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New directions in the theory of electron cooling

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Contents

Abstract. **The theory of electron cooling of ions and positrons is reviewed. Formulas describing the retarding force of ions in an electron beam with an 'oblate' velocity distribution, which is typical for electron cooling, are considered for arbitrary intensities of a magnetic field. Considered for positrons are the cases of intermediate and strong magnetic fields, which are of the greatest practical interest. The friction force and the components of the positron velocity diffusion tensor are calculated. Also discussed is the relaxation of positrons in their electron cooling in positron storage rings and their transition to the stationary distribution. The stationary velocity distribution function for positrons is shown to practically coincide in this case with that for electrons. The feasibility of lowering the transverse electron temperature is analyzed, which is required for decreasing the positron spread in momentum.**

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1. Introduction

The electron cooling method [1] (see also reviews [2, 3]) has been validly applied to decrease the phase volume for beams of particles with mass

$$M \gg m\,, \tag{1}$$

where m is the electron mass. Hereinafter, these particles will be referred to as M particles, and their charge denoted by q. The problem of cooling positrons, when the masses satisfy the condition

$$M = m\,, \tag{2}$$

is new. This problem arose in projects aimed at obtaining antihydrogen and positronium atoms and studying their properties (see review [4] as well as Refs [5, 6]).

One of the key quantities subject to calculation is the friction force $\mathbf{F}$ experienced by a particle moving relative to an electron cloud (beam). This review is concerned primarily with the cooling of light particles (positrons). For the sake of completeness of the picture, given at the beginning is a brief analysis of the theory of cooling of heavy particles (1), which has been well elaborated up to the present time. When inequality (1) holds, collective (cooperative) effects and binary collisions make comparable contributions to the friction force [3, 4]. However, when equality (2) is valid, collective effects play the leading role. The corresponding computation is more arduous, and this is supposedly the reason why the theory of cooling has not been completed for positrons, with only a few papers published on this subject [4–9]. In view of the significance of cooling positrons in the projects drawn up in Refs [4–6], in this review we return to this issue to critically analyze the published results, as well as to summarize and supplement the data required for planning experiments on positrons and positronium atoms.

We discuss in this review the methods for calculating the positron velocity ($\mathbf{V}$) distribution function $\Phi(\mathbf{V}, t)$. This function contains exhaustive information about the positron cooling process. Of special interest is the stationary positron velocity distribution function $\Phi(\mathbf{V})$, which sets in as a result of relaxation, and ways to decrease the positron spread in momentum.

Powerful methods have been elaborated in plasma physics, which may be applied validly to the analysis of the range of phenomena under consideration, and one of the objectives of our review resides in demonstrating this. Another objective lies in giving a list of formulas sufficient for the practical calculations of positron moderation kinetics, along with brief derivations of the formulas. The author endeavored to supplement the formal derivations with qualitative estimates and discussions of the physical meaning of the results obtained. Lastly, the third objective is to outline new results pertaining to a rather interesting field — the physics of anisotropic plasmas.

2. Main effects in the electron cooling of heavy particles

Prior to addressing ourselves to the discussion of the theory of positron cooling, we recall the main ideas advanced and effects predicted theoretically and discovered in the course of investigations into the electron cooling of heavy particles.

An electron cloud is confined by the combination of external electric and magnetic fields which compensate for the Coulomb repulsion between the electrons and thereby are a substitute for the background of positive ions present in ordinary quasineutral plasma. As a result, practically all properties of a cloud coincide with those of the ordinary plasma [1–4].

For a nonzero magnetic field ($\mathbf{H} \neq 0$), the particle trajectories are helical, with the radii equal to the Larmor radii:

$$r_{\mathrm{HM}} = \frac{V_\perp}{\omega_{\mathrm{HM}}}\,, \qquad r_{\mathrm{H}} = \frac{v_\perp}{\omega_{\mathrm{H}}}\,, \qquad \omega_{\mathrm{HM}} = \frac{qH}{Mc}\,,$$
$$\omega_{\mathrm{H}} = \frac{eH}{mc}\,, \qquad V_\perp = |\mathbf{V}_\perp|\,, \qquad v_\perp = |\mathbf{v}_\perp|\,. \tag{3}$$

Here, the vectors $\mathbf{V}_\perp$ and $\mathbf{v}_\perp$ are the respective components of the velocities $\mathbf{V}$ and $\mathbf{v}$ of the particle M and the electron that are perpendicular to the direction of magnetic field $\mathbf{h} = \mathbf{H}/H$ (considered here and almost everywhere below is the rest frame of the beam, so that $\mathbf{V}$ and $\mathbf{v}$ are the velocities in this reference system). Calculations of the friction force $\mathbf{F}$ for helical paths are quite cumbersome, and so as a preliminary step we will analyze the limiting cases most important for practical applications: the cases of a weak (the definition is given below) and an ultimately strong magnetic field (in beam physics, the latter case is also referred to as 'magnetized plasma'). The electron plasma is assumed to be ideal:

$$\xi = \frac{\overline{K}}{\overline{U}} \gg 1\,, \tag{4}$$

where $\overline{K}$ and $\overline{U}$ are the average values of the kinetic and potential energies of the electrons. Furthermore, when describing electron cooling it would suffice to consider the nonrelativistic case:

$$\Delta = \sqrt{\frac{T}{m}} \ll c\,, \tag{5}$$

$$V \ll c\,. \tag{6}$$

In relations (4)–(6), the following notation was used:

$$\overline{U} \sim \frac{e^2}{\overline{R}}\,, \tag{7}$$

$T = 2\overline{K}/3$ is the effective electron temperature, $\overline{R} \sim n^{-1/3}$ is the average interelectron distance, and n is the electron number density [in cm^{-3}] in the rest frame of the beam.

From inequality (4) there follows a conclusion on the smallness of the particle scattering angle θ_{scat}:

$$\theta_{\mathrm{scat}} \ll 1 \tag{8}$$

for a typical collision in an ideal plasma. The duration of the interaction for particles M and m in a collision with an impact parameter $\boldsymbol{\rho}$ is defined by the time $\tau_{\mathrm{coll}} \sim \rho/u$, where $\rho = |\boldsymbol{\rho}|$, $u = |\mathbf{u}|$, and $\mathbf{u} = \mathbf{V} - \mathbf{v}$ is their relative velocity. In this case, the particle m experiences acceleration $a_{\mathrm{m}} \sim qe/(m\rho^2)$ and becomes displaced by a distance $l_{\mathrm{m}} \sim a_{\mathrm{m}}\tau_{\mathrm{coll}}^2 \sim qe/(mu^2)$, and therefore

$$\theta_{\mathrm{scat}}^{(\mathrm{m})} \sim \frac{l_{\mathrm{m}}}{\rho} \sim \frac{e^2}{\rho m u^2} \sim \frac{\overline{U}}{\overline{K}} \ll 1\,.$$

It should also be noted that

$$l_{\mathrm{m}} \ll \rho\,. \tag{9}$$

The force of friction is conveniently resolved into two components:

$$\mathbf{F} = \mathbf{F}_{\mathrm{b}} + \mathbf{F}_{\mathrm{c}}\,, \tag{10}$$

where $\mathbf{F}_{\mathrm{b}}$ is the contribution from the binary collisions of particle M with the electrons, and $\mathbf{F}_{\mathrm{c}}$ is the contribution made by collective interactions, when particle M interacts, under condition (4), simultaneously with a large number of electrons, which is on the order of the number of electrons N_{D} in the Debye sphere ($N_{\mathrm{D}} = \xi^{3/2}$).

First, we discuss the case of a weak magnetic field ($H \to 0$). Let us consider the inertial reference system in which particle M is at rest prior to collision [we emphasize that when condition (1) is satisfied particle M remains practically motionless in this reference system after a collision, as well]. As a result of the interaction with M, particle m acquires the transverse momentum $\Delta p_\perp = 2qe/(\rho u)$ and is deflected by an angle $\theta_{\mathrm{scat}}^{(\mathrm{m})} = \Delta p_\perp/(mu) = 2qe/(\rho m u^2)$. The momentum $\Delta\mathbf{p}_\perp$, which is transverse relative to the vector $\mathbf{u}$, is of no concern to us, because it vanishes on averaging over collisions with different directions of the vector $\boldsymbol{\rho}$. The change in the longitudinal momentum component of particle m, caused by its deflection, is expressed as $\Delta p_{\mathrm{m}\parallel} = mu(1-\cos\theta_{\mathrm{scat}}^{(\mathrm{m})}) \approx mu(\theta_{\mathrm{scat}}^{(\mathrm{m})})^2/2$, or in a vector form as $\Delta \mathbf{p}_{\mathrm{m}\parallel} = (2q^2e^2/mu^3\rho^2)\hat{\mathbf{u}}$. Due to the Galilean invariance, these formulas are valid in any frame of reference, including the laboratory frame. On the strength of momentum conservation we conclude that the change in the momentum of particle M, caused by the deflection of particle m, is given by

$$\Delta \mathbf{p}_{\mathrm{M}}^{(1)} = -\Delta \mathbf{p}_{\mathrm{m}\parallel} = -\frac{2q^2e^2}{mu^3\rho^2}\,\hat{\mathbf{u}}\,. \tag{11}$$

The number density of electrons with velocities $\mathbf{v}$ in the element d^3v is expressed as $\mathrm{d}n = nf(\mathbf{v})\,\mathrm{d}^3v$, where $f(\mathbf{v})$ is their distribution function normalized to unity: $\int f(\mathbf{v})\,\mathrm{d}^3v = 1$. The number of such electrons traversing an impact parameter ring with an area of $2\pi\rho\,\mathrm{d}\rho$ per unit time is $\mathrm{d}n\,u\,2\pi\rho\,\mathrm{d}\rho$, whence we obtain [10, 11]

$$\mathbf{F}_{\mathrm{b}} = \int \mathrm{d}n\,u\,2\pi\rho\,\mathrm{d}\rho\,\Delta\mathbf{p}_{\mathrm{M}}^{(1)} = \frac{4\pi ne^2q^2\Lambda_{\mathrm{b}}}{m}\,\nabla_{\mathbf{V}}\Phi(\mathbf{V})\,, \tag{12}$$

where $\Phi(\mathbf{V})$ is 'the first Trubnikov potential' defined as

$$\Phi(\mathbf{V}) = \int \frac{f(\mathbf{v})\,\mathrm{d}^3\mathbf{v}}{u}\,, \tag{13}$$

Λ_{b} is the Coulomb logarithm for the binary collisions of particles M with the electrons:

$$\Lambda_{\mathrm{b}} = \ln\frac{\overline{R}}{R_{\mathrm{T}}}\,, \tag{14}$$

and R_{T} is the Thomson radius:

$$R_{\mathrm{T}} = \frac{e^2}{T}\,. \tag{15}$$

The main contribution to $\mathbf{F}_{\mathrm{b}}$ is made by collisions with impact parameters $\rho < \overline{R}$. Formula (12) is logarithmically accurate up to $\sim 1/\Lambda_{\mathrm{b}}$. For an isotropic distribution $\Phi(\mathbf{V}) = \Phi(V)$, one finds $\nabla_{\mathbf{V}}\Phi(\mathbf{V}) = \Phi'(V)\hat{\mathbf{V}}$, and $\hat{\mathbf{V}} = \mathbf{V}/V$. In particular, for the Maxwellian isotropic distribution $f(\mathbf{v}) = (m/2\pi T)^{3/2}\exp(-m\mathbf{v}^2/2T)$, we arrive at

$$\Phi'(V) = -\sqrt{\frac{2}{\pi}}\,\frac{1}{\Delta^3V^2}\int_0^V \mathrm{d}v\,v^2\exp\left(-\frac{v^2}{2\Delta^2}\right).$$

In the limiting cases, we thus obtain

$$\Phi'(V) = -\begin{cases} V^{-2}\,, & V \gg \Delta\,,\\ \Delta^{-3}V\,\dfrac{2\sqrt{2}}{3\sqrt{\pi}}\,, & V \ll \Delta\,. \end{cases} \tag{16}$$

In the calculation of $\mathbf{F}_{\mathrm{c}}$, ideal plasma characterized by condition (4) may be treated, in view of inequality $N_{\mathrm{D}} \gg 1$, as a continuous medium whose response to an electromagnetic perturbation with frequency ω and wave vector $\mathbf{k}$ is described by the permittivity tensor [11–17]

$$\varepsilon_{\alpha\beta}(\mathbf{k},\omega) = \delta_{\alpha\beta} + \frac{\omega_{\mathrm{p}}^2}{\omega}\int \frac{v_\alpha\,\partial f(\mathbf{v})/\partial v_\beta}{\omega + \mathrm{i}0 - \mathbf{kv}}\,\mathrm{d}^3v\,, \tag{17}$$

where $\omega_{\mathrm{p}} = \sqrt{4\pi ne^2/m}$ is the plasma frequency.

In the nonrelativistic case (5), one may neglect the transverse electromagnetic field and consider only the longitudinal electric field [11–17]. In this limiting case, for arbitrary intensities of the magnetic field the plasma is similar to an isotropic medium with the permittivity

$$\varepsilon(\mathbf{k},\omega) = \varepsilon_{\alpha\beta}(\mathbf{k},\omega)\hat{k}_\alpha\hat{k}_\beta \equiv \varepsilon_\parallel\,, \tag{18}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. We elucidate this conclusion, because it is of importance for the subsequent discussion. In the nonrelativistic case (5) we have

$$\mathbf{E}(\mathbf{r},t) = -\nabla\varphi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \approx -\nabla\varphi(\mathbf{r},t)\,,$$

and the electric field is therefore longitudinal:

$$\mathbf{E}(\mathbf{k},\omega) = \int \mathrm{d}^3r\,\mathrm{d}t\,\exp(-\mathrm{i}\mathbf{kr} + \mathrm{i}\omega t)\,\mathbf{E}(\mathbf{r},t) = -\mathrm{i}\mathbf{k}\varphi(\mathbf{k},\omega)\,. \tag{19}$$

The Maxwell equations in the plasma may be represented in the form coincident with the form of the equations for a dielectric medium [11–18]. In particular, one finds

$$\nabla\,\mathbf{D}(\mathbf{r},t) = 4\pi\rho_{\mathrm{ex}}(\mathbf{r},t)\,,\qquad D_\alpha(\mathbf{k},\omega) = \varepsilon_{\alpha\beta}(\mathbf{k},\omega)\,E_\beta(\mathbf{k},\omega)\,, \tag{20}$$

where $\rho_{\mathrm{ex}}(\mathbf{r},t)$ is the density of charge introduced into the plasma. From expressions (19), (20) we obtain $\mathrm{i}k_\alpha D_\alpha(\mathbf{k},\omega) = 4\pi\rho_{\mathrm{ex}}(\mathbf{k},\omega)$, i.e., $\mathrm{i}k_\alpha\varepsilon_{\alpha\beta}(\mathbf{k},\omega)\big(-\mathrm{i}k_\beta\varphi(\mathbf{k},\omega)\big) = 4\pi\rho_{\mathrm{ex}}(\mathbf{k},\omega)$, and therefore

$$\varphi(\mathbf{k},\omega) = \frac{4\pi\rho_{\mathrm{ex}}(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)}\,,\qquad \mathbf{E}(\mathbf{k},\omega) = -\frac{4\pi\mathrm{i}\mathbf{k}}{k^2\varepsilon(\mathbf{k},\omega)}\,\rho_{\mathrm{ex}}(\mathbf{k},\omega)\,, \tag{21}$$

where

$$\rho_{\mathrm{ex}}(\mathbf{k},\omega) = \int \mathrm{d}^3r\,\mathrm{d}t\,\exp(-\mathrm{i}\mathbf{kr} + \mathrm{i}\omega t)\,\rho_{\mathrm{ex}}(\mathbf{r},t)\,.$$

Formula (21) is the generalization of the ordinary Coulomb formula

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\varepsilon}\int \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\rho_{\mathrm{ex}}(\mathbf{r}',t)\,\mathrm{d}^3r'\,,$$

which is valid for a dielectric medium in the absence of dispersion (i.e., for ε independent of $\mathbf{k}$, ω).

For physical reasons we rewrite formula (21) in the form

$$\mathbf{E}(\mathbf{k},\omega) = \mathbf{E}_{\mathrm{c}}(\mathbf{k},\omega) + \mathbf{E}_{\mathrm{p}}(\mathbf{k},\omega)\,,$$

where $\mathbf{E}_{\mathrm{c}}(\mathbf{k},\omega) = -(4\pi\mathrm{i}\mathbf{k}/k^2)\,\rho_{\mathrm{ex}}(\mathbf{k},\omega)$ is the strength of an intrinsic field of the charge, and $\mathbf{E}_{\mathrm{p}}$ is the strength of an electric field, defined as

$$\mathbf{E}_{\mathrm{p}}(\mathbf{k},\omega) = -\frac{4\pi\mathrm{i}\mathbf{k}}{k^2}\,\rho_{\mathrm{p}}(\mathbf{k},\omega)\,, \tag{22}$$

of the charges with density

$$\rho_{\mathrm{p}}(\mathbf{k},\omega) = \rho_{\mathrm{ex}}(\mathbf{k},\omega)\left(\frac{1}{\varepsilon(\mathbf{k},\omega)} - 1\right), \tag{23}$$

induced in the plasma (the 'Debye' cloud) by the initial charge of density ρ_{ex}.

For a point charge q proceeding along a trajectory $\mathbf{r} = \mathbf{R}(t)$ we have

$$\rho_{ex}(\mathbf{r}, t) = q\delta\big(\mathbf{r} - \mathbf{R}(t)\big)\,,$$
$$\rho_{ex}(\mathbf{k}, \omega) = q\int_{-\infty}^{+\infty} \mathrm{d}t\, \exp\big(\mathrm{i}\omega t - \mathrm{i}\mathbf{k}\mathbf{R}(t)\big)\,. \qquad (24)$$

The force acting on the charge is expressed as

$$\mathbf{F}_c(t) = q\mathbf{E}_p\big(\mathbf{r} = \mathbf{R}(t), t\big) = q\int \frac{\mathrm{d}^3k\,\mathrm{d}\omega}{(2\pi)^4} \exp\big(\mathrm{i}\mathbf{k}\mathbf{R}(t) - \mathrm{i}\omega t\big)\,\mathbf{E}_p(\mathbf{k}, \omega)\,. \qquad (25)$$

For an immobile charge placed at the origin ($\mathbf{R} = 0$) and $H \to 0$, from expressions (21) and (23), in view of relationships

$$\rho_{ex}(\mathbf{r}, t) = q\delta(\mathbf{r})\,, \qquad \rho_{ex}(\mathbf{k}, \omega) = 2\pi q\delta(\omega)\,, \qquad (26)$$

we obtain for an isotropic electron distribution:

$$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r})\,, \qquad \varphi(\mathbf{r}) = q\,\frac{\exp(-r/r_D)}{r}\,,$$
$$\frac{1}{r_D^2} = \omega_p^2\overline{\left(\frac{1}{v^2}\right)} = 4\pi\omega_p^2\int_0^\infty \mathrm{d}v\, f(v)\,, \qquad (27)$$
$$\rho_p(\mathbf{k}, t) = \frac{q}{1 + k^2 r_D^2}\,, \qquad \rho_p(\mathbf{r}, t) = \frac{q\exp(-r/r_D)}{4\pi k^2 r_D^2 r}\,,$$

where r_D is the Debye radius:

$$r_D \approx \sqrt{\frac{T}{4\pi n e^2}} \qquad (28)$$

(for the Maxwellian distribution, the last relation is exact). Therefore, r_D is the characteristic radius of the Debye cloud. For a uniformly moving charge, one finds

$$\mathbf{R}(t) = X(t)\widehat{V}\,, \qquad X(t) = Vt\,, \qquad \rho(\mathbf{k}, \omega) = 2\pi q\delta(\omega - k_\parallel V)\,, \qquad (29)$$

where $k_\parallel = \mathbf{k}\widehat{\mathbf{V}}$.

In an ideal plasma, the following ratio would hold:

$$\frac{r_D}{\overline{R}} \sim \xi^{1/2} \gg 1\,. \qquad (30)$$

The main contribution to the collective term $\mathbf{F}_c$, as to $\mathbf{F}_b$, is made by the domain of large k values:

$$\frac{1}{r_D} < k < \frac{1}{\overline{R}}\,, \qquad (31)$$

which corresponds to the interaction range $\overline{R} < \rho < r_D$. Whence, and from the expression $\varepsilon(\mathbf{k}, \omega = 0) = 1 + 1/(k^2 r_D^2)$, one can see that

$$|\varepsilon - 1| \ll 1\,, \qquad (32)$$

and in formula (23) we can therefore put

$$\frac{1}{\varepsilon} - 1 \approx 1 - \varepsilon\,. \qquad (33)$$

Then, $\mathbf{F}_c$ is expressed in terms of $\operatorname{Im}\varepsilon$ and, with a logarithmic accuracy ($\sim 1/\Lambda_c$), we have

$$\mathbf{F}_c = \frac{4\pi q^2 e^2 n\Lambda_c}{m}\,\nabla_V\Phi(\mathbf{V})\,, \qquad \Lambda_c = \ln\frac{r_D}{\overline{R}}\,. \qquad (34)$$

The physical meaning of formula (34) is as follows. For $V < \Delta$, the Debye cloud lags behind the particle by a distance $x \sim r_D V/\Delta$. The total charge of the Debye cloud is equal in modulus and opposite in sign to the particle charge: $q_D = -q$ (otherwise, at a long distance from the charge there would be an electric field $(q + q_D)/r^2$ slowly decreasing with distance, which would lead to charge redistribution and eventually result in the compensation for the charge). Assuming for an estimate that the charge q_D is uniformly distributed over the Debye cloud, we obtain from expression (25) the relation $F_c \sim q(x/r_D)(q/r_D^2) \sim q^2V/(\Delta r_D^2)$, $V < \Delta$, which is consistent with expression (34).

For $V > \Delta$, the charge resides at the edge of the cloud whose size is $L \sim V\tau_p = V/\omega_p$ (the electrons respond to the charge field in a time τ_p, during which the charge traverses a distance L). Under the action of the electric field $E \sim q/L^2$ of the charge, the electrons are displaced by a distance $S \sim a_e\tau_p^2/2$ from their initial positions, where $a_e \sim E/m$ is the characteristic electron acceleration. The charge q_D is formed when the electrons enter the Debye domain from its near-surface layer of thickness $\sim S$. They make up a charge $q_D \sim -en_eSL^2 \sim -q$ (as indicated above, the rigorous result is $q_D = -q$). Consequently [see expression (25)], $E_p \sim q_D/L^2$, $F_c = qE_p \sim q^2e^2n/(mV^2)$, $V > \Delta$, which is also consistent with formula (34). Later on, these qualitative estimates will be helpful in elucidating the more complicated cases of a nonzero magnetic field.

Formulas (10), (12), and (34) lead to a well-known remarkable conclusion: the intermediate dimension $\overline{R}$ does not enter into the expression for the total force of friction:

$$\mathbf{F} = \frac{4\pi q^2 e^2 n\Lambda}{m}\,\nabla_V\Phi(\mathbf{V})\,,$$
$$\Lambda = \Lambda_b + \Lambda_c = \ln\frac{\overline{R}}{R_T} + \ln\frac{r_D}{\overline{R}} = \ln\frac{r_D}{R_T}\,. \qquad (35)$$

For this reason, it is unnecessary to strictly define the value of the boundary impact parameter separating the binary and collective processes.

Now let us take into account the 'oblateness' of the electron velocity distribution function [19, 20]

$$f(\mathbf{v}) = G_e(v_\perp)\,g(v_\parallel)\,, \qquad G_e(v_\perp) = \frac{1}{2\pi\Delta_\perp^2}\exp\left(-\frac{v_\perp^2}{2\Delta_\perp^2}\right),$$
$$g(v_\parallel) = \frac{1}{\sqrt{2\pi}\,\Delta_\parallel}\exp\left(-\frac{v_\parallel^2}{2\Delta_\parallel^2}\right), \qquad (36)$$

which is always inherent in coolers' electron beams and plays a critical role in the formation of ultralow-temperature ion beams [2, 3]. In formulas (36) we introduced the notation

$$\Delta_\perp = \sqrt{\frac{T_\perp}{m}}\,, \qquad \Delta_\parallel = \sqrt{\frac{T_\parallel}{m}}\,, \qquad (37)$$

where $T_\perp$and $T_\parallel$ are the temperatures characterizing the electron motion across and along the magnetic field lines that the electron beam travels along. The oblateness emerges due to the effect of kinematic cooling of the accelerated electron beam, which is a corollary of Louiville's theorem. Starting from the cathode with a velocity u_0, the electron traverses a region with a potential drop U, and its velocity becomes equal to

$$u = \sqrt{u_0^2 + \frac{2eU}{m}} \tag{38}$$

(we consider the nonrelativistic electron beams which are employed at the present time). If the initial velocities of two electrons differ by Δu_0, their final velocities will differ according to formula (38) by

$$\Delta u = \frac{u_0 \Delta u_0}{u} \,. \tag{39}$$

Clearly, these relations pertain to the longitudinal motion, and therefore from formula (39) and an estimate $|\Delta u_0| \sim u_0 \sim \sqrt{T_c/m}$, where T_c is the cathode temperature, there follows the approximate relationships:

$$T_\parallel \sim \frac{T_c^2}{\varepsilon} \,, \qquad \varepsilon = \frac{mu^2}{2} \approx eU \,, \qquad T_\perp \approx T_c \,. \tag{40}$$

For typical values of $T_c = 1000$ K and $U = 10$ kV, from formulas (40) we obtain $T_\parallel \sim 0.3$ K. Lower values of $T_\parallel$ cannot be attained in reality, because the electrons in the beam would necessarily move relative to each other. By virtue of the virial theorem, one finds

$$T_\parallel \sim \frac{e^2}{\overline{R}} + \frac{T_c^2}{\varepsilon} \,. \tag{41}$$

To speed up the electron cooling rate, the beam should be as dense as possible; however, owing to Coulomb repulsion the density of a stable beam cannot exceed the value of

$$n \sim 10^9 \text{ cm}^{-3} \,. \tag{42}$$

In this case, $e^2/\overline{R} \sim 1$ K, and therefore, in the typical case, the following estimates hold true:

$$T_\perp \approx T_c \sim 1000 \text{ K} \,, \qquad T_\parallel \sim 1 \text{ K} \,. \tag{43}$$

So, we next assume that

$$\frac{T_\perp}{T_\parallel} \gg 1 \,. \tag{44}$$

It should be noted that for the values given in expression (43) $T_\parallel \sim e^2/\overline{R}$, and as regards the longitudinal electron motion the cloud therefore represents a nonideal plasma:

$$\xi \sim 1 \,, \qquad N_D \sim 1 \,. \tag{45}$$

For this reason, some results will evidently be accurate to an order of magnitude. All calculations will nevertheless be conducted assuming the fulfilment of the ideality criterion (4), which is written out for an anisotropic plasma as

$$\xi_\parallel = \frac{T_\parallel \overline{R}}{e^2} \gg 1 \,. \tag{46}$$

Let us consider the problem of the screening of a point charge q by an electron cloud with an oblate distribution (36). Then, from expressions (17), (18), and (23) for the Fourier component of the charge density in the Debye cloud we obtain [compare with expressions (27)]

$$\rho_p(\mathbf{k}) = \frac{q}{1 + k_\parallel^2 R_\parallel^2 + k_\perp^2 R_\perp^2} \,, \tag{47}$$

where $R_\parallel$ and $R_\perp$ are the 'longitudinal' and 'transverse' Debye radii defined as

$$R_\parallel = \sqrt{\frac{T_\parallel}{4\pi n e^2}} = \frac{\Delta_\parallel}{\omega_p} \,, \qquad R_\perp = \sqrt{\frac{T_\perp}{4\pi n e^2}} = \frac{\Delta_\perp}{\omega_p} \,. \tag{48}$$

Since $R_\parallel \ll R_\perp$, the surfaces of equal charge density $\rho_p(\mathbf{k}) = \text{const}$ in the $\mathbf{k}$-space are strongly elongated along the magnetic field (hereinafter, the z-axis). This signifies that the Debye charge distribution $\rho_p(\mathbf{r})$ in ordinary space will be strongly flattened along the z-axis (the direction '$\parallel$'):

$$\rho_p(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} \, \rho_p(\mathbf{k}) \exp(i\mathbf{kr}) = \frac{q}{4\pi R_\parallel R_\perp^2} \frac{\exp(-S)}{S} \,, \tag{49}$$

where

$$S = \left(\frac{z^2}{R_\parallel^2} + \frac{r_\perp^2}{R_\perp^2} \right)^{1/2} , \qquad z = \mathbf{hr} \,, \qquad \mathbf{r}_\perp = \mathbf{r} - \mathbf{h}z \,.$$

For $R_\parallel = R_\perp$, expression (49) coincides with expression (27). According to expression (49), the Debye cloud of an immobile charge is a strongly flattened ellipsoid of rotation with thickness $\sim 2R_\parallel$ along the axis of rotation (the z-axis):

$$\frac{R_\perp}{R_\parallel} = \frac{\Delta_\perp}{\Delta_\parallel} = \sqrt{\frac{T_\perp}{T_\parallel}} \sim 30 \,. \tag{50}$$

For fast particles with the velocities

$$V > \Delta_\perp \,, \tag{51}$$

for any mass of particle M we have

$$r_{HM} > r_H \,, \tag{52}$$

$$\bar{r}_H = \frac{\Delta_\perp}{\omega_H} \,, \tag{53}$$

and therefore in the calculation of $\mathbf{F}$ the trajectory of particle M may be thought of as being rectilinear. Furthermore, in expression (13) we may put $f(\mathbf{v}) = \delta(\mathbf{v})$, so that

$$\Phi(\mathbf{V}) = \frac{1}{V} \,, \qquad \nabla_\mathbf{V} \Phi(\mathbf{V}) = -\frac{\widehat{\mathbf{V}}}{V^2} \,. \tag{54}$$

Owing to a decrease in F with increasing V, the longest phase comprises the moderation of fast particles, and therefore formulas (12), (34), and (54) are sufficient for estimating the slowing-down time for particles with an initial velocity satisfying inequality (51):

$$\tau_d \sim \frac{MmV^3}{12\pi n e^2 q^2 \Lambda} \,. \tag{55}$$

When defining τ_d more precisely, the effect of a magnetic field on the electron motion must be taken into account (see

Sections 3 and 4). If the condition

$$V < \Delta_{\perp} \tag{56}$$

is fulfilled, the rectilinear trajectory approximation still holds for the heavy particles M (1) and breaks down for positrons (2). For this reason, the velocity range (56) for the positrons will be considered later (beginning with Section 5), while here and in Sections 3 and 4 the motion of particles M is assumed to be rectilinear. In Sections 3 and 4, we will thereby restrict ourselves to the case of heavy particles (1) as well as to case (2), (51) of fast positrons. Jumping ahead, we note that interval (56) (to be more precise, $V_{\perp} \sim \Delta_{\perp}$, $V_{\parallel} \sim \Delta_{\parallel}$) is of greatest importance for the positrons, because this is precisely the range that accommodates the stationary velocity distribution of the positrons, which sets in as a result of their moderation.

