

METHODOLOGICAL NOTES

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Permutation asymmetry of the relativistic velocity addition law and non-Euclidean geometry

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Contents

Abstract. **The asymmetry of the relativistic addition law for noncollinear velocities under the velocity permutation leads to two modified triangles on a Euclidean plane depicting the addition of unpermuted and permuted velocities and the appearance of a nonzero angle ω between two resulting velocities. A particle spin rotates through the same angle ω under a Lorentz boost with a velocity noncollinear to the particle velocity. Three mutually connected three-parameter representations of the angle ω, obtained by the author earlier, express the three-parameter symmetry of the sides and angles of two Euclidean triangles identical to the sine and cosine theorems for the sides and angles of a single geodesic triangle on the surface of a pseudosphere. Namely, all three representations of the angle ω, after a transformation of one of them, coincide with the representations of the area of a pseudospherical triangle expressed in terms of any two of its sides and the angle between them. The angle ω is also symmetrically expressed in terms of three angles or three sides of a geodesic triangle, and therefore it is an invariant of the group of triangle motions over the pseudosphere surface, the group that includes the Lorentz group. Although the pseudospheres in Euclidean and pseudo-Euclidean spaces are locally isometric, only the latter is isometric to the entire Lobachevsky plane and forms a homogeneous isotropic curved 4-velocity space in the flat Minkowski space. In this connection, relativistic physical processes that may be related to the pseudosphere in Euclidean space are especially interesting.**

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1. Introduction

The velocity addition law in special relativity follows from the Lorentz transformation of velocity, under which the transformed velocity is the sum of the velocity undergoing the transformation and the velocity of the Lorentz transformation (boost). As a result, the addition of two relativistic velocities is most easily presented in terms of 4-velocities and is reduced to the vector sum of spatial parts of the corresponding 4-velocities in which, however, the spatial part of the boost velocity is stretched. This relativistic velocity addition law is represented on a Euclidean plane as a modified velocity addition triangle. The vector sum of spatial parts of 4-velocities is changed when permuting the added velocities because the velocity that was regarded as undergoing the transformation before the permutation becomes the boost velocity and acquires stretching, while the old boost velocity becomes the velocity that undergoes transformation and enters the sum without stretching. Ultimately, the relativistic sum of two velocities is not symmetric under the permutation of added velocities if they are noncollinear. Permutation of such velocities leads to two different modified triangles depicting the velocity addition on a Euclidean plane.

Mathematically, this asymmetry is related to the noncommunicativity of Lorentz boosts with noncollinear velocities. The relativistic sum of two velocities can be represented as the velocity given to a particle at rest by two boosts. Hence, the velocities given to a particle at rest by noncollinear boosts depend on the boost order. As a result, the velocity permutation of two noncollinear velocities yields two vector sums that have the same value but different directions. The angle ω between these directions is considered in detail in the present paper. It is through this angle that the spin of a particle directed along its velocity rotates when the particle velocity is changed by a noncollinear Lorentz boost.

In the most general case, the angle ω is determined by three scalar parameters: the absolute values of the added velocities and the angle between them. It can also be represented through three similar parameters of any two velocities of the three connected by a Lorentz transformation. As a result, for any Lorentz transformation of velocity, there are three representations for the angle ω determined by symmetric scalar functions of two vectors taken as independent variables from three velocities connected by the Lorentz transformation. The symmetry of these functions under the permutation of their vector arguments means, in particular, that ω is independent of the permutation of the velocity undergoing transformation and the boost velocity. Therefore, one of the three representations for ω is shared by two Lorentz transformations that differ by the above permutation. Correspondingly, it is shared by both modified velocity addition triangles.

Two other representations of ω, which are related to one addition triangle by a simple but important algebraic transformation, are converted into two representations of ω related to another triangle. It then turns out that all three representations of ω can be described by one symmetric scalar function $\omega(\mathbf{a},\mathbf{b})$ that depends on two vectors having the meaning of the boost velocity and the velocity undergoing transformation (or the inverse boost velocity and the transformed velocity) in one addition velocity triangle or another.

Thus, two representations of the angle ω contain all the information about the sides and angles of both addition velocity triangles. This information can be formulated in the following statement: the absolute values of three nonstretched velocities in two Euclidean triangles and their opposite angles satisfy the sine and cosine theorems for the sides and angles of one geodesic triangle on the surface of a pseudosphere (the surface of constant negative Gaussian curvature). Moreover, three-parameter representations for the angle ω coincide with representations of the area S of this geodesic triangle measured in units of the inverse curvature, and hence, according to the geometry on a pseudosphere, with the defect of the sum of its angles A, A_1, and A_2:

$$\omega = |K|S = \pi - A - A_1 - A_2 .$$

In this way, another three-parameter representation of the angle ω appears, this time in terms of the inner angles of the geodesic triangle (i.e., angles adjacent to boost velocities in two Euclidean triangles). Moreover, the angle ω (an invariant of the relativistic velocity addition under permutation) acquires a purely geometric interpretation as being an invariant of the group of motions of the geodesic triangle on the surface of a pseudosphere. This group of motions contains the Lorentz group.

In Euclidean and pseudo-Euclidean three-dimensional spaces, two-dimensional surfaces with constant negative Gaussian curvature (pseudospheres) have the same internal geometry (metric), but differ substantially in terms of their realizations in these spaces.

A pseudosphere in a pseudo-Euclidean space is a space-like surface, a sheet of a two-sheet hyperboloid with the rotation axis aligned with the time axis. It has an infinite area, and is regular, homogeneous, and isotropic at all of its points; its metric coincides with that of the entire Lobachevsky hyperbolic plane. Space and time coordinates of a point on the surface of the pseudosphere differ from the space and time components of a 4-velocity only by the factor $|K|^{-1/2}$.

In the three-dimensional Euclidean space, as was proved by Hilbert, there is no two-dimensional smooth surface isometric to the entire Lobachevsky plane. A pseudosphere in this space (the Beltrami surface) has a finite area and is isometric only to some region of the Lobachevsky hyperbolic plane. Such constraints on a pseudosphere in Euclidean space come from the impossibility of covering its surface by an infinite Chebyshev network of coordinate lines x, y whose network angle $\phi(x, y)$ would be a regular solution of the nonlinear equation

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = \sin\phi ,$$

satisfying the condition $0 < \phi < \pi$.

Although pseudospheres in the Euclidean and non-Euclidean space are locally isometric and allow representing the relativistic velocity addition by geodesic triangles with identical metric properties on their surfaces, they are arenas of substantially different physical processes. The nonhomogeneity and anisotropy of a pseudosphere in Euclidean space leads to nonlinear wave processes, in contrast to linear processes pertaining to the homogeneous and isotropic surface of a pseudosphere in pseudo-Euclidean space.

2. Velocity addition triangle on a Euclidean plane

As is well known (see, e.g., § 5 in [1]), the velocity $\mathbf{v}$ of a particle is transformed by a Lorentz boost $\mathbf{v}_1$ to the velocity $\mathbf{v}_2$ with the components

$$v_{2x} = \frac{v_x + v_1}{1 + \mathbf{v}\mathbf{v}_1} , \qquad v_{2y} = \frac{v_y}{(1 + \mathbf{v}\mathbf{v}_1)\gamma_1} , \qquad v_{2z} = \frac{v_z}{(1 + \mathbf{v}\mathbf{v}_1)\gamma_1} ,$$

$$\gamma_1 = \frac{1}{\sqrt{1 - v_1^2}} \tag{1}$$

if $\mathbf{v}_1$ is directed along the x axis. These formulas can be combined into one vector relation

$$\mathbf{v}_2 = \frac{1}{1 + \mathbf{v}\mathbf{v}_1} \left\{ \mathbf{v}_1 \left[\frac{\mathbf{v}\mathbf{v}_1}{v_1^2} \left(1 - \frac{1}{\gamma_1} \right) + 1 \right] + \frac{\mathbf{v}}{\gamma_1} \right\} , \tag{2}$$

as was done in [2] or in the author's paper [3]. Here and below, we use the notation from that paper. The asymmetry of the expression for $\mathbf{v}_2$ under the permutation of $\mathbf{v}$ and $\mathbf{v}_1$ is obvious.

However, a more compact expression for the velocity transformation can be obtained if we use spatial parts

$$\mathbf{u} = \mathbf{v}\gamma , \qquad \mathbf{u}_1 = \mathbf{v}_1\gamma_1 , \qquad \mathbf{u}_2 = \mathbf{v}_2\gamma_2 \tag{3}$$

and time components γ, γ_1, and γ_2 of the corresponding 4-velocities $u^\alpha = (\mathbf{u}, \gamma)$, $\gamma = \sqrt{\mathbf{u}^2 + 1}$, and analogously for u_1^α,

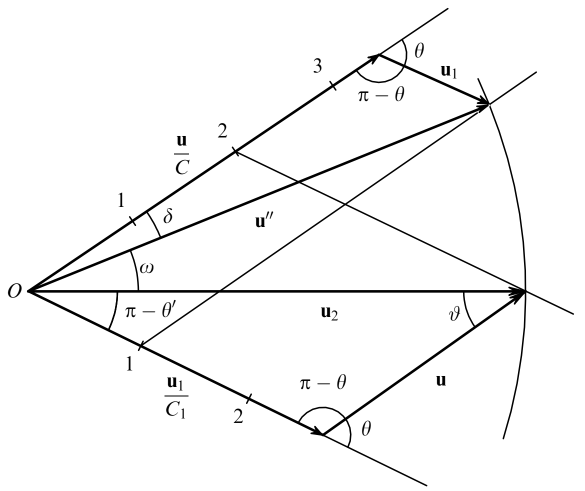

Figure 1. Velocity addition triangles corresponding to formulas (4) and (16).

and u_2^α. Then formula (2) and the inverse transformation become

$$\mathbf{u}_2=\mathbf{u}+\frac{\mathbf{u}_1}{C_1}\,,\qquad \mathbf{u}=\mathbf{u}_2-\frac{\mathbf{u}_1}{C_1}\,,\qquad \mathbf{u}_1=(\mathbf{u}_2-\mathbf{u})C_1\,,\quad(4)$$

where the coefficient

$$C_1=\frac{\gamma_1+1}{\gamma+\gamma\gamma_1+\mathbf{u}\mathbf{u}_1}=\frac{\gamma_1+1}{\gamma_2+\gamma_2\gamma_1-\mathbf{u}_2\mathbf{u}_1}=\frac{\gamma+\gamma_2}{1+\gamma\gamma_2+\mathbf{u}\mathbf{u}_2}\qquad(5)$$

is expressed in terms of three independent parameters, the absolute values of any two velocities participating in the Lorentz transformation and the angle between them. This coefficient coincides with C introduced in [3] but is marked with subscript 1 to stress the permutation asymmetry due to the Lorentz boost velocity $\mathbf{u}_1$.

