the Planck constant in our new TEL system of units is $[\hbar] = ET$, and that of the speed of light is $[c] = LT^{-1}$, as before. The dimensionality of radiation flux is $[q] = EL^{-2}T^{-1}$. Since the dimension of length enters only the speed of light, we have $q = c^{-2}\ldots$, but now the numerator gains in addition a factor T^2. The dimensionality of T^{-3} can be obtained by dividing the expressions by $\hbar^3$: $q = c^{-2}\hbar^{-3}, \ldots$. Replacing E^4 in the numerator by the quantity Θ^4, we ultimately arrive at $q = c^{-2}\hbar^{-3}\Theta^4$. If temperature is measured in degrees kelvin, we need to introduce the Boltzmann constant $k =$

G S Golitsyn A M Obukhov Institute of Atmospheric Physics, Russian Academy of Sciences,
Pyzhevskii per. 3, 119017 Moscow, Russian Federation
Tel. (7-495) 951 55 65
E-mail: gsg@ifaran.ru, mail_adm@ifaran.ru

Received 17 December 2007
Uspekhi Fizicheskikh Nauk **178** (7) 753 – 755 (2008)
DOI: 10.3367/UFNr.0178.200807e.0753
Translated by V I Kisin; edited by A Radzig

1.38×10^{-25} J K^{-1} and then, taking into account that $\Theta = kT$, we obtain

$$q = \frac{k^4 T^4}{c^2 \hbar^3} \,. \tag{2}$$

Obviously, the Stefan – Boltzmann radiation law has been derived for more than a century now by integrating the Planck distribution function over frequency, which gives the numerical factor $\pi^2/60$ in the law (2). Nevertheless, in view of its clarity we consider our derivation methodically useful.

Example 3. Derivation of the equation of state of an ideal gas, i.e., pressure as a function of the number of particles n and temperature. The dimensions are as follows: $[p] = EL^{-3}$, $[n] = L^{-3}$, $[T] = E$. It is immediately clear that

$$p = n\Theta \,, \tag{3}$$

or, if temperature is measured in kelvins, one finds

$$p = nkT \,. \tag{3'}$$

Example 4. The formula for the square of the (isothermal) speed of sound is also obvious:

$$c^2 = \frac{p}{\rho} = \frac{EL^{-3}}{ET^2 L^{-5}} = L^2 T^{-2} \,. \tag{4}$$

Example 5. The kinetic energy K of hurricane. Hurricanes in the tropics stretch across the entire troposphere, i.e., to altitudes of 15 to 18 km and masswise involve nearly three-fourths of the mass M_1 of the atmospheric column. The dimension is $[M_1] = ET^2 L^{-4}$. The convection intensity is dictated by the buoyancy flux $b = g\rho^{-1}\langle \rho' w' \rangle$, where g is the acceleration of gravity, and ρ' and w' are the fluctuations of density and vertical velocity, respectively. The dimension of interest is $[b] = L^2 T^{-3}$, and the value typical of hurricanes is on the order of 3×10^{-2} m^2 s^{-3}. The emerging convective column sucks in air from the surrounding area of hundreds and even thousands of kilometers in diameter, thereby concentrating the angular momentum of air masses on the rotating planet. This causes the buildup of the tangential component of wind and increases heat release from the surface of the ocean, which enhances convection, and so forth. The friction of wind on the surface of sea water poses a limit on the process. The corresponding characteristic on a rotating planet is the Coriolis parameter $l_{\mathrm{C}} = 2\omega \sin\theta$, where θ is the latitude, and $[l_{\mathrm{C}}] = T^{-1}$. The dimension of energy is present only in the dimension of mass of the atmospheric column, which needs to be multiplied by b^2 to eliminate the dimension of L^{-4}; now the denominator contains only T^{-4}, which is eliminated by dividing by l_{C}^4. As a result, we have

$$K = M_1 b^2 l_{\mathrm{C}}^{-4} \,. \tag{5}$$

Formula (5) was first obtained in Ref. [3], where the results of studying convection in rotating fluid were used, containing the velocity squared scale $U^2 = bl_{\mathrm{C}}^{-1}$, and the area scale $S = bl_{\mathrm{C}}^{-3}$. These scales multiplied by the mass M_1 give us formula (5). For $\theta = 20°$, we have $l_{\mathrm{C}} = 0.5 \times 10^{-4}$ s^{-1}, and for $b \approx 3 \times 10^{-2}$ m^2 s^{-3} and $M_1 = 10^4$ kg m^{-2} we find $K = 2 \times 10^{18}$ J. Hurricanes occur at high latitudes as well, where they are considerably smaller: $l_{\mathrm{C}} = 1.37 \times 10^{-4}$ s^{-1} at the latitude 70°, i.e., nearly three times larger than in the tropics, even though the wind force at the 70° latitude may also exceed the hurricane force of 33 m s^{-1}.