The case specified by

$$\bar{r}_{\mathrm{H}} > R_{\perp} \tag{57}$$

will be referred to as the weak-magnetic-field limit. Expression (49) may be rewritten in the form

$$H < H_1 , \qquad \frac{H_1^2}{4\pi} = mc^2 n .$$

With the parameters defined by expression (42), from inequality (57) results the estimate

$$H < H_1 = 40 \text{ G} . \tag{58}$$

In this case, the effect of a magnetic field on the friction force may be neglected and it is defined by expressions (10), (12), and (34). If we take advantage of an analogy with electrostatics, according to expression (13) the vector $-\nabla_{\mathbf{V}} \Phi$ defined in the velocity space is similar to the electric vector of the field produced by a unit charge distributed by law (36). Consequently, for

$$\Delta_{\parallel} < V_{\parallel} < \Delta_{\perp} , \qquad V_{\perp} < \Delta_{\perp} , \tag{59}$$

the 'field' $-\nabla_{\mathbf{V}} \Phi$ is analogous to the electric field near the plane of a uniformly charged disk (with surface density σ):

$$|-\nabla_{\mathbf{V}} \Phi| \sim 2\pi\sigma , \qquad \sigma \sim \frac{1}{\pi \Delta_{\perp}^2} .$$

Hence, one obtains

$$F \sim F_{\mathrm{c}} \sim \frac{nq^2 e^2}{m\Delta_{\perp}^2} , \qquad \Delta_{\parallel} < V < \Delta_{\perp} , \qquad H < H_1 . \tag{60}$$

For the subsequent discussion it might be beneficial and instructive to obtain estimate (60) proceeding directly from the shape of the Debye cloud (49), (50), which may be represented approximately as a uniformly charged disk of charge $q_{\mathrm{D}} = -q$, radius $\sim R_{\perp}$, and thickness $\sim 2R_{\parallel}$:

$$F \sim F_{\mathrm{c}} = qE_{\mathrm{p}} \sim \frac{q^2}{R_{\perp}^2} , \qquad V_{\parallel} \sim \Delta_{\parallel} , \qquad H < H_1 , \tag{61}$$

which coincides with expression (60). Therefore, estimate (60) gives the magnitude of the friction force for a particle velocity $V_{\parallel} \sim \Delta_{\parallel}$, when the particle resides at the edge of the Debye cloud owing to its lag.

With an increase in magnetic field intensity, relation (57) between the parameters is replaced by the following one:

$$R_{\parallel} < \bar{r}_{\mathrm{H}} < R_{\perp} , \tag{62}$$

which corresponds to magnetic field intensities (this domain will be referred to as the 'medium field domain')

$$H_1 < H < H_2 , \tag{63}$$

where

$$\frac{H_2^2}{4\pi} = \frac{T_{\perp}}{T_{\parallel}} mc^2 n , \qquad H_2 \sim 1200 \text{ G} . \tag{64}$$

The electron motion in the direction transverse ($\perp$) to the magnetic field is restricted to a range $\sim \bar{r}_{\mathrm{H}}$, and therefore the lateral dimension of the Debye cloud decreases as the magnetic field is strengthened. Under conditions (62), (63), this dimension is $\sim \bar{r}_{\mathrm{H}}$, and in lieu of expression (61) we therefore obtain

$$F \sim F_{\mathrm{c}} \sim \frac{q^2}{\bar{r}_{\mathrm{H}}^2} , \qquad V_{\parallel} \sim \Delta_{\parallel} , \qquad H_1 < H < H_2 . \tag{65}$$

Finally, when $H > H_2$, the following inequality holds true:

$$\bar{r}_{\mathrm{H}} < R_{\parallel} , \tag{66}$$

and the electrons are therefore 'attached' to the magnetic field lines and, like beads on a thread in tension, may execute only one-dimensional motion along these lines. This motion leads to a redistribution of the electrons along the magnetic field, with the consequence that their concentration is described by the Boltzmann formula

$$n(\mathbf{r}) = n \exp\left(-\frac{u}{T_{\parallel}}\right) \approx n\left(1 - \frac{u}{T_{\parallel}}\right) , \qquad u = -e\varphi , \tag{67}$$

where $\varphi = \varphi(\mathbf{r})$ is the electric potential. From formula (67) and the Poisson equation for φ it follows that the Debye cloud under condition (66) is spherically symmetric and described by formulas (27), (28), in which $R_{\parallel}$ should be substituted for r_{D}. Hence it follows that

$$F \sim F_{\mathrm{c}} \sim \frac{q^2}{R_{\parallel}^2} \sim \frac{nq^2 e^2}{m\Delta_{\parallel}^2} , \qquad V_{\parallel} \sim \Delta_{\parallel} , \qquad H > H_1 . \tag{68}$$

Since the field strength E_{p} attains its maximum at the edge of the Debye cloud, the friction force peaks for $V_{\parallel} \sim \Delta_{\parallel}$. Estimates (61), (65), and (68) may be combined by means of interpolation:

$$F_{\max} \sim F(V_{\parallel} = \Delta_{\parallel}) \sim q^2 \left(\frac{1}{R_{\parallel}^2 + \bar{r}_{\mathrm{H}}^2} + \frac{1}{R_{\perp}^2} \right) . \tag{69}$$

These relationships are plotted in Fig. 1: for weak fields (58), the force of friction is independent of H; for medium fields (63), $F_{\max}$ increases proportionally to the square of the magnetic field intensity; lastly, in the magnetized plasma (66), the force of friction becomes the most strong and ceases to depend on H. The rectilinear trajectory approximation for

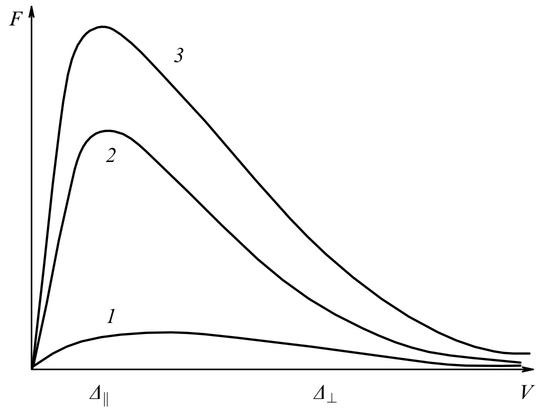

Figure 1. Qualitative form of the velocity dependence of the friction force for a heavy particle ($M \gg m$) for $H < H_1$ (curve *1*), $H_1 < H < H_2$ (curve *2*), and $H > H_2$ (curve *3*).

particle M is assumed to hold true, and therefore Fig. 1 applies only to the case of a heavy particle (in Section 7, we will show that dependences similar to those plotted in Fig. 1 are also valid for positrons, though not for the modulus $F = |\mathbf{F}|$, but for the projection of the force of friction on the direction of magnetic field $F_\parallel$).

To avoid misunderstanding, it should be mentioned that the term 'magnetized plasma' in conventional plasma physics [11 – 16] implies collisionless plasma: $\omega_{\rm H} \gg \nu$, where ν is the frequency of electron Coulomb collisions with electrons and ions. Alternatively, condition (66) is implied in our case covering the physics of charged particle beams. The increase ($\sim H^2$) in the force of friction in the velocity (59) and magnetic field (63) domains (this is referred to as the 'electron magnetization effect' [2, 3]) is extremely important for the electron cooling of beams, because it leads to the shortening of the cooling time, i.e., to a reduction in the coolers' dimensions. As discussed earlier, the collective contribution $F_{\rm c}$ to the force of friction increases due to the reduction in the Debye cloud dimension arising from the increase in magnetic field intensity. Similarly, the $F_{\rm b}$ contribution caused by binary collisions also increases (see Section 3), although its physical reason is different. With shortening $\bar{r}_{\rm H}$, the electron motion perpendicular to the magnetic field is 'confined' to a circle of short radius, and therefore particle M interacts with an electron like with a point particle traveling along the magnetic field at the low velocity $v_\parallel \sim \Delta_\parallel$. When $V_\parallel \sim \Delta_\parallel$, the relative velocity u of the particles is also low and the interaction time is long. That is why, in every collision the particles exchange a greater momentum, i.e., $F_{\rm b}$ increases.

Thus, the oblateness of velocity distribution function (36) for $H > H_1$ leads to an increase in friction force.

3. $\mathbf{F}_{\rm b}$ contribution to the friction force from binary collisions in the rectilinear particle trajectory and electron magnetization approximations

This section, which is partly methodical in character, is dedicated to a rigorous derivation of the formula for the force of friction due to binary collisions in a magnetized plasma. Here (and in Section 4), the trajectory of particle M is assumed to be rectilinear, which is practically always true, both in case (1) and in case (2) for velocities satisfying inequality (51). Furthermore, it is assumed that

$$\bar{r}_{\rm H} < \bar{R}\,, \tag{70}$$

which is why the Larmor radius may be thought of as being zero. Relation (70) will be termed the electron magnetization condition with respect to binary collisions. For parameters (42), (43), this condition is fulfilled when $H > 500$ G. Binary magnetized collisions comprise collisions with an impact parameter ρ (the minimal distance between particle M and the electron) satisfying the condition

$$\bar{r}_{\rm H} < \rho < \bar{R}\,; \tag{71}$$

ordinary binary conditions are those with

$$\rho < \bar{r}_{\rm H}\,, \tag{72}$$

and collective collisions are those with

$$\rho > \bar{R}\,. \tag{73}$$

In this section, we consider cases (71) and (72), with case (71) being considered first.

Let us evaluate the contribution $\mathrm{d}F_{\rm b}$ from the magnetized collisions (71) of particle M with the electrons having velocities in the interval $(v_\parallel, v_\parallel + \mathrm{d}v_\parallel)$. According to expressions (36), the concentration of these electrons equals

$$\mathrm{d}n = ng(v_\parallel)\,\mathrm{d}v_\parallel\,. \tag{74}$$

Since $T_\perp \gg T_\parallel$, we should accept the following expression for the Thomson radius (15) in the case of distribution (36):

$$R_{\rm T} = \frac{e^2}{T_\perp}\,. \tag{75}$$

For $\rho \sim R_{\rm T}$, the angle of electron scattering by particle M is large: $\theta_{\rm scat} \sim 1$. With parameters (42), (43) and a magnetic field strength

$$H \sim 1000\ \mathrm{G}\,, \tag{76}$$

typical for the Low Energy Particle Toroidal Accumulator experiment (Joint Institute for Nuclear Research, Dubna) [4 – 7], the characteristic dimensions [in cm] are as follows:

$$R_{\rm T} \sim 1.5\times10^{-6}\,,\quad \bar{r}_{\rm H} \sim 1\times10^{-3}\,,\quad \bar{R} \sim 2\times10^{-3}\,,$$
$$R_\perp \sim 3\times10^{-2}\,,\quad R_\parallel \sim 1\times10^{-3}\,. \tag{77}$$

One can see that the Thomson radius is short in comparison with the other dimensions. Consequently, $\theta_{\rm scat} \ll 1$ in a typical collision. This signifies that the contributions to $\mathbf{F}_{\rm b}$ from different electrons are independent:

$$\mathbf{F}_{\rm b} = \int \mathrm{d}\mathbf{F}_{\rm b}\,. \tag{78}$$

To calculate $\mathrm{d}\mathbf{F}_{\rm b}$ we move to the reference system K' moving along $\mathbf{H}$ with a velocity $v_\parallel$. In the reference system K', the group of electrons under consideration is immobile and particle M moves with a velocity $\mathbf{u} = \mathbf{V} - v_\parallel\mathbf{h}$. The equations of the particles' motion are of the form

$$M\ddot{\mathbf{r}}_{\rm M} = -qe\,\frac{\mathbf{r}}{r^3}\,,\qquad m\ddot{z}_{\rm e} = qe\,\frac{z}{r^3}\,, \tag{79}$$

where $z_e = \mathbf{hr}_e$, $\mathbf{r}_M$ and $\mathbf{r}_e = z_e\mathbf{h}$ are the respective coordinates of particle M and the electron, $\mathbf{r} = \mathbf{r}_M - \mathbf{r}_e$ and $z = \mathbf{hr}$. From Eqns (79) it follows that

$$\ddot{z} = -qe\,\frac{z}{\mu r^3}\,, \qquad \ddot{\mathbf{r}}_\perp = -qe\,\frac{\mathbf{r}_\perp}{Mr^3}\,, \qquad \ddot{X} = 0\,, \tag{80}$$

where $X = (Mz_M + mz_e)/(M+m)$, and $\mu = Mm/(M+m)$ is the reduced mass of the particles. We rewrite the first two of equations (80) as

$$\ddot{x}_\alpha = -qeA_{\alpha\beta}\,\frac{x_\beta}{r^3}\,, \tag{81}$$

where x_α is a component of vector $\mathbf{r}$, and

$$A_{\alpha\beta} = \frac{1}{M}\,\delta_{\alpha\beta} + \frac{1}{m}\,h_\alpha h_\beta\,; \tag{82}$$

it is assumed that summation is performed over repeating indices.

As stated in Section 2, the interparticle interaction may be thought of as being weak owing to the smallness of R_T, and so we will solve equations (80) by the method of successive approximations, neglecting quantities of the third (and higher) order of smallness:

$$x_{M\alpha} \equiv (\mathbf{r}_M)_\alpha\,, \qquad x_{M\alpha} = x^{(0)}_{M\alpha} + x^{(1)}_{M\alpha} + x^{(2)}_{M\alpha} + \ldots\,, \qquad x^{(0)}_{M\alpha} = (\boldsymbol{\rho} + \mathbf{u}t)_\alpha = x^{(0)}_\alpha\,, \tag{83}$$

where $\boldsymbol{\rho}$ is the impact parameter, and $\boldsymbol{\rho} \perp \mathbf{u}$. We write out the formulas for the M-particle coordinates

$$z_M = X + \frac{m}{M+m}\,z\,, \qquad \mathbf{r}_{M\perp} = \mathbf{r}_\perp$$

in the form

$$x_{M\alpha} = Xh_\alpha + B_{\alpha\beta}x_\beta\,, \qquad B_{\alpha\beta} = \delta_{\alpha\beta} - \frac{M}{M+m}\,h_\alpha h_\beta\,. \tag{84}$$

Since $\dot{X} = \text{const}$ [see expression (80)], then according to formulas (84) the velocity variation of particle M due to the collision is expressed, correct to the uncertainty accepted above, in the form

$$\Delta V_\alpha = \Delta V^{(1)}_\alpha + \Delta V^{(2)}_\alpha\,, \qquad \Delta V^{(1)}_\alpha = B_{\alpha\beta}\Delta\dot{x}^{(1)}_\beta\,, \qquad \Delta V^{(2)}_\alpha = B_{\alpha\beta}\Delta\dot{x}^{(2)}_\beta\,. \tag{85}$$

From expressions (85) it follows that

$$\dot{x}^{(1)}_\alpha(t) \equiv \frac{\mathrm{d}x^{(1)}_\alpha}{\mathrm{d}t} = -qeA_{\alpha\beta}\,J_\beta(t)\,, \qquad \Delta\dot{x}^{(1)}_\alpha \equiv \dot{x}^{(1)}_\alpha(+\infty) = -\frac{2qe}{u\rho}\,A_{\alpha\beta}\,\hat{\rho}_\beta\,, \tag{86}$$

$$J_\beta(t) = \int_{-\infty}^{t} \mathrm{d}t'\,\frac{x^{(0)}_\alpha(t')}{r^3(t')} = f\hat{u}_\alpha + k\hat{\rho}_\alpha\,, \qquad f = -\frac{1}{ur}\,, \qquad k = \frac{1}{ur}\left(1 + \frac{ut}{\rho}\right), \qquad \hat{\boldsymbol{\rho}} = \frac{\boldsymbol{\rho}}{\rho}\,, \qquad \hat{\mathbf{u}} = \frac{\mathbf{u}}{u}\,, \qquad r \equiv r(t) = \sqrt{\rho^2 + u^2t^2}\,, \tag{87}$$

$$\Delta\dot{x}^{(2)}_\alpha = -qeA_{\alpha\beta}\int_{-\infty}^{+\infty} \mathrm{d}t\,x^{(1)}_\gamma(t)\,\frac{\partial}{\partial x_\gamma}\left(\frac{x_\beta}{r^3}\right). \tag{88}$$

In the reference system K', the electrons of the selected group are at rest prior to the interaction with particle M, while particle M moves with a velocity $\mathbf{u}$. We calculate the variation in the velocity of particle M in a path L, i.e., during the time $\Delta t = L/u$:

$$\delta V_\alpha = \sum_L \Delta V_\alpha = \mathrm{d}n\int \mathrm{d}V\,\Delta V_\alpha\,, \tag{89}$$

where summation is performed over the electrons of group (74), which reside in a cylinder of height L and infinite radius, with a symmetry axis coinciding with the unperturbed particle trajectory $x^{(0)}_\alpha(t)$, and $\mathrm{d}V$ is a volume element. Upon integration over the azimuthal angle $\mathrm{d}\varphi$ contained in the element of volume

$$\mathrm{d}V = L\,d^2\rho\,, \qquad \mathrm{d}^2\rho = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,, \tag{90}$$

the term $\Delta V^{(1)}_\alpha$ in expressions (85) vanishes. We next perform integration by parts in expression (88). For the velocity variation we obtain, in view of Eqn (86), the expression

$$\Delta V_\alpha = -q^2e^2B_{\alpha\beta}A_{\beta\gamma}A_{\delta\eta}\int_{-\infty}^{+\infty} \mathrm{d}t\,P_{\gamma\delta}(t)\,J_\eta(t)\,, \tag{91}$$

where

$$P_{\gamma\delta}(t) = \int_{-\infty}^{t} \mathrm{d}t\,\frac{\partial}{\partial x_\gamma}\left(\frac{x_\delta}{r^3}\right) = a\delta_{\gamma\delta} + b(\hat{u}_\gamma\hat{\rho}_\delta + \hat{u}_\delta\hat{\rho}_\gamma) + c\hat{u}_\gamma\hat{u}_\delta + d\,\hat{\rho}_\delta\,\hat{\rho}_\gamma\,,$$

$$a = \frac{1}{u\rho^2}\left(1 + \frac{ut}{r}\right), \qquad b = \frac{\rho}{ur^3}\,,$$

$$c = \frac{t}{r^3} - \frac{1}{u\rho^2}\left(1 + \frac{ut}{r}\right), \qquad d = -\frac{t}{r^3} - \frac{2}{u\rho^2}\left(1 + \frac{ut}{r}\right).$$

In expression (91) we perform averaging over the directions of the unit vector $\hat{\rho}$ perpendicular to the cylinder axis:

$$\big\langle P_{\gamma\delta}(t)\,J_\eta(t)\big\rangle = f\hat{u}_\eta\left[a\delta_{\gamma\delta} + c\hat{u}_\gamma\hat{u}_\delta + \frac{1}{2}\,d(\delta_{\gamma\delta} - \hat{u}_\gamma\hat{u}_\delta)\right] + \frac{bk}{2}\,S_{\gamma\eta\delta}\,, \tag{92}$$

$$S_{\gamma\eta\delta} = \hat{u}_\gamma(\delta_{\eta\delta} - \hat{u}_\eta\hat{u}_\delta) + \hat{u}_\delta(\delta_{\eta\gamma} - \hat{u}_\eta\hat{u}_\gamma)\,. \tag{93}$$

This averaging emerges in the integration over the angle φ in expressions (89), (90):

$$\int \mathrm{d}\varphi \ldots = 2\pi\langle\ldots\rangle\,. \tag{94}$$

The terms odd in t, which make a zero contribution to expression (91), should be discarded in expression (92), following which the expression in the square brackets on the right-hand side of expression (92) vanishes:

$$\langle\Delta V_\alpha\rangle = -q^2e^2B_{\alpha\beta}A_{\beta\gamma}A_{\delta\eta}\,\frac{1}{u^3\rho^2}\,S_{\gamma\eta\delta}\,. \tag{95}$$

From expressions (82), (84), and (95) it follows that

$$M\langle\Delta V_\alpha\rangle = -\frac{q^2e^2}{u^3\rho^2}\,(A\hat{u}_\alpha + B\hat{h}_\alpha)\,, \tag{96}$$

where

$$A = \frac{2}{M} + \frac{1}{m}\left(1 - 2(\hat{\mathbf{u}}\mathbf{h})^2\right), \qquad B = \frac{1}{m}\,\hat{\mathbf{u}}\mathbf{h}\,.$$

By summing up the contributions from electrons with different velocities $v_\parallel$ [see formula (78)], we obtain from expressions (89), (90), (94), and (96) the following expression for the contribution to $\mathbf{F}_\mathrm{b}$ from magnetized binary collisions (71):

$$\mathbf{F}_\mathrm{Mb}(\mathbf{V}) = \sum_L \frac{M\,\delta\mathbf{V}}{\Delta t} = -2\pi n q^2 e^2 \Lambda_\mathrm{Mb} \int_{-\infty}^{+\infty} \frac{\mathrm{d}v_\parallel\, g(v_\parallel)}{u^2} (A\hat{\mathbf{u}} + B\mathbf{h})\,, \qquad (97)$$

where $\Lambda_\mathrm{Mb} = \ln(\overline{R}/\bar{r}_\mathrm{H})$. In particular, for $V \gg \Delta_\parallel$, one obtains

$$\mathbf{F}_\mathrm{Mb}(\mathbf{V}) = -\frac{2\pi n q^2 e^2 \Lambda_\mathrm{Mb}}{V^2} (A\widehat{\mathbf{V}} + B\mathbf{h})\,; \qquad (98)$$

here and in expressions for A and B, $\widehat{\mathbf{V}} = \mathbf{V}/V$ should be used in lieu of $\hat{\mathbf{u}}$. For $V \gg \Delta_\parallel$ and $M \gg m$, we arrive at

$$\mathbf{F}_\mathrm{Mb}(\mathbf{V}) = -\frac{2\pi n q^2 e^2 \Lambda_\mathrm{Mb}}{m V^2} \left[\widehat{\mathbf{V}}\left(1 - 2(\widehat{\mathbf{V}}\mathbf{h})^2\right) + \mathbf{h}(\widehat{\mathbf{V}}\mathbf{h})\right], \quad (99)$$

or in components:

$$F_\mathrm{Mb}^\parallel = \mathbf{h}\mathbf{F}_\mathrm{Mb} = -\frac{4\pi n q^2 e^2 \Lambda_\mathrm{Mb}}{m V^3}\, V_\parallel\, \frac{V_\perp^2}{V^2}\,,$$
$$F_\mathrm{Mb}^\perp = -\frac{2\pi n q^2 e^2 \Lambda_\mathrm{Mb}}{m V^3}\, \mathbf{V}_\perp\, \frac{V^2 - 2V_\parallel^2}{V^2}\,. \qquad (100)$$

The total contribution from binary collisions is defined by the equation

$$\mathbf{F}_\mathrm{b} = \mathbf{F}_\mathrm{Mb} + \mathbf{F}_\mathrm{sb}\,, \qquad (101)$$

where the contribution $\mathbf{F}_\mathrm{sb}$ due to simple binary collisions (72) is given by formula (12) in which $\Lambda_\mathrm{sb} = \ln(\bar{r}_\mathrm{H}/R_\mathrm{T})$ should be substituted for Λ_b, and R_T is defined by formula (75).

From formulas (12), (60), and (98) result the conclusion that the following relation holds for velocities satisfying inequalities (59):

$$\frac{F_\mathrm{Mb}}{F_\mathrm{sb}} \sim \frac{T_\perp}{T_\parallel} \gg 1\,, \qquad (102)$$

which is a consequence of the strong anisotropy of electron distribution (44) and the electron magnetization effect (see the end of Section 2) which manifests itself in binary collisions in the case (59). We therefore may put

$$\mathbf{F}_\mathrm{b} \approx \mathbf{F}_\mathrm{Mb}\,. \qquad (103)$$

Clearly, the method of successive approximations employed in this section may also be applied to the derivation of formula (12). This formula is simpler to derive by considering the center-of-mass system of particle M and an electron of group (74): in the center-of-mass system, owing to the interaction, the particle momenta rotate through some angle, remaining opposite in direction and equal in modulus μu.

4. $\mathbf{F}_\mathrm{c}$ contribution to the friction force from collective interactions in the rectilinear particle trajectory approximation

In this section, we will obtain the expression for $\mathbf{F}_\mathrm{c}$ in the case of the rectilinear motion of particle M and $\mathbf{H} \neq 0$.

According to expressions (22), (25), and (29), for rectilinear uniform motion one has

$$\mathbf{F}_\mathrm{c} = -\frac{\mathrm{i}q^2}{2\pi^2} \int \mathrm{d}^3k\, \frac{\mathbf{k}}{k^2} \left[\frac{1}{\varepsilon(\mathbf{k}, \mathbf{kV})} - 1\right] = \frac{q^2}{2\pi^2} \int \mathrm{d}^3k\, \frac{\mathbf{k}}{k^2}\, \mathrm{Im}\,\frac{1}{\varepsilon}\,, \qquad (104)$$

where use was made of the property $\varepsilon(-\mathbf{k}, -\mathbf{kV}) = \varepsilon^*(\mathbf{k}, \mathbf{kV})$. Formula (104) is valid under the conditions specified in Section 3, as well as in the cases where the main contribution to the integral in formula (104) is made by the collective interaction domain $k < 1/\overline{R}$. As in the case of $H = 0$ described by formula (34), the main logarithmic contribution to $\mathbf{F}_\mathrm{c}$ is made by the 'nonresonance' domain (31) in which approximation (33) is valid, and therefore

$$\mathbf{F}_\mathrm{c} = \mathbf{F}_\mathrm{n} + \mathbf{F}_\mathrm{r}\,. \qquad (105)$$

Here, $\mathbf{F}_\mathrm{n}$ is the 'nonresonance' term:

$$\mathbf{F}_\mathrm{n} = -\frac{q^2}{2\pi^2} \int \mathrm{d}^3k\, \frac{\mathbf{k}}{k^2}\, \mathrm{Im}\,\varepsilon\,, \qquad (106)$$

and $\mathbf{F}_\mathrm{r}$ is the 'resonance' term:

$$\mathbf{F}_\mathrm{r} = -\frac{\mathrm{i}q^2}{2\pi^2} \int_\mathrm{res} \mathrm{d}^3k\, \frac{\mathbf{k}}{k^2}\, \frac{1}{\varepsilon} \qquad (107)$$

which emerges due to the pole

$$\varepsilon(\mathbf{k}, \omega) = 0\,. \qquad (108)$$

Under the action of the field of particle M, consistent, collective motion of the electrons occurs. A part of the energy of this motion is converted to the energy of their chaotic thermal motion by the Landau damping mechanism [11 – 16], which gives rise to the force of friction (106). Force (107) emerges due to the Cherenkov radiation of collective waves by the particle, the waves carrying away its energy into the plasma interior.

In formula (104), one may perceive an analogy with ionization losses and losses due to Cherenkov radiation in the travel of a charged particle through a material [18]. For nonrelativistic (in the rest frame of the beam) velocities (6), only longitudinal waves, whereby $\mathbf{E}(\mathbf{k}, \omega) \parallel \mathbf{k}$, can be emitted, and their frequency and wave vector therefore obey the dispersion relation (108) [11 – 18]. The emission of Langmuir plasma waves makes the main contribution to the force $\mathbf{F}_\mathrm{r}$:

$$\omega \sim \omega_\mathrm{p}\,. \qquad (109)$$

The emission of high-frequency ($\omega \sim \omega_\mathrm{H} \gg \omega_\mathrm{p}$) cyclotron waves may be neglected (see Section 5). According to Refs [12, 13], it can be shown that

$$\mathrm{Im}\,\varepsilon(\mathbf{k}, \omega) = -\frac{\pi\omega_\mathrm{p}^2}{k^2} \sum_{l=-\infty}^{\infty} \int \mathrm{d}^3v\, J_l^2\left(\frac{k_\perp v_\perp}{\omega_\mathrm{H}}\right) \times \delta(\omega - l\omega_\mathrm{H} - k_\parallel v_\parallel) \left(k_\parallel \frac{\partial f}{\partial v_\parallel} + \frac{l\omega_\mathrm{H}}{v_\perp} \frac{\partial f}{\partial v_\perp}\right), \qquad (110)$$

where J_l is the Bessel function. Since

$$\frac{k_\perp v_\perp}{\omega_{\rm H}} < \frac{v_\perp}{\omega_{\rm H}\overline{R}} \sim \frac{\overline{r}_{\rm H}}{\overline{R}}\,, \tag{111}$$

the terms with $l \neq 0$ in expression (110) for magnetized electrons may be omitted in accordance with inequality (70):

$$\operatorname{Im}\varepsilon(\mathbf{k},\omega) \approx \frac{\pi m\omega_{\rm p}^2\omega}{T_\parallel |k_\parallel| k^2}\, g\left(\frac{\omega}{k_\parallel}\right).$$

Hence, and from expression (106), it follows that

$$\mathbf{F}_{\rm n} = -\frac{q^2\omega_{\rm p}^2}{2\pi\Delta_\parallel^2}\int {\rm d}^3k\,\frac{\mathbf{k}(\mathbf{kV})}{k^4|k_\parallel|}\, g\left(\frac{\mathbf{kV}}{k_\parallel}\right), \tag{112}$$

where the function $g(v_\parallel)$ was defined by formula (36). In the spherical coordinate system in the $\mathbf{k}$-space, the volume element is ${\rm d}^3k = k^2\,{\rm d}k\,{\rm d}\Omega_{\mathbf{k}}$, and therefore we perform, in view of $\mathbf{k} = \hat{\mathbf{k}}k$, integration over ${\rm d}k$ in expression (106):

$$\int \frac{{\rm d}k}{k} = \ln\frac{R_\parallel}{\overline{R}} \equiv \Lambda_{\rm n}\,, \tag{113}$$

where we accepted, in agreement with inequality (31) and the reasoning about the character of Debye screening outlined at the end of Section 2, the following limits of integration:

$$k_{\max} = \frac{1}{\overline{R}}\,, \qquad k_{\min} = \frac{1}{R_\parallel}\,. \tag{114}$$

Finally, we arrive at

$$\mathbf{F}_{\rm n} = -\frac{2q^2e^2n\Lambda_{\rm n}V}{m\Delta_\parallel^2}\,\mathbf{I}\,, \tag{115}$$

$$\mathbf{I} = \int {\rm d}\Omega_{\mathbf{k}}\,\frac{\hat{\mathbf{k}}(\hat{\mathbf{k}}\hat{\mathbf{V}})}{|\hat{\mathbf{k}}\mathbf{h}|}\,\frac{1}{\sqrt{2\pi}\,\Delta_\parallel}\exp\left(-\frac{(\hat{\mathbf{k}}\mathbf{V})^2}{2\Delta_\parallel^2\hat{k}_\parallel^2}\right). \tag{116}$$

For $\mathbf{V} \parallel \mathbf{H}$, it is readily shown that

$$\mathbf{F}_{\rm n} = -\frac{2\sqrt{2\pi}\,q^2e^2n\Lambda_{\rm n}}{m\Delta_\parallel^3}\,\mathbf{V}\exp\left(-\frac{V_\parallel^2}{2\Delta_\parallel^2}\right). \tag{117}$$

For $V \gg \Delta_\parallel$, the main contribution to expression (115) is made by those directions of $\hat{\mathbf{k}}$ for which $|\hat{\mathbf{k}}\hat{\mathbf{V}}| \ll 1$. Let the axis k_z be directed parallel to $\mathbf{V}$. From formula

$$(\hat{\mathbf{k}}\hat{\mathbf{V}})\exp\left(-\frac{V^2k_z^2}{2\Delta_\parallel^2k_\parallel^2}\right) \approx -\sqrt{\frac{\pi}{2}}\,\frac{\Delta_\parallel|k_\parallel|}{kV}\,\delta'(k_z)\,,$$

which is valid for $V \gg \Delta_\parallel$, we find

$$F_{{\rm n}\parallel} = -\frac{4\pi nq^2e^2\Lambda_{\rm n}}{mV^2}\,\frac{V_\perp^2}{V^2}\,V_\parallel\,, \tag{118}$$

$$\mathbf{F}_{{\rm n}\perp} = -\frac{2\pi nq^2e^2\Lambda_{\rm n}}{mV^2}\,\frac{V_\perp^2 - 2V_\parallel^2}{V^3}\,\mathbf{V}_\perp\,. \tag{119}$$

Expressions (118) and (119) were obtained by Derbenev and Skrinskii [21] in a different way.

Comparing formulas (100), (103) and (118), (119) we notice that effect (35) of the vanishing of the intermediate dimension $\overline{R}$ from the expression for the total force of friction takes place in the presence of a magnetic field, as well.

