13. Concluding remarks

The '$E = mc^2$ problem': could it be avoided? One is tempted to think that the '$E = mc^2$ problem' would not arise from the first place if the quantity E/c^2 — the proportionality coefficient between velocity and momentum — were identified with a new physical quantity christened as, say, 'inertia' or 'iner'; it would be identical to mass as momentum tended to zero. As a result, mass would become 'rest inertia'. Likewise, another new quantity could be introduced — 'heaviness' or 'grav' — $p_i p_k/E$ reducing to mass at zero momentum. But physicists preferred 'to refrain from multiplying entities' and from introducing new physical quantities. They formulated instead new, more general relations between old quantities, for example $E^2 - \mathbf{p}^2c^2 = m^2c^4$ and $\mathbf{p} = \mathbf{v}E/c^2$.

Unfortunately, many authors attempt to retain even in relativistic physics such nonrelativistic equations as $\mathbf{p} = m\mathbf{v}$, and such nonrelativistic glued-up concepts as 'mass is a measure of inertia' and 'mass is a measure of gravitation'; as a result, they prefer to use the notion of velocity-dependent mass.

It is amazing how again and again a physicist would choose the first of these paths (new equations) in his research papers and the second one (old glued-up concepts) in science-popularizing and pedagogical activities. This could of course only produce unbelievable confusion in the minds of those who read popular texts and blindly follow the authority.

On the reliability of science. An opinion that has become widely publicized recently is that science in general and physics in particular are untrustworthy. Many popularizers of science create the impression that the theory of relativity proved Newton's mechanics wrong just as chemistry proved alchemy wrong and astronomy proved astrology wrong. Such declarations are a crude distortion of the essence of scientific revolutions. Newton's mechanics remains a correct science today, in the XXIst century, and will continue to be correct forever. The discovery of the theory of relativity only put bounds on the domain of applicability of Newton's mechanics to velocities much smaller than the speed of light c. It also demonstrated its approximate nature in this domain (to within corrections of the order of v^2/c^2).

Similarly, the discovery of quantum mechanics put bounds on the domain of applicability of classical mechanics to phenomena for which the quantity of action is large in comparison with the quantum of action $\hbar$. Quite to the contrary, the domain where astrology and alchemy exist is that of prejudice, superstition, and ignorance. It is rather funny that those who compare Newton's mechanics with astrology typically believe that mass depends on velocity.

Recent publications. Additional information on the aspects discussed above can be found in [11, 12].

On the title. My good friend and expert in the theory of relativity read the slides of this talk and advised me to drop Pythagoras's name from the title. I chose not to follow his advice as in the relativity-related literature I had never come across a discussion of right-angled triangles without the approximate extraction of square roots.