Example 6. The kinetic energy K_0 of circulation of the entire planetary atmosphere. This quantity is calculated [4] from the mean density of solar energy flux q with the dimension $EL^{-2}T^{-1}$, the heat capacity of the atmosphere c_p with the dimension $L^2 T^{-2} K^{-1}$, where K has the dimension of temperature in degrees kelvin, the size of the planet (its diameter) r, and the σ constant in the Stefan – Boltzmann radiation law that ensures a stable temperature regime of the planet: $[\sigma] = ET^{-1}L^{-2}K^{-4}$. We seek $[K_0] = E$. We wish to compose combinations that successively eliminate the dimensions of K, T, and L. We will write out the result as a product of factors implementing this procedure:

$$E = \left(\frac{\sigma^{1/4}}{c_p}\right)^{1/2} (q^{7/4})^{1/2} r^3 = \left(\frac{\sigma^{1/4} q^{7/4}}{c_p}\right)^{1/2} r^3 \,. \tag{6}$$

If we followed the familiar formula in the MLTK system of units, we would have a linear set of four algebraic equations with four unknowns. In fact, our TEL system is simpler and can serve as a test for the standard procedure. For planets that rotate sufficiently fast, formula (6) must be multiplied by a rotation scaling parameter $\Pi = \omega r/c$, where c is the speed of sound [4].

Example 7. The shape of the integral energy spectrum of galactic cosmic rays (CRs) in the energy range $10 \leqslant E < 3 \times 10^5$ GeV. The lower bound on energy emerges because CRs of solar origin are predominant at lower energies, while at the upper bound the Larmor radius is found to be comparable to the thickness of the galactic disk and particles cease to be effectively confined by the galactic magnetic field [5]. The primary sources of energy are supernovas exploding 2 – 3 times per century. Their power is $G \sim 3 \times 10^{33}$ W, and $[G] = E/T$. The bulk energy density w of CRs is about 0.5 eV cm^{-3}, i.e., $[w] = EL^{-3}$, as it is for pressure. The integral energy spectrum $N(\geqslant E)$ is the number of particles with energy $\geqslant E$ recorded per unit time per unit area per unit solid angle: $[N(\geqslant E)] = L^{-2}T^{-1}$ and $N(\geqslant E) = f(G, w, E)$. Only energy density has the dimension of length, so that $N(\geqslant E) \sim w^{2/3}$, only power G possesses the time dimension to the required power, so that $N(\geqslant E) \sim w^{2/3}G$, and to eliminate the energy dimension it is necessary that the following relation be hold:

$$N(\geqslant E) \sim w^{2/3} G E^{-5/3} \,. \tag{7}$$

The experimental value of the exponent in the function $N(\geqslant E)$ is $E^{-1.7}$. A complete phenomenological derivation of the energy spectrum $N(\geqslant E)$ using the Fermi mechanism for accelerating CR particles can be found in Ref. [6]. For the energy range $3 \times 10^5 \leqslant E < 10^8$ GeV, the spectrum becomes steeper and the exponent approaches -2.1. In Ref. [6] its value was found using an estimate of the bulk density of energy w for high-energy cosmic rays, which yielded $-19/9 = -(2 + 1/9)$.