Let us calculate $\mathbf{F}_{\rm r}$ for $V \gg \Delta_\parallel$ for magnetized electrons obeying inequality (66). It this case, we may put $T_\parallel = 0$ and, in addition, assume that the electrons can travel only along the magnetic field lines, which is described by the system of equations

$$m\,\frac{{\rm d}v}{{\rm d}t} = e\,\frac{\partial\varphi}{\partial z}\,, \qquad \frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) = 0\,, \qquad \Delta\varphi = 4\pi e(n - n_0)\,,$$

where the z-axis is aligned with $\mathbf{H}$, and φ is the electric potential. Upon linearization with the result that ${\rm d}v/{\rm d}t \approx \partial v/\partial t$, $n = n_0 + n_1$ and moving to the Fourier components $(\mathbf{k},\omega)$ we obtain

$$v = -{\rm i}\omega\xi = \frac{ek_z}{m\omega}\,\varphi = \frac{{\rm i}e}{m\omega}\,E_z\,, \tag{120}$$

where ξ is the electron displacement along the direction of the magnetic field. From formula (120) we find the electric induction vector

$$\mathbf{D} = \mathbf{E} + 4\pi(-e)n\xi\mathbf{h} = \mathbf{E} - \frac{\omega_{\rm p}^2}{\omega^2}\,\mathbf{h}E_z\,.$$

Hence we obtain the expressions for the permittivity tensor of the magnetized electron cloud:

$$\varepsilon_{\alpha\beta}(\mathbf{k},\omega) = \delta_{\alpha\beta} - \frac{\omega_{\rm p}^2}{\omega^2}\,h_\alpha h_\beta\,, \tag{121}$$

and the longitudinal permittivity (18):

$$\varepsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{\rm p}^2}{\omega^2}\cos^2\alpha\,, \tag{122}$$

where α is the angle between $\mathbf{k}$ and $\mathbf{H}$. Expression (122) may easily be obtained from the permittivity tensor of a cold plasma [11–18]:

$$\varepsilon_{xx} = \varepsilon_{yy} = 1 - \frac{\omega_{\rm p}^2}{\omega^2 - \omega_{\rm H}^2}\,, \quad \varepsilon_{zz} = 1 - \frac{\omega_{\rm p}^2}{\omega^2}\,, \quad \varepsilon_{xz} = \varepsilon_{yz} = 0\,.$$

From here we find

$$\varepsilon(\mathbf{k},\omega) = 1 - \omega_{\rm p}^2\left(\frac{\sin^2\alpha}{\omega^2 - \omega_{\rm H}^2} + \frac{\cos^2\alpha}{\omega^2}\right). \tag{123}$$

In the typical case of a nonideal plasma with the parameters (45), we have $R_\parallel \sim \overline{R}$, and therefore from relationship

$$\frac{\omega_{\rm H}}{\omega_{\rm p}} \sim \frac{\Delta_\perp}{\overline{r}_{\rm H}\omega_{\rm p}} \sim \frac{R_\parallel}{\overline{r}_{\rm H}}\,\frac{\Delta_\perp}{\Delta_\parallel}$$

we conclude that

$$\frac{\omega_{\rm H}}{\omega_{\rm p}} \gg 1\,. \tag{124}$$

For the most significant frequencies (109), from formula (123) and inequality (124) there follows the resultant expression (122). The derivation given earlier makes it more lucid.

According to the causality principle, the poles of $\varepsilon(\mathbf{k},\omega)$ may lie only in the lower semiplane of ω [18], and therefore the pole in expression (107) should be detoured by the rule

$$\omega \to \omega + {\rm i}0\,, \tag{125}$$

whence we arrive at the expression for the friction force

$$\mathbf{F}_\mathrm{r} = -\frac{q^2}{2\pi}\int \mathrm{d}^3k\;\frac{\mathbf{k}(\mathbf{kV})^2\,\mathrm{sign}\,(\mathbf{kV})}{k^2}\;\delta\left((\mathbf{kV})^2 - \omega_\mathrm{p}^2\,\frac{(\mathbf{kh})^2}{k^2}\right). \tag{126}$$

Here, the δ function is representative of the plasmon emission with the dispersion law

$$\omega(\mathbf{k}) = \omega_\mathrm{p}|\hat{\mathbf{k}}\mathbf{h}| = \omega_\mathrm{p}\,\frac{|k_\parallel|}{k}\,, \tag{127}$$

which follows from formulas (108) and (122). Let the k_z-axis be directed along $\mathbf{V}$, and the k_y-axis perpendicular to the vectors $\mathbf{H}$ and $\mathbf{v}$. We next move to spherical coordinates and perform integration over $\mathrm{d}k$ to obtain

$$k = \frac{\omega_\mathrm{p}}{V}\,\frac{|\hat{\mathbf{k}}\mathbf{h}|}{|\hat{\mathbf{k}}\widehat{\mathbf{V}}|}\,, \tag{128}$$

$$\mathbf{F}_\mathrm{r} = -\frac{q^2e^2n}{mV^2}\int \mathrm{d}\Omega_\mathbf{k}\,\hat{\mathbf{k}}\,\mathrm{sign}\,(\mathbf{kV})\,\frac{(\mathbf{kh})^2}{(\hat{\mathbf{k}}\widehat{\mathbf{V}})^2}\,,\quad V > \varDelta_\parallel\,. \tag{129}$$

From expression (126) and relations $V > \varDelta_\parallel$, $R_\parallel = \varDelta_\parallel/\omega_\mathrm{p}$ result the conclusion that

$$k < \frac{1}{\overline{R}}\,, \tag{130}$$

which corresponds to the range of distances $r > \overline{R}$, i.e., the collective interaction range. Therefore, the calculation of $\mathbf{F}_\mathrm{r}$ is noncontradictory. When

$$\mathbf{V}\parallel\mathbf{H}\,, \tag{131}$$

from expressions (100), (118), (119), and (129) we obtain

$$\mathbf{F}_\mathrm{Mb} = \mathbf{F}_\mathrm{n} = 0\,,\qquad F_\mathrm{sb} \ll F_\mathrm{r}\,,$$
$$\mathbf{F} \approx \mathbf{F}_\mathrm{r} \approx -\frac{2\pi q^2e^2n}{mV^2}\,\widehat{\mathbf{V}}\,,\quad \mathbf{V}\parallel\mathbf{H}\,,\quad V \gg \varDelta_\parallel\,. \tag{132}$$

Therefore, the friction force induced in the particle motion along the field in a magnetized plasma is almost completely determined by the Cherenkov radiation of plasmons. To understand the physical cause of this effect we will consider the collision of two particles, which is described by the equations

$$M\ddot{z}_1 = -\frac{qez}{r^3}\,,\qquad m\ddot{z}_1 = \frac{qez}{r^3}\,, \tag{133}$$

where $z = z_1 - z_2$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and $r = \sqrt{\rho^2 + z^2}$. In equations (133) we separate off the center-of-mass motion to obtain

$$V_0 = \dot{X} = \mathrm{const}\,,\qquad X = \frac{Mz_1 + mz_2}{M+m}\,,\qquad \frac{\mu\dot{z}^2}{2} + \frac{qe}{r} = \mathrm{const}\,.$$

From these equations it follows that for $q > 0$ (the case considered in our review) the energies of the particles which experience collisions remain invariable: $E_1' = E_1$, $E_2' = E_2$, where $E_1 = M\dot{z}_1^2/2$, and $E_2 = m\dot{z}_2^2/2$. A typical binary collision takes place for an impact parameter $\rho \sim \overline{R}$. During the strongest interaction of these particles, which proceeds for $r \sim \overline{R}$, at a distance of about $\overline{R}$ from them there is at least one, a third, particle, the interaction with which must be taken into account for this reason. It interacts with the fourth one, etc. Therefore, particle M generates a collective excitation — a plasmon, with the consequential change in the particle's energy.

For an arbitrary direction of $\mathbf{V}$ and for $V > \varDelta_\parallel$, it can be shown that

$$(\mathbf{F}_\mathrm{r})_z = \widehat{\mathbf{V}}\mathbf{F}_\mathrm{r} \approx -\frac{2\pi q^2e^2n}{mV^2}\left[\left(\varLambda_1 - \frac{3}{2}\right)\sin^2\beta + 1\right],$$
$$(\mathbf{F}_\mathrm{r})_x \approx -\frac{4\pi q^2e^2n\varLambda_1}{mV^2}\,\sin\beta\cos\beta\,,$$

where $\varLambda_1 = \ln\left[V/(\omega_\mathrm{p}\overline{R})\right]$, β is the angle between the vectors $\mathbf{V}$ and $\mathbf{H}$, and the orientation of coordinate axes was specified above.

Formula (132) is important from the standpoint of determining the stationary positron distribution over longitudinal velocities; that is why in Appendix A1 it is derived in a simpler and physically more lucid way in comparison with the permittivity formalism employed above. The results obtained in this section will be applied to the analysis of positron motion.

5. Friction force acting on a light particle in an isotropic plasma

Prior to turning our attention to the complicated case of a nonzero magnetic field, in this section we will consider a substantially simpler case of light particle deceleration in an isotropic plasma ($T_\perp = T_\parallel \equiv T$) at a zero magnetic field.

For a light particle ($M \sim m$), an additional term $\mathbf{F}^{(2)}$ in the force of friction emerges, which is associated with its trajectory changing due to interaction with electrons. To calculate the required force of friction, we shall reason in much the same way as we did in Section 2.

Let us consider an inertial reference system 'm' in which particle m is at rest prior to a collision. By virtue of inequality (9) it may be assumed that the particle resides at the origin of this system throughout the collision time. In the same fashion as in the derivation of formula (11) we can find $\Delta\mathbf{p}_\mathrm{M}^{(2)} = -[2q^2e^2/(Mu^3\rho^2)]\hat{\mathbf{u}}$, and for the total variation of the momentum of particle M we therefore have

$$\Delta\mathbf{p}_\mathrm{M} = \Delta\mathbf{p}_\mathrm{M}^{(1)} + \Delta\mathbf{p}_\mathrm{M}^{(2)} = -\frac{2q^2e^2}{\mu u^3\rho^2}\,\hat{\mathbf{u}}\,. \tag{134}$$

The result for the binary-collision contribution to the force of friction is as follows:

$$\mathbf{F}_\mathrm{b} = \mathbf{F}_\mathrm{b}^{(1)} + \mathbf{F}_\mathrm{b}^{(2)} = \frac{4\pi ne^2q^2\varLambda_\mathrm{b}}{\mu}\,\nabla_\mathbf{V}\varPhi(\mathbf{V})\,. \tag{135}$$

Here, $\mathbf{F}_\mathrm{b}^{(1)}$ is the 'dynamic' force of friction, which is given by formula (12), and

$$\mathbf{F}_\mathrm{b}^{(2)} = \frac{4\pi ne^2q^2\varLambda_\mathrm{b}}{M}\,\nabla_\mathbf{V}\varPhi(\mathbf{V})\,.$$

The force $\mathbf{F}_\mathrm{b}^{(1)}$ acts on an infinitely heavy particle ($M \to \infty$), and the electrons therefore exert no effect on the motion of this particle.

We now evaluate the collective term $\mathbf{F}_\mathrm{c}$ for a light particle:

$$\mathbf{F}_\mathrm{c} = q\Big[\mathbf{E}'\big(\mathbf{R}(t),t\big) + \mathbf{E}_\mathrm{p}\big(\mathbf{R}(t),t\big)\Big]\,. \tag{136}$$

Unlike the calculation performed in Section 2, here account is taken of the plasma fluctuation electric field $\mathbf{E}'(\mathbf{R}(t),t)$ which acts on the particle at its location $\mathbf{R}(t)$. The particle travels according to the law $\mathbf{R}(t)=\mathbf{r}(t)+\xi(t), \mathbf{r}(t)=(0,0,Vt)\equiv \mathbf{V}t$, where $\xi(t)$ is a small deflection caused by the fluctuation field. We retain in expression (136) the quantities of the zero- and first-order of smallness in fluctuations to obtain

$$\ddot{\xi}_\alpha(t)\approx\frac{q}{M}\,E'_\alpha\big(\mathbf{r}(t),t\big)\,. \tag{137}$$

Therefore, we have an expression similar to expression (135): $\mathbf{F}_\mathrm{c}=\mathbf{F}_\mathrm{c}^{(1)}+\mathbf{F}_\mathrm{c}^{(2)}$. Here, $\mathbf{F}_\mathrm{c}^{(1)}$ is the dynamic force defined by relationships (22)–(25), (34); $\mathbf{F}_\mathrm{c}^{(2)}$ is the fluctuation term:

$$\mathbf{F}_\mathrm{c}^{(2)}=(0,0,F_{\mathrm{c}z}^{(2)})\,,$$
$$F_{\mathrm{c}z}^{(2)}=q\big\langle E'_z\big(\mathbf{R}(t),t\big)\big\rangle\approx q\left\langle \xi_\alpha(t)\left(\frac{\partial E'_z}{\partial x_\alpha}\right)_{\mathbf{r}(t),\,t}\right\rangle. \tag{138}$$

In expression (138), statistical averaging over a plasma state unperturbed by the field of a particle is performed in connection with the fact that we consider the motion of a beam of particles M as a whole, rather than the motion of every particle in it. From expressions (137) and (138), on passing to the Fourier components of the function $\xi(t)$ and the electric field strength

$$E'_\alpha(\mathbf{r},t)=\int \mathrm{d}Q\,E'_\alpha(\mathbf{k},\omega)\exp(\mathrm{i}\mathbf{kr}-\mathrm{i}\omega t)\,,\qquad \mathrm{d}Q=\frac{\mathrm{d}^3k\,\mathrm{d}\omega}{(2\pi)^4}\,,$$

we obtain the expression

$$F_{\mathrm{c}z}^{(2)}=-\frac{\mathrm{i}q^2}{M}\int\frac{\mathrm{d}Q\,k_\parallel}{(\Omega-\mathrm{i}0)^2}\,(E^2)_{\mathbf{k},\omega}=M\,\frac{\partial D_{z\beta}(V)}{\partial V_\beta}\,. \tag{139}$$

Here, $\Omega=\omega-k_\parallel V$, $k_\parallel=\mathbf{kV}/V$, $D_{\alpha\beta}$ is the tensor of the diffusion coefficients of particle M in the velocity space in the isotropic plasma considered in this section ([14, 22]; see also Appendices A4–A6):

$$D_{\alpha\beta}(\mathbf{V})=-\frac{\mathrm{i}q^2}{M^2}\int \mathrm{d}Q\,E_{\alpha\beta}(\mathbf{k},\omega)\,\frac{1}{\Omega-\mathrm{i}0}$$
$$=\frac{q^2}{2M^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{k_\alpha k_\beta}{k^2}\,(E^2)_{\mathbf{k},\mathbf{kV}}$$
$$=\frac{2nq^2e^2}{M^2}\int \mathrm{d}^3v\,f(\mathbf{v})\int \mathrm{d}^3k\,\frac{k_\alpha k_\beta\,\delta(\mathbf{ku})}{k^4\big|\varepsilon(\mathbf{k},\mathbf{kV})\big|^2}$$
$$=\sqrt{\frac{2}{\pi}}\,\frac{nq^2e^2}{M^2\Delta}\int \mathrm{d}^3k\,\frac{k_\alpha k_\beta}{k^5\big|\varepsilon(\mathbf{k},\mathbf{kV})\big|^2}\exp\left[-\frac{(\mathbf{kV})^2}{2k^2\Delta^2}\right], \tag{140}$$

and $\mathbf{u}=\mathbf{V}-\mathbf{v}$. In expressions (139) and (140) we introduced the correlation function for the components of an electric field strength (see Appendix A3)

$$E_{\alpha\beta}(\mathbf{k},\omega)=\frac{k_\alpha k_\beta}{k^2}\,(E^2)_{\mathbf{k},\omega}\,,$$
$$(E^2)_{\mathbf{k},\omega}=-\frac{8\pi T}{\omega}\,\mathrm{Im}\,\frac{1}{\varepsilon(\mathbf{k},\omega)}$$
$$=\frac{32\pi^3ne^2}{k^2\big|\varepsilon(\mathbf{k},\omega)\big|^2}\int \mathrm{d}^3v\,f(\mathbf{v})\,\delta(\mathbf{ku})\,. \tag{141}$$

Account was also taken of the property $E_{\alpha\beta}(\mathbf{k},\omega)=E^*_{\alpha\beta}(-\mathbf{k},-\omega)$ and the Sokhotskii formula

$$\frac{1}{\Omega+\mathrm{i}0}=P\left(\frac{1}{\Omega}\right)-\pi\mathrm{i}\delta(\Omega)\,. \tag{142}$$

According to papers [12, 14–16] [see also formulas (17) and (18)], the longitudinal permittivity of an isotropic electron plasma is given by

$$\varepsilon(\mathbf{k},\omega)=1+\frac{\omega_\mathrm{p}^2}{k^2}\int \mathrm{d}^3v\,\frac{\big(\mathbf{k}\nabla_\mathbf{v}f(\mathbf{v})\big)}{\omega-\mathbf{kv}+\mathrm{i}0}$$
$$=1+\frac{m\omega_\mathrm{p}^2}{Tk^2}\left[1-Z\left(\frac{\omega}{\sqrt{2}\,k\Delta}\right)\right]. \tag{143}$$

Here, the notation was introduced as follows:

$$Z(x)=X(x)-\mathrm{i}Y(x)\,,$$
$$X(x)=2x\exp(-x^2)\int_0^x \mathrm{d}t\,\exp t^2\,, \tag{144}$$
$$Y(x)=\sqrt{\pi}\,x\exp(-x^2)\,.$$

Let us present an asymptotic expression required for the subsequent discussion:

$$X(x)\approx 1+\frac{1}{2x^2}\,,\qquad x\gg 1\,. \tag{145}$$

We point out an analog of the Einstein relation in the velocity space:

$$(\mathbf{F}_\mathrm{c}^{(1)})_\alpha=-\frac{M^2}{T}\,D_{\alpha\beta}V_\beta\,, \tag{146}$$

which follows from expressions (22)–(25), (140), and (141) [see also formula (104)].

Formula (140) is simplified for an ideal plasma satisfying condition (4). For brevity of writing, we assume *a priori* that the main contribution is made by the range of wave numbers $k\gg 1/r_\mathrm{D}$, which corresponds to the range of impact parameters $\rho\ll r_\mathrm{D}$. Then, putting $\varepsilon(\mathbf{k},\mathbf{kV})\approx 1$ in accordance with inequality (32), in view of the relation $\delta(\mathbf{ku})=\delta(k_\parallel)/u$, where $k_\parallel=\mathbf{k}\hat{\mathbf{u}}$, we arrive at the expression contained in the Landau collision integral (for details, see Lifshitz and Pitaevskii [15, § 46]):

$$B_{\alpha\beta}=\frac{2q^2e^2}{u}\int \mathrm{d}^2k_\perp\,\frac{k_{\perp\alpha}k_{\perp\beta}}{k_\perp^4}\approx\frac{2\pi q^2e^2\Lambda_\mathrm{c}}{u}\,(\delta_{\alpha\beta}-\hat{u}_\alpha\hat{u}_\beta)\,,$$
$$D_{\alpha\beta}(\mathbf{V})=\frac{n}{M^2}\int \mathrm{d}^3v\,f(\mathbf{v})B_{\alpha\beta}=\frac{2\pi nq^2e^2\Lambda_\mathrm{c}}{M^2}\,\frac{\partial^2\langle u\rangle}{\partial V_\alpha\,\partial V_\beta}\,. \tag{147}$$

Here, $\langle u\rangle=\int \mathrm{d}^3v\,f(\mathbf{v})u$ is the second Trubnikov potential. From expression (139) we then obtain

$$(\mathbf{F}_\mathrm{c}^{(2)})_\alpha=M\,\frac{\partial D_{\alpha\beta}(V)}{\partial V_\beta}=F_\mathrm{c}^{(2)}\,\widehat{V}_\alpha\,,$$
$$F_\mathrm{c}^{(2)}=\frac{4\pi ne^2q^2\Lambda_\mathrm{c}}{M}\,\Phi'(V)\,. \tag{148}$$

Assumption (31) is correct provided that $\Lambda_\mathrm{c}\gg 1$, which is fulfilled for ideal plasmas. From expressions (28) and (148) we find the collective force of friction:

$$\mathbf{F}_\mathrm{c}=\frac{4\pi ne^2q^2\Lambda_\mathrm{c}}{\mu}\,\nabla_\mathbf{V}\Phi(\mathbf{V})\,. \tag{149}$$

From formulas (135) and (149) we obtain the resultant expression for the total force of friction:

$$\mathbf{F} = \frac{4\pi n e^2 q^2 \Lambda}{\mu} \nabla_{\mathbf{V}} \Phi(\mathbf{V}) \,. \tag{150}$$

Two remarkable facts are worthy of mention:

(A) the vanishing [compare with expression (35)] of the intermediate dimension $\bar{R}$ from the total force (150), and the appearance of the resultant Coulomb logarithm Λ;

(B) the union of the masses of the particle being decelerated (M) and the plasma particles (m) into the reduced mass μ in the formula for the total force of friction.

Fact A is well known in plasma physics [10–16]: every plasma particle resides in the fluctuation electric field produced by all other particles, while binary collisions may be treated as the shortest-term fluctuations. By comparing the volumes of the calculations presented in Sections 3 and 4, it may be concluded that this property is methodically important: it permits considering only collective interactions which are simpler to evaluate. Binary collisions then are automatically included if the cutoff in the resultant logarithmically diverging expressions is performed not at the shortest 'collective' dimension $\bar{R}$, but at the shortest 'binary' dimension R_{T} [as indicated earlier, $\bar{r}_{\mathrm{H}}$ should be taken instead of R_{T} in the conditions of fulfilling inequality (44)]. This substantially simplifies calculations in the most complicated and practically important case of a nonzero magnetic field ($\mathbf{H} \neq 0$), for which the applicability of this procedure was proved by Montgomery et al. [23]. Furthermore, it is shown in Section 7 that the role played by binary collisions becomes progressively less significant in the very cases most important for positron moderation: (1) with increasing the magnetic field intensity; (2) with an increase in the degree of anisotropy $T_\perp/T_\parallel$, and (3) with a decrease in the mass of particle M.

Fact B is rather surprising, because it is also valid for collective interactions whereat particle M interacts simultaneously with a large number of plasma particles. The reason lies with the specific character of Coulomb interaction. In Section 6 it is shown, however, that the effect of electron-mass replacement by the reduced mass takes place only in the absence of a magnetic field.

6. Kinetics of the electron cooling of light particles. Initial equations

For a nonzero magnetic field, the motion of particles is substantially complicated and it is required to find a rigorous approach to their moderation analysis with a view of strictly taking into account the momentum conservation which leads to the appearance of a reduced mass in the formulas. All necessary information about moderation kinetics is contained in the distribution function $\Phi(\mathbf{V}, t)$ in particle velocities $\mathbf{V}$. The starting point when writing the equation for the function $\Phi(\mathbf{V}, t)$ is assumption (46) of anisotropic plasma ideality. According to N N Bogoliubov's fundamental conclusion [24], the system of collisionless Vlasov–Maxwell equations, i.e., the self-consistent field approximation, is applicable in this case to the description of plasma and the charged particles in it owing to the long-range nature of Coulomb interactions. According to Rostoker [25] as well as Klimontovich and Silin [26], this field may be resolved into two components — the large- and small-scale components — and averaging may be performed over the fast fluctuations of the small-scale field, which gives a collisional term in the Landau form on the right-hand side of the Vlasov equation. With the inclusion of plasma polarization, i.e., the larger-scale field, this term may be represented in the form of the Balescu–Lenard collision integral. The Vlasov equation reduces to the Fokker–Planck equation (see, for instance, Aleksandrov and Rukhadze [14, § 9.4] and Lifshitz and Pitaevskii [15, § 47], which is due to the smallness of velocity variations $\Delta\mathbf{V}$ in individual scattering events. This is exactly the desired equation for $\Phi(\mathbf{V}, t)$:

$$\begin{gathered} \frac{\partial \Phi(\mathbf{V}, t)}{\partial t} + \frac{q}{M} (\mathbf{V} \times \mathbf{H}) \nabla_{\mathbf{V}} \Phi(\mathbf{V}, t) = \mathrm{St}\,(\Phi)\,, \\ \mathrm{St}\,(\Phi) = -\frac{\partial j_\alpha}{\partial V_\alpha}\,, \end{gathered} \tag{151}$$

where j_α is the flux density of particles M in the velocity space:

$$j_\alpha = A_\alpha(\mathbf{V})\, \Phi(\mathbf{V}, t) - D_{\alpha\beta}(\mathbf{V}) \frac{\partial \Phi(\mathbf{V}, t)}{\partial V_\beta}\,, \tag{152}$$

$$A_\alpha(\mathbf{V}) = \frac{F_\alpha^{(1)}}{M}\,, \tag{153}$$

where $\mathbf{F}^{(1)}(\mathbf{V})$ is the dynamic force introduced earlier.

Indeed, let us consider a beam of particles M with the initial distribution

$$\Phi(\mathbf{V}', t = 0) = \delta(\mathbf{V}' - \mathbf{V})\,. \tag{154}$$

At the instant of time $t > 0$, the beam velocity equals

$$\mathbf{V}(t) = \langle \mathbf{V}' \rangle = \int \mathrm{d}^3 V' \, \mathbf{V}' \Phi(\mathbf{V}', t)\,,$$

and the acceleration is expressed as

$$\begin{aligned} (\mathbf{a})_\alpha &= \frac{\mathrm{d}\big(\mathbf{V}(t)\big)_\alpha}{\mathrm{d}t} = \frac{(\mathbf{F})_\alpha}{M} = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 V' \, V'_\alpha \Phi(\mathbf{V}', t) \\ &= \int \mathrm{d}^3 V' \, V'_\alpha \frac{\partial \Phi(\mathbf{V}', t)}{\partial t} = -\int \mathrm{d}^3 V' \, V'_\alpha \frac{\partial j_\beta}{\partial V'_\beta} = \int \mathrm{d}^3 V' \, j_\alpha \\ &= \int \mathrm{d}^3 V' \left[A_\alpha(\mathbf{V}')\, \Phi(\mathbf{V}', t) - D_{\alpha\beta}(\mathbf{V}') \frac{\partial \Phi(\mathbf{V}', t)}{\partial V'_\beta} \right] \\ &= \int \mathrm{d}^3 V' \, A_\alpha(\mathbf{V}')\, \Phi(\mathbf{V}', t) + \int \mathrm{d}^3 V' \, \Phi(\mathbf{V}', t) \frac{\partial D_{\alpha\beta}(\mathbf{V}')}{\partial V'_\beta}\,. \end{aligned}$$

Hence, and from expression (154) at $t = 0$, it follows that

$$(\mathbf{a})_\alpha = \frac{(\mathbf{F})_\alpha}{M} = A_\alpha(\mathbf{V}) + \frac{\partial D_{\alpha\beta}(\mathbf{V})}{\partial V_\beta}\,. \tag{155}$$

By comparing this relation with the results outlined in Section 5, we ascertain the meaning of vector $\mathbf{A}(\mathbf{V})$ and its linkage (153) with the dynamic force.

The components $D_{\alpha\beta}(\mathbf{V})$ of the M-particle diffusion tensor in the velocity space, which are highly complicated in form for $\mathbf{H} \neq 0$, were calculated by Rostoker and Rosenbluth [27]. However, it is unnecessary here to know all of the components, because we are concerned with the case of azimuthal and axial symmetries: $\Phi = \Phi(\mathbf{V}, t) = \Phi(V_\perp, V_\parallel, t) = \Phi(V_\perp, -V_\parallel, t)$. We write down equations (151) and (152), in view of the relationship

$(\mathbf{V}\times\mathbf{H})\,\nabla_{\mathbf{V}}\Phi(V_\perp, V_\parallel, t) = 0$, in the form

$$\frac{\partial\Phi(\mathbf{V},t)}{\partial t} = \mathrm{St}\,(\Phi)\,, \tag{156}$$

$$\mathrm{St}\,(\Phi) = -\frac{\partial j_\alpha}{\partial V_\alpha} = -\frac{\partial j_\parallel}{\partial V_\parallel} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}(V_\perp j_\perp)\,, \tag{157}$$

$$j_\parallel = A_\parallel \Phi(\mathbf{V},t) - D_\parallel\frac{\partial\Phi}{\partial V_\parallel} - D_{\mathrm{LT}}\frac{\partial\Phi}{\partial V_\perp}\,,$$
$$j_\perp = A_\perp \Phi(\mathbf{V},t) - D_\perp\frac{\partial\Phi}{\partial V_\perp} - D_{\mathrm{LT}}\frac{\partial\Phi}{\partial V_\parallel}\,. \tag{158}$$

Here, $A_\parallel$ and $A_\perp$ are the longitudinal and transverse 'dynamic' accelerations; $D_\parallel = D_{\alpha\beta}h_\alpha h_\beta$ and $D_\perp = D_{\alpha\beta}\widehat{V}_{\perp\alpha}\widehat{V}_{\perp\beta}$ are the coefficients of positron diffusion in the longitudinal and transverse velocities, and $D_{\mathrm{LT}} = D_{\alpha\beta}\widehat{V}_{\perp\alpha}h_\beta$ is the nondiagonal element of the matrix of diffusion coefficients $D_{\alpha\beta}$, which is responsible for the longitudinal–transverse relaxation and describes the establishment of equilibrium between the longitudinal and transverse degrees of freedom in positron motion. It is shown in Section 10 that the terms with D_{LT} in equations (158) may be neglected in the case of interest (44), and hereinafter we therefore assume that

$$j_\parallel \approx A_\parallel \Phi(\mathbf{V},t) - D_\parallel\frac{\partial\Phi}{\partial V_\parallel}\,,\qquad j_\perp \approx A_\perp \Phi(\mathbf{V},t) - D_\perp\frac{\partial\Phi}{\partial V_\perp}\,. \tag{159}$$

To determine the relations for an axially symmetric distribution, which are analogous to relations (153) and (155), in lieu of distribution (154) we should consider the initial velocity distribution

$$\Phi(V'_\parallel, V'_\perp, 0) = \frac{1}{2\pi V_\perp}\,\delta(V'_\parallel - V_\parallel)\,\delta(V'_\perp - V_\perp)\,. \tag{160}$$

This distribution corresponds to particles with equal values of $(V_\parallel, V_\perp)$. We repeat the calculation to obtain

$$a_\perp(V_\parallel, V_\perp) = \frac{F_\perp(V_\parallel, V_\perp)}{M} = \frac{\mathrm{d}}{\mathrm{d}t}\langle V_\perp\rangle_{t=0}$$
$$= A_\perp(V_\parallel, V_\perp) + \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\big(V_\perp D_\perp(V_\parallel, V_\perp)\big)\,,$$
$$a_\parallel(V_\parallel, V_\perp) = \frac{F_\parallel(V_\parallel, V_\perp)}{M} = \frac{\mathrm{d}}{\mathrm{d}t}\langle V_\parallel\rangle_{t=0}$$
$$= A_\parallel(V_\parallel, V_\perp) + \frac{\partial D_\parallel(V_\parallel, V_\perp)}{\partial V_\parallel}\,. \tag{161}$$

Hence, it is clear that $F_\perp^{(1)}$ and $F_\parallel^{(1)}$ are the dynamic forces defined as

$$F_\perp^{(1)}(V_\parallel, V_\perp) = MA_\perp(V_\parallel, V_\perp)\,,$$
$$F_\parallel^{(1)}(V_\parallel, V_\perp) = MA_\parallel(V_\parallel, V_\perp)\,, \tag{162}$$

which will be calculated in Section 7.

7. Dynamic force for a positron

In the positron storage ring of the LEPTA experiment [6], a volume of $\sim 10^4$ cm^3 will contain $\sim 10^8$ positrons. Since the particle number density in the positron beam is rather high, viz.