For the Lorentz boost velocity $\mathbf{u}_1$ in (4), it is natural (but not necessary) to take the last representation in (5) as the coefficient C_1, such that the right-hand side of $\mathbf{u}_1$ depend only on $\mathbf{u}$ and $\mathbf{u}_2$. A similar argument also applies to expressions for the velocities $\mathbf{u}_2$ and $\mathbf{u}$ in Eqn (4), in which the first and the second representations from (5) can be respectively used as C_1.

We also give expressions for the time components of 4-velocities u^α, u_1^α, and u_2^α in terms of the spatial parts of the other two velocities participating in the Lorentz transformation:

$$\gamma_2=\gamma\gamma_1+\mathbf{u}\mathbf{u}_1\,,\qquad \gamma=\gamma_2\gamma_1-\mathbf{u}_2\mathbf{u}_1\,,$$
$$\gamma_1=\frac{(\gamma+\gamma_2)^2}{1+\gamma\gamma_2+\mathbf{u}\mathbf{u}_2}-1\,.\qquad(6)$$

The last expression allows representing C_1 as a function of only the absolute values of all the three velocities:

$$C_1=\frac{\gamma_1+1}{\gamma+\gamma_2}\,.\qquad(7)$$

We note that in passing from the direct to the inverse Lorentz transformation, i.e., under changing $\mathbf{u}_1\leftrightarrow-\mathbf{u}_1$, $\mathbf{u}\leftrightarrow\mathbf{u}_2$, the expressions for the velocities $\mathbf{u}_2$ and $\mathbf{u}$ in Eqn (4) transform into each other, and the expression for $\mathbf{u}_1$ transforms into itself. A similar asymmetry holds for the corresponding expressions for time components (6), as well as for expressions (5) for C_1.

Formulas (4) and (5) represent the relativistic velocity addition law, whose only difference from the ordinary addition of 3-vectors is that only one added velocity (the Lorentz boost one) enters stretched by $1/C_1$ times. The bottom part of Fig. 1 shows the relativistic velocity addition triangle for $\mathbf{u}$ and $\mathbf{u}_1$ when their values and the angle between them are $u=2$, $u_1=1$, and $\theta=60°$. Then the dilation factor is $1/C_1=2.65$. It is easy to see that this triangle describes all formulas (4) for a corresponding change in the directions of $\mathbf{u}_1$ and $\mathbf{u}$.

Thus, the relations derived above demonstrate the peculiar active role of the Lorentz boost velocity $\mathbf{u}_1$, which is proportional to the difference between the transformed velocity and the velocity undergoing transformation. This special role disappears in the nonrelativistic approximation, when $\gamma=\gamma_1=\gamma_2=1$ and u, u_1, and u_2 are small compared to 1. Then $C_1=1$ and the above formulas become $\mathbf{v}_2=\mathbf{v}+\mathbf{v}_1$.

3. Particle spin rotation angle under a Lorentz transformation of its velocity

Wigner showed that the angle between the spin of a massive particle and its velocity is not Lorentz invariant [4, 5]. For example, if a particle has a velocity $\mathbf{v}$ and the spin directed along $\mathbf{v}$, and a Lorentz boost gives it a velocity $\mathbf{v}_1$ in the direction normal to $\mathbf{v}$, then the particle velocity rotates through the angle ϑ,

$$\sin\vartheta=\frac{v_1}{v_2}=\frac{u_1\gamma}{u_2}\,,\qquad(8)$$

and becomes equal to $\mathbf{v}_2$, while the spin rotates through the smaller angle

$$\omega=\vartheta-\delta=\arcsin\frac{uu_1}{1+\gamma\gamma_1}\,,\qquad \sin\delta=\frac{u_1}{u_2}\qquad(9)$$

(see Refs [4, 5]). But if the particle moves with the speed of light, $v=v_2=1$, then $\delta=0$ and the spin rotation angle coincides with the velocity rotation angle. In this case, the angle between spin and velocity is Lorentz invariant, as is the value of the spin itself along the particle motion direction.

In the more general case where the velocities $\mathbf{v}$ and $\mathbf{v}_1$ are not orthogonal, the spin rotation was considered by Stapp [6], the author [7], and some others. We here give three closely related representations for the angle ω obtained in [7]. Each of them is explicitly expressed in terms of three independent parameters, the absolute values of any two velocities among $\mathbf{u}$, $\mathbf{u}_1$, $\mathbf{u}_2$ connected by the Lorentz transformation, and the angle between these two velocities:

$$\begin{aligned}\omega&=2\arctan\frac{\big|[\mathbf{u}\mathbf{u}_1]\big|}{(\gamma+1)(\gamma_1+1)+\mathbf{u}\mathbf{u}_1}\\&=2\arctan\frac{\big|[\mathbf{u}_1\mathbf{u}_2]\big|}{(\gamma_1+1)(\gamma_2+1)-\mathbf{u}_1\mathbf{u}_2}\\&=2\arctan\frac{\big|[\mathbf{u}_2\mathbf{u}]\big|}{(\gamma_2+1)(\gamma+1)+\mathbf{u}_2\mathbf{u}}\,.\end{aligned}\qquad(10)$$

All three formulas are given as Eqns (32), (36)–(38) in [7], and Eqns (20)–(22) in [3]. They are directly related to the non-Euclidean geometry on the surface with negative constant Gaussian curvature, i.e., to the Lobachevsky geometry. We consider this relation in what follows.

4. Transforming an asymmetric system of expressions for the angle ω into a symmetric one

We write three expressions for ω in terms of the *internal* angles $\pi - \theta$ and $\pi - \theta'$ of the Euclidean velocity addition triangle (see Fig. 1, the lower triangle). In this case, the asymmetry produced by the Lorentz boost velocity $\mathbf{u}_1$ is more evident:

$$\tan\frac{\omega}{2} = \frac{uu_1 \sin(\pi - \theta)}{(\gamma + 1)(\gamma_1 + 1) - uu_1 \cos(\pi - \theta)}$$
$$= \frac{u_2 u_1 \sin(\pi - \theta')}{(\gamma_2 + 1)(\gamma_1 + 1) - u_2 u_1 \cos(\pi - \theta')}$$
$$= \frac{uu_2 \sin\vartheta}{(\gamma + 1)(\gamma_2 + 1) + uu_2 \cos\vartheta}\,. \qquad (11)$$

Indeed, in the first two expressions for ω, the cosines of the internal angles $\pi - \theta$ and $\pi - \theta'$ of the triangle adjacent to the active side enter with a negative sign, but in the third formula, the cosine of the internal angle ϑ opposite to the active side enters with a positive sign.

This asymmetry of the expressions for ω is related to the special role of the Lorentz boost velocity $\mathbf{u}_1$ in the Euclidean velocity addition triangle $\mathbf{u}_1 = (\mathbf{u}_2 - \mathbf{u})C_1$. Changing the boost velocity sign preserves the active role of this velocity and only changes the roles of $\mathbf{u}$ and $\mathbf{u}_2$ as the velocity undergoing the transformation and the resultant velocity, without changing their values or directions [see Eqn (4)]. The velocities $\mathbf{u}$ and $\mathbf{u}_2$ play a passive role in the considered triangle. In the first case, the active vector $\mathbf{u}_1/C_1$ acts on the passive vector $\mathbf{u}$ and transforms it into the passive vector $\mathbf{u}_2$. In the second case, the active vector $-\mathbf{u}_1/C_1$ acts on the passive vector $\mathbf{u}_2$ and transforms it into the passive vector $\mathbf{u}$ (see Fig. 1, the lower triangle).

To reveal the hidden symmetry in (11), it is significant that the third expression for ω in Eqn (11) can be identically transformed into the first two with the old velocities u and u_2 but the new angle $\delta = \vartheta - \omega$ instead of ϑ and with the minus sign before $\cos\delta$:

$$\tan\frac{\omega}{2} = \frac{uu_2 \sin\delta}{(\gamma + 1)(\gamma_2 + 1) - uu_2 \cos\delta}\,, \qquad \delta = \vartheta - \omega\,. \quad (12)$$

Indeed, because

$$\tan\frac{\omega}{2} \equiv \frac{\sin\omega}{1 + \cos\omega} = \frac{uu_2 \sin\vartheta}{(\gamma + 1)(\gamma_2 + 1) + uu_2 \cos\vartheta}\,, \qquad (13)$$

we have

$$(\gamma + 1)(\gamma_2 + 1)\sin\omega + uu_2 \cos\vartheta \sin\omega$$
$$= uu_2 \sin\vartheta + uu_2 \sin\vartheta \cos\omega$$

or

$$(\gamma + 1)(\gamma_2 + 1)\sin\omega = uu_2 \sin(\vartheta - \omega)$$
$$+ uu_2 \sin(\vartheta - \omega)\cos\omega + uu_2 \cos(\vartheta - \omega)\sin\omega\,,$$

whence

$$\sin\omega\big[(\gamma + 1)(\gamma_2 + 1) - uu_2 \cos(\vartheta - \omega)\big]$$
$$= uu_2 \sin(\vartheta - \omega)(1 + \cos\omega)\,,$$

which is formula (12). Thus, the last expression in (11) can be substituted by Eqn (12) such that all three representations for $\tan(\omega/2)$ eventually differ only by a cycle change of variables:

$$(u, u_1, \pi - \theta) \to (u_1, u_2, \pi - \theta') \to (u_2, u, \delta)\,. \qquad (14)$$

We note that the sum of the angles here is $\pi - \omega$, i.e., smaller than π.