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References

1. Okun L B "Formula Einshteina: $E_0 = mc^2$. 'Ne smeetsya li Gospod' Bog'?" *Usp. Fiz. Nauk* **178** 541 (2008) ["The Einstein formula $E_0 = mc^2$ 'Isn't the Lord laughing'?" *Phys. Usp.* **51** (5) (2008)]
2. Klein F *Elementarmathematik vom höeheren Standpunkte aus. Erster Band, Arithmetik, Algebra, Analysis* 3 Auflage (Berlin: Verlag von Julius Springer, 1924) [Translated into English: *Elementary Mathematics from an Advanced Standpoint, Arithmetics, Algebra, Analysis* (New York: Dover Publ., 2007); Translated into Russian (Moscow: Nauka, 1987)]
3. Poincaré H "Sur la dynamique de l'électron" *Rendiconti del Circolo Matematico di Palermo* **21** 129 (1906) [Translated into Russian: "O dinamike elektrona" ("On the dynamics of electron") *Izbrannye Trudy* (Selected Works) Vol. 3 (Moscow: Nauka, 1974) p. 433]
4. Minkowski H "Raum und Zeit" *Phys. Z.* **10** 104 – 111 (1909) [Translated into Russian: "Prostranstvo i Vremya" ("Space and time"), in Lorentz H A, Poincaré H, Einstein A, Minkowski H *Printsip Otnositel'nosti. Sbornik Rabot Klassikov Relyativizma* (The Principle of Relativity. Collected Papers of Classics of Relativism) (Eds V K Frederiks, D D Ivanenko) (Moscow – Leningrad: ONTI, 1935) pp. 181 – 203]
5. Hawking S *The Universe in a Nutshell* (New York: Bantam Books, 2001) [Translated into Russian (Translated from English by A Sergeev) St.-Petersburg: Amfora, 2007)]
6. Landau L D, Lifshitz E M *Teoriya polya* (The Classical Theory of Fields) (Moscow: Nauka, 1988) [Translated into English (Amsterdam: Reed Elsevier, 2000)]
7. Landau L D, Lifshitz E M *Mekhanika* (Mechanics) (Moscow: Nauka, 1988) [Translated into English (Amsterdam: Elsevier Sci., 2003)]
8. Okun L B "The concept of mass" *Phys. Today* **42** (6) 31 – 36 (1989); Okun L B "Putting to rest mass misconceptions" *Phys. Today* **43** (5) 13, 15, 115, 117 (1990)
9. Pound R V, Rebka G A (Jr.) "Apparent weight of photons" *Phys. Rev. Lett.* **4** 337 – 341 (1960)
10. Pauli W, Jung C *Atom and Archetype: Pauli/Jung Letters. 1932 – 1958*. (Ed. C A Meyer) (Princeton, NJ: Princeton Univ. Press, 2001); Meier C A (Herausgegeben) *Wolfgang Pauli, C. G. Jung: Ein Briefwechsel 1932 – 1958* (Berlin: Springer-Verlag, 1992)
11. Okun L B "Chto takoe massa? (Iz istorii teorii otnositel'nosti)" ('What is mass? (From the history of relativity theory)'), in *Issledovaniya po Istorii Fiziki i Mekhaniki*. 2007 (Research on the History of Physics and Mechanics. 2007) (Executive Ed. G M Idlis) (Moscow: Nauka, 2007)
12. Okun L B "The evolution of the concepts of energy, momentum and mass from Newton and Lomonosov to Einstein and Feynman", in *Proc. of the 13th Lomonosov Conf. August 23, 2007* (Singapore: World Scientific) (in press)

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DOI: 10.3367/UFNr.0178.200806l.0663

Bjorken and Regge asymptotics of scattering amplitudes in QCD and in supersymmetric gauge models

L N Lipatov

1. Introduction

We review theoretical approaches to the investigation of deep-inelastic lepton – hadron interactions and high-energy hadron – hadron scattering in the Regge kinematics. It is demonstrated that the gluon in QCD is Reggeized and the Pomeron is a composite state of the Reggeized gluons. Remarkable properties of the BFKL equation for the

Pomeron wave function in QCD and supersymmetric gauge theories are outlined. It is shown that by the AdS/CFT correspondence, the BFKL Pomeron is equivalent to the Reggeized graviton in the $N = 4$ extended supersymmetric model. The maximal transcendentality and integrability properties realized in this model allow calculating the anomalous dimension of twist-2 operators up to 4 loops.

2. Deep-inelastic ep scattering

The inclusive electron – proton scattering in the Bjorken kinematics (see Fig. 1),

$$2pq \sim Q^2 = -q^2 \to \infty\,, \quad x = \frac{Q^2}{2pq}\,, \quad 0 \leqslant x \leqslant 1\,, \tag{1}$$

is very important because it gives direct information about the distribution $n^{\mathrm{q}}(x)$ of quarks inside the proton as a function of their energy ratio $x = |\mathbf{k}|/|\mathbf{p}|$ ($|\mathbf{p} \to \infty|$). Indeed, in the framework of the Feynman – Bjorken quark – parton model [1, 2], we can obtain the following simple expression for the structure functions $F_{1,2}(x)$ of this process:

$$\frac{1}{x}\,F_2(x) = 2F_1(x) = \sum_{i=\mathrm{q},\bar{\mathrm{q}}} Q_i^2\, n^i(x)\,, \tag{2}$$

where the quark charges are $Q_{\mathrm{u}} = 2/3$, $Q_{\mathrm{d}} = -1/3$.