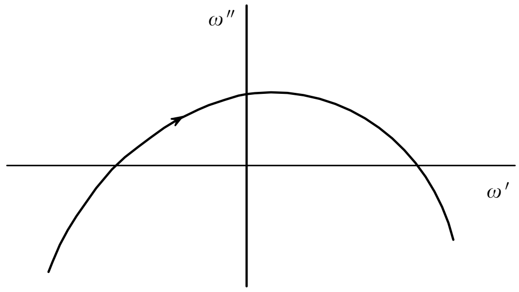

Figure 2. Qualitative form of the trajectories of the roots $\omega = \omega(k_\parallel)$ of equation (166) with varying $k_\parallel$.

$$N \sim 10^4\ \mathrm{cm}^{-3}\,, \tag{163}$$

above all it is required to elucidate how the positrons are decelerated: as a single beam, collectively, or independently of one another. In other words, it is necessary to investigate the possibility of emerging a beam instability accompanied by a strong enhancement of the deceleration of positrons which would excite the electron Langmuir oscillations. The dispersion law $\omega = \omega(k_\parallel)$ governing these oscillations for cold electron and positron beams is obtained from the equation [11, 15, 16]

$$\frac{\omega_{\mathrm{p}}^2}{\omega^2} + \frac{\Omega_{\mathrm{p}}^2}{(\omega - k_\parallel V_0)^2} = 1\,, \tag{164}$$

where $\Omega_{\mathrm{p}} = \sqrt{4\pi N e^2/m}$, and V_0 is the positron beam velocity (as usual, in the rest frame of the electron beam). Owing to the typical temperatures involved (43), the cold electron approximation is justified (since $\mathbf{V}_0 \parallel \mathbf{H}$, it is only the longitudinal temperature $T_\parallel$ that matters). The generalization of formula (164) for an arbitrary distribution $F(V_\parallel)$ of the positrons in longitudinal velocities $V_\parallel$ takes the form

$$\frac{\omega_{\mathrm{p}}^2}{\omega^2} + \Omega_{\mathrm{p}}^2\int_{-\infty}^{\infty}\mathrm{d}V_\parallel\,\frac{F(V_\parallel)}{(\omega - k_\parallel V_\parallel)^2} = 1\,, \tag{165}$$

where the function $F(V_\parallel)$ is normalized by the condition $\int_{-\infty}^{\infty}\mathrm{d}V_\parallel\,F(V_\parallel) = 1$. Expression (165) may be obtained from the Vlasov equation written out for electrons and positrons. For a positron velocity distribution of the form

$$F(V_\parallel) = \frac{\Delta}{\pi\big[(V_\parallel - V_0)^2 + \Delta^2\big]}\,,$$

from expressions (125) and (165) we obtain

$$\frac{\omega_{\mathrm{p}}^2}{\omega^2} + \frac{\Omega_{\mathrm{p}}^2}{(\omega - k_\parallel V_0 + \mathrm{i}k_\parallel\Delta)^2} = 1\,. \tag{166}$$

With a variation in $k_\parallel$, the roots $\omega = \omega' + \mathrm{i}\omega''$ of equation (166) move as depicted in Fig. 2. The electric field of the Langmuir oscillations excited by the positron beam depends on the time as $\exp(-\mathrm{i}\omega t)$, and therefore the instability develops if only one of the roots takes the positive value: $\omega'' > 0$. The instability increment is $\gamma = \max\big[\omega''(k_\parallel)\big]$. An instability emerges when the particle trajectory with the greatest imaginary part ω'' is tangent to the ω'-axis (see

Fig. 2):

$$\omega''(k_\parallel) = 0\,, \qquad \frac{\mathrm{d}\omega''(k_\parallel)}{\mathrm{d}k_\parallel} = \mathrm{Im}\left(\frac{\mathrm{d}\omega}{\mathrm{d}k_\parallel}\right) = 0\,. \tag{167}$$

From the three equations (166), (167), it follows that this critical trajectory emerges for the parameter value

$$\varDelta = V_0\sqrt{\frac{N}{n}}\,. \tag{168}$$

The critical trajectory is tangent to the ω'-axis for a wave vector $k_\parallel = \omega_\mathrm{p}/(V_0\sqrt{2})$ at a point $\omega' = \omega_\mathrm{p}/\sqrt{2}$. Therefore, the condition for the development of the beam instability is as follows:

$$\varDelta < V_0\sqrt{\frac{N}{n}}\,. \tag{169}$$

From formulas (42), (163), and (169) results the conclusion that the positrons in the LEPTA experiment, which is characterized by fulfilling the condition $V_0 \sim \varDelta$, are decelerated independently of each other, because beam instability is absent.

In this section we will consider positrons with velocities from the range (56) which accommodates their stationary distribution. Therefore, for a typical $\mathrm{e}^+ + \mathrm{e}^-$ collision [see formulas (3), where $M = m$ in this case] one has

$$r_\mathrm{Hp} \sim \bar{r}_\mathrm{H}\,. \tag{170}$$

That is why, owing to inequality (70), there are two characteristic domains, depending on the Larmor-circle impact parameter ρ_0 (the distance between the straight lines described by the Larmor circles prior to the electron–positron collision), which are specified as follows:
the collective interaction domain, viz.

$$\rho_0 > \overline{R}\,, \tag{171}$$

and the binary collision domain, viz.

$$\rho_0 < \overline{R}\,. \tag{172}$$

The latter may be additionally divided into two domains:
the one pertinent to nonoverlapping Larmor circles, viz.

$$\bar{r}_\mathrm{H} < \rho_0 < \overline{R}\,, \tag{173}$$

and the other with overlapping Larmor circles, viz.

$$\rho_0 < \bar{r}_\mathrm{H}\,. \tag{174}$$

Formula (10) is valid in this instance because positrons experience collisions of a different type. In this section we will calculate the collective force $\mathbf{F}_\mathrm{c}$ which corresponds to the interactions specified by inequality (171).

A positron moves along a helical line $\mathbf{R}(t) = V_\parallel t\,\mathbf{h} + \mathbf{R}_\perp(t)$, for which it follows from expressions (24) that

$$\rho(\mathbf{k},\omega) = 2\pi q \sum_{S=-\infty}^{\infty} \delta(\Omega - \omega_\mathrm{H} S)\, J_S(k_\perp r_\mathrm{Hp})\,, \tag{175}$$

where $\Omega = \omega - k_\parallel V_\parallel$, and J_S is the Bessel function. Due to the smallness of the positron Larmor radius and the electron magnetization [see condition (170)], we may retain in expression (175) only the monopole ($S = 0$) and dipole ($S = \pm 1$) terms and discard the quadrupole and higher terms ($|S| \geqslant 2$):

$$\rho(\mathbf{k},\omega) \approx \rho_\mathrm{M} + \rho_\mathrm{d}\,. \tag{176}$$

Here, ρ_M is the charge density for the zero size of the positron Larmor radius:

$$\rho_\mathrm{M}(\mathbf{k},\omega) \approx 2\pi q \delta(\Omega)\,, \tag{177}$$

and ρ_d is the charge density produced by the rotating dipole moment of the Larmor positron motion:

$$\rho_\mathrm{d}(\mathbf{k},\omega) \approx 2\pi q J_1(k_\perp r_\mathrm{Hp})\big[\delta(\Omega - \omega_\mathrm{H}) + \delta(\Omega + \omega_\mathrm{H})\big]\,. \tag{178}$$

In accordance with expressions (177) and (178), one has

$$\mathbf{F}_\mathrm{c} = \mathbf{F}_\mathrm{cM} + \mathbf{F}_\mathrm{cd}\,. \tag{179}$$

Below we shall show that the main contribution to $\mathbf{F}_\mathrm{cM}$ is made by the entire collective interaction domain (130), i.e., $k_\perp \sim |k_\parallel| < 1/\overline{R}$. That is why, owing to relations (70) and (170) for positrons we arrive at the relation $k_\perp r_\mathrm{Hp} < \bar{r}_\mathrm{H}/\overline{R} \ll 1$ that is similar to inequality (111) for electrons, which allowed putting equal to zero the argument of the Bessel function in expression (177). We will ascertain that the main contribution to the force $\mathbf{F}_\mathrm{cd}$ is made by only a part of domain (130), namely

$$k_\perp \sim \frac{1}{r_\mathrm{Hp}} \sim \frac{1}{\bar{r}_\mathrm{H}}\,, \qquad |k_\parallel| \ll k_\perp\,, \tag{180}$$

which is why the Bessel function is retained in expression (178).

The term (177) describes a point charge q moving with velocity $V_\parallel$ along the magnetic field, and therefore formula (104) holds true, in which $\omega = V_\parallel k_\parallel$:

$$\mathbf{F}_\mathrm{cM} = F_\mathrm{cM}\mathbf{h}\,, \qquad F_\mathrm{cM} = \frac{q^2}{2\pi^2}\int \frac{\mathrm{d}^3k\,k_\parallel}{k^2}\,\mathrm{Im}\,\frac{1}{\varepsilon}\,. \tag{181}$$

Let us calculate $\mathbf{F}_\mathrm{cM}$ for magnetized electrons (66) at an arbitrary value of $V_\parallel$. Formulas (121), (122) are easily generalized to the case of $T_\parallel \neq 0$:

$$\begin{aligned}
&\varepsilon_{\alpha\beta}(\mathbf{k},\omega) = \delta_{\alpha\beta} + h_\alpha h_\beta\, Q(\mathbf{k},\omega)\,,\\
&\varepsilon(\mathbf{k},\omega) = 1 + \cos^2\alpha\, Q(\mathbf{k},\omega)\,,\\
&Q(\mathbf{k},\omega) = \frac{\omega_\mathrm{p}^2}{\omega}\int_{-\infty}^{\infty} \frac{\mathrm{d}v_\parallel\, v_\parallel}{\omega + \mathrm{i}0 - k_\parallel v_\parallel}\,\frac{\mathrm{d}g(v_\parallel)}{\mathrm{d}v_\parallel}\,.
\end{aligned} \tag{182}$$

The quantities $\varepsilon_{\alpha\beta}$ are calculated, as in the derivation of formulas (121) and (122), on the basis of the Vlasov kinetic equation for the distribution function $f(v_\parallel, \mathbf{r}, t)$ of magnetized electrons:

$$\frac{\partial f}{\partial t} + v_\parallel\,\frac{\partial f}{\partial z} - \frac{eE_\parallel(\mathbf{r},t)}{m}\,\frac{\partial f}{\partial v_\parallel} = 0\,,$$

and the relationship

$$\frac{\partial \xi(\mathbf{r},t)}{\partial t} = \bar{v}_\parallel(\mathbf{r},t) = \int \mathrm{d}v_\parallel\, v_\parallel f(v_\parallel,\mathbf{r},t)\,.$$

Formulas (182) also follow from the rigorous expression for $\varepsilon \equiv \varepsilon_{\parallel}$, which is valid for arbitrary values of the magnetic field intensity [12, 13, 28]:

$$\varepsilon(\mathbf{k},\omega) = 1 + \frac{\omega_{\mathrm{p}}^2}{k^2}\sum_{l=-\infty}^{\infty}\int \mathrm{d}^3 v\, J_l^2\left(\frac{k_\perp v_\perp}{\omega_{\mathrm{H}}}\right) \times \frac{k_\parallel\, \partial f/\partial v_\parallel + l\omega_{\mathrm{H}}/v_\perp\, \partial f/\partial v_\perp}{\omega - \mathrm{i}0 - l\omega_{\mathrm{H}} - k_\parallel v_\parallel}\,, \tag{183}$$

where f is the electron velocity distribution function. For an 'oblate' distribution, f is defined by formulas (36). In the frequency range $|\omega| \sim \omega_{\mathrm{p}}$, which is most significant for the force $\mathbf{F}_{\mathrm{cM}}$, the terms with $l \neq 0$ in expression (183), which are exponentially small, may be neglected, whence we obtain formulas (182). From these formulas we have

$$\varepsilon(\mathbf{k}, k_\parallel V_\parallel) = 1 - \frac{\omega_{\mathrm{p}}^2}{V_\parallel \varDelta_\parallel^2 k^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}v_\parallel\, v_\parallel^2 g(v_\parallel)}{V_\parallel - v_\parallel + \mathrm{i}0\,\mathrm{sign}\,k_\parallel}\,. \tag{184}$$

Hence, it is clear that $\mathrm{Im}\,\varepsilon$ and, consequently, $\mathrm{Im}\,(1/\varepsilon)$ are the odd functions of $k_\parallel$, and therefore expression (181) is brought to the form

$$F_{\mathrm{cM}} = \frac{2q^2}{\pi}\,\mathrm{Im}\left[\int_0^\infty \mathrm{d}k_\parallel\, k_\parallel \int_{k_\parallel}^\infty \frac{\mathrm{d}k}{k\varepsilon(k)}\right], \tag{185}$$

where it was taken into account that owing to azimuthal symmetry it is possible to perform integration over the angles, which reduces to the change: $\mathrm{d}^3k \to 2\pi k_\perp\,\mathrm{d}k_\perp k_\parallel$, and use was made of the relation $k_\perp\,\mathrm{d}k_\perp = k\,\mathrm{d}k$ valid at $k_\parallel = \mathrm{const}$. According to formula (184), for $k_\parallel > 0$ it is easily shown that

$$\varepsilon(k) = 1 + \frac{a}{k^2}\,, \tag{186}$$

where

$$a = \frac{1}{V_\parallel R_\parallel^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}v_\parallel\, g(v_\parallel) v_\parallel^2}{v_\parallel - V_\parallel - \mathrm{i}0} = \frac{1}{R_\parallel^2}\left[1 - X\left(\frac{V_\parallel}{\sqrt{2}\,\varDelta_\parallel}\right) + \mathrm{i}\,Y\left(\frac{V_\parallel}{\sqrt{2}\,\varDelta_\parallel}\right)\right], \tag{187}$$

and functions X and Y are defined by formulas (144).

In expression (185) we perform integration by parts:

$$F_{\mathrm{cM}} = \frac{q^2}{\pi}\,\mathrm{Im}\int_0^\infty \frac{\mathrm{d}k_\parallel\, k_\parallel}{\varepsilon(k_\parallel)} = \frac{q^2}{\pi}\,\mathrm{Im}\int_0^\infty \mathrm{d}k_\parallel\, k_\parallel\left(\frac{1}{\varepsilon(k_\parallel)} - 1\right) = -\frac{q^2}{\pi}\,\mathrm{Im}\left(a\int_0^\infty \frac{\mathrm{d}k_\parallel\, k_\parallel}{k_\parallel^2 + a}\right).$$

From inequality (130) it is clear that the upper integration limit ∞ should be replaced by the quantity $1/\overline{R}$, then one finds

$$F_{\mathrm{cM}} = -\frac{q^2}{\pi}\,\mathrm{Im}\,(a\ln p)\,, \tag{188}$$

where $p = \ln\left[1 + 1/(a\overline{R}^{\,2})\right]$. According to expression (187), the quantities a and p are complex functions of the real parameter $x = V_\parallel/(\sqrt{2}\,\varDelta_\parallel)$. The qualitative form of the trajectory $p = p(x) = p_1(x) + \mathrm{i}p_2(x)$ is plotted in Fig. 3. In the limiting case of $V_\parallel \ll \varDelta_\parallel$, one has

$$a \approx \frac{1}{R_\parallel^2}\left(1 + \frac{\mathrm{i}\sqrt{\pi}\,V_\parallel}{\sqrt{2}\,\varDelta_\parallel}\right),$$

$$F_{\mathrm{cM}} \approx -\frac{\sqrt{2\pi}\,q^2 e^2 n V_\parallel}{m\varDelta_\parallel^3}\ln\left(1 + \frac{R_\parallel^2}{\overline{R}^{\,2}}\right).$$

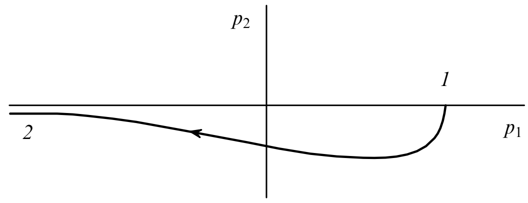

Figure 3. Trajectory of the value of function $p = p(x)$. Point *1* corresponds to $x = 0$ (at this point $p_1 = 1 + R_\parallel^2/\overline{R}^{\,2}$ and $p_2 = 0$). Domain *2* in the curve corresponds to $x \to +\infty$.

For longitudinal velocities $V_\parallel \gg \varDelta_\parallel$ (see Fig. 3), $\ln p \approx -\pi\mathrm{i}$, and in accordance with formula (132) we therefore arrive at

$$F_{\mathrm{cM}} \approx -\frac{2\pi q^2 e^2 n}{m V_\parallel^2}\,. \tag{189}$$

For arbitrary values of $V_\parallel$, the force F_{cM} is calculated by the following formulas:

$$F_{\mathrm{cM}} = -\frac{2q^2e^2n}{m\varDelta_\parallel^2}\left[-\varphi(1 - X) + \frac{1}{2}\,Y\ln\left(p_1^2 + p_2^2\right)\right],$$

$$\varphi = \frac{\pi}{2} - \arctan\frac{p_1}{p_2}, \qquad p_1 = 1 + \frac{R_\parallel^2(1 - X)}{\overline{R}^{\,2} D}\,,$$

$$p_2 = \frac{R_\parallel^2 Y}{\overline{R}^{\,2} D}\,, \qquad D = (1 - X)^2 + Y^2\,.$$

Let us now evaluate the force $\mathbf{F}_{\mathrm{cd}}$. We begin by calculating the work done by this force in a unit time:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E_\perp}{\mathrm{d}t} + \frac{\mathrm{d}E_\parallel}{\mathrm{d}t} = q\langle \mathbf{V}\mathbf{E}_{\mathrm{p}}\rangle\,,$$

where

$$E_\perp = \frac{mV_\perp^2}{2}\,, \qquad E_\parallel = \frac{mV_\parallel^2}{2}\,, \qquad \frac{\mathrm{d}E_\perp}{\mathrm{d}t} = q\langle \mathbf{V}_\perp \mathbf{E}_{\mathrm{p}}\rangle\,, \qquad \frac{\mathrm{d}E_\parallel}{\mathrm{d}t} = q\langle V_\parallel E_{\mathrm{p}\parallel}\rangle\,, \tag{190}$$

and $\mathbf{E}_{\mathrm{p}}$ is the plasma electric field strength at the positron location [see formula (25)]. The angular brackets in formulas (190) denote averaging over the Larmor period, which gives relationships (see also Shafranov [12])

$$\frac{\mathrm{d}E_\parallel}{\mathrm{d}t} = V_\parallel F_{\mathrm{cM}}\,, \tag{191}$$

$$\frac{\mathrm{d}E_\perp}{\mathrm{d}t} = V_\perp F_{\mathrm{cd}} = 4\pi q^2 \sum_{S=-\infty}^{\infty}\omega_{\mathrm{H}} S\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2}\,\mathrm{Im}\left(\frac{1}{\varepsilon_S}\right)J_S^2(k_\perp r_{\mathrm{Hp}}) \approx 4\pi q^2\omega_{\mathrm{H}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2}\,R(\mathbf{k})\,J_1^2(k_\perp r_{\mathrm{Hp}})\,, \tag{192}$$

where $\varepsilon_S \equiv \varepsilon(\mathbf{k}, k_\parallel V_\parallel + \omega_H S)$, and

$$R(\mathbf{k}) = \operatorname{Im}\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_{-1}}\right). \tag{193}$$

In this calculation, use was made of expressions (22), (23), and (175). On averaging over the Larmor period there remains a nonzero contribution to Eqn (191) from only the term with $S = 0$, i.e., from the monopole charge density (177). The main contribution to Eqn (192) is made by the terms with $S = \pm 1$, i.e., the dipole density (178). The simplifications in Eqn (192) were made in view of relation (176).

Therefore, the friction force $\mathbf{F}_{cM}$ is aligned with the magnetic field [see formulas (181) and (189)], and it therefore lowers the energy $E_\parallel$ of longitudinal positron motion. The averaged force $\mathbf{F}_{cd}$ exerts no effect on $E_\parallel$ and lowers $E_\perp$, i.e., it is directed transversely to the magnetic field:

$$\mathbf{F}_{cd} = F_{cd}\widehat{\mathbf{V}}_\perp, \qquad \widehat{\mathbf{V}}_\perp = \frac{\mathbf{V}_\perp}{V_\perp},$$
$$F_{cd} = \frac{4\pi q^2 \omega_H}{V_\perp}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2}\,R(\mathbf{k})\,J_1^2(k_\perp r_{Hp}), \tag{194}$$

where use was made of relationship (192). The main contribution to the magnitude of ε_S is made by the resonance term with $l = S$ in expression (183). However, the resultant expression remains highly complicated, and so first we consider the limiting case of $V_\parallel \gg \Delta_\parallel$, where we may put $T_\parallel \to 0$, which gives expression (123). Hence, in view of expressions (124) and (125), by neglecting the terms of order ω_p^2/ω_H^2 we obtain

$$\operatorname{Im}\frac{1}{\varepsilon(\mathbf{k},\omega)} = -\pi \operatorname{sign}(\omega)\,\omega_p \sin\alpha\,\delta\big(\omega^2 - \Omega^2(\mathbf{k})\big), \tag{195}$$

where $\Omega(\mathbf{k})$ is the dispersion law for 'fast' cyclotron waves:

$$\Omega(\mathbf{k}) = \sqrt{\omega_H^2 + \omega_p^2\sin^2\alpha} \approx \omega_H + \frac{\omega_p^2}{2\omega_H}\sin^2\alpha, \tag{196}$$

in the limiting case [11–13, 16, 17, 29]. From relations (193)–(195) we have

$$R(\mathbf{k}) = -\frac{\pi\omega_p^2\sin^2\alpha}{2\omega_H V_\parallel}\big[\delta(k_\parallel - p) + \delta(k_\parallel + p)\big], \quad p = \frac{\omega_p^2\sin^2\alpha}{2\omega_H V_\parallel}, \tag{197}$$

$$F_{cd} = -\frac{4\pi q^2 e^2 n}{m V_\perp V_\parallel}\,f_0, \tag{198}$$

where

$$f_0 = \int_0^\infty \frac{dk_\perp\,k_\perp^3}{(k_\perp^2 + p^2)^2}\,J_1^2(k_\perp r_{Hp}).$$

We take into consideration that $\alpha \approx \pi/2$ [see expressions (202)] and then obtain

$$p = \frac{\omega_p^2}{2\omega_H V_\parallel}, \qquad p r_{Hp} = \frac{\omega_p^2 V_\perp}{2\omega_H^2 V_\parallel}.$$

For a steady-state positron distribution in the typical case (see Section 9), namely

$$V_\perp \sim \Delta_\perp, \qquad V_\parallel \sim \Delta_\parallel, \tag{199}$$

in view of inequality (124) we therefore arrive at

$$p r_{Hp} \sim \frac{\Delta_\parallel}{\Delta_\perp} \ll 1. \tag{200}$$

Consequently, we may put $p = 0$ in formula (198), which gives $f_0 = 1/2$, and also

$$F_{cd} = -\frac{2\pi q^2 e^2 n}{m V_\perp V_\parallel}. \tag{201}$$

According to formulas (197) and (198), the main contribution to F_{cd} comes from the domain in the wave-vector space defined by the relations

$$k_\perp \sim \frac{1}{r_{Hp}} \sim \frac{1}{\bar r_H}, \quad |k_\parallel| = p \ll k_\perp, \quad \sin^2\alpha = \frac{k_\perp}{k} \approx 1. \tag{202}$$

This domain corresponds to the spatial domain in which the distance ρ_0 between the orbit centers of the interacting positron and electron and the distance z between them along the magnetic field satisfy the relations

$$\rho_0 \sim r_{Hp} \sim \bar r_H, \qquad |z| \sim \frac{1}{p} \sim d = \frac{2\omega_H V_\parallel}{\omega_p^2}. \tag{203}$$

Consequently, such electrons reside within a thin cylinder of radius $\sim r_{Hp}$ at a distance $\sim d \gg r_{Hp}$ from the positron. Since $d \gg \overline{R}$, these electrons are located in the collective interaction domain, which lends validity to result (201) obtained under the assumption of a continuous electron liquid. The physical interpretation of this cylindrical domain is discussed in Appendix A2. Hence, it is also clear that this domain contains a large number of electrons, which jointly make up the force $\mathbf{F}_{cd}$, thereby testifying to its collective nature.

The asymptotic expressions (189), (201) were first obtained by Artamonov and Derbenev [9] in a different way in comparison with that devised in this section, which justifies presenting at length the derivation of these formulas, which are significant to the subsequent discussion. Equally important is the elucidation of the limits of validity of these formulas and their substantiation given in our work, which is not done in due measure in Ref. [9]. In particular, formula (201) proves to be correct for velocities $V_\parallel \gg \Delta_\parallel \ln(T_\perp/T_\parallel)$ and, furthermore, for $V_\perp \gg \Delta_\perp$. Consequently, formula (201) is inapplicable in the most important domain (56), and therefore we will derive a more general formula below.

To analyze the positron moderation kinetics requires knowing the force F_{cd} for arbitrary values of $V_\parallel$ and $V_\perp$ (as well as F_{cM} and the diffusion coefficients $D_\parallel$ and $D_\perp$ in the velocity space). From the property

$$\varepsilon(\mathbf{k},\omega) = \varepsilon^*(-\mathbf{k},-\omega) \tag{204}$$

and formulas (193), (194) it follows that

$$F_{cd} = \frac{2\pi q^2\omega_H}{\pi V_\perp}\int_0^\infty \frac{dk_\perp}{k_\perp}\,J_1^2(k_\perp r_{Hp})A_0, \tag{205}$$

$$A_0 = \int_{-\infty}^\infty dk_\parallel\,\operatorname{Im}\left(\frac{1}{\varepsilon_1} - 1\right), \tag{206}$$

where it was taken into account that $k^2 \approx k_\perp^2$ according to expressions (202). In the expression $\varepsilon(\mathbf{k}, \omega_H + k_\parallel V_\parallel) \equiv \varepsilon_1$, we retain the term with a resonance denominator [see formula (183)]:

$$\varepsilon_1 \approx 1 - \frac{m\omega_{\rm p}^2}{k_\perp^2} \int {\rm d}^3 v\, J_1^2\left(\frac{k_\perp v_\perp}{\omega_{\rm H}}\right) f(\mathbf{v}) L\,, \tag{207}$$

$$L = \frac{k_\parallel v_\parallel / T_\parallel + \omega_{\rm H}/T_\perp}{k_\parallel (V_\parallel - v_\parallel) + {\rm i}0}\,. \tag{208}$$

We assume *a priori* that the main contribution to $\mathbf{F}_{\rm cd}$ (205) is made by the 'tube' (202), (203) (which will be borne out below) and notice that the first term in the numerator of formula (208) may be neglected in the case (66) of magnetized plasma of interest. Indeed, the estimates give

$$\frac{|k_\parallel v_\parallel| T_\perp}{T_\parallel \omega_{\rm H}} \sim \frac{\Delta_\parallel T_\perp}{d T_\parallel \omega_{\rm H}} \sim \frac{\bar r_{\rm H}^2}{R_\parallel^2} \ll 1\,. \tag{209}$$

Integration in expression (207) with respect to ${\rm d}^2 v_\perp$ [see formulas (36)] is performed by the formula

$$\int {\rm d}^2 v_\perp\, G(v_\perp)\, J_1^2\left(\frac{k_\perp v_\perp}{\omega_{\rm H}}\right) = \exp(-\beta^2)\, I_1(\beta^2)\,, \tag{210}$$

where $\beta = k_\perp \bar r_{\rm H}$. The subsequent integration with respect to the longitudinal electron velocity ${\rm d}v_\parallel$ gives

$$\varepsilon_1 = 1 - \frac{b_0}{k_\parallel}\,, \qquad b_0 = \frac{x - {\rm i}sY}{d}\, P_1(\beta^2)\,, \tag{211}$$

where the functions X and Y are defined by formulas (144), $s = {\rm sign}\, k_\parallel$, the parameter d is given by formula (203), and

$$P_1(x) = \frac{2\exp(-x)\, I_1(x)}{x}\,, \tag{212}$$

where I_1 is the modified first-order Bessel function of the first kind. We bring expression (206) to the form

$$A_0 = \int_0^\infty {\rm d}k_\parallel\ {\rm Im}\left(\frac{b_0}{k_\parallel - b_0} + \frac{b_0^*}{k_\parallel + b_0^*}\right).$$

For the upper limit we must take $k_{\parallel\,\max} = 1/\bar r_{\rm H}$:

$$A_0 = {\rm Im}\left[b_0 \ln\left(1 - \frac{1}{b_0 \bar r_{\rm H}}\right) - b_0^* \ln\left(1 + \frac{1}{b_0^* \bar r_{\rm H}}\right)\right]. \tag{213}$$

With a logarithmic accuracy [$\sim 1/\ln(T_\perp/T_\parallel)$], we may put $P_1(\beta^2) \to 1$ in the arguments of the logarithms in expression (213) and obtain

$$F_{\rm cd} = -\frac{4q^2 e^2 n}{m V_\perp V_\parallel}\, \Phi\left(\frac{V_\perp}{\Delta_\perp}\right) G_1\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right),$$

$$G_1(x) = X(x)(\varphi_1 - \varphi_2) + Y(x)(\kappa_1 + \kappa_2)\,,$$

$$\varphi_1 = \frac{\pi}{2} + \arctan \frac{X - (X^2 + Y^2)\delta/x}{Y}\,,$$

$$\varphi_2 = \arctan \frac{Y}{(X^2 + Y^2)\delta Y/x + X}\,, \tag{214}$$

$$\kappa_1 = \frac{1}{2} \ln\left\{\left[1 - \frac{Xx}{(X^2 + Y^2)\delta}\right]^2 + \left[\frac{Xx}{(X^2 + Y^2)\delta}\right]^2\right\},$$

$$\kappa_2 = \frac{1}{2} \ln\left\{\left[1 + \frac{Xx}{(X^2 + Y^2)\delta}\right]^2 + \left[\frac{Xx}{(X^2 + Y^2)\delta}\right]^2\right\},$$

$$\delta = \frac{\omega_{\rm p}^2 \Delta_\perp}{2\sqrt{2}\,\omega_{\rm H}^2 \Delta_\parallel}\,, \qquad \Phi(z) = 2\int_0^\infty \frac{{\rm d}\beta}{\beta^3}\, J_1^2(z\beta) \exp(-\beta^2)\, I_1(\beta^2)\,.$$

It should be noted that $\delta \sim \Delta_\parallel/\Delta_\perp \ll 1$. In the limiting cases, the functions G_1 and Φ from formulas (214) reduce to the expressions

$$G_1 \approx \pi\,, \quad V_\parallel \gg \Delta_\parallel\,, \quad G_1 \approx \sqrt{\frac{\pi}{2}}\, \frac{V_\parallel}{\Delta_\parallel} \ln\left(\frac{T_\perp}{T_\parallel}\right), \quad V_\parallel \ll \Delta_\parallel\,,$$

$$\Phi \approx \frac{1}{2}\,, \quad V_\perp \gg \Delta_\perp\,, \quad \Phi \approx \frac{1}{4}\left(\frac{V_\perp}{\Delta_\perp}\right)^2, \quad V_\perp \ll \Delta_\perp\,.$$

8. Estimate of the contribution from binary positron – electron collisions

Let us discuss here binary collisions (172) which are described by the equations

$$\ddot{\mathbf{r}}_1 = \omega_{\rm H}(\dot{\mathbf{r}}_1 \times \mathbf{h}) - \frac{e^2}{m}\frac{\mathbf{r}}{r^3}\,, \qquad \ddot{\mathbf{r}}_2 = -\omega_{\rm H}(\dot{\mathbf{r}}_2 \times \mathbf{h}) + \frac{e^2}{m}\frac{\mathbf{r}}{r^3}\,, \tag{215}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the respective radius vectors of the positron and the electron, and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = (x, y, z)$.