5. Permutation of added velocities and the three-parameter symmetry of ω

In the plane of vectors $\mathbf{u}$ and $\mathbf{u}_1$, we introduce a vector $\mathbf{u}''$ with the length $u'' = u_2$ directed at the angle δ to $\mathbf{u}$; this allows rewriting formula (12) in the form

$$\tan\frac{\omega}{2} = \frac{\big|[\mathbf{u}''\mathbf{u}]\big|}{(\gamma'' + 1)(\gamma + 1) - \mathbf{u}''\mathbf{u}}\,. \qquad (15)$$

The appearance of such a formula is not accidental. It emerges if we the permute the velocities $\mathbf{u}$ and $\mathbf{u}_1$ in velocity addition law (4), i.e., if we consider $\mathbf{u}$ as the Lorentz boost velocity and $\mathbf{u}_1$ as the velocity undergoing the transformation. Obviously, instead of addition rule (4), we then find the addition law

$$\mathbf{u}'' = \mathbf{u}_1 + \frac{\mathbf{u}}{C}\,, \qquad \mathbf{u}_1 = \mathbf{u}'' - \frac{\mathbf{u}}{C}\,, \qquad \mathbf{u} = (\mathbf{u}'' - \mathbf{u}_1)C\,, \quad (16)$$

in which the new vector $\mathbf{u}''$ plays the role of the velocity undergoing transformation and the coefficient C is given by

$$C = \frac{\gamma + 1}{\gamma_1 + \gamma\gamma_1 + \mathbf{u}\mathbf{u}_1} = \frac{\gamma + 1}{\gamma'' + \gamma\gamma'' - \mathbf{u}\mathbf{u}''} = \frac{\gamma_1 + \gamma''}{1 + \gamma_1\gamma'' + \mathbf{u}_1\mathbf{u}''}\,, \qquad (17)$$

which differs from (5) by the change $\mathbf{u} \leftrightarrow \mathbf{u}_1$, $\mathbf{u}_2 \leftrightarrow \mathbf{u}''$. The coefficient C, similarly to the boost velocity $\mathbf{u}$ in (16), has no index but differs from C introduced in Ref. [3], which is now denoted as C_1 (see above). For C, the following expression also holds:

$$C = \frac{\gamma + 1}{\gamma_1 + \gamma_2} \qquad (18)$$

[cf. Eqn (7)].

We note that the product of two dilatations $1/C$ and $1/C_1$ is always greater than or equal to one:

$$\frac{1}{CC_1} = \frac{(\gamma_1 + \gamma_2)(\gamma + \gamma_2)}{(\gamma + 1)(\gamma_1 + 1)} \geqslant 1\,, \qquad (19)$$

which is not the case for each of them separately.

Velocity addition law (16) is also demonstrated by a modified triangle in Fig. 1 (the upper triangle) with the same velocity values $u = 2$, $u_1 = 1$ and the angle between them $\theta = 60^\circ$ as used in rule (4). But now the boost velocity $\mathbf{u}$ participates in this triangle stretched by $1/C = 1.72$ times, in contrast to $1/C_1 = 2.65$ for the boost velocity $\mathbf{u}_1$ in the triangle corresponding to rule (4).

In Eqns (17) and (18), we again have

$$\gamma = \gamma_1\gamma_2 - \mathbf{u}_1\mathbf{u}_2\,, \qquad \gamma_2 = \gamma'' = \gamma\gamma_1 + \mathbf{u}\mathbf{u}_1\,, \qquad (20)$$

and γ_1, in contrast to Eqn (6), takes the form of the scalar product of two 4-vectors,

$$\gamma_1 = \gamma\gamma'' - \mathbf{u}\mathbf{u}'' , \tag{21}$$

if we use $\mathbf{u}''$. This is the standard form of the time component of the 4-velocity $\mathbf{u}_1, \gamma_1$ undergoing transformation expressed in terms of the 4-velocity $\mathbf{u}, \gamma$ of the Lorentz boost and the 4-velocity $\mathbf{u}'', \gamma''$ transformed in accordance with law (16). The more complicated expression (6) for γ_1 is due to addition law (4), in which γ_1 is the time component of the 4-velocity $\mathbf{u}_1, \gamma_1$ of the Lorentz boost and not of the 4-velocity undergoing the transformation or the transformed one.

As a result, velocity addition law (16), like law (4), is characterized by the same angle ω, which is now represented by the expressions

$$\begin{aligned}\omega &= 2\arctan\frac{\left|[\mathbf{u}\mathbf{u}_1]\right|}{(\gamma+1)(\gamma_1+1)+\mathbf{u}\mathbf{u}_1}\\ &= 2\arctan\frac{\left|[\mathbf{u}\mathbf{u}'']\right|}{(\gamma''+1)(\gamma+1)-\mathbf{u}''\mathbf{u}}\\ &= 2\arctan\frac{\left|[\mathbf{u}''\mathbf{u}_1]\right|}{(\gamma''+1)(\gamma_1+1)+\mathbf{u}''\mathbf{u}_1} .\end{aligned} \tag{22}$$

Under the permutation $\mathbf{u} \leftrightarrow -\mathbf{u}$, $\mathbf{u}_1 \leftrightarrow \mathbf{u}''$, which does not change addition law (16), the first and the second expressions for ω transform into each other, while the third expression is preserved.

These three formulas for ω are related to the corresponding representations of ω in Eqn (10) by the permutation $\mathbf{u} \leftrightarrow \mathbf{u}_1$, $\mathbf{u}'' \leftrightarrow \mathbf{u}_2$, with the first of them, as functions of $\mathbf{u}$, $\mathbf{u}_1$, being simply identical due to the symmetry under this permutation. We emphasize again that the third expression for ω in (10) or (11) obtained for the velocity triangle $\mathbf{u}_1 = (\mathbf{u}_2 - \mathbf{u})C_1$ is identically transformed into the second formula for ω in Eqn (22) obtained for the velocity triangle $\mathbf{u} = (\mathbf{u}'' - \mathbf{u}_1)C$. Similarly, it can be shown that the third formula for ω in Eqn (22) is identical to the second formula for ω in Eqn (10) or (11).

Thus, the global symmetry of ω under all possible velocity permutations is reflected by three expressions: by the first and second ones in (10) and by the second one in (22). Each of them is represented by the same scalar function $\omega(\mathbf{a}, \mathbf{b})$ symmetrically depending on two velocity vectors:

$$\begin{aligned}&\omega(\mathbf{a}, \mathbf{b}) = 2\arctan\frac{\left|[\mathbf{a}\mathbf{b}]\right|}{(a^0+1)(b^0+1)+\mathbf{a}\mathbf{b}} ,\\ &a^0 = \sqrt{\mathbf{a}^2+1} , \qquad b^0 = \sqrt{\mathbf{b}^2+1} .\end{aligned} \tag{23}$$

The symmetry can then be expressed as the chain of three equalities:

$$\omega(\mathbf{u}, \mathbf{u}_1) = \omega(\mathbf{u}_2, -\mathbf{u}_1) = \omega(\mathbf{u}'', -\mathbf{u}) = \omega(\mathbf{u}_1, \mathbf{u}) . \tag{24}$$

Indeed, the first equality reflects the symmetry of addition law (4) under the permutation $\mathbf{u}_1 \leftrightarrow -\mathbf{u}_1$, $\mathbf{u} \leftrightarrow \mathbf{u}_2$ relating the direct and inverse Lorentz transformations with velocities $\pm\mathbf{u}_1$. The second equality reflects the symmetry under the permutation $\mathbf{u}_1 \leftrightarrow \mathbf{u}$, $\mathbf{u}_2 \leftrightarrow \mathbf{u}''$ relating addition laws (4) and (16). Finally, the third equality reflects the symmetry of addition law (16) under the permutation $\mathbf{u} \leftrightarrow -\mathbf{u}$, $\mathbf{u}'' \leftrightarrow \mathbf{u}_1$, i.e., the direct and inverse Lorentz transformations with velocities $\pm\mathbf{u}$. At the same time, this permutation brings us back to the original function with permuted arguments, which confirms its symmetry.

We note that the arguments of ω in equalities (24) have the meaning of the velocity undergoing the transformation and the boost velocity belonging to one velocity addition triangle or another.

The function $\omega(\mathbf{a}, \mathbf{b})$ can be regarded as the function

$$\omega(a, b, \zeta) = 2\arctan\frac{ab\sin\zeta}{(a^0+1)(b^0+1)+ab\cos\zeta} \tag{25}$$

of three variables: the absolute values a and b of two velocities and the angle ζ between them, which is external in the corresponding Euclidean velocity triangle if one of the a, b velocities is active, and internal if both the a, b velocities are passive. Such a function was used by the author in Eqns (36)–(38) in [7]. Three terms in equality (24) are written in terms of this function as

$$\omega(\mathbf{u}, \mathbf{u}_1) = \omega(u, u_1, \theta) , \tag{26}$$

$$\omega(\mathbf{u}_2, -\mathbf{u}_1) = \omega(u_1, u_2, \theta') , \tag{27}$$

$$\omega(\mathbf{u}'', -\mathbf{u}) = \omega(u_2, u, \pi-\delta) , \qquad u'' = u_2 , \tag{28}$$

where θ, θ', and $\pi - \delta$ are the external angles of two Euclidean velocity triangles (see Fig. 1). Equality (24) itself is now rewritten as an equality of the right-hand sides of formulas (26)–(28).

Below, however, instead of the external angles θ, θ', and $\pi - \delta$, it is more convenient to use the internal angles adjacent to them, $\pi - \theta$, $\pi - \theta'$, and δ of the same triangles. Then, after expressing $\omega(a, b, \zeta)$ through 'another' function $f(a, b, \pi - \zeta)$ depending on the angle adjacent to ζ,

$$\begin{aligned}\omega(a, b, \zeta) &= f(a, b, \pi-\zeta)\\ &= 2\arctan\frac{ab\sin(\pi-\zeta)}{(a^0+1)(b^0+1)-ab\cos(\pi-\zeta)} ,\end{aligned} \tag{29}$$

three-parameter symmetry (24) can be represented by the equality of three expressions for the angle ω:

$$\omega = f(u, u_1, \pi-\theta) = f(u_1, u_2, \pi-\theta') = f(u_2, u, \delta) . \tag{30}$$

Each of these expressions is determined by three independent parameters: the absolute values of the passive and active velocities and the internal angle between them, i.e., by the parameters of the two Euclidean velocity triangles (4) and (16). It is essential that three parameters determining the function f in Eqn (30) determine two other three parameters uniquely owing to three equations (34), (35) given below.

We note that the function f of the same arguments as in Eqn (30) has already appeared in the two equations (11) and (12) as the term $\tan(f/2)$. The third representation for the angle ω in Eqn (11), written as $\omega(u_2, u, \vartheta)$, is transformed into

$$\omega(u_2, u, \vartheta) = \omega(u_2, u, \pi-\delta) = f(u_2, u, \delta) , \qquad \delta = \vartheta - \omega \tag{31}$$

using Eqns (12) and (29). The first equality here is nontrivial. Essentially, it is a functional equation for ω. It relates passive velocities and the internal angle between them in velocity addition triangle (4) and the passive and active velocities and the external angle between them in velocity triangle (16).