Example 8. The number of tsunamis as a function of the height of wave breaking on the coast. Tsunamis are generated by underwater earthquakes (EQs), the majority of which occur near mid-ocean ridges where the Earth's crust is thinner. The source of energy in geodynamics is the geothermal flux of power $F = 4.5 \times 10^{13}$ W [7]. The measure of intensity of an earthquake is the energy E it releases. The geothermal flux generates convection in the mantle and splits

the crust into tectonic plates. Owing to the spatial nonuniformity of convection, plate velocities (not faster than several centimeters per year) create stress at the plate edges and an EQ is a way of stress relaxation. The dimension of the cumulative number $N(\geqslant E)$ of EQs is that of frequency. Recalling that $[F] = ET^{-1}$, we immediately obtain

$$N(\geqslant E) = \frac{F}{E} f(\Pi_i)\,, \tag{8}$$

where Π_i are scaling parameters. The basic parameter for EQs is the ratio of fault length $L = (E/\sigma)^{1/3}$, where σ is the stress released in the course of the earthquake, to crust thickness H. An analysis of EQ catalogues shows that when $\Pi = L/H \geqslant O(1)$, then the function $f(\Pi) \to \text{const}$. The value of this constant is close to 0.4 [8]. The oceanic crust is thin, so for underwater EQs we have $N(\geqslant E) \sim E^{-1}$. Obviously, other conditions being equal, the height h of the tsunami wave is proportional to the EQ energy. Hence, the expected number of tsunami waves should be $N(\geqslant E) \sim h^{-1}$. Indeed, according to Ref. [8], the observational data point to a power law with the exponent equal to -1.01.

Cumulative distributions of landslides and mud flows also have exponents close to -1, depending on their mass M: $N(\geqslant M) \propto M^{-n}$, where $0.95 \leqslant n \leqslant 1.1$, and the number of lakes depending on their area S: $N(\geqslant S) \propto S^{-0.95}$, and so forth. There is a physical explanation: such natural objects and phenomena are formed in the process of the long-term reaction of a system to random factors whose correlation time is short in comparison with the response time of the system. The correlation function of random forces is then a delta function, and their spectrum is that of white noise, and we observe random walk behavior in the momentum space. This explains many statistical patterns in nature [9], justifies our dimensional analysis, and makes it possible to establish limits to its applicability. For instance, this approach elegantly explains the main results of the theory of locally uniform and isotropic Kolmogorov–Obukhov turbulence [9].

It appears that the analysis given here on the basis of the TEL system of units is advisable on methodological grounds. The author uses it successfully in his lectures on natural phenomena at the Physics Department of Moscow State University and at the Aerocosmic Department of the Moscow Institute of Physics and Technology. The simplicity never fails to impress.

This work was partly supported by the RAS Presidium Program 'Mathematical Methods of Nonlinear Dynamics' and the grant NSh 4166 2006.5 of the President, Russian Federation.

References

1. Sedov L I *Metody Podobiya i Razmernosti v Mekhanike* (Similarity and Dimensional Methods in Mechanics) (Moscow: Nauka, 1972) [Translated into English (Moscow: Mir Publ., 1982)]
2. Barenblatt G I *Podobie, Avtomodel'nost', Promezhutochnaya Asimptotika* (Similarity, Self-Similarity, and Intermediate Asymptotics) (Leningrad: Gidrometeoizdat, 1st ed. — 1978, 2nd ed. — 1982) [Translated into English (New York: Consultants Bureau, 1979); *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge: Cambridge Univ. Press, 1996)]
3. Golitsyn G S *Dokl. Ross. Akad. Nauk* **354** 535 (1997) [*Dokl. Earth Sci.* **354** 633 (1997)]
4. Golitsyn G S *Icarus* **13** 1 (1970) [Translated into Russian as *Vvedenie v Dinamiku Planetnykh Atmosfer* (Leningrad: Gidrometeoizdat, 1973)]
5. Berezinskii V S et al. *Astrofizika Kosmicheskikh Luchei* (Astrophysics of Cosmic Rays) (Ed. V L Ginzburg) 2nd ed. (Moscow: Fizmatlit, 1990) [Translated into English (Ed. V L Ginzburg) (Amsterdam: North-Holland, 1990)]
6. Golitsyn G S *Pis'ma Astron. Zh.* **31** 500 (2005) [*Astron. Lett.* **31** 446 (2005)]
7. Schubert G, Turcotte D L, Olson P *Mantle Convection in the Earth and Planets* (Cambridge: Cambridge Univ. Press, 2001)
8. Golitsyn G S *Comput. Seismol.* (32) 138 (2001)
9. Golitsyn G S *Dokl. Ross. Akad. Nauk* **398** 177 (2004) [*Dokl. Phys.* **49** 501 (2004)]