We separate off the center-of-mass motion:

$$\mathbf{v} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2\,, \qquad \mathbf{V}_{\rm c} = \frac{\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2}{2}\,.$$

From equations (215) there follow the equations

$$\mathbf{V}_{\rm c}(t) = \frac{1}{2}\,\omega_{\rm H}(\mathbf{r} \times \mathbf{h}) + \mathbf{V}_0\,, \tag{216}$$

where $\mathbf{V}_0 = \mathbf{V}_{0\perp} + \mathbf{V}_{0\parallel} = {\rm const}$ is the constant of integration, and

$$\dot{\mathbf{v}} = 2\omega_{\rm H}(\mathbf{V}_{\rm c} \times \mathbf{h}) - \frac{2e^2}{m}\frac{\mathbf{r}}{r^3}\,. \tag{217}$$

The constant $\mathbf{V}_{0\parallel}$ corresponds to the center-of-mass uniform motion along the magnetic field. To find the $\mathbf{V}_{0\perp}$ constant, we will consider the motion of the particles during the time $t \to -\infty$, i.e., prior to collision, when they travel along two approaching Larmor orbits still unperturbed by the Coulomb interaction of the particles. Let the center of the positron orbit initially travel along the z-axis, and the center of the electron orbit travel along the straight line parallel to the z-axis and intersecting the x-axis at a point $x = \rho_0$. By writing out in explicit form the time dependence of the coordinates of the particles, it is easy to obtain the relations

$$V_{{\rm c}x} = \frac{1}{2}\,\omega_{\rm H}\, y\,, \qquad V_{{\rm c}y} = -\frac{1}{2}\,\omega_{\rm H}(x + \rho_0)\,. \tag{218}$$

Hence, and from Eqn (216), it follows that

$$V_{0x} = 0\,, \quad V_{0y} = -\frac{1}{2}\,\omega_{\rm H}\rho_0\,, \quad \mathbf{V}_0 = V_{0\parallel}\mathbf{h} + \frac{1}{2}\,\omega_{\rm H}(\boldsymbol{\rho}_0 \times \mathbf{h})\,,$$

which, in view of Eqns (216) and (217), gives

$$\mathbf{V}_{\rm c} = \frac{1}{2}\,\omega_{\rm H}(\mathbf{R}_\perp \times \mathbf{h}) + V_{0\parallel}\mathbf{h}\,, \tag{219}$$

$$\ddot{\mathbf{R}}_\perp + \left(\omega_{\rm H}^2 + \frac{2e^2}{m r^3}\right)\mathbf{R}_\perp = \frac{2e^2}{m}\frac{\boldsymbol{\rho}_0}{r^3}\,, \tag{220}$$

$$r = \sqrt{z^2 + r_\perp^2} = \sqrt{z^2 + (\boldsymbol{\rho}_0 - \mathbf{R}_\perp)^2}\,, \qquad \ddot z = -\frac{2e^2}{m}\frac{z}{r^3}\,, \tag{221}$$

where a two-dimensional radius vector $\mathbf{R}_\perp = \mathbf{r}_\perp + \boldsymbol{\rho}_0$ is introduced in place of the radius vector $\mathbf{r}_\perp = (x, y)$. Equation (217) is preferably written in the form (220) for distant binary collisions satisfying condition (173): $R_\perp \ll \rho_0$, $r = (z^2 + \rho_0^2)^{1/2}$. In the collision of Larmor circles, the Coulomb particle interaction potential $|U| \sim e^2/\bar{r}_{\rm H}$ may be treated as a weak perturbation, because for a typical collision [see expressions (77)] the following estimate holds true:

$$\frac{|U|}{E_1} \sim \frac{|U|}{T_\perp} \sim \frac{R_{\rm T}}{\bar{r}_{\rm H}} \sim 1.5 \times 10^{-3} \,. \tag{222}$$

Typically, such a collision lasts for a time $\tau_{\rm c} \sim \rho_0/\Delta_\parallel \sim \bar{r}_{\rm H}/\Delta_\parallel$ and constitutes an adiabatic process:

$$\xi_0 = \omega_{\rm H}\tau_{\rm c} \sim \frac{\omega_{\rm H}\bar{r}_{\rm H}}{\Delta_\parallel} \sim \frac{\Delta_\perp}{\Delta_\parallel} \gg 1 \,. \tag{223}$$

The attraction of circles (221) decreases $\tau_{\rm c}$ by only a factor of ~ 1.5, and therefore the conclusion about adiabaticity, due to which the Larmor circles can hardly exchange their internal energy, remains valid:

$$\frac{\Delta E_\perp}{E_\perp} \sim \exp(-\xi_0) \ll 1 \,. \tag{224}$$

It is pertinent to note that the oscillator model considered in Appendix A2 [see formula (A2.20)] turns out to be incorrect in this respect. According to formula (A2.21), in a system of two interacting oscillators there are two close eigenfrequencies $\omega_{1,2}$, which gives rise to the beating effect.

The beating is due to the resonance exchange of internal energy occurring in a time $\sim 1/|\omega_1 - \omega_2|$. The two frequencies correspond to two types of natural oscillations of the oscillators: in-phase ($\xi_1 = \xi_2$) and antiphased ($\xi_1 = -\xi_2$) oscillations. Two opposite charges rotating in the magnetic field oscillate relative to one another only in antiphase, and like charges oscillate in phase. That is why, as is clear from Eqn (220), in these cases there is only one eigenmode and, consequently, the beating effect reflecting the resonance energy exchange is absent.

So, both in the case (173) and in the case (174), the transverse-energy exchange in a typical binary collision is negligible, which is also true of the longitudinal-energy exchange [see the reasoning following formula (133)]. The energy exchange in binary collisions (172) takes place only in the case (174) as a result of infrequent hard collisions with $\rho \sim R_{\rm T}$ ('Thomson' collisions), which make a contribution (12) to the friction force. In this case, due to the magnetization of binary collisions (66), $\bar{r}_{\rm H}$ must be taken instead of $\bar{R}$ in formula (11): $\Lambda_{\rm b} = \ln(\bar{r}_{\rm H}/R_{\rm T})$. This conclusion is consistent with the data of numerical computations [7]. Owing to the smallness of $R_{\rm T}$ and the high transverse particle energy $\varepsilon_\perp \sim T_\perp$, this contribution is small in comparison with the collective contribution (179) and can be neglected:

$$\frac{F_{\rm c\perp}}{F_{\rm b\perp}} \sim \sqrt{\frac{T_\perp}{T_\parallel}} \,, \qquad \frac{F_{\rm c\parallel}}{F_{\rm b\parallel}} \sim \frac{T_\perp}{T_\parallel} \,.$$

To summarize this section, we add that the positron trajectory in the domain of typical velocities $V_\parallel \sim \Delta_\parallel \ll V_\perp \sim \Delta_\perp$ (see Section 3) is a short-pitch helix:

$$l = \frac{2\pi V_\parallel}{\omega_{\rm H}} \sim \frac{2\pi\Delta_\parallel}{\omega_{\rm H}} \,, \qquad \frac{l}{r_{\rm Hp}} \sim \frac{2\pi\Delta_\parallel}{\omega_{\rm H} r_{\rm Hp}} \sim \frac{\Delta_\parallel}{\Delta_\perp} \ll 1 \,.$$

Since $l/R_{\rm T} \sim 100$, despite the shortness of the pitch l of helix the particles can experience only one Thomson encounter in the collision of Larmor circles (174), and therefore multiple collisions of close particles in the helical trajectories cannot occur.

9. Qualitative discussion of the stationary positron velocity distribution $\Phi(\mathbf{V})$

According to the conclusions arrived at Refs [8, 9, 21, 30] (see Sections 7 and 8), the principal mechanisms responsible for the loss of longitudinal positron energy $E_\parallel$ are the Cherenkov emission of plasmons and the Landau damping of the perturbations generated in the plasma by a moving positron. As a result of the inverse process — plasmon absorption — positrons will gain energy. The combined effect of these two oppositely directed processes leads to the establishment of stationary distribution over longitudinal velocities $V_\parallel$. It is worth noting here that the positron residence time in the storage ring is long enough for the stationary velocity distribution to set in. By contrast, the initial distribution (36) of electrons, which execute only about a half turn in the ring, is hardly changed.

The plasmon frequency is estimated as $\omega_{\rm p} \sim 4 \times 10^8$ s^{-1}, its energy $\hbar\omega_{\rm p} \sim 3 \times 10^{-3}$ K $\ll E_\parallel$, and therefore classical mechanics applies to the description of these processes [the evaluation of the direct process (see Section 7) was made in its framework]. Like everywhere above, the plasma is assumed to be ideal. For the velocity distribution (36), plasma ideality criterion (4) is written out as inequality (46). Owing to criterion (46), the number of electrons in the Debye sphere is also large:

$$N_{\rm D} \sim nR_\parallel^3 \sim \xi_\parallel^{3/2} \gg 1 \,. \tag{225}$$

The equilibrium, thermal, plasma oscillation spectrum is established in a characteristic time $\omega_{\rm p}^{-1} \sim 10$ ns, which is short in comparison with the time of cooperative motion of electrons and positrons in a cooler (~ 400 ns for the LEPTA experiment). The plasma oscillations are characterized by the time, length, and volume scales defined, respectively, as

$$\tau_{\rm p} \sim \frac{1}{\omega_{\rm p}} \,, \qquad l \sim R_\parallel \,, \qquad V_{\rm p} \sim R_\parallel^3 \,. \tag{226}$$

In ideal plasma satisfying inequality (46), the particles are virtually free to move, and therefore particle number fluctuations in a volume $V_{\rm p}$ are of the order of $\delta N \sim \sqrt{N_{\rm D}}$. Consequently, the characteristic fluctuation amplitudes of the electric potential and field strength in the plasma are given by

$$\delta\varphi \sim \frac{e\sqrt{N_{\rm D}}}{R_\parallel} \sim \sqrt{\frac{T_\parallel}{R_\parallel}} \,, \qquad E \sim \frac{\delta\varphi}{R_\parallel} \sim \frac{T_\parallel^{1/2}}{R_\parallel^{3/2}} \,. \tag{227}$$

We initially consider particle M of large mass (1). For an estimate we assume that the electric field (227) exists for a time $\sim \tau_{\rm p}$, following which it changes direction — and everything is repeated. Under the action of this field, particle M executes Brownian motion in the velocity space with the diffusion coefficient

$$D \sim \frac{(\Delta\mathbf{V})^2}{\tau_{\rm p}} \sim \frac{ne^4}{M^2\Delta_\parallel} \,, \tag{228}$$

where $|\Delta\mathbf{V}| \sim eE\tau_{\rm p}/M$ is the particle velocity variation in one fluctuation 'period'. The contribution to D from binary collisions with electrons is smaller than estimate (228) by a factor of $\Delta_\perp/\Delta_\parallel$ owing to the smallness of the Thomson radius (75). If initially at $t = 0$ the particle is at rest, its velocity will increase with time by the law

$$\overline{V^2} \sim Dt \tag{229}$$

characteristic for Brownian movement until its increase is suppressed by the force of friction. According to Refs [9, 30], whose results were outlined in Section 7, for $V < \Delta_\parallel$ and magnetized electrons this force is estimated as

$$F \sim -\frac{ne^4 V}{m\Delta_\parallel^3} \,. \tag{230}$$

According to expressions (229), (230), the average particle energy varies according to the law

$$\frac{\mathrm{d}\bar{E}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M\overline{V^2}}{2}\right) \sim \langle MD\rangle + \langle\mathbf{FV}\rangle \sim \frac{ne^4}{M\Delta_\parallel} - \frac{ne^4}{m\Delta_\parallel^3}\,\overline{V^2}\,. \tag{231}$$

From this results the conclusion that the steady-state $(\mathrm{d}\bar{E}/\mathrm{d}t = 0)$ energy is given to an order of magnitude by [2, 3]

$$\bar{E} \sim \bar{E}_\parallel \sim \bar{E}_\perp \sim T_\parallel \,. \tag{232}$$

Estimate (232) applies to both degrees of freedom in the heavy-particle case (1). For positrons [case (2)], formula (230) is valid only for $F_\parallel$, and therefore

$$\bar{E}_\parallel \sim T_\parallel \,. \tag{233}$$

According to the results presented in Section 7, for positrons moving with transverse velocities $V_\perp < \Delta_\perp$, one finds

$$F_\perp \sim \frac{ne^4 V_\perp}{m\Delta_\perp^2\Delta_\parallel}\,, \qquad V_\perp < \Delta_\perp\,, \qquad |V_\parallel| \sim \Delta_\parallel\,. \tag{234}$$

To estimate the coefficient $D_\perp$ of diffusion in the velocity component $\mathbf{V}_\perp$, we take advantage of the physical notion outlined in Appendix A2. Owing to screening, a positron interacts only with the electrons located in the 'tube'. Let us consider one of these electrons. Its field is also screened, and at the location of the positron it is defined by formula (A2.15) in which in lieu of $\exp(-\mathrm{i}\omega_{\rm H}t)$ we must borrow $\exp(-\mathrm{i}\omega t)$, thereby taking into account their frequency mismatch

$$\Delta\omega \sim \frac{\omega_{\rm p}^2}{\omega_{\rm H}} \tag{235}$$

[see expressions (A2.17), (A2.18), and (A2.26)]. Therefore, the total electric field strength at the location of the positron is expressed in the form

$$E_\perp(t) \sim \sum_a E_a \cos(\omega_a t + \varphi_a)\,, \tag{236}$$

where $E_a \sim E_0 = e/(d_0 r_{\rm H})$ [see expression (A2.15)], and φ_a is the phase of the Larmor gyration of the ath electron, the total number of electrons in the tube being

$$N_{\rm t} = nV_{\rm t} \sim \xi_\parallel^{3/2}\frac{\omega_{\rm H}}{\omega_{\rm p}} \gg 1\,. \tag{237}$$

Owing to the frequency mismatch for ω_a [see expression (235)], it may be assumed that the electric field $E_\perp(t)$ consists of independent trains of length $\tau_{\rm c} \sim 1/\Delta\omega$. From the oscillator model [see formula (A2.3)] it follows that the positron velocity $V_\perp$ changes in a time $\tau_{\rm c}$ by

$$\Delta V_\perp \sim \frac{eE_0\tau_{\rm c}\sum_a\cos\varphi_a}{m}\,,$$

and thus

$$D_\perp \sim \frac{\Delta V_\perp^2}{\tau_{\rm c}} \sim \frac{e^2E_0^2\tau_{\rm c}}{m^2}\,N_{\rm t} \sim \frac{ne^4}{m^2\Delta_\parallel}\,. \tag{238}$$

The square of electric field amplitude E_0^2 is found from the estimate

$$\overline{E_\perp^2} \sim E_0^2 N_{\rm t} \sim \frac{ne^2}{d_0}\,. \tag{239}$$

As would be expected, owing to the randomness of the phases φ_a the electrons located in the tube make independent contributions to the positron diffusion coefficient $D_\perp$, as well as to $\overline{E_\perp^2}$. From expressions (234) and (238) and the equation

$$\frac{\mathrm{d}\bar{E}_\perp}{\mathrm{d}t} = \langle 2mD_\perp\rangle + \langle\mathbf{F}_\perp\mathbf{V}_\perp\rangle$$

we obtain

$$\overline{V_\perp^2} \sim \Delta_\perp^2\,, \qquad \bar{E}_\perp \sim T_\perp\,. \tag{240}$$

Estimates (233) and (240) were first given (without proof) in Ref. [9].

In concluding this section we explain the relation $\bar{E}_\perp \sim T_\parallel$ [see estimates (232)] valid for heavy particles. The transverse electron degree of freedom will be considered as a heater with a temperature $T_\perp$, and the longitudinal one as a cooler with a temperature $T_\parallel$. If the friction force is 'disengaged', the ions will heat up to a temperature $T_\perp$ in a time $\tau_{\rm h}$ defined by relation (229): $D\tau_{\rm h} \sim T_\perp/M$. Hence, one finds

$$\tau_{\rm h} \sim \frac{Mm\Delta_\parallel\Delta_\perp^2}{ne^4}\,. \tag{241}$$

If diffusion is 'disengaged' $(D \to 0)$, the ion velocity according to relation (230) will diminish by the law

$$M\,\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{ne^4V}{m\Delta_\parallel^3}\,,$$

whence we obtain an estimate for the time of ion cooling to the temperature $T_\parallel$:

$$\tau_{\rm cool} \sim \frac{Mm\Delta_\parallel^3}{ne^4}\,. \tag{242}$$

In the general case, the average energy $E_\perp$ of the ion motion perpendicular to the magnetic field obeys the equation

$$\frac{\mathrm{d}\bar{E}_\perp}{\mathrm{d}t} = \lambda_{\rm h}(T_\perp - \bar{E}_\perp) - \lambda_{\rm cool}(\bar{E}_\perp - T_\parallel)\,, \tag{243}$$

where $\lambda_{\rm h} = 1/\tau_{\rm h}$, and $\lambda_{\rm cool} = 1/\tau_{\rm cool}$. In the stationary case defined as $d\overline{E}_\perp/dt = 0$, it is apparent that

$$\overline{E}_\perp = \frac{\lambda_{\rm h} T_\perp + \lambda_{\rm cool} T_\parallel}{\lambda_{\rm h} + \lambda_{\rm cool}}\,,$$

$$\frac{\Delta\overline{E}_\perp}{T_\parallel} = \frac{\lambda_{\rm h}}{\lambda_{\rm h} + \lambda_{\rm cool}}\,\frac{T_\perp - T_\parallel}{T_\parallel}\,,$$

$$\Delta\overline{E}_\perp = \overline{E}_\perp - T_\parallel\,.$$

Since $\lambda_{\rm h}/\lambda_{\rm cool} \sim T_\parallel/T_\perp \ll 1$, then $\overline{E}_\perp \sim T_\parallel$. In other words, because the ion cooling proceeds much faster than the heating, the transverse ion temperature $\overline{E}_\perp$ turns out to be close to the cooler temperature.

10. Stationary positron distribution function

In the stationary state, the velocity distribution function is $\Phi(\mathbf{V}, t) = \Phi(\mathbf{V})$, and therefore

$$\mathrm{St}\,(\Phi) = 0\,. \qquad (244)$$

According to estimates (233) and (240), one has

$$\overline{E}_\parallel \ll \overline{E}_\perp\,, \qquad (245)$$

and so from expressions (159) and (244) there follows the relation

$$|j_\parallel| \sim \frac{\Delta_\parallel}{\Delta_\perp}|j_\perp| \ll |j_\perp|\,. \qquad (246)$$

Owing to inequalities (245) and (246), the approximate solution of Eqn (244) is written in the form

$$\Phi(\mathbf{V}) = G(V_\perp)\,g_0(V_\parallel; V_\perp)\,, \qquad (247)$$

where the longitudinal velocity distribution function $g_0(V_\parallel; V_\perp)$ of the positrons satisfies the equation

$$A_\parallel(V_\parallel, V_\perp)\,g_0(V_\parallel; V_\perp) - D_\parallel(V_\parallel, V_\perp)\,\frac{\partial g_0(V_\parallel; V_\perp)}{\partial V_\parallel} = 0 \qquad (248)$$

and the normalization condition

$$\int_{-\infty}^{\infty} g_0(V_\parallel; V_\perp)\,dV_\parallel = 1\,.$$

The semicolon in the argument of g_0 emphasizes the circumstance that $|\partial g_0/\partial V_\parallel| \gg |\partial g_0/\partial V_\perp|$, which underlies the emergence of the solution to Eqn (244) in the form of expression (247).

In essence, approximation (247) corresponds to the method of separation of fast and slow variables (by way of example, see the problem of an atom in an ultrastrong magnetic field [31, 32]). The equation for the positron distribution function $G(V_\perp)$ in transverse velocities is obtained by integrating equation (244) with respect to $dV_\parallel$:

$$A_\perp(V_\perp)\,G(V_\perp) - D_\perp(V_\perp)\,\frac{dG(V_\perp)}{dV_\perp} = 0\,, \qquad (249)$$

where

$$A_\perp(V_\perp) = \int_{-\infty}^{\infty} dV_\parallel\,g_0(V_\parallel; V_\perp) \times \left[A_\perp(V_\parallel, V_\perp) + \frac{\partial D_\perp(V_\parallel, V_\perp)}{\partial V_\perp}\right] - \frac{dD_\perp(V_\perp)}{dV_\perp}\,,$$

$$D_\perp(V_\perp) = \int_{-\infty}^{\infty} dV_\parallel\,g_0(V_\parallel; V_\perp)\,D_\perp(V_\parallel, V_\perp)\,.$$

The solution of equation (248) assumes the form

$$g_0(V_\parallel; V_\perp) = B(V_\perp)\exp\left[\int_0^{V_\parallel} \frac{A_\parallel(V'_\parallel, V_\perp)}{D_\parallel(V'_\parallel, V_\perp)}\,dV'_\parallel\right] = B(V_\perp)\exp\left[\int_0^{V_\parallel} \frac{F_\parallel^{(1)}(V'_\parallel, V_\perp)}{MD_\parallel(V'_\parallel, V_\perp)}\,dV'_\parallel\right], \qquad (250)$$

where $B(V_\perp)$ is the normalization constant which depends on $V_\perp$ as a parameter. Expression (250) takes into account relation (162). Similarly, from equation (249) we have

$$G(V_\perp) = C_0\exp\left[\int_0^{V_\perp} dV'_\perp\,\frac{A_\perp(V'_\perp)}{D_\perp(V'_\perp)}\right] = C_0\exp\left[\int_0^{V_\perp} dV'_\perp\,\frac{F_\perp^{(1)}(V'_\perp)}{MD_\perp(V'_\perp)}\right], \qquad (251)$$

where C_0 is another normalization constant defined by the condition

$$\int G(V_\perp)\,d^2V_\perp = 2\pi\int_0^{\infty} G(V_\perp)V_\perp\,dV_\perp = 1\,.$$

In the anisotropic case (44), simple manipulations with the use of formulas (162), (181), and (183), as well as (A2.17), (A2.19), and (A5.16), lead to the conclusion that the coefficients $A_\parallel$, $D_\parallel$, $A_\perp$, $D_\perp$ obey the relationship similar to relationship (146):

$$A_\parallel(V_\parallel, V_\perp) = -\frac{MV_\parallel}{T_\parallel}\,D_\parallel(V_\parallel, V_\perp)\,. \qquad (252)$$

Similarly proven is the formula

$$A_\perp(V_\perp) = -\frac{MV_\perp}{T_\perp}\,D_\perp(V_\perp)\,. \qquad (253)$$

From formulas (250)–(253) we conclude that the stationary positron velocity distribution ($M = m$) coincides with the electron one and is given by the expression [30]

$$g_0(V_\parallel; V_\perp) = g(V_\parallel) = \left(\frac{M}{2\pi T_\parallel}\right)^{1/2}\exp\left(-\frac{MV_\parallel^2}{2T_\parallel}\right), \quad G(V_\perp) = G_{\rm e}(V_\perp) = \frac{M}{2\pi T_\perp}\exp\left(-\frac{MV_\perp^2}{2T_\perp}\right). \qquad (254)$$

This conclusion comes as no surprise. Indeed, according to the results presented in Sections 6 and 8, the energy transfer from transversely moving electrons to longitudinally traveling positrons (LT transitions) may be neglected. [The LT-transition suppression effect in a strong magnetic field was discovered at the MOSOL (abbr. from 'model of solenoid') facility at the G I Budker Institute of Nuclear Physics of the

Siberian Branch of the Russian Academy of Sciences in an experiment reported by Kudelainen et al. [20].] In this approximation, the energy exchange takes place only in longitudinal – longitudinal (LL) and transverse – transverse (TT) transitions. In each of these degrees of freedom, the electrons have an equilibrium Maxwellian distribution with the corresponding temperature, which results in the establishment of the same distribution for the positrons.

The case of heavy particles (see the end of Section 9) is quite a different matter. The trajectories of these particles are practically rectilinear and the electrons are 'magnetized', i.e., may travel only along the magnetic field, like beads threaded on a needle. This picture is correct for magnetized electrons satisfying inequality (70), when the characteristic distance between the interacting particle M and electron exceeds the Larmor radius of the electron orbit. In this case, the ions exchange their longitudinal and transverse energies with the longitudinal electron motion, but do not exchange them with the transverse one. From these considerations it is clear that the stationary velocity distribution for heavy particles (1) is, in accordance with estimate (232), of the form

$$\mathrm{d}W = \left(\frac{M}{2\pi T_{\parallel}}\right)^{3/2} \exp\left[-\frac{M(V_{\parallel}^2 + V_{\perp}^2)}{2T_{\parallel}}\right] \mathrm{d}^3 V . \qquad (255)$$

To summarize this section, let us estimate the uncertainty of the resultant expressions (254). To the first approximation in the small coefficient D_{LT}, instead of equation (248) we obtain

$$A_{\parallel}(V_{\parallel}, V_{\perp})\, g_0(V_{\parallel}; V_{\perp}) - D_{\parallel}(V_{\parallel}, V_{\perp})\, \frac{\partial g_0(V_{\parallel}; V_{\perp})}{\partial V_{\parallel}} - D_{\mathrm{LT}}\left(-\frac{MV_{\perp}}{T_{\perp}}\right) g_0(V_{\parallel}; V_{\perp}) = 0 . \qquad (256)$$

Owing to adiabaticity criterion (224) of collisions with respect to the transverse motion, coefficients D_{LT} for magnetized electrons (66) with an oblate velocity distribution (36) are exponentially small in distant collisions (see also Ref. [33]). The main contribution to them is made by collisions with impact parameters $\rho \leqslant \bar{r}_{\mathrm{H}}$, when binary collisions are possible, in which the adiabatic invariant may not be conserved. According to the drift approximation [34] (see also Ref. [15, § 60]), one has

$$D_{\mathrm{LT}} \approx \frac{2\pi n q^2 e^2 \Lambda_1 V_{\perp} V_{\parallel}}{M^2 (V_{\perp}^2 + V_{\parallel}^2)^{3/2}} , \qquad \Lambda_1 = \ln \frac{\bar{R}}{\bar{r}_{\mathrm{H}}} . \qquad (257)$$

From Eqns (256) and (252) we obtain the relation for the longitudinal positron temperature $T_{\mathrm{p}\parallel}$:

$$\frac{1}{T_{\mathrm{p}\parallel}} - \frac{1}{T_{\parallel}} \sim -\frac{D_{\mathrm{LT}}}{D_{\parallel}\sqrt{T_{\perp} T_{\parallel}}} .$$

Hence, and from formulas (257) and (A5.6) (see Appendix A5), we arrive at an estimate

$$\frac{T_{\mathrm{p}\parallel} - T_{\parallel}}{T_{\parallel}} \sim \left(\frac{T_{\parallel}}{T_{\perp}}\right)^{3/2} , \qquad (257')$$

which signifies that the result (254) is rather accurate. The coefficients D_{LT} and $D_{\parallel}$ are determined by pair collisions and collective processes, respectively, and the expression (257) provides in essence a quantitative description of how the part played by binary collisions decreases in importance with increasing temperature anisotropy $T_{\perp}/T_{\parallel}$.

The resultant expression (254) applies to the magnetized electron beam (66) with an oblate distribution (36), (44), the case most interesting from the practical standpoint. Upon lowering the magnetic field intensity, condition (66) is violated and in this case the stationary positron velocity distribution is substantially different from the electron one.

11. Transverse cooling of electrons. Qualitative analysis

As pointed out earlier, heavy particles under condition (70) are cooled to a low temperature $T_{\parallel}$ — the lower of two temperatures, $T_{\parallel}$ and $T_{\perp}$, characterizing the electron beam. This is like nature's present to accelerator scientists. For positrons, the attainable longitudinal temperature $T_{\parallel}$ is low, but the transverse temperature $T_{\perp}$ turns out to be high.

For several applications it is desirable to raise the degree of positron cooling, which reduces to the task of lowering the transverse electron temperature. The obvious idea is to make electrons dump the transverse energy in the course of cyclotron radiation emission. An isolated electron radiates very slowly: the characteristic time amounts to $\tau \sim 200(H_0/H)^2$ [s], where $H_0 = 1000$ G. To enhance the radiation it would be reasonable to employ collective processes — the maser cyclotron instability of the electron plasma. Many reefs are encountered in the analysis of this practically important process, and so we will discuss it using several approaches.

In the development of the cyclotron instability by the Gaponov-Grekhov mechanism [35, 36] there emerges a grouping of electrons in phase (angle φ) of their Larmor gyration, which occurs owing to the dependence of the Larmor frequency on the electron velocity due to relativistic effects:

$$\Omega_{\mathrm{H}} = \frac{\omega_{\mathrm{H}}}{\gamma} = \omega_{\mathrm{H}}\sqrt{1 - \frac{v^2}{c^2}} \approx \omega_{\mathrm{H}} - \frac{\omega_{\mathrm{H}} v^2}{2c^2} \approx \omega_{\mathrm{H}} - \frac{\omega_{\mathrm{H}} v_{\perp}^2}{2c^2} \equiv \Omega_{\mathrm{H}}(v_{\perp}) . \qquad (258)$$

The rotating dipole moment of the resultant bunches is proportional to the number of particles in them, while the cyclotron radiation intensity, which is proportional to the square of this number, rises sharply. The process is concluded by superradiation in which the transverse energy is carried away by photons [37 – 41]. It is significant that for coolers $\omega_{\mathrm{H}}/\omega_{\mathrm{p}} \sim 100 \gg 1$, and therefore the transverse energy is really radiated and not transferred to plasma oscillations, as suggested by the theory of cyclotron waves [29] (a critical viewpoint is set forth by Parkhomchuk [42]). Let us estimate the time τ of transverse cooling of electrons and the increment $\gamma = 1/\tau$ of the maser instability.

The gyrating electrons generate in the plasma a circularly polarized cyclotron extraordinary wave with a gyrating electric field, following the electrons, with components $\mathbf{E}_{\mathrm{d}} = (E_{\mathrm{d}} \cos\omega t, E_{\mathrm{d}} \sin\omega t, 0)$ (the z-axis is aligned with the magnetic field $\mathbf{H}$). The field amplitude E_{d} is assumed to be constant for the present. We consider an individual electron with the velocity $\mathbf{v} = (v\cos\varphi, v\sin\varphi, 0)$, where φ is the angle between $\mathbf{v}$ and the x-axis [due to inequality (44) we neglect the longitudinal electron velocity]. The angle between $\mathbf{v}$ and $\mathbf{E}$ is

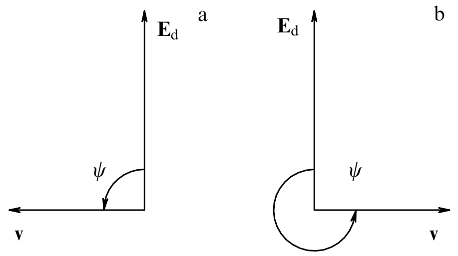

Figure 4. Possible mutual orientations of the gyrating electric field $\mathbf{E}_d$ and electron velocity $\mathbf{v}$ in stationary states. The magnetic field is pointed 'towards us'. In the laboratory system of coordinates, the electron and the field rotate counterclockwise.

$\psi = \varphi - \omega t$. When $\psi \neq \pi/2, 3\pi/2$, the electric field does work

$$W_E = -eE_d v \cos\psi \quad (259)$$

in a unit time on the electron by changing its velocity and, owing to dependence (258), its Larmor frequency. This changes the ψ angle. The angle under consideration remains invariable only in two cases: for $\psi = \pi/2$ (Fig. 4a), and $\psi = 3\pi/2$ (Fig. 4b). In these cases, the electron, apart from the specific angle between vectors **v** and **E**, should have a quite specific 'resonance' velocity v_0, for which it gyrates with the same frequency as the electric field of the cyclotron wave generated: $\Omega_H(v_0) = \omega$.

We consider the $\psi = \pi/2$ case. For $v > v_0$, according to formula (258) one finds $\Omega_H(v) < \omega$, and therefore the ψ angle for this electron decreases, and hence the deceleration regime sets in, because for $0 < \psi < \pi/2$ the power is negative: $W_E < 0$. As a result, the electron velocity lowers to the resonance value v_0. Reasoning of this kind leads to a conclusion that the state $\psi = \pi/2$ is stable and the state $\psi = 3\pi/2$ is unstable. Consequently, the electrons group with time into the states close to $\psi = \pi/2$. This is just the phasing according to Gaponov-Grekhov.