6. Theorems of sines and cosines in the geometry of geodesic triangles on a pseudosphere

The numerators of the arctan arguments in Eqns (10) and (22) are the lengths of equal vectors

$$[\mathbf{u}\mathbf{u}_1] = [\mathbf{u}_2\mathbf{u}_1] = [\mathbf{u}\mathbf{u}_2]C_1 = [\mathbf{u}\mathbf{u}''] = [\mathbf{u}''\mathbf{u}_1]C\,, \tag{32}$$

and the denominators are the equal quantities

$$\begin{aligned}(\gamma+1)(\gamma_1+1)+\mathbf{u}\mathbf{u}_1 &= (\gamma_2+1)(\gamma_1+1)-\mathbf{u}_2\mathbf{u}_1\\ &= C_1\big((\gamma+1)(\gamma_2+1)+\mathbf{u}\mathbf{u}_2\big) = (\gamma''+1)(\gamma+1)-\mathbf{u}''\mathbf{u}\\ &= C\big((\gamma''+1)(\gamma_1+1)+\mathbf{u}''\mathbf{u}_1\big)\,,\end{aligned} \tag{33}$$

where $\mathbf{u}$ and $\mathbf{u}_2$ are related by the Lorentz transformation with the velocity $\mathbf{u}_1$ [see Eqn (4)], and $\mathbf{u}_1$ and $\mathbf{u}''$ are related by the Lorentz transformation with the velocity $\mathbf{u}$ [see Eqn (16)].

The equality of the lengths of vectors, which do not contain C_1 and C in Eqn (32), can be expressed as

$$\frac{u}{\sin(\pi-\theta')} = \frac{u_1}{\sin\delta} = \frac{u_2 = u''}{\sin(\pi-\theta)}\,. \tag{34}$$

On the other hand, the quantities in Eqn (33) are different representations of the sum $1+\gamma+\gamma_1+\gamma_2, \gamma_2 = \gamma''$ in terms of the lengths of two vectors and the angle between them; therefore, it follows from Eqn (33), in particular, that

$$\gamma = \gamma_1\gamma_2 - \mathbf{u}_1\mathbf{u}_2\,, \qquad \gamma_1 = \gamma\gamma'' - \mathbf{u}\mathbf{u}''\,, \qquad \gamma_2 = \gamma'' = \gamma\gamma_1 + \mathbf{u}\mathbf{u}_1\,. \tag{35}$$

Because $\mathbf{u}$, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}''$ are the spatial parts of 4-velocities and γ, γ_1, and $\gamma_2 = \gamma''$ are their time components, these quantities can be characterized by hyperbolic angles α, α_1, and α_2 such that

$$\begin{aligned}&u = \sinh\alpha\,, \quad \gamma = \cosh\alpha\,; \qquad u_1 = \sinh\alpha_1\,, \quad \gamma_1 = \cosh\alpha_1\,;\\ &u_2 = u'' = \sinh\alpha_2\,, \quad \gamma_2 = \gamma'' = \cosh\alpha_2\,.\end{aligned} \tag{36}$$

Then, Eqn (34) takes the form of the 'theorem of sines' for a triangle in the hyperbolic Lobachevsky geometry (the geometry on a pseudosphere),

$$\frac{\sinh\alpha}{\sin(\pi-\theta')} = \frac{\sinh\alpha_1}{\sin\delta} = \frac{\sinh\alpha_2}{\sin(\pi-\theta)}\,, \tag{37}$$

and the three relations in (35) take the form of the 'theorem of cosines' in this geometry:

$$\begin{aligned}\cosh\alpha &= \cosh\alpha_1\cosh\alpha_2 - \sinh\alpha_1\sinh\alpha_2\cos(\pi-\theta')\,,\\ \cosh\alpha_1 &= \cosh\alpha_2\cosh\alpha - \sinh\alpha_2\sinh\alpha\cos\delta\,,\\ \cosh\alpha_2 &= \cosh\alpha\cosh\alpha_1 - \sinh\alpha\sinh\alpha_1\cos(\pi-\theta)\,.\end{aligned} \tag{38}$$

These two theorems uniquely determine the geodesic velocity triangle on the surface of a pseudosphere up to its motions as a whole along the surface (i.e., without cuts or folds and with the lengths and angles preserved).

7. A pseudosphere in Euclidean space

A pseudosphere, or the Beltrami surface, is the surface of constant negative curvature formed by rotation of a planar curve, the tractrix, around its asymptote. The tractrix (or the equitangential curve) is the curve with an asymptote and a cusp, and has the property that the segment of the tangent from the cusp to the asymptote has a constant length, denoted by a in Fig. 2 and below.

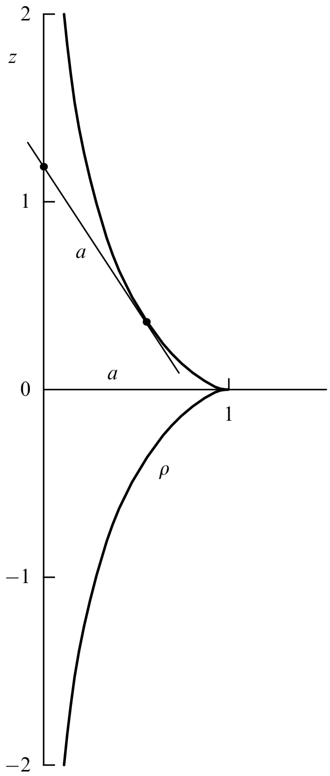

Figure 2. A tractrix with the parameter $a = 1$.

In cylindrical coordinates z, ρ, φ, the equation of a pseudosphere with the rotation axis along z can be written as

$$\begin{aligned}&z = \pm\left(a\ln\frac{a+\sqrt{a^2-\rho^2}}{\rho} - \sqrt{a^2-\rho^2}\right),\\ &0 < \rho \leqslant a\,, \qquad -\pi < \varphi \leqslant \pi\,.\end{aligned} \tag{39}$$

For a constant φ and variable ρ, this equation describes the meridians of the pseudosphere — tractrices that are also geodesic arcs; for a constant ρ and variable φ, it describes the parallels of the pseudosphere — circles (Fig. 3). The Gaussian curvature of the pseudosphere is $-1/a^2$. The surface area and volume of the pseudosphere are $4\pi a^2$ and $(2/3)\pi a^3$, respectively. Below, however, we mostly consider the upper half of the pseudosphere, for which $z > 0$. The area and volume of this Beltrami funnel are $2\pi a^2$ and $(1/3)\pi a^3$.

Equation (39) for the pseudosphere can also be written in the parametric form

$$\begin{aligned}&x = a\sin u\cos\varphi\,, \qquad y = a\sin u\sin\varphi\,,\\ &z = a\left(\ln\cot\frac{u}{2} - \cos u\right), \qquad 0 < u < \pi\,, \qquad -\pi < \varphi \leqslant \pi\end{aligned} \tag{40}$$

if the radius of the latitudinal circle is represented as $\rho = a\sin u$, where $0 < u < \pi$. Then the upper and lower Beltrami funnels, which constitute a pseudosphere, correspond to values u in the respective ranges $0 < u \leqslant \pi/2$ and

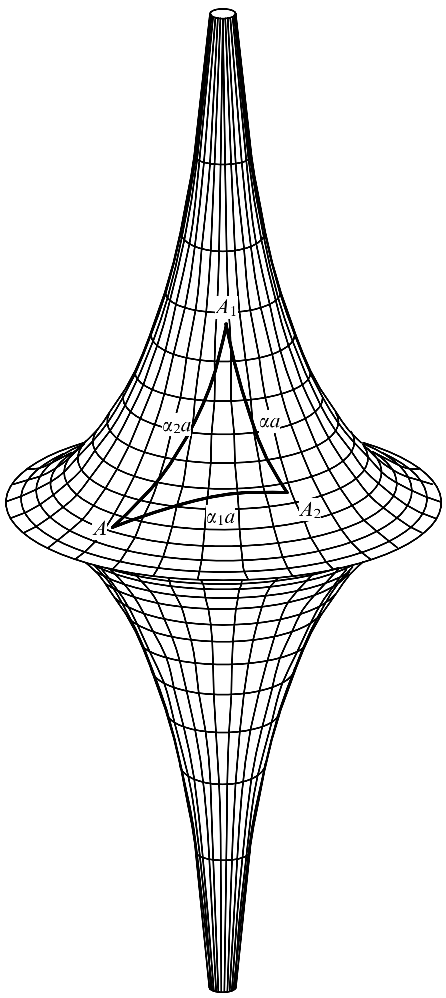

Figure 3. Pseudosphere in Euclidean space with a geodesic triangle AA_1A_2.

$\pi/2 \leqslant u < \pi$, and their common latitudinal circle $\rho = a$ (the cuspidal edge) corresponds to $u = \pi/2$. Although all functions (40) determining the surface are analytic in the parameter range of u and φ (and even in the extended interval $-\infty < \varphi < \infty$ for φ), the surface is not regular at the cuspidal edge, it cannot be extended beyond the cuspidal edge continuously with a continuous change of the tangent plane.

A geodesic triangle on the pseudosphere with the Gaussian curvature $K = -1/a^2$ (i.e., a triangle whose sides are geodesic arcs) with the internal angles $A = \pi - \theta'$, $A_1 = \delta$, and $A_2 = \pi - \theta$ has opposite sides with the lengths αa, $\alpha_1 a$, and $\alpha_2 a$, where the hyperbolic angles α, α_1, and α_2 and the angles A, A_1, and A_2 satisfy theorems (37) and (38) (see Fig. 3).

Mapping the Euclidean velocity triangle $\mathbf{u}_1 = (\mathbf{u}_2 - \mathbf{u})C_1$ on the pseudospherical surface preserves the internal angles of the triangle $\pi - \theta$ and $\pi - \theta'$ that are opposite to its passive sides $\mathbf{u}_2$ and $\mathbf{u}$, as well as the lengths of these sides, which are measured by the hyperbolic angles α_2 and α. This means that they are identically mapped into the angles $A_2 \equiv \pi - \theta$ and $A \equiv \pi - \theta'$ and their opposite sides $\alpha_2 a$ and αa of a geodesic triangle on the pseudospherical surface. At the same time, the angle ϑ that is opposite to the active side $\mathbf{u}_1/C_1$ of the Euclidean triangle, and the length of this side are not conserved; they are transformed into the angle $A_1 \equiv \delta = \vartheta - \omega$ and the length $\alpha_1 a$ of the opposite side of the geodesic triangle.

Similarly, the Euclidean velocity triangle $\mathbf{u} = (\mathbf{u}'' - \mathbf{u}_1)C$ is mapped onto the surface of a pseudosphere such that its internal angles $\pi - \theta$ and δ that are opposite to the passive sides $\mathbf{u}''$ and $\mathbf{u}_1$, and the lengths of these sides measured by the hyperbolic angles α_2 and α_1 are mapped identically into the angles $A_2 \equiv \pi - \theta$ and $A_1 \equiv \delta$ and their opposite sides $\alpha_2 a$ and $\alpha_1 a$ of the geodesic triangles. But the angle $\theta - \delta$ opposite to the active side $\mathbf{u}/C$ of this triangle and its length are not preserved and are transformed into the angle $A \equiv \theta - \delta - \omega = \pi - \theta'$ and the length αa of its opposite side of the geodesic triangle.

Therefore, two Euclidean velocity triangles are mapped into one geodesic triangle on the pseudosphere surface with the internal angles $A_2 = \pi - \theta$, $A_1 = \delta$, and $A = \pi - \theta'$ and the lengths $\alpha_2 a$, $\alpha_1 a$, and αa of their opposite sides. This is a direct consequence of the fact that the angles and velocities of two Euclidean triangles that differ by the permutation of added velocities $\mathbf{u}_1 \leftrightarrow \mathbf{u}$, satisfy the theorems of sines in (37) and cosines in (38) for the geometry on a pseudosphere.

In the geometry on a pseudosphere, it is proved that the area S of a geodesic triangle is proportional to the defect of the sum of its angles [8]:

$$S = a^2(\pi - A - A_1 - A_2)\,. \tag{41}$$

Because the sum of the angles of a Euclidean velocity triangle $\mathbf{u}_1 = (\mathbf{u}_2 - \mathbf{u})C_1$ is π $(\pi - \theta + \pi - \theta' + \vartheta = \pi)$, the defect of the sum of angles of the geodesic triangle is positive and is equal to

$$\pi - A - A_1 - A_2 = \pi - (\pi - \theta') - \delta - (\pi - \theta) = \vartheta - \delta = \omega\,, \tag{42}$$

i.e., coincides with the angle ω between the vectors $\mathbf{u}_2$ and $\mathbf{u}''$ (see Fig. 1) that emerged because the relativistic velocity addition law for $\mathbf{u}$ and $\mathbf{u}_1$ is not symmetric under their permutation,

$$\mathbf{u}_2 = \mathbf{u} + \frac{\mathbf{u}_1}{C_1}\,, \qquad \mathbf{u}'' = \mathbf{u}_1 + \frac{\mathbf{u}}{C}\,, \tag{43}$$

or, alternatively, because of the noncommutativity of Lorentz boosts with noncollinear velocities.

Thus, the angle ω reflecting the asymmetry of the relativistic velocity addition law on a Euclidean plane is represented symmetrically by the defect of the sum of angles of the geodesic triangle on a pseudosphere.

8. The area of a geodesic triangle on a pseudosphere

Another important three-parameter formula for the area S of a triangle in the Lobachevsky space, i.e., the area of a geodesic triangle on the surface of a pseudosphere with the curvature $-1/a^2$, can be found in [9]. In our notation, this formula can be written as

$$\sin\frac{S}{2a^2} = \frac{\sinh(\alpha_1/2)\sinh(\alpha_2/2)\sin A}{\cosh(\alpha/2)}\,. \tag{44}$$

Two other formulas for S are obtained from here by cyclic permutation of the hyperbolic angles α, α_1, α_2 and the angles A, A_1, A_2. However, these formulas express S/a^2 in terms of four parameters, although in fact only three of them are independent. In addition, halves of the hyperbolic angles enter here, which is inconvenient for the relation with velocities [see Eqn (36)].

Nevertheless, using Eqn (44) allows obtaining the corresponding expression for

$$\cos\frac{S}{2a^2} = \frac{1}{\cosh(\alpha/2)}\left(\cosh\frac{\alpha_1}{2}\cosh\frac{\alpha_2}{2} - \sinh\frac{\alpha_1}{2}\sinh\frac{\alpha_2}{2}\cos A\right), \tag{45}$$

which also contains four parameters, and then the expression for

$$\tan\frac{S}{2a^2} = \frac{\sinh(\alpha_1/2)\sinh(\alpha_2/2)\sin A}{\cosh(\alpha_1/2)\cosh(\alpha_2/2) - \sinh(\alpha_1/2)\sinh(\alpha_2/2)\cos A}, \tag{46}$$

which contains only three parameters. Next, to pass from the hyperbolic half-angles to the whole angles, it is sufficient to multiply both the numerator and the denominator of the last expression by $4\cosh(\alpha_1/2)\cosh(\alpha_2/2)$ and then use the known formulas for double angles. We thus obtain

$$\tan\frac{S}{2a^2} = \frac{\sinh\alpha_1 \sinh\alpha_2 \sin A}{(\cosh\alpha_1 + 1)(\cosh\alpha_2 + 1) - \sinh\alpha_1 \sinh\alpha_2 \cos A}. \tag{47}$$

This expression and two others, which are obtained from it by a cyclic change of the hyperbolic and ordinary angles, coincide with the first two expressions for $\tan(\omega/2)$ in Eqn (11) and Eqn (12), considering parameterization (36) of the components of 4-velocities by the hyperbolic angles. The equality $S/a^2 = \omega$ of the geodesic triangle area to the angle ω is also demonstrated by formula (30).

Thus, two of the three three-parameter representations of ω (the rotation angle of the spin of a particle under noncollinear Lorentz transformations) obtained in [7] coincide identically with the two representations for the area S/a^2 of the velocity triangle on the pseudospherical surface, and the third one coincides identically with the third representation for S/a^2 after transforming the angle ϑ to the angle $\delta = \vartheta - \omega$, as we did above. At the same time, just the third representation in Eqn (11) is special due to its asymmetry with respect to the first two representations, because it contains the angle ϑ opposite to the Lorentz boost velocity, which is the source of asymmetry in the Euclidean velocity addition triangle (see Fig. 1). None of the angles of the geodesic triangle has the physical meaning similar to that of the angle ϑ, the rotation angle of a particle velocity under the Lorentz transformation with the velocity noncollinear to the particle velocity. Hence, this representation directly answers the most interesting question on the relation of a particle spin rotation angle to its velocity rotation angle under a Lorentz transformation of the velocity.

We note two important particular cases that follow from the third representation.

1. The boost changes the particle velocity only in direction. Then $u = u_2$ and

$$\omega = 2\arctan\frac{\sin\vartheta}{(\gamma+1)/(\gamma-1) + \cos\vartheta}. \tag{48}$$

If the angle ϑ is small, then

$$\omega = \left(1 - \frac{1}{\gamma}\right)\vartheta. \tag{49}$$

2. The particle speed is close to the speed of light. Then $u = u_2 = \gamma = \gamma_2 \gg 1$ and

$$\omega \approx 2\arctan\frac{\sin\vartheta}{1+\cos\vartheta} = \vartheta, \tag{50}$$

irrespective of the angle ϑ, i.e., the spin and velocity of an ultrarelativistic particle rotate through the same angle.

9. A pseudosphere in the pseudo-Euclidean space

In the three-dimensional Euclidean space, a two-dimensional sphere with a radius a centered at the coordinate origin is described by the equation

$$x^2 + y^2 + z^2 = a^2. \tag{51}$$

In the pseudo-Euclidean space, to which we can pass via the replacement $z \to \mathrm{i}t$, this surface is transformed into a one-sheet hyperboloid

$$x^2 + y^2 - t^2 = a^2, \tag{52}$$

which is sometimes referred to as a real-radius sphere in a pseudo-Euclidean space [10].

Making the radius a purely imaginary, $a \to \mathrm{i}a$, yields the equation

$$x^2 + y^2 - t^2 = -a^2, \tag{53}$$

describing a sphere with a purely imaginary radius in the pseudo-Euclidean space. It is also referred to as a pseudosphere of radius a in the pseudo-Euclidean space R^3_1 (see [11]). Equation (53), rewritten in the more customary form

$$t^2 = a^2 + x^2 + y^2, \tag{54}$$

represents a two-sheet hyperboloid in three-dimensional space (Fig. 4).

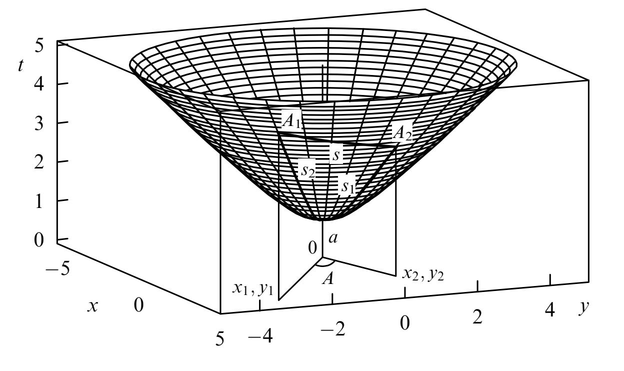

Figure 4. The nothern sheet $t = \sqrt{a^2 + x^2 + y^2}$ of a pseudopshere in pseudo-Euclidean space with the geodesic triangle AA_1A_2. The vertex A is placed at the pole $x = y = 0$, $t = a$. The angles are denoted by the same letters as the vertices, and s, s_1, and s_2 are pseudospherical arc lengths. The parameter $a = 1$.

For an ordinary sphere, the length of a geodesic arc (the arc of a large circle) s connecting two arbitrary points A_1 and A_2 on the surface, the spherical radius a, and the angle α between the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ from the center of the sphere $x = y = z = 0$ to these points are related as

$$\alpha = \frac{s}{a}\,, \qquad \cos\alpha = \frac{\mathbf{r}_1\mathbf{r}_2}{\sqrt{\mathbf{r}_1^2}\sqrt{\mathbf{r}_2^2}} = \frac{\mathbf{r}_1\mathbf{r}_2}{a^2} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{a^2}\,. \tag{55}$$

Passing to the pseudo-Euclidean space, $z_1z_2 \to -t_1t_2$, and to the imaginary-radius sphere, $a \to \mathrm{i}a$, yields

$$\begin{aligned} &\alpha \to -\mathrm{i}\,\frac{s}{a} = -\mathrm{i}\alpha\,,\\ &\cos\alpha \to \cosh\alpha = \frac{t_1t_2 - x_1x_2 - y_1y_2}{a^2} = \gamma_1\gamma_2 - \mathbf{u}_1\mathbf{u}_2 \end{aligned} \tag{56}$$

instead of Eqn (55). This formula gives the shortest distance s between the points A_1 and A_2 on a pseudosphere in the pseudo-Euclidean space.