Let the electrons be uniformly distributed over angle ψ and possess equal velocities $v = v_0 \sim \Delta_\perp$ at $t = 0$. In a time τ they will acquire the characteristic spread $|\Delta v| \sim (eE_d/m)\tau$ in velocity and, due to formula (258), in the Larmor frequency:

$$|\Delta\Omega_H| \sim \frac{\omega_H v_0 |\Delta v|}{c^2} \sim \frac{\omega_H v_0 e E_d \tau}{mc^2} \,. \quad (260)$$

The phasing, which is the limiting, slowest stage of the entire maser action [see estimate (271)], will occur in a time $\tau \sim 1/|\Delta\Omega_H|$, which gives, in combination with expression (260), the desired estimate

$$\gamma = \frac{1}{\tau} \sim \frac{\Delta_\perp}{c}\,\omega_p \,. \quad (261)$$

In passing from expression (260) to expression (261) we took into account the relation $\mathbf{E}_d \approx -4\pi\mathbf{P}$, where $\mathbf{P}$ is the polarization vector, which reduces to the following estimate

$$E_d \sim 4\pi n e r_H \sim 4\pi n e \frac{\Delta_\perp}{\omega_H} \sim 0.5\ \text{V cm}^{-1} \,. \quad (262)$$

The resultant expression (261) is of significance to subsequent estimates, and so we will derive it in a rigorous manner, with exact dimensionless factors. The dispersion relations $\omega = \omega(\mathbf{k})$ for plasma waves are derived from the Vlasov–Maxwell equations. For a nonzero magnetic field, these dispersion relations are highly complicated, and therefore we will consider the simplest case in which the wave propagates along the magnetic field: $k_\perp = 0$, $k_\| = k$. For the transverse cyclotron wave of frequency ω we have the equation (see, for instance, Refs [16, § 2.7], [29, 43]):

$$\omega^2 - k^2c^2 \approx \frac{\omega_p^2}{2}\int d^3v\, v_\perp \left[\frac{\omega - kv_z}{kv_z - \omega + \omega_H}\,\frac{\partial f(\mathbf{v})}{\partial v_\perp} + \frac{k^2 v_\perp f(\mathbf{v})}{(kv_z - \omega + \omega_H)^2}\right] . \quad (263)$$

It should be noted that the left-hand side of this equation already contains the speed of light, which permits putting $\Omega_H \approx \omega_H$ on the right-hand side. From expressions (36) and (263) in the limit $\Delta_\| \to 0$ [see inequality (44)] we obtain

$$\omega^2 - k^2c^2 = \frac{\omega_p^2\omega}{\omega - \omega_H} + \frac{\omega_p^2 k^2 \Delta_\perp^2}{(\omega - \omega_H)^2} \,. \quad (264)$$

We *a priori* assume that

$$\frac{\omega^2}{k^2c^2} \ll 1 \,. \quad (265)$$

This enables simplifying equation (264) and obtaining the desired result which confirms initial assumption (265):

$$\omega(k) = \omega'(k) + i\gamma(k)\,, \qquad \omega'(k) = \omega_H - \frac{\omega_p^2\omega_H}{2k^2c^2}\,,$$
$$\gamma(k) = \frac{\Delta_\perp}{c}\,\omega_p\sqrt{1 - \left(\frac{k_{cr}}{k}\right)^4}\,, \qquad k_{cr} = \sqrt{\frac{\omega_p\omega_H}{2c\Delta_\perp}} \,. \quad (266)$$

Therefore, in the anisotropic electron plasma (44) existing in coolers masing emerges with the emission of waves with wave vectors $k > k_{cr} \sim 7\ \text{cm}^{-1}$ (the corresponding wavelengths $\lambda < \lambda_{cr} \sim 1$ cm). The transverse electron temperature decreases by the law

$$T_\perp = T_\perp^0(1 + \gamma_0 t)^{-2} = T_\perp^0\left(1 + \frac{x}{L}\right)^{-2} ,$$

where $\gamma_0 = (\omega_p/c)(T_\perp^0/m)^{1/2}$ is the increment for some initial temperature $T_\perp^0$, $L = u/\gamma_0$ is the characteristic cooling length, t is the time, and x is the distance traversed by the electron beam.

Let us specify the conditions under which the transverse cooling of electrons becomes possible. It is possible to select the most advantageous values of electron velocity and magnetic field strength H. Under variations in the magnetic field intensity H and the electron energy $\varepsilon = mu^2/2$ (in the laboratory system of coordinates), the following quantities are conserved:

$$nuS = \text{const}\,, \qquad SH = \text{const}\,, \qquad \frac{\Delta_\perp^2}{H} = \text{const}\,,$$

where S is the beam cross section. Hence, we obtain the following operational formula:

$$L = L_0\left(\frac{\varepsilon}{\varepsilon_0}\right)^{3/4} , \qquad L_0 = 50\ \text{m}\,, \qquad \varepsilon_0 = 10\ \text{keV}\,. \quad (267)$$

We note that the value of L is independent of the magnetic field. For $\varepsilon = 0.1$ keV, from formula (267) we find that $L = 1.5$ m. One can see that the possibility for the transverse cooling of the electron beam exists only very early in its acceleration.

Now let us ascertain that, as was assumed in this section, phasing is the limiting stage of the entire masing process. Consider an electron cloud with diameter $D \sim 1$ cm and particle number density $n \sim 10^9$ cm^{-3}, which is embedded in the magnetic field of intensity $H \sim 1000$ G. The length of the cyclotron waves radiated is $\lambda = 2\pi c/\omega_{\rm H} \sim 10$ cm. Since

$$D \ll \lambda \,, \tag{268}$$

the time lag in the bunch domain may be neglected. In a time $\tau \sim c/\Delta_{\perp}\omega_{\rm H}$, phasing, i.e., the grouping of electrons in the cyclotron gyration angle φ, occurs. There forms a collective gyrating dipole moment $d \sim er_{\rm H}N$, where $N \sim nD^3$ is the number of particles in the bunch. The particle cloud radiates the energy [44]

$$I = \frac{2(\ddot{\mathbf{d}})^2}{3c^3} \sim \frac{\omega_{\rm H}^4 e^2 r_{\rm H}^2 N^2}{c^2} \sim \frac{\omega_{\rm H}^4 e^2 r_{\rm H}^2 n^2 D^6}{c^2} \tag{269}$$

per unit time. The cloud energy $E_{\rm e} \sim m\Delta_{\perp}^2 nD^3$ decreases by the law

$$\frac{{\rm d}E_{\rm e}}{{\rm d}t} = -I(t) \,, \tag{270}$$

and therefore the radiation time is $\tau_{\rm R} \sim c^3/\omega_{\rm p}^2\omega_{\rm H}^2 D^3$ for a fully completed phasing. For the time ratio we obtain

$$\frac{\tau}{\tau_{\rm R}} \sim \frac{\omega_{\rm p}\omega_{\rm H}^2 D^3}{c^2\Delta_{\perp}} \sim 100 \gg 1 \,. \tag{271}$$

Hence, the characteristic time of $\Delta_{\perp}$ decreasing, i.e., the sought-for time of the transverse cooling of electrons, as assumed above, is equal to τ. The kinetic energy of the particles lowers because each of them experiences the force $\mathbf{F} = (-e)\mathbf{E}_{\rm c}$. The collective field $\mathbf{E}_{\rm c} = \mathbf{E}_{\rm d} + \mathbf{E}_{\rm R}$, where $\mathbf{E}_{\rm d}$ is the total average field (262) produced by the gyrating dipoles, and $\mathbf{E}_{\rm R} = 2\dddot{\mathbf{d}}/3c^3$ is the field strength of radiative friction [43]. According to formula (269), the electric field does the work

$$\langle W \rangle = \langle \dot{\mathbf{r}}(-e)\mathbf{E}_{\rm c} \rangle \sim \frac{1}{N}\langle \dot{\mathbf{d}}\mathbf{E}_{\rm R} \rangle \sim \frac{1}{c^3 N}\langle \dot{\mathbf{d}}\dddot{\mathbf{d}} \rangle = -\frac{1}{c^3 N}\langle (\ddot{\mathbf{d}})^2 \rangle \sim \frac{I}{N}$$

in a unit time on each particle (for a bunch grouped in phase, the work of the dipole field $\mathbf{E}_{\rm d}$ is equal to zero).

The cyclotron instability of electrons with the anisotropic velocity distribution considered in Refs [38, 39] is well known (see, for instance, Ref. [16]). This instability also develops in an unbounded plasma. Its emergence does not necessitate resonators (although, as assumed in Refs [38, 39], their use may speed up the process). This is the first, and longest, part of transverse electron cooling, which is linear in a collective electric field strength. The cooling is concluded with a nonlinear stage — the loss of transverse electron energy in the superradiation regime [40, 41].

In the preparation of this review, the author found a paper by Golubev and Shalashov [45] concerned with essentially the same process — the pulsed maser radiation of an electron bunch with an anisotropic distribution. In this process, like in the process considered in this section, the kinetic electron energy is also converted to the energy of electromagnetic waves emitted by the plasma.

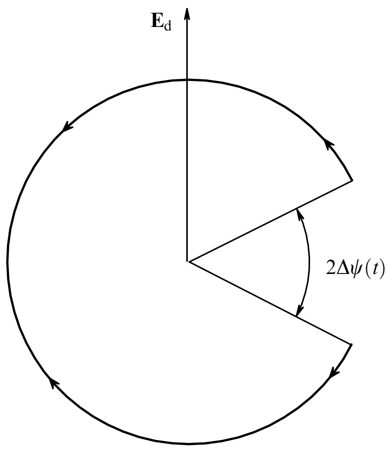

Figure 5. Qualitative form of the electron velocity distribution in directions at an arbitrary instant of time.

Proceeding from the picture outlined above we shall investigate the kinetics of transverse electron cooling in greater detail. There is a strongly anisotropic electron velocity distribution (36), (44). Initially, the electrons are uniformly distributed in phase ψ, and the directions of their velocities may therefore be depicted by points uniformly distributed over a circumference. To make an estimate, the subsequently emerging phasing will be represented as 'turning' the points-filled circumference into a sector $-\pi/2 + \Delta\psi(t) < \psi < 3\pi/2 - \Delta\psi(t)$ (Fig. 5). According to relation (271), it would suffice to assume that $\Delta\psi(t) \ll 1$. At $t = 0$, a nonuniform distribution emerges due to a typical fluctuation, whereat the numbers of particles on the left and on the right (see Fig. 5) differ by $\sim \sqrt{N}$, which corresponds to

$$\Delta\psi(0) \sim \frac{1}{\sqrt{N}} \,, \qquad E_{\rm d}(0) \sim \frac{ne\Delta_{\perp}(0)}{\sqrt{N}} \,, \tag{272}$$

where $\Delta_{\perp}(0) = \Delta_{\perp}$ is the initial transverse velocity spread. In accordance with formula (259), the moduli of particle velocities in the upper semiplane in Fig. 5 by the instant of time t decrease by $\Delta v(t)$, and those in the lower semiplane increase by the same value. From formulas (259) and (260) we obtain approximate equations

$$\frac{{\rm d}\Delta v(t)}{{\rm d}t} \sim \frac{eE_{\rm d}(t)}{m} \,, \qquad \frac{{\rm d}\Delta\psi(t)}{{\rm d}t} \sim \frac{\omega_{\rm H}}{c^2}\,\Delta_{\perp}(t)\,\Delta v(t) \,. \tag{273}$$

To these we should add the equations

$$E_{\rm d}(t) \sim 4\pi ne\,\frac{\Delta_{\perp}(t)}{\omega_{\rm H}}\,\Delta\psi(t) \,, \qquad \frac{{\rm d}\Delta_{\perp}(t)}{{\rm d}t} \sim -\frac{Ne^2\omega_{\rm H}^2}{mc^3}\,\Delta_{\perp}(t)\big[\Delta\psi(t)\big]^3 \,, \tag{274}$$

the former following from an estimate for the polarization vector $P \sim ne\,\Delta\psi(t)\Delta_{\perp}(t)/\omega_{\rm H}$, and the latter from equation (270). We solve Eqns (273) and (274) to obtain

$$\Delta_{\perp}(t) \approx \frac{\Delta_{\perp}}{1 + A\exp(2\gamma t)} \,, \qquad I(t) \approx Nm\Delta_{\perp}^2\gamma\,\frac{A\exp(2\gamma t)}{\big(1 + A\exp(2\gamma t)\big)^2} \,, \tag{275}$$

where $A = \omega_p c/(\omega_H \Delta_\perp \sqrt{N}) \sim 0.01$. From formulas (275) follows the conclusion that the characteristic transverse cooling time is estimated as

$$t_{cool} \sim \frac{1}{2\gamma} \Lambda_{cool} \sim 4 \ \mu s\,, \qquad \Lambda_{cool} = \ln \frac{1}{A}\,. \tag{276}$$

The radiation intensity $I(t)$ initially increases exponentially $[\sim \exp(2\gamma t)]$ to attain at $t_{max} = \ln(1/2A)/2\gamma$ its peak value $I_{max} = (4/27) Nm\Delta_\perp^2 \gamma$, and then decreases exponentially $[\sim \exp(-2\gamma t)]$. One can see that I_{max} is smaller than the quantity (269) in magnitude. The reason lies with the incompleteness of phasing owing to the fast emission of cyclotron waves ($\Delta\psi \leqslant 0.3$).

The logarithm in formula (276) describes the well-known effect of superradiation pulse delay [37] and emerges owing to the weakness of the initial fluctuation electric field (272) which brings about the electron phasing. Hence, it is clear that the transverse electron cooling time t_{cool} would shorten by a factor of order $\Lambda_{cool} \sim 5$ should we generate a circularly polarized 'seed' electromagnetic wave with the electric vector $\mathbf{E}$ comparable in magnitude with the vector of the intrinsic bunch field (262):

$$E \leqslant E_d \sim 0.5 \ \text{V cm}^{-1}\,. \tag{277}$$

This issue invites additional study with recourse to the methods set forth in Section 12.

Ikegami [46–48] came up with the idea of cooling electrons by an electromagnetic wave with $E \sim 100$ V cm^{-1}. In this case, the intrinsic bunch field $\mathbf{E}_d$ (262), and even more so the radiative friction field $\mathbf{E}_R$, may be neglected. The motion of each electron is then described by the Hamiltonian [43]

$$H(\mathbf{r}, \mathbf{p}, t) = c\sqrt{\left(\mathbf{p} + \frac{e}{c}\,\mathbf{A}(\mathbf{r}, t)\right)^2 + m^2c^2} + e\varphi(\mathbf{r}, t)\,,$$

where $\mathbf{A}(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$ are the vector and electric potentials of the wave. That is why the Liouville theorem applies here and the phase volume of the bunch is conserved (generally speaking, six-dimensional, but four-dimensional in the case (44) of interest). Therefore, the cooling of electrons by the method suggested in Refs [46–48] does not take place [49, 50]. In essence, cyclotron resonance occurs, with the result that the electron velocity attains a value of $v = eE_0/\left[m(\omega_H - \Omega)\right]$. In this steady-state regime, the electron velocity is perpendicular to the electric field and therefore no longer increases. When the radiative friction is taken into account, the angle between the velocity and the electric vector is slightly different from the right angle, and the field therefore does work on the particles, being completely converted to the energy of the radiated electromagnetic waves. Preliminary numerical calculations (see Section 12) suggest that the wave field strength E should satisfy condition (277) for cooling the electrons, because the cooling slows down when the field strength is increased further.

It is noteworthy that the maser mechanism of transverse electron cooling discussed in this section is well known in plasma physics, where it is now referred to as the 'Bernstein mode instability'. This mechanism was supposedly first discussed by Sagdeev and Shafranov [51]. At the present time, this mechanism is believed to be the principal radiation mechanism of some astrophysical objects [13, 52–54].

12. Role of the dipole–dipole interaction of electron Larmor orbits in the transverse cooling effect

In Section 11 we showed that no cooling of electrons occurs under an external electric field alone. In this case, the electrons participate simultaneously in two motions: external and internal (Fig. 6). The external motion constitutes the collective circular motion following the electric field. The internal one is depicted by the closed curve with arrows in Fig. 6. On switching off the field, the external motion adiabatically vanishes, and the internal one remains.

The cooling of electrons represents the lowering of the energy of internal motion. If the interelectron interaction is neglected, electron motions become independent. External motion produces a collective gyrating dipole moment. The energy losses due to the emerging radiation are made up by the work of the external field. The dipole moment corresponding to the internal motion is equal to zero, and therefore the energy of this motion is invariable and no cooling occurs.

The conclusion about the occurrence of cooling through a spontaneous mechanism considered in Section 11 was relied on the self-consistent field ($\mathbf{E}_d$) approximation (SCFA) (262). The action of this field is approximately equivalent to the action of all other particles on a given particle. The field is the same for each of them, and therefore within the SCFA we are once again dealing with Hamiltonian motion, and hence there is no cooling. In this approximation there once again emerge independent external and internal electron motions, which do not exchange energy.

The correct answer consists in the following: the cooling does take place and all SCFA-based estimates of its characteristics given in Section 11 are valid. We are led to this conclusion when account is taken of fluctuation effects which are ignored in the SCFA. The fluctuation electric field $E_{fl} \sim ner_H$ is primarily produced by the particles nearest to a given electron (we take an interest in the resonance field produced by the gyrating dipole moments of the Larmor orbits). Since $E_{fl} \sim E_d$, there is no small parameter allowing us to separate the external and internal motions. The excessive energy is rapidly 'pumped' from the internal motion into the external one and is carried away by electromagnetic waves. Below, we shall show that this process is so efficient that the bulk of electron energy is emitted by superradiation (SR) in the form of a short pulse.

Therefore, the conclusions drawn by proceeding from the SCFA should be treated with caution. Among the SCFA-

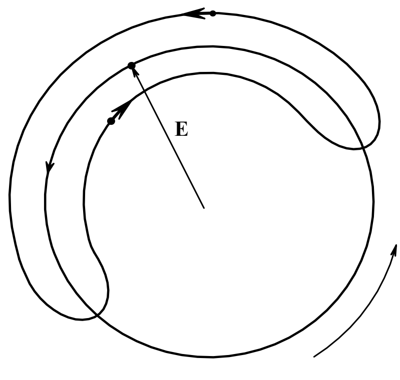

Figure 6. Motion of bunch particles in the co-moving inertial frame of reference on imposition of gyrating electric field **E**.

related artifacts is, for instance, the phenomenon of incomplete energy emission in a system of charged oscillators, which was pointed out in Ref. [37]. The author of Ref. [37] also suggested that the metastable states of this system may be disrupted by the dipole – dipole interaction of the oscillators, which was borne out by numerical calculations [40, 41].

Because of the absence of a small parameter, the SCFA-reliant predictions may be verified only by way of numerical computation, the data of which are discussed in this section. This computation amply, without approximations, takes into account the dipole – dipole interaction of Larmor electron orbits. We emphasize that this interaction is characterized by the spatial scale of the order of $\overline{R} = n^{-1/3}$ and is therefore the small-scale factor which is, according to Van der Meer [49], required for particle cooling.

To discuss the part played by the dipole – dipole interaction we consider a system of charged nonlinear oscillators of size (268), which is completely similar to the electron bunch embedded in the magnetic field and considered in Section 11 [37, 40, 41].

Let particles of charge e and mass m be located at points with the coordinates $\mathbf{r}_a + \boldsymbol{\xi}_a$ $(a = 1, 2, \ldots, N)$ at the ends of springs with the coefficients of stiffness k, fixed at points $\mathbf{r}_a$ at which there are also the compensating charges $-e$. The equation of oscillator motion has the form [44]

$$\ddot{\boldsymbol{\xi}}_a + \omega_0^2(1 + \gamma\xi_a^2)\boldsymbol{\xi}_a = -\frac{2e^2\omega_0^2}{3mc^3}\sum_b \dot{\boldsymbol{\xi}}_b + \frac{e^2}{m}\sum_{b\neq a}\boldsymbol{\nabla}_a \times \left(\boldsymbol{\nabla}_a \times \frac{\boldsymbol{\xi}_b(t_{ab})}{r_{ab}^3}\right). \quad (278)$$

Here, $\boldsymbol{\nabla}_a = \partial/\partial\mathbf{r}_a$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$, $t_{ab} = t - r_{ab}/c$ is the time lag, $\omega_0 = \sqrt{k/m}$ is the oscillator eigenfrequency, and γ is the nonlinearity parameter. Upon substitution of $\boldsymbol{\xi}_a = b\left[\mathbf{F}_a(t)\exp(-\mathrm{i}\omega t) + \mathbf{F}_a^*(t)\exp(\mathrm{i}\omega t)\right]$, where b is the characteristic initial amplitude of oscillator oscillations, the system of equations (278) takes on the form

$$\dot{\mathbf{F}}_a + \mathrm{i}\delta\left(|\mathbf{F}_a|^2 - 1\right)\mathbf{F}_a = \mathrm{i}\beta\sum_{b\neq a}\boldsymbol{\nabla}_a \times \left[\boldsymbol{\nabla}_a\frac{\exp(\mathrm{i}\mathbf{k}\mathbf{r}_{ab})}{r_{ab}} \times \mathbf{F}_b(t)\right] - \frac{1}{2}\,\beta_0\sum_b \mathbf{F}_b\,. \quad (279)$$

We omitted in equation (279) the second derivatives of the functions $\mathbf{F}_a(t)$ which vary slowly in comparison with the exponents $\exp(\pm\mathrm{i}\omega t)$, and selected the frequency $\omega = \omega_0 + \delta$, where $\delta = 3\gamma\omega_0 b^2/2$. It should be noted that $\delta < 0$ corresponds to the case of particles gyrating in a magnetic field. For a system of small size, we obtain from expression (279) the following equation

$$\dot{\mathbf{F}}_a + \mathrm{i}\delta\left(|\mathbf{F}_a|^2 - 1\right)\mathbf{F}_a = \mathrm{i}\beta\sum_{b\neq a}\frac{3\mathbf{n}_{ab}(\mathbf{n}_{ab}\mathbf{F}_b) - \mathbf{F}_b}{r_{ab}^3} - \frac{1}{2}\,\beta_0\sum_b \mathbf{F}_b\,, \quad (280)$$

where $\beta = e^2/(2m\omega_0)$, $\beta_0 = 2e^2\omega_0^2/(3mc^3)$, and $\mathbf{n}_{ab} = \mathbf{r}_{ab}/r_{ab}$. The first term on the right-hand side of equation (280) describes the dipole – dipole interaction of the oscillators, and the second term describes the radiative friction.

Following Il'inskii and Maslova [55], we consider one-dimensional oscillators, i.e., we assume that the dipoles oscillate along the **x**-axis and the vectors $\mathbf{F}_a$ are therefore parallel to this axis: $\mathbf{F}_a = F_a\mathbf{i}$, $\mathbf{i} = (1, 0, 0)$. At an arbitrary instant of time t we have $F_a(t) = \rho_a(t)\exp\left(\mathrm{i}\varphi_a(t)\right)$. The atomic dipole moments are expressed as $\mathbf{d}_a(t) = e\boldsymbol{\xi}_a(t) = eb\mathbf{i}\rho_a\cos(\omega t + \varphi_a)$. The radiation intensity averaged over the fast dipole oscillations is given by the expression

$$I(t) = e^2\omega^4 b^2\sum_{a,b}|F_a||F_b|\,\frac{\cos(\varphi_a - \varphi_b)}{3c^3}\,. \quad (281)$$

The problem therefore reduces to the numerical solution of a set of equations (280) for a system of N oscillators which are located in a body of arbitrary shape, with randomly specified initial phases $\varphi_a(0)$.

To explain the results of calculations, we consider the complex plane $(x, y) = \left(\mathrm{Re}(F), \mathrm{Im}(F)\right)$. The state of the oscillator system is depicted by N points with the coordinates (x_a, y_a). The movements of the points in this plane obey the equations that follow from equation (280):

$$\mathbf{v}_a = \boldsymbol{\omega}(\boldsymbol{\rho}_a) \times \boldsymbol{\rho}_a + \mathbf{f} + \sum_b \mathbf{d}(\boldsymbol{\rho}_a, \boldsymbol{\rho}_b; \mathbf{r}_a, \mathbf{r}_b)\,. \quad (282)$$

Here, $\boldsymbol{\rho}_a = \left(\mathrm{Re}(F_a), \mathrm{Im}(F_a), 0\right)$, $\mathbf{v}_a = \dot{\boldsymbol{\rho}}_a$, $\mathbf{f} = -\beta_0\sum_a \boldsymbol{\rho}_a/2$, $\boldsymbol{\omega}(\boldsymbol{\rho}) = \left(0, 0, -\delta(\rho^2 - 1)\right)$, and $\mathbf{d}(\boldsymbol{\rho}_a, \boldsymbol{\rho}_b; \mathbf{r}_a, \mathbf{r}_b)$ is the term which takes into account the dipole – dipole interaction (not given due to its awkwardness). Vector $-\mathbf{f}$ is proportional to the total system's dipole moment $\mathbf{d} = eb\sum_a \boldsymbol{\rho}_a/2$. Initially, the points are uniformly scattered over the circumference of unit radius $\rho = 1$, and therefore $\omega(\rho_a) = 0$ at $t = 0$.

For definiteness, we assume that $\gamma > 0$, i.e., the oscillation frequency rises with increasing amplitude (the picture of the emerging phenomena is independent of the sign of parameter γ). Owing to distribution density fluctuations of the initial oscillator phases $\varphi_a(0)$, the initial magnitude of vector **f** is nonzero. At $t = 0$, from equations (282) it follows that $\mathrm{d}\mathbf{d}/\mathrm{d}t = -\mathbf{d}/\tau_{\mathrm{SR}}$, where $\tau_{\mathrm{SR}} = 1/(N\beta_0)$ is the characteristic SR pulse length in the oscillator model being considered [37].

According to expression (282), the system begins moving with a velocity **f** in the direction opposite to the dipole moment **d**. In a time $\sim\tau_{\mathrm{SR}}$, the system of points as a whole shifts by a distance of order $\mathbf{d}(0)/(Ne)$ (Fig. 7a). As a result of this motion, half of the points find themselves in the domain $\rho > 1$, where $\omega > 0$, and the other half in the $\rho < 1$ domain, where $\omega < 0$. The points located outside the unit circumference will start rotating clockwise, and those located inside it, counterclockwise. This rotation with different angular velocities results in the formation of a bunch of points (Fig. 7b). The onset of bunch formation corresponds to the first SR peak (Fig. 8), during which the bulk of energy stored in the oscillators is radiated. This takes place for $t \sim 10\tau_{\mathrm{SR}}$, which is consistent with the time delay $t_0 \sim \tau_{\mathrm{SR}}\ln N$ for two-level atoms [56]. In a half turn, the above groups of points find themselves on opposite sides of the origin (Fig. 7c). At this instant of time, the system's dipole moment reaches its minimum, which corresponds to the first minimum in Fig. 8. Subsequently, everything is repeated, resulting in SR pulses far lower in intensity. Repetitive pulses are characteristic of the SR in small-sized classical systems [55]. They do not occur in quantum systems consisting of two-level atoms [56].

In Refs [40, 41] it was also shown that suppression of SR occurs with increasing oscillator concentration n, as suggested in Ref. [37]. This is attributable to the chaotic nature of the electric fields of the dipoles, which disrupts the phasing. As applied to the cooling of electrons, SR suppression takes place at practically unattainable number densities $n \geqslant 10^{12}\ \mathrm{cm}^{-3}$.

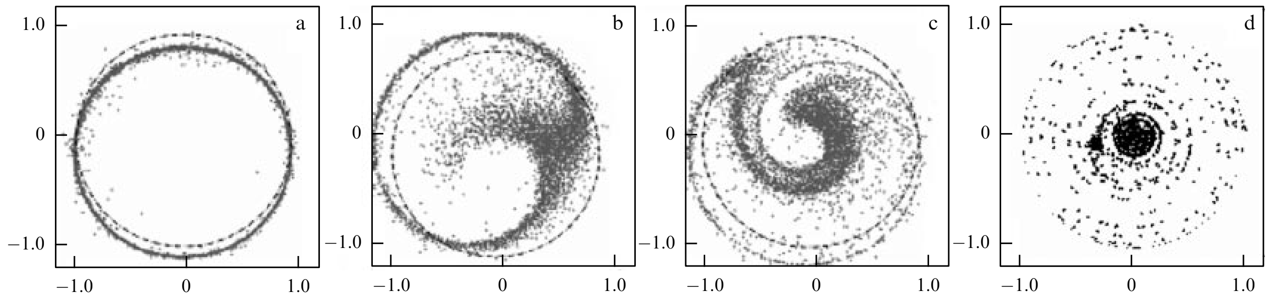

Figure 7. Temporal evolution of the phase distribution of oscillators (depicted by points). Plotted on the axes are the oscillator coordinates $\boldsymbol{\rho}_a = \left(\mathrm{Re}\,(F_a),\ \mathrm{Im}\,(F_a)\right)$ [see formula (282)]. The dashed circumference has a unit radius. The number of oscillators is $N = 5 \times 10^3$.

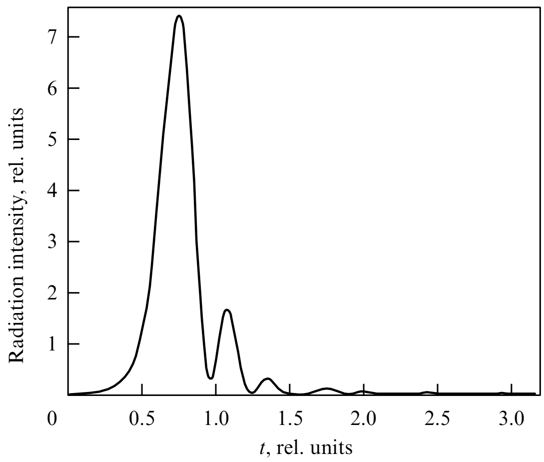

Figure 8. Time dependence of the radiation intensity for $N = 5 \times 10^3$.

The problem of radiation by a system of charged oscillators is easy to solve in the framework of the SCFA [37]. The answer is as follows: for a low anisotropy of the initial phases $\varphi_a(0)$, only a small fraction of the energy is radiated in a time $\sim \tau_{\mathrm{SR}}$, following which the radiation terminates. Therefore, the electric field fluctuations produced by the nearest dipoles play a crucial role in converting the entire energy stored by the oscillators to radiation. It is pertinent to note that these fluctuations are, in terms customary to plasma physics scientists, nothing more nor less than the energy transfer by cyclotron waves inside the plasma.

Another point of view on the transverse cooling effect can also be useful. According to formula (266), the amplitudes of cyclotron waves exponentially increase with frequencies ω next to ω_{H}, but slightly below this value. The source of energy for these waves is the excess kinetic energy of transverse motion of electrons. According to equation (2.150) given in monograph [59], these waves have negative energies. Due to the nonequilibrium state of electrons, the energy of plasma without waves is larger than its energy with due regard for the wave energy.

13. Conclusions

At the beginning of this review we discussed at a qualitative level the principal effects concerned with the electron cooling of heavy particle beams, which is required for the understanding of a more intricate problem of the deceleration of light particles — positrons. Special emphasis was placed on the electron magnetization effect, i.e., the increase in particle-decelerating friction force with increasing magnetic field intensity. It was explained that, apart from the 'freezing' of the transverse electron motion, the decisive role in the magnetization effect is played by the 'oblateness' of electron velocity distribution (36) and the new properties of charge screening in the electron cloud inherent in this oblateness: with increasing magnetic field intensity, the Debye cloud shrinks, with a consequential rise in the strength of decelerating electric field which acts on the charge from the plasma. This is a collective effect in which a large number of electrons participate in the case of an ideal plasma.

In the review it was shown that the part played by collective effects in electron cooling increases in importance: (a) with a decrease in mass of the particle being moderated; (b) with a strengthening of the electron cloud anisotropy $T_\perp/T_\parallel$, and (c) with increasing magnetic field intensity. Hence, it follows that the behavior of positrons in coolers is entirely determined by collective effects. This difference from the case of heavy-particle deceleration invites the development of a new theory, the groundwork for which has been laid by recent papers [8, 9, 21, 30, 38–41]. These papers were discussed in our review, as were new results obtained on the basis of the generalization of the results of these works.