The last expression in Eqn (56) relates the coordinates of the points A_1 and A_2 on the pseudosphere directly to the dimensionless components of 4-velocities:

$$u_{1x} = \frac{x_1}{a}\,, \qquad u_{1y} = \frac{y_1}{a}\,, \qquad \gamma_1 = \frac{t_1}{a}\,, \tag{57}$$

and similarly for $(\mathbf{u}_2, \gamma_2)$.

Placing A_2 at the 'north pole' of the pseudosphere, $N(x = y = 0, t = a)$, we use Eqn (56) to obtain

$$\cosh\alpha_1 = \gamma_1\,, \qquad \mathbf{u}_1 = \mathbf{e}_1\sinh\alpha_1\,, \qquad \mathbf{e}_1^2 = 1\,. \tag{58}$$

Similarly, placing the point A_1 at the pole N, we obtain

$$\cosh\alpha_2 = \gamma_2\,, \qquad \mathbf{u}_2 = \mathbf{e}_2\sinh\alpha_2\,, \qquad \mathbf{e}_2^2 = 1 \tag{59}$$

from Eqn (56). Vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ have unit length and lie in the Euclidean plane tangent to the pseudosphere at the point N. Therefore, the distances of the points A_1 and A_2 from the north pole N are $s_1 = \alpha_1 a$ and $s_2 = \alpha_2 a$, and the distance between the points is $s = \alpha a$.

Placing the vertex A of the triangle AA_1A_2 at the north pole N of a pseudosphere and letting A denote the angle between the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, we obtain

$$\cosh\alpha = \cosh\alpha_1\cosh\alpha_2 - \sinh\alpha_1\sinh\alpha_2\cos A \tag{60}$$

from Eqns (56), (58), and (59). This is the first line of the 'theorem of cosines' for the triangle AA_1A_2 on a pseudosphere.

A surface of constant Gaussian curvature (positive or negative) has one important property. The motion of any figure completely lying on such a surface, for example, the triangle AA_1A_2, preserves all lengths and values of all angles, in spite of possible deformation of the moved figure.

By moving the triangle AA_1A_2 along the surface of the pseudosphere such that the vertex A_1 coincides with the north pole $x = y = 0$, $t = a$ (instead of the vertex A), we find the second line of the theorem of cosines:

$$\cosh\alpha_1 = \cosh\alpha_2\cosh\alpha - \sinh\alpha_2\sinh\alpha\cos A_1\,. \tag{61}$$

Similarly, moving the triangle AA_1A_2 such that the vertex A_2 coincides with the north pole of the pseudosphere yields the third line of the theorem of cosines:

$$\cosh\alpha_2 = \cosh\alpha\cosh\alpha_1 - \sinh\alpha\sinh\alpha_1\cos A_2\,. \tag{62}$$

Using the theorem of cosines, it is straightforward to obtain the theorem of sines in (37). Indeed, using $\cos A$ from Eqn (60), we can represent the ratio $\sin A/\sinh\alpha$ as

$$\begin{aligned} \frac{\sin A}{\sinh\alpha} &\equiv \frac{\sqrt{1-\cos^2 A}}{\sinh\alpha}\\ &= \frac{\sqrt{1 + 2\cosh\alpha\cosh\alpha_1\cosh\alpha_2 - \cosh^2\alpha - \cosh^2\alpha_1 - \cosh^2\alpha_2}}{\sinh\alpha\sinh\alpha_1\sinh\alpha_2}\,, \end{aligned} \tag{63}$$

i.e., as a totally symmetric expression in α, α_1, α_2; the same must be true for the other two terms in the theorem of sines.

In turn, formula (63) allows finding one more three-parameter representation for the angle ω, symmetric with respect to the hyperbolic angles α, α_1, and α_2 or the time components γ, γ_1, and γ_2:

$$\begin{aligned} \omega &= 2\arctan\frac{uu_1\sin A_2}{(\gamma+1)(\gamma_1+1) - uu_1\cos A_2}\\ &= 2\arctan\frac{\sqrt{1 + 2\gamma\gamma_1\gamma_2 - \gamma^2 - \gamma_1^2 - \gamma_2^2}}{1 + \gamma + \gamma_1 + \gamma_2}\,. \end{aligned} \tag{64}$$

Each of these expressions for the angle ω is explicitly represented in terms of either the lengths of two sides and the angle between them or the lengths of three sides of the geodesic triangle, i.e., the parameters that are invariant under the motion of the triangle over the pseudospherical surface. Therefore, ω is an invariant of this group of motions, which contains the Lorentz group. This statement also follows from the representation for ω in (42) in terms of the angles of the geodesic triangle.

10. The metric of the surface of a pseudosphere

The quantity $\gamma = \gamma_1\gamma_2 - \mathbf{u}_1\mathbf{u}_2 = -u_1^{\alpha}u_{2\alpha}$, being the scalar product of two 4-velocities, is clearly Lorentz invariant. The length $u = \sqrt{\gamma^2 - 1}$ of the spatial part of the 4-velocity $u^{\alpha} = (\mathbf{u}, \gamma)$ is also Lorentz invariant. The ratio $v = u/\gamma$ can be regarded as the invariant value of the relative 3-velocity of two particles with velocities u_1^{α} and u_2^{α} (see §12 in Ref. [1]). The same is true for the values γ_1 and $\gamma_2 = \gamma''$ in Eqn (35).

We consider the square of the length of $\mathbf{u}$ in the case where the vector $\mathbf{u}_2$ is very close to $\mathbf{u}_1$: $\mathbf{u}_2 = \mathbf{u}_1 + \mathrm{d}\mathbf{u}_1$. It is easy to see that the invariant γ then differs from unity by a value of the second order in $\mathrm{d}\mathbf{u}_1$:

$$\begin{aligned} \gamma &= \gamma_1\gamma_2 - \mathbf{u}_1\mathbf{u}_2 = \gamma_1\sqrt{1 + (\mathbf{u}_1 + \mathrm{d}\mathbf{u}_1)^2} - \mathbf{u}_1^2 - \mathbf{u}_1\,\mathrm{d}\mathbf{u}_1\\ &\approx 1 + \frac{1}{2}\left(\mathrm{d}\mathbf{u}_1^2 - \frac{(\mathbf{u}_1\,\mathrm{d}\mathbf{u}_1)^2}{1+\mathbf{u}_1^2}\right) + \ldots\,. \end{aligned} \tag{65}$$

Because $u^2 = \gamma^2 - 1$, using parameterization (36) for u and the relation $s = \alpha a$, we obtain

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\mathbf{u}_1^2 - \frac{(\mathbf{u}_1\,\mathrm{d}\mathbf{u}_1)^2}{1+\mathbf{u}_1^2}\right)\,. \tag{66}$$

With $\mathbf{u}_1$ regarded as an N-dimensional vector, formula (66) for the square of the length element $\mathrm{d}s^2$ represents the metric of an N-dimensional homogeneous and isotropic surface with the negative Gaussian curvature $K = -1/a^2$. This surface is embedded into a flat $(N+1)$-dimensional pseudo-Euclidean space with the metric

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x}^2 - \mathrm{d}t^2 \tag{67}$$

(see §3, chapter 13 in [12] and §111 in [1]). In our case, $N = 2$, and for clarity it is convenient to write metric (66) in the 'polar' coordinates

$$u_{1x} = \frac{r}{a}\cos\varphi\,, \qquad u_{1y} = \frac{r}{a}\sin\varphi\,. \tag{68}$$

Then,

$$\mathrm{d}s^2 = a^2\,\mathrm{d}\alpha^2 = \frac{\mathrm{d}r^2}{1 + r^2/a^2} + r^2\,\mathrm{d}\varphi^2\,, \tag{69}$$

and hence the ratio of the circumference $2\pi r$ of the latitudinal circle to the distance from this circle to the pole,

$$s = \int_0^r \frac{\mathrm{d}r}{\sqrt{1 + r^2/a^2}} = a\,\mathrm{arsinh}\,\frac{r}{a}\,, \tag{70}$$

is greater than 2π.

11. Pseudospheres in Euclidean and pseudo-Euclidean spaces

We now consider the relation between the pseudosphere in a Euclidean space (PSE) and the pseudosphere in a pseudo-Euclidean space (PSPE). Both pseudospheres have the same constant Gaussian curvature $K = -1/a^2$. The coordinates x, y, t of the PSPE surface are directly related to the components of 4-velocities [see Eqn (57)], but the relation of the PSE surface coordinates to velocities has yet to be established. For this, we compare the basic metric form of a PSE written in cylindrical coordinates $\rho, \varphi, z = z(\rho)$ [see Eqn (39)],

$$\mathrm{d}s^2 = \frac{a^2}{\rho^2}\,\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\varphi^2\,, \qquad 0 < \rho \leqslant a\,, \qquad -\pi < \varphi \leqslant \pi\,, \tag{71}$$

with the PSPE metric in (66).

Changing coordinates ρ, φ to ξ, η,

$$\rho = \frac{a^2}{\eta}\,, \qquad \varphi = \frac{\xi}{a}\,, \tag{72}$$

we can rewrite metric (71) in the form

$$\mathrm{d}s^2 = a^2\,\frac{\mathrm{d}\xi^2 + \mathrm{d}\eta^2}{\eta^2}\,, \tag{73}$$

which is the metric of the Klein model in Lobachevsky geometry [11]. In this model, metric (73) is defined on the Euclidean halfplane $-\infty < \xi < \infty$, $\eta > 0$ with the Cartesian orthogonal coordinates ξ, η. The Euclidean halfplane endowed with this metric turns out to be a complete metric manifold with the constant negative curvature $K = -1/a^2$. At the same time, the PSE surface, which has the same curvature, is isometrically mapped only onto the part

$$-\pi a < \xi = a\varphi \leqslant \pi a\,, \qquad a \leqslant \eta = \frac{a^2}{\rho} < \infty \tag{74}$$

of the Euclidean halfplane $\xi, \eta > 0$ bounded by the lines $\xi = \pm\pi a$ and $\eta = a$.