The particle energy dissipation takes place within the Debye sphere by the Landau damping mechanism which essentially consists in the following. The domain in which the electron motion is perturbed by the particle field moves through the plasma together with the particle. The formation of this electron cloud is explicable on the basis of the notion that the particle emits and absorbs virtual plasmons which do not escape to infinity. A part of the energy of this collective motion is converted into the energy of single-particle chaotic thermal motion, which generates the friction force acting on the particle.

When a particle travels along a magnetic field with a velocity exceeding the characteristic longitudinal electron velocity $\Delta_\parallel = \sqrt{T_\parallel/m}$, the principal mechanism through which the particle loses energy is the emission of real plasmons which propagate through distances far greater than the Debye radius, i.e., the Cherenkov radiation of plasmons. These mechanisms play the leading part in the loss of the longitudinal energy of positrons in a cloud of magnetized electrons, and that is why they have been studied in detail in our review.

Another contribution to the force of friction comes from binary collisions of a particle with electrons which practically execute one-dimensional motion in a strong magnetic field along the lines of force. According to Refs [2, 3], in the motion of a heavy particle at some angle to the magnetic lines of force there emerges asymmetry in the dependence on the time during which the particle experiences a force from an electron, which determines the electron-to-particle momentum transfer. This asymmetry is related to the electron acceleration under the action of the electric field of the particle. This effect was comprehensively considered in Section 3 on the basis of the solution of the equations of motion by the method of successive approximations in the strength of the particle's electric field.

For positrons, this mechanism turns out to be insignificant, like binary collisions with electrons in general. For them, the main part is played by collective effects: the energy of longitudinal positron motion is lost and gained as a result of the emission and absorption of real and virtual plasmons, while the energy of transverse motion is transferred in similar processes with the excitation of cyclotron waves, because positrons gyrate at resonance with electrons in the magnetic field.

The review is concerned with a detailed qualitative and quantitative study of these processes. We revealed a peculiar character of screening of the variable part of the electromagnetic field of a positron gyrating in an electron cloud. This field is confined in a 'tube' (203) extended along the magnetic field. Outside of the tube, the field decays by a power law. The bulk of the energy of transverse positron motion is absorbed in the tube through the Landau damping mechanism. A small fraction of the energy 'flows out' of the tube and reaches the quasistatic region (A2.10), where diverging cyclotron waves are generated due to the action of retardation effects. This effect of screening of the gyrating charge field explains, in our view, the suppression of cyclotron radiation at the fundamental harmonic in a dense plasma, reported in Ref. [28].

In our review we calculated the components of the dynamic force of friction and the coefficients of longitudinal and transverse positron diffusion in the velocity space, which are required for the analysis of positron moderation kinetics in coolers. It was shown that, as a result of moderation, a stationary positron velocity distribution function is gained in the storage ring, which is practically coincident with the electron one.

The review concluded with an analysis of the feasibility of lowering the energy of transverse electron motion in the cooler, which would allow decreasing the momentum spread in a positron beam.

Also given and analyzed in our review were new results that are of significance in planning positron beam experiments characteristic for the LEPTA facility [6].

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14. Appendices

A1. Cherenkov deceleration as the excitation of plasma oscillators

Let us calculate the force $\mathbf{F}_{\mathrm{r}}$ from formula (132) employing the method borrowed from Ginzburg [17], which will enable gaining a deeper insight into its nature. We assume that particle M travels along the lines of force of a uniform magnetic field $\mathbf{H}$ (the z-axis), and electrons are magnetized. In accordance with expression (25), one finds

$$F_{\mathrm{r}} = -q \left.\frac{\partial \varphi_{\mathrm{p}}}{\partial z}\right|_{z=Vt} . \tag{A1.1}$$

The linearized equations of motion of the magnetized electrons are of the form

$$\begin{gathered} m\frac{\partial v}{\partial t} = -qe\frac{z-Vt}{r^3} + e\frac{\partial \varphi_{\mathrm{p}}}{\partial z}, \qquad \frac{\partial n_1}{\partial t} + n\frac{\partial v}{\partial z} = 0, \\ n_{\mathrm{e}} = n + n_1, \qquad |n_1| \ll n, \\ \Delta\varphi_{\mathrm{p}} = 4\pi e n_1, \qquad r = \sqrt{\rho^2 + (z-Vt)^2}, \\ \rho = \sqrt{x^2+y^2}, \qquad \dot{z} = v. \end{gathered} \tag{A1.2}$$

Here, $\mathbf{r} = (x, y, z)$ are the electron coordinates; the electrons are treated as a liquid moving along the magnetic lines of force with a velocity $\mathbf{v}(\mathbf{r}, t)$. In Eqn (A1.2) we move to the Fourier components in spatial coordinates, for instance, as in the case of velocity:

$$v(\mathbf{r}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, v(\mathbf{k}, t) \exp(\mathrm{i}\mathbf{kr}) .$$

We eliminate v and n_1 to obtain the equation for $\varphi_{\mathrm{p}}(\mathbf{k}, t)$:

$$\frac{\partial^2 \varphi_{\mathrm{p}}}{\partial t^2} + \omega^2(\mathbf{k})\varphi_{\mathrm{p}} = -q\,\frac{4\pi k_{\parallel}^2 \omega_{\mathrm{p}}^2}{k^4} \exp(-\mathrm{i}k_{\parallel} Vt), \tag{A1.3}$$

where $\omega(\mathbf{k})$ is the plasmon frequency defined by formula (127). Equation (A1.3) has a clear physical meaning: a moving particle M excites 'plasma oscillators' [if the plasma is treated as a mechanical system with eigenfrequencies $\omega(\mathbf{k})$].

Let us consider the case in which the particle is immobile when the time $t < 0$, and moves with a velocity V for $t > 0$. This signifies that Eqn (A1.3) is to be solved subject to the initial conditions $\varphi_{\mathrm{p}} = 0$ and $\partial\varphi_{\mathrm{p}}/\partial t = 0$ at $t = 0$. The solution assumes the form

$$\varphi_{\mathrm{p}}(\mathbf{k}, t) = \frac{2\pi k_{\parallel}^2 \omega_{\mathrm{p}}^2}{\omega_k k^4}\, Q(\mathbf{k}, t), \tag{A1.4}$$

where $\omega_k \equiv \omega(\mathbf{k})$,

$$Q(\mathbf{k}, t) = \frac{\exp(-\mathrm{i}k_{\parallel} Vt) - \exp(-\mathrm{i}\omega_k t)}{\omega_k - k_{\parallel} V} + \frac{\exp(-\mathrm{i}k_{\parallel} Vt) - \exp(-\mathrm{i}\omega_k t)}{\omega_k + k_{\parallel} V} .$$

We are concerned with the stationary Debye cloud, i.e., the case when $t \gg \omega_{\mathrm{p}}^{-1}$. Then, expression (A1.4) may be rearranged with the use of the formula from Appendix A1 in Davydov's monograph [57]:

$$\lim_{t\to\infty} \frac{1 - \exp(-\mathrm{i}\alpha t)}{\alpha} = \frac{1}{\alpha - \mathrm{i}0} .$$

According to this formula, for $t \to \infty$ we have

$$\varphi_{\mathrm{p}}(\mathbf{k}, t) \to \frac{2\pi k_{\parallel}^2 \omega_{\mathrm{p}}^2}{\omega_k k^4}\left(\frac{1}{\omega_k - k_{\parallel} V - \mathrm{i}0} + \frac{1}{\omega_k + k_{\parallel} V + \mathrm{i}0}\right) \times \exp(-\mathrm{i}k_{\parallel} Vt) .$$

Hence, and from expression (A1.1), it follows that

$$F_{\mathrm{r}} = -q \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, k_{\parallel} \varphi_{\mathrm{p}}(\mathbf{k}, t) \exp\big(\mathrm{i}\mathbf{k}\mathbf{R}(t)\big)$$
$$= \frac{q^2\omega_{\mathrm{p}}^2}{4\pi} \int \mathrm{d}^3 k\, \frac{k_{\parallel}^3}{\omega_k k^4} \left[\delta(\omega_k - k_{\parallel} V) - \delta(\omega_k + k_{\parallel} V)\right], \tag{A1.5}$$

where $\mathbf{R}(t) = \mathbf{V}t$. Integration employing the formula

$$\delta\left(\omega_{\mathrm{p}} \frac{|k_{\parallel}|}{k} - \lambda k_{\parallel} V\right) = \frac{k}{\omega_{\mathrm{p}}|k_{\parallel}|}\, \delta\left(k - \frac{\omega_{\mathrm{p}}}{V}\, \lambda \operatorname{sign} k_{\parallel}\right)$$

gives the resultant formula (132). Here, $\lambda = +1$ and $\lambda = -1$ correspond to the first and second δ functions in formula (A1.5). Hence, we conclude that, first, the relation $\operatorname{sign} k_{\parallel} = \lambda$ is fulfilled and, second, the magnitude of the wave vector of the emitted plasmons is $k = \omega_{\mathrm{p}}/V$.

A2. Screening of the field of a positron rotating in a magnetic field in a magnetized electron cloud

We consider an immobile ($V_{\parallel} = 0$) positron rotating in the Larmor orbit and embedded in a cloud of magnetized electrons with a strongly anisotropic velocity distribution (44). It radiates cyclotron waves with a frequency $\omega \approx \omega_{\mathrm{H}}$ and a wavelength

$$\lambda_0 = \frac{2\pi c}{\omega_{\mathrm{H}} n_0}\,, \tag{A2.1}$$

where n_0 is the refractive index of the plasma at this frequency (and for a given type of waves).

Expressions for n_0 are given in many books (see, for instance, Refs [11–17, 58]); they are complicated, however, and for greater clarity we therefore restrict ourselves to an estimate of this quantity. We neglect the spatial dispersion (which is equivalent to the cold plasma case) and consider an electron moving in uniform fields: a constant magnetic field, and the alternating electric field

$$\mathbf{E}(t) = \mathbf{E}_0 \cos \omega t = \operatorname{Re}\left[\mathbf{E}_0 \exp(-\mathrm{i}\omega t)\right].$$

This motion is described by the equations

$$m\ddot{\mathbf{r}} - \frac{e}{c}\,(\dot{\mathbf{r}} \times \mathbf{H}) = -e\mathbf{E}(t)\,.$$

Hence we obtain, assuming the presence of weak damping, the solution of these equations for steady motion:

$$\mathbf{r}(t) = \frac{e}{m(\omega^2 - \omega_{\mathrm{H}}^2)} \left(\mathbf{E} - \mathrm{i}\,\frac{\omega_{\mathrm{H}}}{\omega}\,(\mathbf{h} \times \mathbf{E})\right) + \frac{e}{m\omega^2}\,(\mathbf{E}\mathbf{h})\,\mathbf{h}\,.$$

Just as in the derivation of formula (121), we introduce the polarization (**P**) and electric induction (**D**) vectors, as is done in the plasma theory by analogy with dielectrics [59]. Taking into account the relation $D_\alpha = \varepsilon_{\alpha\beta} E_\beta$, we arrive at formula (123) which describes cold plasmas. From this formula follows the sought-for estimate:

$$n_0 = \sqrt{\varepsilon}\,, \qquad |\varepsilon - 1| \sim \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{H}}|\Delta\omega|}\,,$$
$$\Delta\omega = \omega - \omega_{\mathrm{H}}\,, \qquad |\Delta\omega| \ll \omega_{\mathrm{H}}\,. \tag{A2.2}$$

Estimate (A2.2) may be obtained in a still simpler way by treating each electron as a one-dimensional (for simplicity) oscillator described by the equation

$$\ddot{\xi} + \omega_{\mathrm{H}}^2 \xi = -\frac{e}{m}\, E\,. \tag{A2.3}$$

The solution of this equation, viz.

$$\xi = \frac{e}{m(\omega^2 - \omega_{\mathrm{H}}^2)}\, E \approx \frac{e}{2m\omega_{\mathrm{H}}\Delta\omega}\, E\,,$$

leads to estimate (A2.2) once again.

For typical parameters (42), (44) of electron cooling, the cyclotron resonance width is determined by the Doppler mechanism, and so

$$\Delta\omega \sim \Delta\omega_{\mathrm{D}} = b_{\parallel}\omega_{\mathrm{H}}\,, \qquad b_{\parallel} = \frac{\Delta_{\parallel}}{c}\,,$$
$$|\varepsilon - 1| \sim \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{H}}^2}\,\frac{1}{\beta_{\parallel}} \sim 10^2\,, \qquad \beta_{\parallel} = \frac{\Delta_{\parallel}}{c}\,, \tag{A2.4}$$
$$n_0 = \sqrt{\varepsilon} \sim \frac{\omega_{\mathrm{p}}}{\omega_{\mathrm{H}}\sqrt{\beta_{\parallel}}} \sim 10\,, \qquad \lambda_0 \sim 1\ \mathrm{cm}\,.$$

For a short r_{Hp} inherent in the case of fulfilling relation (170) and typical of magnetized electrons, dipole term (178), in which we must put $V_{\parallel} = 0$, $\Omega = \omega$, is responsible for the excitation of cyclotron waves. For the electric potential in the plasma we obtain the following expressions

$$\varphi(\mathbf{k}, \omega) = \frac{8\pi^2 q}{k^2} \left[\frac{\delta(\omega - \omega_{\mathrm{H}})}{\varepsilon(\mathbf{k}, \omega_{\mathrm{H}})} + \frac{\delta(\omega + \omega_{\mathrm{H}})}{\varepsilon(\mathbf{k}, -\omega_{\mathrm{H}})}\right] J_1(k_{\perp} r_{\mathrm{Hp}})\,,$$
$$\varphi(\mathbf{r}, t) = \int \frac{\mathrm{d}^3 k\, \mathrm{d}\omega}{(2\pi)^4}\, \varphi(\mathbf{k}, \omega) \exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\mathbf{r})$$
$$= \frac{2q}{\pi}\, \operatorname{Re}\left[J \exp(-\mathrm{i}\omega_{\mathrm{H}} t)\right], \tag{A2.5}$$

$$J = \int \frac{\mathrm{d}^3 k}{k^2}\, \frac{\exp(\mathrm{i}\mathbf{k}\mathbf{r})}{\varepsilon(\mathbf{k}, \omega_{\mathrm{H}})}\, J_1(k_{\perp} r_{\mathrm{Hp}})\,. \tag{A2.6}$$

At $V_{\parallel} = 0$, we retain the resonance terms ($S = \pm 1$) in expression (183) and put $J_1(k_{\perp} V_{\perp}/\omega_{\mathrm{H}}) \approx k_{\perp} V_{\perp}/\omega_{\mathrm{H}}$, which gives an uncertainty of about 20%. In line with the results outlined in Section 7, we *a priori* assume that the domain (202), (203) is most important for $|\omega| \approx \omega_{\mathrm{H}}$ (this will be borne out by the calculation). Then we arrive at the expressions

$$\varepsilon(\mathbf{k}, \omega_{\mathrm{H}}) \approx 1 + \frac{\mathrm{i}}{|k_{\parallel}| d_0}\,, \tag{A2.7}$$

$$d_0 = \frac{\omega_{\mathrm{H}} \Delta_{\parallel}}{\omega_{\mathrm{p}}^2 \sqrt{2\pi}} \sim 0.04\ \mathrm{cm}\,, \tag{A2.8}$$

$$J \approx 2\pi I_1 I_2\,, \tag{A2.9}$$

$$I_1 = \int_0^{\infty} \frac{\mathrm{d}k_{\perp}}{k_{\perp}}\, J_0(k_{\perp}, r_{\perp})\, J_1(k_{\perp} r_{\mathrm{Hp}})$$
$$\approx \left(1 + \frac{1}{3}\,\alpha_0^2 + A_0 \alpha_0^4\right)^{-3/4},$$

$$I_2 = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_{\parallel}\, |k_{\parallel}|}{|k_{\parallel}| + \mathrm{i}/d_0}\, \exp(\mathrm{i}k_{\parallel} z)$$
$$= -\frac{\mathrm{i}}{d_0} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_{\parallel}\, \exp(\mathrm{i}k_{\parallel} z)}{|k_{\parallel}| + \mathrm{i}/d_0} \approx \frac{2\mathrm{i}}{\alpha_0}\, \ln\left(\frac{\beta_0}{1 + \beta_0^3}\right),$$

$$A_0 = \left(\frac{2}{3}\right)^{4/3}, \qquad \alpha_0 = \frac{r_{\perp}}{r_{\mathrm{Hp}}}\,, \qquad \beta_0 = \frac{|z|}{d_0}\,.$$

The interpolation formulas for integrals I_1 and I_2 given above are exact in the limiting cases of small and large values of the parameters α_0 and β_0. From expressions (A2.5) and (A2.9) it follows that in the quasistatic domain

$$r < \lambda_0 \,, \tag{A2.10}$$

where retardation may be neglected ($c \to \infty$), the electric potential produced in the plasma by a positron rotating in the Larmor orbit is confined in the region (203). This region is a narrow long tube of radius $\sim r_{\mathrm{Hp}}$ extended along the magnetic field through a distance $\sim d_0$ on either side of the positron, with

$$\frac{d_0}{r_{\mathrm{Hp}}} \sim \frac{\Delta_\perp}{\Delta_\parallel} = \sqrt{\frac{T_\perp}{T_\parallel}} \gg 1 \,. \tag{A2.11}$$

The electric potential decreases with distance to the tube axis as $\sim r_\perp^{-3}$. Inside the tube, i.e., at distances $r_\perp \leqslant r_{\mathrm{Hp}}$ from its axis, the potential at a distance $|z|$ from the positron is of the form

$$\varphi \sim \frac{q d_0}{|z|^2} \,, \qquad |z| \gg d_0 \,, \tag{A2.12}$$

$$\varphi \sim \frac{q}{d_0} \ln\left(\frac{d_0}{|z|}\right) , \qquad r_0 \ll |z| \ll d_0 \,. \tag{A2.13}$$

Formulas (A2.12) and (A2.13) break down at distances $|z| < r_0 \sim (r_{\mathrm{Hp}}^2 d_0)^{1/3}$. In this case, we are approximately dealing with a quadrupole field, because the dipole moment of the positron orbit is perpendicular to the magnetic field, i.e., to the direction toward the observation point:

$$\varphi \sim \frac{q r_{\mathrm{Hp}}^2}{|z|^3} \,, \qquad r_{\mathrm{Hp}} < |z| < r_0 \,. \tag{A2.14}$$

Let us estimate the total dipole moment $\mathbf{D}'$ of the electrons in the tube. To do this we proceed from the simplest oscillator model defined by Eqn (A2.3), in which for the electric field we must substitute the quantity following from expressions (A2.5), (A2.9), and (A2.13):

$$E_\perp = -\frac{\partial \varphi}{\partial r_\perp} \sim \frac{q}{d_0 r_{\mathrm{Hp}}} \exp\left(-\mathrm{i}\omega_{\mathrm{H}} t\right) . \tag{A2.15}$$

From expressions (A2.3) and (A2.15) we obtain an estimate for the induced dipole moment of one electron:

$$d_1 \sim e\xi_1 \sim \frac{e^3}{m d_0 r_{\mathrm{Hp}} \omega_{\mathrm{H}} \Delta\omega} \,. \tag{A2.16}$$

For $\Delta\omega$ in formula (A2.16) we must substitute the quantity

$$\Delta\omega = \max\left(\Delta\omega_{\mathrm{dd}}, \Delta\omega_{\mathrm{D}}\right) . \tag{A2.17}$$

Here, $\Delta\omega_{\mathrm{dd}}$ is the characteristic change in the electron Larmor frequency [see formula (196)]:

$$\Delta\omega_{\mathrm{dd}} \sim \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{H}}} \,, \tag{A2.18}$$

due to the dipole – dipole interaction with the neighboring electrons:

$$V_{\mathrm{dd}} \sim e^2 \,\frac{\xi_1 \xi_2}{r^3(t)} \sim e^2 n \xi_1 \xi_2 \,, \tag{A2.19}$$

where $\mathbf{r}(t)$ is the difference between the coordinates of the centers of the Larmor circles for the orbits of these electrons. Estimate (A2.18) will be obtained by treating the two electrons as two interacting oscillators:

$$\begin{aligned} m\ddot{\xi}_1 + m\omega_{\mathrm{H}}^2 \xi_1 &= -\frac{\partial V_{\mathrm{dd}}}{\partial \xi_1} = -\frac{e^2}{r^3}\,\xi_2 \,, \\ m\ddot{\xi}_2 + m\omega_{\mathrm{H}}^2 \xi_2 &= -\frac{\partial V_{\mathrm{dd}}}{\partial \xi_2} = -\frac{e^2}{r^3}\,\xi_1 \,. \end{aligned} \tag{A2.20}$$

According to equations (A2.20), for an invariable distance $r(t) = \mathrm{const} \sim \bar{R}$ there are two eigenmodes with the close frequencies

$$\omega_{1,2} = \sqrt{\omega_{\mathrm{H}}^2 \pm \frac{e^2}{m r^3}} \approx \omega_{\mathrm{H}} \pm \frac{e^2}{2 m r^3 \omega_{\mathrm{H}}} \,, \tag{A2.21}$$

whence follows estimate (A2.18).

The characteristic broadening of the cyclotron resonance arising from the Doppler effect is given by

$$\Delta\omega_{\mathrm{D}}' \sim |k_\parallel| \Delta_\parallel \,. \tag{A2.22}$$

We emphasize that $\Delta\omega_{\mathrm{D}}' \neq \Delta\omega_{\mathrm{D}}$ [see estimates (A2.4)]. These frequency mismatches are different due to the fact that the outgoing cyclotron waves are formed too far away — at the boundary $r \sim \lambda_0$ between the quasistatic (A2.10) and wave ($r > \lambda_0$) zones. In this connection, it is pertinent to note that

$$\frac{\lambda_0}{d_0} \sim n_0 \gg 1 \,. \tag{A2.23}$$

In the field formation in the domain of interest, cyclotron waves with the wavelengths

$$\lambda \sim d_0 \ll \lambda_0 \,, \qquad |k_\parallel| = \frac{2\pi}{\lambda} \sim \frac{1}{d_0} \,, \tag{A2.24}$$

constructively interfere, thus making the main contribution, and from equation (A2.20) and relation (A2.22) we therefore conclude that

$$\Delta\omega_{\mathrm{D}} \sim \frac{\Delta_\parallel}{d_0} \,. \tag{A2.25}$$

From relation (A2.25) and formulas (A2.8), (A2.18) it follows that

$$\Delta\omega_{\mathrm{D}}' \sim \Delta\omega_{\mathrm{dd}} \,, \qquad \Delta\omega \sim \Delta\omega_{\mathrm{D}}' \sim \Delta\omega_{\mathrm{dd}} \,. \tag{A2.26}$$

The tube volume is $V_{\mathrm{t}} \sim r_{\mathrm{Hp}}^2 d_0$. From relations (A2.16), (A2.18), and (A2.26) we have

$$|\mathbf{D}'| \sim d_1 n V_{\mathrm{t}} \sim e r_{\mathrm{Hp}} \,. \tag{A2.27}$$

These estimates enable determining the nature of the positron electromagnetic field in the quasistatic zone. As indicated in Section 2, the static field (the field of an immobile charge, dipole, quadrupole, etc.) in the cloud of magnetized electrons with an anisotropic velocity distribution is screened over a distance $\sim R_\parallel$. The portion of the positron field that oscillates at a frequency $\approx \omega_{\mathrm{H}}$ penetrates the plasma much deeper: over a distance $\sim d_0 \gg R_\parallel$. The electron motion is perturbed as a result of collision of the electron and positron Larmor circles ($\rho_0 \sim \bar{r}_{\mathrm{H}} \sim r_{\mathrm{Hp}}$). Since these perturbations are produced by one and the same positron, they are correlated for different electrons that experienced the above collision. This coherent perturbation of the Larmor electron motion is transferred by the electrons some distance L along the magnetic lines of force.

To estimate the distance L, we will consider two electrons that have experienced a collision with a positron. Owing to a small difference (A2.17) in their eigenfrequencies, the coherence between these electrons vanishes when the phase difference of their Larmor motion amounts to

$$\Delta\varphi \sim \Delta\omega \frac{L}{v_\parallel} \sim \frac{\omega_{\rm p}^2 L}{\omega_{\rm H} v_\parallel} \sim \pi\,. \tag{A2.28}$$

Hence follows an estimate for L, which coincides with the estimate for d_0 [see formula (A2.8)]. This coherent perturbation of the electron motion gives rise to a self-consistent electric potential in the tube. However, at this point we would do well to explain that the perturbation of the Larmor motion takes place due to collective interaction rather than to binary interaction (see Section 7).

In the plasma there occurs complete screening of not only the constant part of the positron field, but of its time-varying part, as well: estimate (A2.27) should be perceived as $\mathbf{D} + \mathbf{D}' \approx 0$. From the change in behavior of φ as a function of z for $|z| \sim d_0$ [see formulas (A2.12), (A2.13)] it is clear that the characteristic size along the z-axis, over which this screening is realized, amounts to $\sim d_0$. This screening strongly suppresses the cyclotron emission for a large value of the refractive index n_0 [see estimates (A2.4)], which was pointed out by Ginzburg and Zheleznyakov [28] (some manifestations of this effect are also discussed in Ref. [37]). According to Ref. [28], the suppression is attributable to the noncoincidence of the senses of gyration of the electron and the cyclotron wave polarization vectors. As we saw, however, the cyclotron radiation of the positron embedded in the electron cloud is also suppressed. Therefore, this effect is independent of the sense of gyration of a radiating particle, and so the screening-reliant explanation given above is, in our opinion, preferable to that of Ref. [28].

Thus, provided that

$$n_0 \sim \frac{\omega_{\rm p}}{\omega_{\rm H}} \sqrt{\frac{c}{\varDelta_\parallel}} \gg 1\,, \tag{A2.29}$$

i.e., for a sufficiently high plasma density, there emerges a tube (203) inside the quasistatic zone (A2.10). The energy of transverse particle motion is transferred over it and almost completely absorbed in it (through the mechanism of Landau damping). The retardation of signals in the tube is insignificant, and so the energy loss process is nonrelativistic [the speed of light does not enter into formula (201)]. These particle energy losses were termed polarization losses [12].

Owing to screening, the intensity of an electromagnetic field at a distance $r \sim \lambda$ from a positron turns out to be approximately $\lambda_0/d_0 \sim n_0$ times lower than the field intensity of the positron in vacuum. For this reason, the intensity of the cyclotron waves that go to infinity is $\sim n_0^2$ times lower than in vacuum. These waves are generated due to the retardation effect, and their intensity is therefore much lower than the energy absorbed in the tube in a unit time.

A3. Correlation functions for electric field intensities in an ideal nonrelativistic plasma in a magnetic field

In uniform plasmas, the correlation function $\langle E_\alpha(\mathbf{r},t)\,E_\beta(\mathbf{r}',t')\rangle$ depends only on the differences $\mathbf{r}-\mathbf{r}'$, $t-t'$:

$$\big\langle E_\alpha(\mathbf{r},t)\,E_\beta(\mathbf{r}',t')\big\rangle = E_{\alpha\beta}(\mathbf{r}-\mathbf{r}';t-t') = \int \mathrm{d}Q\, E_{\alpha\beta}(\mathbf{k},\omega)\exp\big[\mathrm{i}\mathbf{k}(\mathbf{r}-\mathbf{r}') - \mathrm{i}\omega(t-t')\big]\,, \tag{A3.1}$$

where $\mathrm{d}Q = \mathrm{d}^3k\,\mathrm{d}\omega/(2\pi)^4$.

To calculate the tensor $E_{\alpha\beta}(\mathbf{k},\omega)$, we adopt the method devised in Ref. [12]. A formula similar to formula (A3.1) applies to charge density fluctuations:

$$\big\langle\rho(\mathbf{r},t)\,\rho(\mathbf{r}',t')\big\rangle = R(\mathbf{r}-\mathbf{r}',t-t') = \int \mathrm{d}Q\, R(\mathbf{k},\omega)\exp\big[\mathrm{i}\mathbf{k}(\mathbf{r}-\mathbf{r}') - \mathrm{i}\omega(t-t')\big]\,. \tag{A3.2}$$

From expressions (A3.1), (A3.2) and the Fourier transforms

$$\rho(\mathbf{r},t) = \int \mathrm{d}Q\,\rho(\mathbf{k},\omega)\exp(\mathrm{i}\mathbf{k}\mathbf{r} - \mathrm{i}\omega t)\,,$$

$$E_\alpha(\mathbf{r},t) = \int \mathrm{d}Q\,E_\alpha(\mathbf{k},\omega)\exp(\mathrm{i}\mathbf{k}\mathbf{r} - \mathrm{i}\omega t)$$

we obtain

$$\begin{aligned} &\big\langle E_\alpha(\mathbf{k},\omega)\,E_\beta(\mathbf{k}',\omega')\big\rangle = S\,E_{\alpha\beta}(\mathbf{k},\omega)\,, \\ &G \equiv \big\langle\rho(\mathbf{k},\omega)\,\rho(\mathbf{k}',\omega')\big\rangle = S\,R(\mathbf{k},\omega)\,, \end{aligned} \tag{A3.3}$$

where $S = (2\pi)^4\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')$. From formulas (21) and (A3.3) it follows that

$$E_{\alpha\beta}(\mathbf{k},\omega) = \frac{16\pi^2 k_\alpha k_\beta}{k^4\big|\varepsilon(\mathbf{k},\omega)\big|^2}\,R(\mathbf{k},\omega)\,, \tag{A3.4}$$

where use was made of the property $\varepsilon(-\mathbf{k},-\omega) = \varepsilon^*(\mathbf{k},\omega)$.