If we now relate the coordinates ξ, η to the coordinates x, y, t on the PSPE surface by the transformation

$$\frac{\xi}{a} = \frac{y}{t + x}\,, \qquad \frac{\eta}{a} = \frac{a}{t + x}\,, \qquad t = \sqrt{a^2 + x^2 + y^2}\,, \tag{75}$$

it is straightforward to show that in the new variables, the metric

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x}^2 - \frac{(\mathbf{x}\,\mathrm{d}\mathbf{x})^2}{a^2 + \mathbf{x}^2}\,, \qquad \mathbf{x} = x\mathbf{i} + y\mathbf{j}\,, \tag{76}$$

coincides with the PSPE metrics in (66). We note that metric (76) is determined by the Cartesian coordinates x, y on the entire Euclidean plane of these coordinates. At the same time, the coordinates ρ and φ of metric (71) are in the range

$$0 < \frac{\rho}{a} = \frac{t + x}{a} < 1\,, \qquad -\pi < \varphi = \frac{y}{t + x} \leqslant \pi\,. \tag{77}$$

This means that the PSE surface is isometrically mapped to only some part of the PSPE surface, or, in other terms, only to the interval

$$-\infty < x < 0\,, \qquad |y| < \pi(t + x)\,, \tag{78}$$

of the entire Euclidean x, y plane endowed with the PSPE metric.

To find this part of the Euclidean plane, it is convenient to use the relation between x, y and ξ, η that is inverse to (75):

$$x = \frac{a^2 - \xi^2 - \eta^2}{2\eta}\,, \qquad y = \frac{a\xi}{\eta}\,, \qquad t = \frac{a^2 + \xi^2 + \eta^2}{2\eta}\,. \tag{79}$$

It is then easy to show that the part of the x, y plane onto which the PSE is mapped is bounded below with respect to x by two hyperbolas $x_1(y)$, $y \gtrless 0$, with the common asymptote $y = 0$, and is bounded above with respect to x by the parabola $x_2(y)$:

$$x_1(y) < x < x_2(y)\,, \qquad -\pi a < y < \pi a\,, \tag{80}$$

where

$$x_1(y) = -\frac{\pi^2 - 1}{2\pi}\,|y| - \frac{\pi a^2}{2|y|}\,, \qquad x_2(y) = -\frac{y^2}{2a} \tag{81}$$

(see also Fig. 5). The hyperbolas and the parabola are the maps of the boundaries $\xi = \pm\pi a$ and $\eta = a$ of the PSE mapping domain on the Euclidean halfplane in the Klein model.

The PSPE has the maximum symmetry, i.e., its surface is isotropic and symmetric: the neighborhoods of any points on the surface are geometrically similar to each other. This follows because metric (76) is invariant under rotations around the t axis and 'quasitranslations' (as termed by Weinberg [12])

$$\mathbf{x}' = \mathbf{x} + \mathbf{q}\left(\sqrt{a^2 + \mathbf{x}^2} + \frac{\mathbf{q}\mathbf{x}}{1 + \sqrt{1 + \mathbf{q}^2}}\right), \tag{82}$$

which take the point $\mathbf{x} = a\mathbf{u}$ into the point $\mathbf{x}' = a\mathbf{u}'$. In particular, any point $\mathbf{x}$ can be translated to the coordinate origin $\mathbf{x}' = 0$ by choosing $\mathbf{q}$ equal to $-\mathbf{x}/a$. In fact, Eqn (82) is

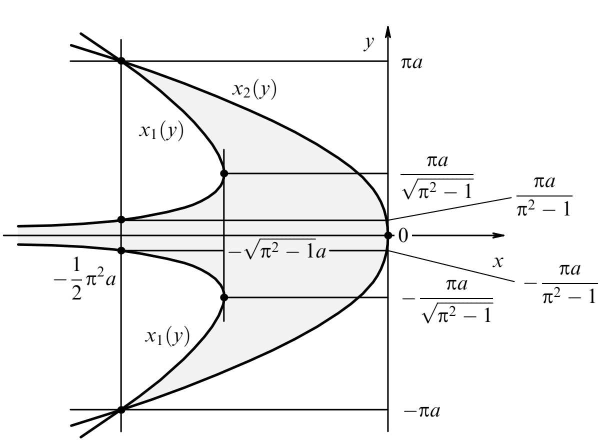

Figure 5. The region of isometric mapping of the Beltrami funnel (cut along the meridian) on the x, y plane with pseudosphere metric (76) in the pseudo-Euclidean space.

the transformation of the velocity $\mathbf{u}$ into $\mathbf{u}'$ by the Lorentz boost with the velocity $\mathbf{q}$:

$$\mathbf{u}' = \mathbf{u} + \frac{\mathbf{q}}{C_q}\,, \qquad C_q = \frac{\gamma_q + 1}{\gamma + \gamma\gamma_q + \mathbf{uq}}\,, \qquad \gamma_q = \sqrt{\mathbf{q}^2 + 1} \quad (83)$$

[cf. Eqns (4) and (5)]. It can be shown straightforwardly that Lorentz transformation (83) of all velocities determining γ, γ_1, and γ_2 in Eqn (35) does not change these factors:

$$\begin{aligned} &\gamma(\mathbf{u}_1, \mathbf{u}_2) = \gamma(\mathbf{u}'_1, \mathbf{u}'_2)\,, \qquad \gamma_1(\mathbf{u}, \mathbf{u}'') = \gamma_1(\mathbf{u}', \mathbf{u}''')\,, \\ &\gamma_2(\mathbf{u}, -\mathbf{u}_1) = \gamma_2(\mathbf{u}', -\mathbf{u}'_1)\,. \end{aligned} \qquad (84)$$

In accordance with representation (64), this implies that the angle ω is also invariant under transformation (83).

Unlike the PSPE surface, the PSE surface is generally not isotropic or homogeneous, its principal curvatures k_1, k_2 being dependent on ρ:

$$k_1 = \frac{\rho}{a\sqrt{a^2 - \rho^2}}\,, \qquad k_2 = -\frac{\sqrt{a^2 - \rho^2}}{a\rho}\,, \qquad -\pi < \varphi \leqslant \pi\,, \qquad (85)$$

and, moreover, they turn to $0, -\infty$ and $\infty, 0$ for $\rho = 0$ and $\rho = a$, although the Gaussian curvature remains constant everywhere on the surface, being equal to $K = k_1 k_2 = -1/a^2$. Hence, the PSE surface is isometric to the PSPE one only in some region [see Eqn (74)]. The area of this region is $2\pi a^2$, while the area of the PSPE surface, as the area of the Lobachevsky planes, is infinite. This is in accordance with the theorem of Hilbert, who showed that there is no complete and regular surface in Euclidean space that would be isometric to the entire Lobachevsky plane [13].

12. The Hilbert theorem

Because the real physical space is pseudo-Euclidean, representing relativistic velocities by segments of geodesics and the velocity addition law by the geodesic triangle on a homogeneous and isotropic PSPE surface seems to be adequate for the physical reality. The momenta $\mathbf{p} = m\mathbf{u}$ and the energies $E = m\gamma$ of free relativistic particles in pseudo-Euclidean Minkowski space follow just these rules. And the coordinates x, y, t of a point on the PSPE surface, where $t = \sqrt{a^2 + x^2 + y^2}$, are straightforwardly (one can say directly) related to components of the 4-velocity $u^\alpha = (\mathbf{u}, \gamma)$:

$$u_x = \frac{x}{a}\,, \qquad u_y = \frac{y}{a}\,, \qquad \gamma = \frac{t}{a}\,. \qquad (86)$$

On the other hand, a hyperbolic geometry, i.e., a metric with a constant negative Gaussian curvature, is also induced on a Beltrami surface embedded into a Euclidean space (we here return to the surface shown in Fig. 3). Because the Beltrami surface has the finite area $4\pi a^2$ and the finite volume $(2/3)\pi a^3$, it is interesting to know whether this means that there are finite Euclidean volumes of the same order πa^3 in which physical objects are characterized not by velocities but by purely spatial quantities — proper volumes, distances from each other, and purely spatial distributions.

It is known from differential geometry that if the Chebyshev coordinate line network x, y is chosen on a surface in a Euclidean space, the metric (the first quadratic form) on this surface is given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + 2\cos\phi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2\,, \qquad (87)$$

where the network angle $\phi(x, y)$ is by definition in the range $0 < \phi < \pi$ [14]. Then, from the famous Gauss formula that relates the curvature K to the coefficients of the first quadratic form, it follows that for the surface with $K = -1/a^2$ and metric (87), the network angle satisfies the nonlinear sine-Gordon equation [13]:

$$a^2\,\frac{\partial^2\phi}{\partial x\,\partial y} = \sin\phi\,. \qquad (88)$$

On the other hand, setting the second quadratic form for the PSE

$$\mathbf{N}\,\mathrm{d}^2\mathbf{r} = \frac{a}{\rho\sqrt{a^2 - \rho^2}}\,\mathrm{d}\rho^2 - \frac{\rho\sqrt{a^2 - \rho^2}}{a}\,\mathrm{d}\varphi^2 \qquad (89)$$

to zero yields two families of asymptotic lines on this surface:

$$\begin{aligned} \varphi_1 &= \mp\,\mathrm{arcosh}\,\frac{a}{\rho} - c_1 = \ln\tan\frac{u}{2} - c_1\,, \\ \varphi_2 &= \pm\,\mathrm{arcosh}\,\frac{a}{\rho} + c_2 = -\ln\tan\frac{u}{2} + c_2\,. \end{aligned} \qquad (90)$$

Here, $\rho = a\sin u$, $0 < u < \pi$ and the upper and lower signs before arcosh and the ranges $0 < u \leqslant \pi/2$, $\pi/2 \leqslant u < \pi$ respectively relate to the upper and lower Beltrami funnels. We recall that

$$\mathrm{arcosh}\,\frac{a}{\rho} = \ln\left(\frac{a}{\rho} + \sqrt{\frac{a^2}{\rho^2} - 1}\right), \qquad \frac{a}{\rho} \geqslant 1\,.$$

Using Eqns (90) for the asymptotic lines, we tacitly assume that the PSE surface, as shown in Fig. 3, is cut by the meridian $\varphi = \pm\pi$ and that infinitely extending sheets of the so-called universal covering surface of the PSE are glued to the edges of this cut. This universal covering surface, which has the same curvature $K = -1/a^2$, covers the PSE surface an infinite number of times, with its kth layer $(2k - 1)\pi < \varphi \leqslant (2k + 1)\pi$, where k is any integer, being considered as lying above the previous $(k - 1)$th one. Thus, Eqns (90) represent the asymptotic PSE lines on its universal covering surface.

With the parameter u increasing in the range $0 < u < \pi$, the asymptotic lines of the two families, by rotating in the opposite directions, move over the surface of the upper funnel and approach the cuspidal edge; at $u = \pi/2$, they touch it at the points $\varphi_1 = -c_1$, $\varphi_2 = c_2$, and touch each other if $c_2 = -c_1$, and then, as u increases further, go away from the edge over the surface of the lower funnel by rotating in the same direction. Intriguingly, the asymptotic lines, while being spatial curves, have no singularities for all values of the parameter u in the interval $0 < u < \pi$, although the PSE surface on which they locate has a singular cuspidal edge at $u = \pi/2$.

It is even more interesting that two asymptotic lines of the first and second family that go down along the surface of the upper funnel and have a joint touch point with each other and with the cuspidal edge ($c_2 = -c_1$ for these lines) can be considered a single line that goes down from the infinity $u = 0$, $\varphi = -\infty$ over the surface of the upper funnel towards the cuspidal edge as a line of the first family, and after touching the cuspidal edge goes up over the surface of the same funnel towards the infinity $u = 0$, $\varphi = +\infty$ as a line of the second family. As a result, the upper funnel is covered by a one-parameter network of asymptotic lines that start and end at $u = 0$, $\varphi = \mp\infty$ and touch the cuspidal edge at the points $u = \pi/2$, $\varphi = c$ determined by the value of the parameter $c = -c_1 = c_2$, $-\infty < c < \infty$. Thus, the cuspidal edge is the natural boundary of the asymptotic network for the upper funnel. The same is true for the asymptotic network on the lower funnel, because it is mirror symmetric to the upper one.

Now, if we use the coordinates x, y (instead of cylindrical coordinates ρ, φ or variables u, φ, $u \in (0, \pi)$) defined as

$$x = \frac{1}{2}\, a\left(\ln\tan\frac{u}{2} - \varphi\right), \qquad y = \frac{1}{2}\, a\left(\ln\tan\frac{u}{2} + \varphi\right) \tag{91}$$

in length element (71) on the PSE surface and allow the angle φ to range from $-\infty$ to $+\infty$, this element takes form (87), where $\cos\phi = \cos 2u$. This means that the variables x and y in Eqn (91), which according to Eqn (90) form the asymptotic line network on the PSE surface (more precisely, on its universal covering surface), coincide with the variables x and y of the coordinate lines of the Chebyshev network. In that case, the network angle $\phi(x, y)$ of the Chebyshev network must also coincide with the angle between the asymptotic lines at the same point on the surface. The last angle can be found using the Euler formula for the normal curvature in the θ direction:

$$k(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta\,. \tag{92}$$

Here, k_1 and k_2 are principal curvatures (85) and the angle θ is referenced to the meridian direction. By definition, the curvature $k(\theta)$ vanishes in the asymptotic directions, and for the angles θ_1 and θ_2 of these directions to the meridian, we obtain

$$\begin{aligned} &\tan\theta_{1,2} = \pm\sqrt{-\frac{k_1}{k_2}} = \pm\frac{\rho}{\sqrt{a^2 - \rho^2}}\,, \\ &\theta_{1,2} = \pm\arctan\frac{\rho}{\sqrt{a^2 - \rho^2}}\,. \end{aligned} \tag{93}$$

Thus, the angle between two asymptotic directions on the PSE surface is

$$2\theta_1 = 2\arctan\frac{\rho}{\sqrt{a^2 - \rho^2}} = 2\arcsin\frac{\rho}{a}\,. \tag{94}$$

It is determined only by the radius of a latitudinal circle, irrespective of whether it belongs to the upper or lower Beltrami funnel, and lies in the range $0 < 2\theta_1 < \pi$.

On the other hand, the equation $\cos\phi = \cos 2u$ relating the Chebyshev angle ϕ to the radius $\rho = a\sin u$ or the parameter u, which is $u = \arcsin(\rho/a)$ and $u = \pi - \arcsin(\rho/a)$ for the upper and lower funnels, respectively, has two solutions:

$$(1)\ \ \phi = 2u\,, \qquad (2)\ \ \phi = 2\pi - 2u\,. \tag{95}$$

Using the first of these solutions for the upper funnel and the second for the lower funnel, we obtain the same result:

$$\begin{aligned} &(1)\ \ \phi = 2u = 2\arcsin\frac{\rho}{a}\,, \\ &(2)\ \ \phi = 2\pi - 2u = 2\pi - 2\left(\pi - \arcsin\frac{\rho}{a}\right) = 2\arcsin\frac{\rho}{a}\,, \end{aligned} \tag{96}$$

coinciding with Eqn (94).

Thus, u increases in the range $0 < u < \pi$, the network angle ϕ first increases linearly, reaches the forbidden value $\phi = \pi$ for the Chebyshev network at the boundary $u = \pi/2$ of the upper and lower funnels, and then decreases linearly to zero. Therefore, the Chebyshev network, as well as the PSE surface itself, has singularities at the boundary of the upper and lower funnels (a jump of the derivative $\partial\phi/\partial u$) and decomposes into two regular networks covering each of the funnels separately. This fact allowed Hilbert to argue that in a Euclidean space, there is no analytic surface with constant negative curvature without singularities that is regular everywhere, i.e., which is isometric to the entire Lobachevsky plane.

Nevertheless, the sine-Gordon equation has an infinite number of solutions that are regular everywhere on the PSE surface (more precisely, on its universal covering surface). However, they no longer have the interpretation of the network angle of the Chebyshev or asymptotic network. We mention only one such solution.

13. A pseudosphere in Euclidean space as an arena for extended relativistic objects

Instead of x and y we introduce the dimensionless variables ξ and φ using formulas (91):

$$x + y = a\ln\tan\frac{u}{2} = a\xi\,, \qquad y - x = a\varphi\,. \tag{97}$$

Similarly to x and y, they are Cartesian variables in the ξ, φ plane. Then sine-Gordon equation (88) takes the form

$$\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\varphi^2} = \sin\phi\,. \tag{98}$$

Its simplest solution

$$\phi_\beta(\xi, \varphi) = 4\arctan\left[\exp\left(\varepsilon\,\frac{\xi - \beta\varphi}{\sqrt{1 - \beta^2}}\right)\right], \quad \varepsilon = \pm 1\,, \tag{99}$$

which is regular everywhere on the plane ξ, φ, is called a soliton ($\varepsilon = 1$) or antisoliton ($\varepsilon = -1$). It contains the parameter β, $|\beta| < 1$, which can be called the velocity of a soliton (antisoliton) if ξ and φ are respectively interpreted as space and time coordinates. It seems very significant that the angle coordinate φ of a point on the universal covering PSE surface plays the role of time. The space coordinate is determined by the radius of a latitudinal circle and is equal to

$$\xi = \mp \operatorname{arcosh} \frac{a}{\rho} \tag{100}$$

for the upper and lower Beltrami funnels. With changing ξ at a fixed φ, the soliton solution smoothly increases from 0 to 2π and the antisoliton solution decreases from 2π to 0. Here, the main change in $\phi_\beta(\xi, \varphi)$ occurs in the region $\xi \approx \beta\varphi$ with the extension $\Delta\xi \sim (1-\beta^2)^{1/2}$, which is contracted as the velocity increases (the Lorentz contraction of the soliton size). Under the Lorentz transformation with the velocity β_1, the solution $\phi_\beta(\xi, \varphi)$, which is a relativistic scalar, preserves its functional dependence on the Lorentz-transformed coordinates ξ', φ' and the Lorentz-transformed velocity β': $\phi_\beta(\xi, \varphi) = \phi_{\beta'}(\xi', \varphi')$, because

$$\frac{\xi - \beta\varphi}{\sqrt{1-\beta^2}} = \frac{\xi' - \beta'\varphi'}{\sqrt{1-\beta'^2}}\,, \qquad \beta' = \frac{\beta + \beta_1}{1 + \beta\beta_1}\,. \tag{101}$$

The transformed velocity β' is the relativistic sum of the velocities β and β_1.

Thus, the PSE surface (more precisely, its universal covering surface) turns out to be the arena of the existence and motion of extended relativistic objects. Extensive literature is devoted to this issue (see, e.g., Refs [15, 16]).

Because the sine-Gordon equation appeared in the geometric problem of embedding a two-dimensional surface with constant negative curvature into the three-dimensional Euclidean space, it follows that in physical problems described by the sine-Gordon equation, the physical variable playing the role of time and the physical unit of the velocity measure become interesting. In any case, it is difficult to expect the dimensionless velocity β occurring in these problems to be measured in units of the speed of light, which emerged because the real four-dimensional space is pseudo-Euclidean.

14. Conclusion

To conclude, we note that the three-parameter representations of the angle ω obtained by the author in 1961 in the non-Euclidean hyperbolic geometry are representations for the area (times the curvature) of the pseudospherical triangle of velocities related by a Lorentz transformation. They demonstrate the purely geometric non-Euclidean origin of this angle, which is the object of the internal geometry of a curved surface, and its relativistic invariance.

But the simplicity of the reflection of the discussed symmetry by *two Euclidean* velocity triangles shown in Fig. 1 should not be underestimated. Indeed, the 'sine theorems' corresponding to these two triangles,

$$\frac{u}{\sin(\pi - \theta')} = \frac{u_2}{\sin(\pi - \theta)} = \frac{u_1}{C_1 \sin\vartheta}\,, \tag{102}$$

$$\frac{u_1}{\sin\delta} = \frac{u''}{\sin(\pi - \theta)} = \frac{u}{C\sin(\theta - \delta)}\,, \tag{103}$$

explicitly include the sine theorem

$$\frac{u}{\sin A} = \frac{u_1}{\sin A_1} = \frac{u_2}{\sin A_2} \tag{104}$$

for one geodesic triangle on a pseudosphere, because $u'' = u_2$ [see also Eqns (34), (37)]. The three expressions for the 'cosine theorem,'

$$u^2 = \frac{u_1^2}{C_1^2} + u_2^2 - 2\,\frac{\mathbf{u}_1\mathbf{u}_2}{C_1}\,, \tag{105}$$

$$u_1^2 = u''^2 + \frac{u^2}{C^2} - 2\,\frac{\mathbf{u}''\mathbf{u}}{C}\,, \tag{106}$$

$$u_2^2 = u^2 + \frac{u_1^2}{C_1^2} + 2\,\frac{\mathbf{u}\mathbf{u}_1}{C_1}\,, \tag{107}$$

with the angles adjacent to the boost velocities in these two Euclidean triangles, after simple transformations, are reduced to expressions

$$\gamma = \gamma_1\gamma_2 - \mathbf{u}_1\mathbf{u}_2\,, \quad \gamma_1 = \gamma''\gamma - \mathbf{u}''\mathbf{u}\,, \quad \gamma_2 = \gamma\gamma_1 + \mathbf{u}\mathbf{u}_1\,, \tag{108}$$

i.e., to the cosine theorem for the same pseudospherical triangle [see also Eqns (35), (38)].

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