For a system of N point electrons in a volume V, the charge density is expressed as

$$\rho(\mathbf{r},t) = -e\sum_{a=1}^{N}\delta\big(\mathbf{r}-\mathbf{r}_a(t)\big)\,.$$

Here, $\mathbf{r}_a(t)$ is the radius vector of the ath electron. The Fourier component of the density assumes the form

$$\rho(\mathbf{k},\omega) = -e\sum_{a=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}t\,\exp\big[\mathrm{i}\omega t - \mathrm{i}\mathbf{k}\mathbf{r}_a(t)\big]\,.$$

Hence, and from formulas (A3.3), we find

$$G = e^2\sum_{a,a'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}t'\,\exp(\mathrm{i}\omega t + \mathrm{i}\omega' t')\,F_{aa'}\,, \tag{A3.5}$$

where

$$F_{aa'} = \big\langle\exp\big[-\mathrm{i}\mathbf{k}\mathbf{r}_a(t) - \mathrm{i}\mathbf{k}'\mathbf{r}_{a'}(t')\big]\big\rangle\,. \tag{A3.6}$$

In the ideal plasma approximation, the terms with $a \neq a'$, corresponding to different electrons, may undergo factorization into two independent components:

$$F_{aa'} = \big\langle\exp\big[-\mathrm{i}\mathbf{k}\mathbf{r}_a(t)\big]\big\rangle\big\langle\exp\big[-\mathrm{i}\mathbf{k}'\mathbf{r}_{a'}(t')\big]\big\rangle\,.$$

Since all electrons are equivalent, the factor $Q = \langle\exp[-\mathrm{i}\mathbf{k}\mathbf{r}_a(t)]\rangle$ is independent of the electron number a, and we therefore omit the subscript a:

$$Q = \left\langle \exp\left[-\mathrm{i}\mathbf{k}\mathbf{r}(t)\right]\right\rangle . \tag{A3.7}$$

Let $\mathbf{r}_0$ and $\mathbf{v}_0$ be the initial (at $t = 0$) electron position and velocity, so that

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \dot{\mathbf{r}}(t_1, \mathbf{v}_0)\,\mathrm{d}t_1 . \tag{A3.8}$$

The angular brackets in expressions (A3.5), (A3.6) imply averaging over the initial data $(\mathbf{r}_0, \mathbf{v}_0)$. In the ideal plasma approximation, when calculating the velocity $\dot{\mathbf{r}}(t, \mathbf{v})$ of an electron its interaction with other electrons should be neglected. First we perform averaging over $\mathbf{r}_0$:

$$Q = \left\langle \exp\left[-\mathrm{i}\mathbf{k}\int_0^t \dot{\mathbf{r}}(t_1, \mathbf{v})\,\mathrm{d}t_1\right]\right\rangle_{\mathbf{v}_0} Q_{\mathbf{r}_0} ,$$

where averaging over electron velocities $\mathbf{v}_0$ (the subscript in $\mathbf{v}_0$ is subsequently omitted) is performed in the first factor on the right-hand side. Averaging over the initial electron position $\mathbf{r}_0$ is performed in the second factor:

$$Q_{\mathbf{r}_0} = \frac{1}{V}\int \mathrm{d}^3 r_0 \,\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r}_0\right) = \delta_{\mathbf{k},0} = \frac{(2\pi)^3}{V}\,\delta(\mathbf{k}) . \tag{A3.9}$$

Since we are concerned with the $\mathbf{k} \neq 0$ case, then $Q_{\mathbf{r}_0} = 0$. And so there are no correlations for different electrons ($a \neq a'$): $F_{aa'} = 0$. Hence we conclude that $F_{aa'} = \delta_{aa'}F$, where $F = \left\langle \exp\left[-\mathrm{i}\mathbf{k}\mathbf{r}(t) - \mathrm{i}\mathbf{k}'\mathbf{r}(t')\right]\right\rangle$. Similarly to Q, the function F is independent of the electron number. Averaging over the initial position $\mathbf{r}_0$ yields the following relationships:

$$\frac{1}{V}\int \mathrm{d}^3 r_0 \,\exp\left[-\mathrm{i}(\mathbf{k}+\mathbf{k}')\,\mathbf{r}_0\right] = \frac{(2\pi)^3}{V}\,\delta(\mathbf{k}+\mathbf{k}') ,$$

$$F = \frac{(2\pi)^3}{V}\,\delta(\mathbf{k}+\mathbf{k}')F_{\mathbf{k}} , \qquad F_{\mathbf{k}} = \left\langle \exp\left(-\mathrm{i}\mathbf{k}\mathbf{R}(t, t')\right)\right\rangle_{\mathbf{v}} = F_1 F_2 , \tag{A3.10}$$

$$F_1 = \left\langle \exp\left(-\mathrm{i}\mathbf{k}_\perp \mathbf{R}_\perp(t, t')\right)\right\rangle_{\mathbf{v}_\perp} , \tag{A3.11}$$

$$F_2 = \left\langle \exp\left(-\mathrm{i}k_\parallel v_\parallel (t - t')\right)\right\rangle_{v_\parallel} , \tag{A3.12}$$

$$\mathbf{R}(t, t') = \int_{t'}^t \dot{\mathbf{r}}(t_1, \mathbf{v})\,\mathrm{d}t_1 = \mathbf{r}(t) - \mathbf{r}(t') = \mathbf{R}_\perp(t, t') + v_\parallel (t - t')\,\mathbf{h} .$$

Let two particles begin their motion at $t = 0$ from the same point with equal values of $v_\perp \equiv |\mathbf{v}_\perp|$ and $v_\parallel$, but with different directions of vectors $\mathbf{v}_\perp$. Then, the vector $\mathbf{R}(t, t')$ for the second particle is obtained from the similar vector for the first particle by rotating it (about the axis parallel to the magnetic field) through the angle which the vectors $\mathbf{v}_\perp$ of these two particles make between themselves. Consequently, averaging over the directions of $\mathbf{v}_\perp$ in expression (A3.11) is equivalent to averaging over the directions of the vector $\mathbf{R}_\perp$:

$$F_1 = \left\langle \exp\left(-\mathrm{i}\mathbf{k}_\perp \mathbf{R}_\perp(t, t')\right)\right\rangle_{\varphi, v_\perp} = \left\langle J_0(k_\perp R_\perp)\right\rangle_{v_\perp} . \tag{A3.13}$$

For a group of electrons with equal $v_\perp$, the magnitude of $\mathbf{R}_\perp(t, t')$ is equal to the length of the chord connecting two points in the circumference of radius $r_{\mathrm{H}} = v_\perp/\omega_{\mathrm{H}}$ and circular measure $\varphi = \omega_{\mathrm{H}}(t - t')$ of the arc:

$$R_\perp(t, t') = 2r_{\mathrm{H}} \sin\left[\frac{1}{2}\,\omega_{\mathrm{H}}(t - t')\right] . \tag{A3.14}$$

One can see from relations (A3.11)–(A3.14) that $F_{\mathbf{k}}$ in expression (A3.10) depends on t and t' only in the combination $\tau = t - t'$: $F_{\mathbf{k}} \equiv F_{\mathbf{k}}(\tau)$. This property permits, in view of expression (A3.9), bringing expression (A3.5) to the form

$$G = S\,\frac{e^2}{V}\sum_a F_{\mathbf{k}}(\omega) , \tag{A3.15}$$

where

$$F_{\mathbf{k}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, F_{\mathbf{k}}(\tau)\exp\left(\mathrm{i}\omega t\right) . \tag{A3.16}$$

Summation over the electron number a gives N, and therefore by comparing formulas (A3.15) and (A3.3) we obtain $R(\mathbf{k}, \omega) = ne^2 F_{\mathbf{k}}(\omega)$. Hence, and from relations (A3.4), (A3.13), and (A3.14), it follows that

$$E_{\alpha\beta}(\mathbf{k}, \omega) = \frac{16\pi^2 ne^2 k_\alpha k_\beta}{k^4 \left|\varepsilon(\mathbf{k}, \omega)\right|^2}\,F_{\mathbf{k}}(\omega) , \qquad F_{\mathbf{k}}(\omega) = \left\langle \int_{-\infty}^{\infty} \mathrm{d}\tau\, \exp\left(\mathrm{i}\Omega\tau\right) J_0\left(z \sin\frac{\omega_{\mathrm{H}}\tau}{2}\right)\right\rangle_{\mathbf{v}} , \tag{A3.17}$$

where $\Omega = \omega - k_\parallel v_\parallel$, and $z = 2k_\perp r_{\mathrm{H}}$.

We introduce a variable $\varphi = \omega_{\mathrm{H}}\tau$ and divide the domain of integration with respect to $\mathrm{d}\varphi$ into intervals of length 2π ($\varphi = 2\pi l + \psi$, $0 < \psi < 2\pi$). In each interval, the integration with respect to $\mathrm{d}\psi$ is taken. In view of the formula

$$\sum_{l=-\infty}^{\infty} \exp\left(\mathrm{i}\beta_0 l\right) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\beta_0 - 2\pi l) ,$$

we arrive at

$$F_{\mathbf{k}}(\omega) = \sum_{l=-\infty}^{\infty} \left\langle \delta(\Omega - \omega_{\mathrm{H}} l) \int_0^{2\pi} \mathrm{d}\varphi\, \exp\left(\mathrm{i}l\varphi\right) J_0\left(y \sin\frac{\varphi}{2}\right)\right\rangle ,$$

where $y = 2k_\perp r_{\mathrm{H}}$. From the formula (see Prudnikov et al. [60])

$$\int_0^{\pi} \mathrm{d}x\, \cos\left(2lx\right) J_0(c \sin x) = \pi J_l^2\left(\frac{c}{2}\right)$$

we find

$$F_{\mathbf{k}}(\omega) = 2\pi \sum_{l=-\infty}^{\infty} \left\langle \delta(\omega - k_\parallel v_\parallel - \omega_{\mathrm{H}} l)\right\rangle_{v_\parallel} \left\langle J_l^2(k_\perp r_{\mathrm{H}})\right\rangle_{\mathbf{v}_\perp} . \tag{A3.18}$$

For $|\omega| \sim \omega_{\mathrm{p}} \ll \omega_{\mathrm{H}}$, it would suffice to keep the term with $l = 0$ in formula (A3.18):

$$F_{\mathbf{k}}(\omega) \approx 2\pi\left\langle \delta(\omega - k_\parallel v_\parallel)\right\rangle_{v_\parallel} \left\langle J_0^2(k_\perp r_{\mathrm{H}})\right\rangle_{\mathbf{v}_\perp} = \frac{2\pi}{|k_\parallel|}\, g\left(\frac{\omega}{k_\parallel}\right) P_0(\beta^2) , \tag{A3.19}$$

where $\beta = k_\perp \bar{r}_{\mathrm{H}}$, and $P_0(x) = \exp\left(-x\right) I_0(x)$; here, I_0 is the modified zero-order Bessel function of the first kind. Furthermore, according to expression (183) we have

$$\varepsilon(\mathbf{k}, \omega) \approx 1 + \frac{P_0(\beta^2)}{k^2 R_\parallel^2}\left[1 - X\left(\frac{\omega}{\sqrt{2}\,k_\parallel \Delta_\parallel}\right) + \mathrm{i}\,\mathrm{sign}\left(k_\parallel\right) Y\left(\frac{\omega}{\sqrt{2}\,k_\parallel \Delta_\parallel}\right)\right] , \tag{A3.20}$$

where the functions X and Y are defined by expressions (144). For $|\omega| \approx \omega_{\rm H}$, it would suffice to retain the terms with $l = \pm 1$ in formula (A3.18), which gives

$$F_{\mathbf{k}}(\omega) \approx \frac{2\pi}{|k_\parallel|}\left[g\left(\frac{\omega-\omega_{\rm H}}{k_\parallel}\right) + g\left(\frac{\omega+\omega_{\rm H}}{k_\parallel}\right)\right] \times \exp\left(-k_\perp^2 \bar{r}_{\rm H}^2\right) I_1\left(k_\perp^2 \bar{r}_{\rm H}^2\right). \qquad \text{(A3.21)}$$

As in Section 7, for $|\omega| \approx \omega_{\rm H}$ in the tube (202) we find that

$$\varepsilon(\mathbf{k},\omega) \approx 1 - \frac{\omega_{\rm p}^2 P_1(\beta^2)}{2\sqrt{2}\,\omega_{\rm H}|k_\parallel|\Delta_\parallel x_1}\left[X(x_1) - \mathrm{i}\,\mathrm{sign}\,(\omega)\,Y(x_1)\right], \qquad \text{(A3.22)}$$

where $x_1 = |\omega - \omega_{\rm H}|/(\sqrt{2}\,k_\parallel \Delta_\parallel)$, and the function P_1 is given by formula (212).

A4. Positron diffusion in the velocity space, occurring in an isotropic electron plasma

The tensor of the diffusion coefficients of particle M in the velocity space, which enters into the Fokker–Planck equation (152), is defined as

$$D_{\alpha\beta}(\mathbf{V}) = \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\langle \Delta V_\alpha \Delta V_\beta\rangle, \qquad \text{(A4.1)}$$

where ΔV_α is the variation of the particle velocity in a time Δt, caused by the fluctuation part of the electric field $\mathbf{E}'(\mathbf{r},t)$: $V_\alpha(t+\Delta t) = V_\alpha(t) + \Delta V_\alpha$. To find $D_{\alpha\beta}(\mathbf{V})$, let us consider the quantity $G_{\alpha\beta}(t) = \langle V_\alpha(t)\, V_\beta(t)\rangle$. The field fluctuations in the intervals $(-\infty, t)$ and $(t, t+\Delta t)$ are independent, and therefore

$$\langle V_\alpha(t)\Delta V_\beta\rangle = 0,$$
$$G_{\alpha\beta}(t+\Delta t) = \langle V_\alpha(t+\Delta t)\, V_\beta(t+\Delta t)\rangle = G_{\alpha\beta}(t) + 2\Delta t D_{\alpha\beta}.$$

Hence it follows that

$$D_{\alpha\beta} = \frac{1}{2}\frac{\mathrm{d}G_{\alpha\beta}(t)}{\mathrm{d}t} = \frac{1}{2}\left[\left\langle \dot{V}_\alpha(t)\, V_\beta(t)\right\rangle + \left\langle V_\alpha(t)\, \dot{V}_\beta(t)\right\rangle\right] = \frac{q}{2M}(K_{\alpha\beta} + K_{\beta\alpha}),$$

where $K_{\alpha\beta} = \langle E'_\alpha(\mathbf{V}t, t)\, V_\beta(t)\rangle$.

Let the particle velocity be $\mathbf{V}^{(i)}$ for the time $t \to -\infty$. Then, one has

$$\mathbf{V}(t) = \mathbf{V}^{(i)} + \frac{q}{M}\int_{-\infty}^{t} \mathbf{E}'(\mathbf{V}t_1, t_1)\,\mathrm{d}t_1.$$

Since $\langle E'_\alpha(\mathbf{V}t,t)\, V_\beta^{(i)}\rangle = 0$, we arrive at

$$K_{\alpha\beta} = \frac{q}{M}\int_{-\infty}^{t}\mathrm{d}t_1\,\langle E'_\alpha(\mathbf{V}t,t)\,E'_\beta(\mathbf{V}t_1,t_1)\rangle = \frac{q}{M}\int \mathrm{d}Q\, E_{\alpha\beta}(\mathbf{k},\omega)\int_0^\infty \mathrm{d}\tau\, \exp\left[-\mathrm{i}(\Omega - \mathrm{i}0)\,\tau\right] = -\frac{\mathrm{i}q}{M}\int\frac{\mathrm{d}Q}{\Omega - \mathrm{i}0}\,E_{\alpha\beta}(\mathbf{k},\omega).$$

Consequently, the desired tensor is given by

$$D_{\alpha\beta} = -\frac{\mathrm{i}q}{M^2}\int\frac{\mathrm{d}Q}{\Omega-\mathrm{i}0}\,E_{\alpha\beta}(\mathbf{k},\omega). \qquad \text{(A4.2)}$$

A5. Coefficient $D_\parallel$ of longitudinal positron diffusion in the velocity space, occurring in electron plasmas

According to the results outlined in Section 7, the longitudinal positron diffusion depends primarily on the fluctuations in the electric field with the characteristic frequencies of the order of the Langmuir frequency [see relations (226)]. Let us obtain the expression for the longitudinal diffusion coefficient, as was done in Appendix A4:

$$D_\parallel(V_\parallel, V_\perp) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\langle V_\parallel^2(t)\rangle\right) = \langle \dot{V}_\parallel(t)\, V_\parallel(t)\rangle = \frac{q^2}{M^2}\int_{-\infty}^{t}\mathrm{d}t'\,\langle E'_z(V_\parallel t, t)\,E'_z(V_\parallel t', t')\rangle = \frac{q^2}{M^2}\int_{-\infty}^{0}\mathrm{d}t'\,\langle E_z(0,0)\,E_z(V_\parallel t', t')\rangle. \qquad \text{(A5.1)}$$

The expression for the correlation function of the electric field intensities was derived in Appendix A3, whence follows the formula

$$D_\parallel = \frac{q^2}{M^2}\int_{-\infty}^{0}\mathrm{d}t'\int \mathrm{d}Q\, E_{zz}(\mathbf{k},\omega)\exp\left[\mathrm{i}(k_\parallel V_\parallel - \omega)\,t'\right] = -\frac{\mathrm{i}q^2}{M^2}\int \mathrm{d}Q\, E_{zz}(\mathbf{k},\omega)\,\frac{1}{\omega - k_\parallel V_\parallel - \mathrm{i}0}, \qquad \text{(A5.2)}$$

where $\mathrm{d}Q = \mathrm{d}^3k\,\mathrm{d}\omega/(2\pi)^4$. As in the case of the longitudinal force $\mathbf{F}_{\rm cM}$ (see Section 7), domain (130) makes the main contribution to the longitudinal diffusion coefficient $D_\parallel$. In combination with inequality (66) this gives $k\bar{r}_{\rm H} \ll 1$, and we may therefore put $\bar{r}_{\rm H} = 0$ in formula (A3.19):

$$F_{\mathbf{k}}(\omega) \approx \frac{2\pi}{|k_\parallel|}\,g\left(\frac{\omega}{k_\parallel}\right). \qquad \text{(A5.3)}$$

According to expression (A3.17),

$$E_{zz}(\mathbf{k},\omega) = \frac{16\pi^2 n e^2 k_\parallel^2}{k^4|\varepsilon(\mathbf{k},\omega)|^2}\,F_{\mathbf{k}}(\omega). \qquad \text{(A5.4)}$$

We transform the integrand in formula (A5.2) using the Sokhotskii formula:

$$\frac{1}{\omega - k_\parallel V_\parallel - \mathrm{i}0} = P\left(\frac{1}{\omega - k_\parallel V_\parallel}\right) + \pi\mathrm{i}\delta(\omega - k_\parallel V_\parallel).$$

The first term on the right-hand side of this formula makes a zero contribution to $D_\parallel$, since $E_{zz}(-\mathbf{k},-\omega) = E_{zz}(\mathbf{k},\omega)$ according to formulas (A5.3) and (A5.4). We perform integration with respect to $\mathrm{d}\omega$ to obtain

$$D_\parallel = \frac{q^2}{2M^2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,E_{zz}(\mathbf{k}, k_\parallel V_\parallel) = \frac{2nq^2e^2}{M^2}\,g(V_\parallel)\int\frac{\mathrm{d}^3k\,|k_\parallel|}{k^4|\varepsilon(\mathbf{k},k_\parallel V_\parallel)|^2}. \qquad \text{(A5.5)}$$

According to formula (A3.20), one finds

$$\varepsilon(\mathbf{k}, k_\parallel V_\parallel) = 1 + \frac{1}{k^2 R_\parallel^2}\left[1 - X\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right) + \mathrm{i}\,\mathrm{sign}\,(k_\parallel)\,Y\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right)\right].$$

The singularity in ε for $k \to 0$ is nothing but the Debye screening of the positron field in the magnetized electron cloud (66). Due to this screening, integral (A5.5) converges for short wave vectors $k \to 0$. However, it diverges logarithmically when $k \to \infty$, since we put $\bar{r}_H = 0$ and $\beta = 0$ in formulas (A3.19) and (A3.20), in line with inequality (66). In the positron velocity range (199), which is of prime interest, in expression (A5.5) we may put $\varepsilon = 1$ with a logarithmical accuracy of order $1/\Lambda_\parallel$, where $\Lambda_\parallel = \ln(k_{\max}/k_{\min}) = \ln(R_\parallel/\bar{r}_H)$, which gives

$$D_\parallel \approx \frac{4\pi n q^2 e^2}{M^2} \Lambda_\parallel g(V_\parallel), \qquad V_\parallel \leqslant \Delta_\parallel \ln\frac{T_\perp}{T_\parallel}, \qquad V_\perp \leqslant \Delta_\perp. \tag{A5.6}$$

For positron velocities $V_\parallel \geqslant \Delta_\parallel \ln(T_\perp/T_\parallel)$, formula (A5.6) breaks down and from expression (A5.5) we obtain

$$D_\parallel \approx \frac{2\pi n q^2 e^2 T_\parallel}{m M^2 V_\parallel^3}. \tag{A5.7}$$

In this case, the main contribution to the longitudinal diffusion coefficient $D_\parallel$ is made by the sharp peak in the integrand of expression (A5.5), which is generated by the smallness of the quantity $|\varepsilon(\mathbf{k}, k_\parallel V_\parallel)|^2$ in a denominator, which corresponds to the Cherenkov emission of plasmons. A simpler way of obtaining formula (A5.7) consists in the use of expressions (189) and (252).

In the LEPTA Project, according to relations (77) one has $\bar{r}_H \sim R_\parallel$, and therefore the uncertainty of formula (A5.6) is significant: $\sim 100\%$. If we do not invoke the magnetization condition (66), the calculation becomes tedious. However, the calculation is not fundamentally different from that outlined above, and so below we will highlight only the main steps and omit the details.

When we abandon the magnetization approximation which corresponds to the limiting case of $r_H \to 0$, $r_{Hp} \to 0$, in lieu of expressions (A5.1) and (A5.2) we obtain

$$D_\parallel = \frac{q^2}{M^2}\int_{-\infty}^{t} dt' \left\langle E_z\big(\mathbf{r}(t), t\big)\, E_z\big(\mathbf{r}(t'), t'\big)\right\rangle$$
$$= \frac{q^2}{M^2}\int_{-\infty}^{t} dt' \int dQ\, E_{zz}(\mathbf{k}, \omega)$$
$$\times \exp\left[i\mathbf{k}_\perp\big(\mathbf{r}_\perp(t) - \mathbf{r}_\perp(t')\big)\right] \exp(-i\Omega\tau),$$

where $\tau = t - t'$, and $\Omega = \omega - k_\parallel V_\parallel$. The vector $\mathbf{r}_\perp(t)$ gyrates with the Larmor frequency, and therefore we transform the exponents in this expression using the formula

$$\exp\big(i\mathbf{k}_\perp \mathbf{r}_\perp(t)\big) = \sum_{S=-\infty}^{\infty} J_S(k_\perp r_{Hp}) \exp(iS\omega t). \tag{A5.8}$$

We make the change $t' = t - \tau$ and integrate with respect to $d\tau$ over the interval $(0, +\infty)$. Furthermore, considering that the time scale $T_H = 2\pi/\omega_H$ is short in comparison with the typical relaxation time of the positron distribution function $\Phi(\mathbf{V}, t)$, we average $D_\parallel$ over the Larmor period, which amounts to the integral

$$\frac{1}{T_H}\int_0^{T_H} dt \exp\left[i\omega_H(S - S')\,t\right] = \delta_{SS'}.$$

Instead of expression (A5.5) we obtain

$$D_\parallel = \frac{\pi q^2}{M^2}\sum_{S=-\infty}^{\infty}\int dQ\, J_S^2(k_\perp r_{Hp})\, E_{zz}(\mathbf{k}, \omega)$$
$$\times\, \delta(\omega - k_\parallel V_\parallel - \omega_H S)$$
$$\approx \frac{q^2}{2M^2}\int d^3k\, J_0^2(k_\perp r_{Hp})\, E_{zz}(\mathbf{k}, k_\parallel V_\parallel), \tag{A5.9}$$

where the exponentially small terms with $S \neq 0$ are omitted [which is related to the factor $g(\omega/k_\parallel)$ in formula (A3.19)]. Hence, and from formula (A5.4), it follows that

$$D_\parallel = \frac{2nq^2e^2}{M^2}\, g(V_\parallel)\int \frac{d^3k\, |k_\parallel|\, J_0^2(k_\perp r_{Hp})\, P_0(\beta^2)}{k^4 \left|\varepsilon(\mathbf{k}, k_\parallel V_\parallel)\right|^2}. \tag{A5.10}$$

According to formula (A3.20), one has

$$\left|\varepsilon(\mathbf{k}, k_\parallel V_\parallel)\right|^2 = \left(1 + \frac{p}{k^2}\right)^2 + \frac{q^2}{k^4}, \tag{A5.11}$$
$$p = R_\parallel^{-2} P_0(\beta^2)\left[1 - X\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right)\right],$$
$$q = R_\parallel^{-2} P_0(\beta^2)\, Y\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right).$$

The integrand in expression (A5.9) is even in $k_\parallel$, which permits reducing the range of integration with respect to $dk_\parallel$ to the interval $(0, +\infty)$. Since $k_\parallel\, dk_\parallel = k\, dk$, integrating over the azimuthal angle gives $d^3k\, k_\parallel \to 2\pi k_\perp\, dk_\perp k\, dk$. Expression (5.9) is brought to the double integral

$$\int_0^\infty k_\perp\, dk_\perp \int_{k_\perp}^\infty \frac{dk}{k^3}\ldots,$$

which, upon integration by parts, gives

$$D_\parallel = \frac{4\pi n q^2 e^2}{M^2}\, g(V_\parallel)\int_0^\infty \frac{dk_\perp\, J_0^2(k_\perp r_{Hp})\, P_0(\beta^2)}{k_\perp\left[(1 + p/k_\perp^2)^2 + q^2/k_\perp^4\right]}.$$

On moving to a dimensionless variable we arrive at the final formula which is suitable for describing the LEPTA experiment:

$$D_\parallel = \frac{4\pi n q^2 e^2}{M^2}\, g(V_\parallel)\int_0^\infty \frac{d\beta\, J_0^2(\gamma\beta)\, P_0(\beta^2)}{\beta\left[(1 + k_1/\beta^2)^2 + k_2^2/\beta^4\right]}, \tag{A5.12}$$
$$k_1 = P_0(\beta^2)\left[1 - X\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right)\right]\left(\frac{\bar{r}_H}{R_\parallel}\right)^2,$$
$$k_2 = P_0(\beta^2)\, Y\left(\frac{V_\parallel}{\sqrt{2}\,\Delta_\parallel}\right)\left(\frac{\bar{r}_H}{R_\parallel}\right)^2, \qquad \gamma = \frac{r_{Hp}}{\bar{r}_H}.$$

A6. Coefficient $D_\perp$ of transverse positron diffusion in the velocity space, occurring in a magnetized electron cloud

In Section 7 and Appendix A5 we calculated the longitudinal force of friction $\mathbf{F}_{cM}$ and the diffusion coefficient $D_\parallel$. The calculation is correct with the condition (46) for the ideality of plasma with an oblate velocity distribution. Unlike the longitudinal force $\mathbf{F}_{cM}$, the transverse force $\mathbf{F}_{cd}$ was calculated in Section 7 under the additional assumption (66) that the electrons were magnetized. To avoid exceeding the accuracy, the diffusion coefficient $D_\perp$ will be calculated in the same approximations (46), (66).

Much as we did in Appendices A4 and A5 we will derive the expression for the transverse diffusion coefficient:

$$D_\perp(V_\parallel, V_\perp) = \widehat{V}_{\perp\alpha}\widehat{V}_{\perp\beta}D_{\alpha\beta} = \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle V_\perp^2(t)\right\rangle$$
$$= \frac{1}{2}\left\langle \dot{\mathbf{V}}_\perp(t)\,\mathbf{V}_\perp(t)\right\rangle \equiv \frac{1}{2}\left\langle \dot{V}_a(t)\,V_a(t)\right\rangle . \quad \text{(A6.1)}$$

Hereinafter, Latin letters denote two-dimensional indices $a = (x, y) \equiv (1, 2)$; summation is performed, as usual, over repetitive indices, and $V_a \equiv (\mathbf{V}_\perp)_a$. The components of velocity $\mathbf{V}_\perp$ obey the equation of motion in which the fluctuation part of the field enters:

$$\dot{V}_a = \omega_{\mathrm{H}}\varepsilon_{ab}V_b + \frac{q}{M}E'_a , \quad \text{(A6.2)}$$

where ε_{ab} is an absolutely antisymmetric tensor in the two-dimensional (x, y) space, and

$$\varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{ab\gamma}h_\gamma .$$

From expression (A6.1) and equation (A6.2) it follows that

$$D_\perp = \frac{q}{2M}\left\langle V_a(t)\,E'_a\big(\mathbf{r}(t), t\big)\right\rangle , \quad \text{(A6.3)}$$

where $\mathbf{r}(t)$ is the unperturbed helical positron trajectory (neglecting fluctuations). From equation (A6.2) we have

$$V_a(t) = \int_{-\infty}^{\infty} G_{ab}(t - t')\,\frac{q}{M}\,E'_b\big(\mathbf{r}(t'), t'\big)\,\mathrm{d}t' , \quad \text{(A6.4)}$$

where $G_{ab}(\tau) = \left[\delta_{ab}\cos(\omega_{\mathrm{H}}\tau) + \varepsilon_{ab}\sin(\omega_{\mathrm{H}}\tau)\right]\theta(\tau)$ is the Green function of equation (A6.2), $\dot{G}_{ab} - \omega_{\mathrm{H}}\varepsilon_{ac}G_{cb} = \delta_{ab}\delta(\tau)$, and $\tau = t - t'$. From expressions (A3.1) and (A6.3) it follows that

$$D_\perp = \frac{q^2}{2M^2}\int \mathrm{d}t'\,\mathrm{d}Q\,G_{ab}E_{ab}(\mathbf{k}, \omega)\exp\left[\mathrm{i}\mathbf{k}\big(\mathbf{r}(t) - \mathbf{r}(t')\big) - \mathrm{i}\omega\tau\right] . \quad \text{(A6.5)}$$

The subsequent transformations are similar to those which led us from formula (A5.7) to expression (A5.9):

$$D_\perp = -\frac{\mathrm{i}q^2}{2M^2}\sum_{S=-\infty}^{\infty}\int \mathrm{d}Q\,J_S^2(k_\perp r_{\mathrm{Hp}})\,E_{aa}(\mathbf{k}, \omega)$$
$$\times\left[\frac{1}{\Omega - \mathrm{i}0 - (S+1)\omega_{\mathrm{H}}} + \frac{1}{\Omega - \mathrm{i}0 - (S-1)\omega_{\mathrm{H}}}\right] ,$$

where $\Omega = \omega - k_\parallel V_\parallel$, and the symmetry of the electric field correlation function (A3.4) with respect to indices has been taken into account. From the evenness of formula (A3.4) with respect to a change of sign of all arguments it follows that

$$D_\perp = \frac{\pi q^2}{4M^2}\sum_{S=-\infty}^{\infty}\int \mathrm{d}Q\,J_S^2(k_\perp r_{\mathrm{Hp}})\,E_{aa}(\mathbf{k}, \omega)$$
$$\times\left\{\delta\big[\Omega - (S+1)\omega_{\mathrm{H}}\big] + \delta\big[\Omega - (S-1)\omega_{\mathrm{H}}\big]\right\} . \quad \text{(A6.6)}$$

In this sum we retain the most significant terms:

$$D_\perp = D_\perp^{(1)} + D_\perp^{(2)} . \quad \text{(A6.7)}$$

Here, $D_\perp^{(1)}$ corresponds to the term with $S = 0$:

$$D_\perp^{(1)} = \frac{\pi q^2}{4M^2}\int \mathrm{d}Q\,J_0^2(k_\perp r_{\mathrm{Hp}})\,E_{aa}(\mathbf{k}, \omega)$$
$$\times\left[\delta(\Omega - \omega_{\mathrm{H}}) + \delta(\Omega + \omega_{\mathrm{H}})\right] , \quad \text{(A6.8)}$$

and $D_\perp^{(2)}$ corresponds to the terms with $S = \pm 1$:

$$D_\perp^{(2)} = \frac{\pi q^2}{2M^2}\int \mathrm{d}Q\,J_1^2(k_\perp r_{\mathrm{Hp}})\,E_{aa}(\mathbf{k}, \omega)\,\delta(\Omega) . \quad \text{(A6.9)}$$

The tube (165) makes the main contribution to the term $D_\perp^{(1)}$. From expression (A6.8) we obtain, in view of expressions (A3.17), (A3.21), and (A3.22), the following formula, which is logarithmically accurate to the terms of order $1/\ln(T_\perp/T_\parallel) \sim 0.15$:

$$D_\perp^{(1)} = \frac{\pi n q^2 e^2}{M^2}\,g(V_\parallel)\ln\frac{T_\perp}{T_\parallel} . \quad \text{(A6.10)}$$

The domain $|k_\parallel| \sim k_\perp \sim 1/R_\parallel$ makes the main contribution to the term $D_\perp^{(2)}$. For magnetized electrons (66), from expressions (A3.17), (A3.19), (A3.20), and (A6.9) we obtain

$$D_\perp^{(2)} \sim \frac{n q^2 e^2}{M^2}\,g(V_\parallel)\ln\frac{R_\parallel}{\bar{r}_{\mathrm{H}}} . \quad \text{(A6.11)}$$

For the LEPTA experiment, one has $\bar{r}_{\mathrm{H}} \sim R_\parallel$, and so from formulas (A6.10) and (A6.11) we conclude that

$$\frac{D_\perp^{(2)}}{D_\perp^{(1)}} \sim \frac{1}{\ln(T_\perp/T_\parallel)} . \quad \text{(A6.12)}$$

Consequently, with the same logarithmic accuracy we have

$$D_\perp \approx D_\perp^{(1)} . \quad \text{(A6.13)}$$

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