It turns out that the partonic picture is also valid in renormalizable field theories if the parton transverse momenta $\mathbf{k}_\perp$ are restricted by an ultraviolet cut-off $k_\perp^2 < \Lambda^2 \sim Q^2$ [3]. In these theories, the running coupling constant $\alpha = g^2/(4\pi)$ in the leading logarithmic approximation (LLA) is

$$\alpha(Q^2) = \frac{\alpha_\mu}{1 + \beta\alpha_\mu/(4\pi)\ln(Q^2/\mu^2)}\,, \tag{3}$$

where α_μ is its value at the renormalization point μ. In quantum electrodynamics (QED) and quantum chromodynamics (QCD), the coefficients β have opposite signs,

$$\beta_{\mathrm{QED}} = -n_{\mathrm{e}}\,\frac{4}{3}\,, \quad \beta_{\mathrm{QCD}} = \frac{11}{3}\,N_{\mathrm{c}} - n_{\mathrm{f}}\,\frac{2}{3}\,, \tag{4}$$

where N_{c} is the rank of the gauge group ($N_{\mathrm{c}} = 3$ for QCD), and n_{e} and n_{f} are the numbers of leptons and quarks, which can be considered massless for a given Q^2.

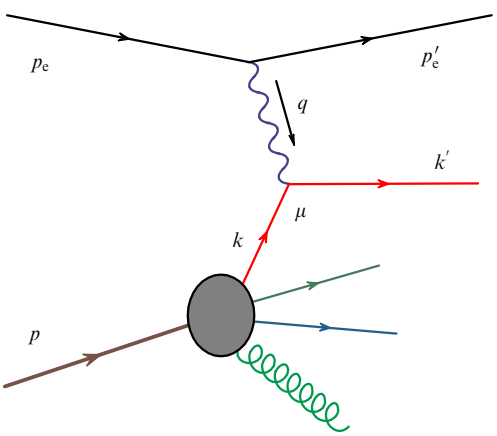

Figure 1.

Landau and Pomeranchuk argued that because of the negative sign of β_{QED}, a Landau pole is generated in the photon propagator, which leads to the vanishing of the physical electric charge in the local limit. On the other hand, in QCD, the non-Abelian interaction disappears at large Q^2 and, as a result of the asymptotic freedom, we have an approximate Bjorken scaling: the structure functions depend on Q^2 only logarithmically [4]. Thus, the experiments on deep-inelasic ep scattering performed at SLAC at the end of the 1960s discovered that the Landau 'zero charge' problem is absent in strong interactions.

In the infinite-momentum frame $|\mathbf{p}| \to \infty$, it is helpful to introduce the Sudakov variables for parton momenta as

$$\mathbf{k}_i = \beta_i\,\mathbf{p} + \mathbf{k}_i^\perp\,, \quad (\mathbf{k}_i^\perp, \mathbf{p}) = 0\,, \quad \sum_i \mathbf{k}_i = \mathbf{p}\,. \tag{5}$$

The parton distributions are defined in terms of the proton wave function Ψ_m as

$$n^i(x) = \sum_m \int \prod_{r=1}^{m-1} \frac{\mathrm{d}\beta_r\,\mathrm{d}^2k_r^\perp}{(2\pi)^2}\,|\Psi_m|^2 \sum_{r\in i} \delta(\beta_r - x)\,. \tag{6}$$

They are functions of $\Lambda \sim Q$ because the factor $|\Psi_m|^2 \sim \prod_{r=1}^m Z_r$ depends on Λ through the wave-function renormalization constants $\sqrt{Z_r}$ and Λ is the upper limit in integrals over the transverse momenta $k_r^\perp$. With the cascade-type dynamics of the parton number growth and with Λ taken into account, we can obtain the evolution equations of Dokshizer, Gribov, Lipatov, Altarelli, and Parisi (DGLAP) [3, 5] in the LLA,

$$\frac{\mathrm{d}}{\mathrm{d}\xi(Q^2)}\,n_i(x) = -w_i\,n_i(x) + \sum_r \int_x^1 \frac{\mathrm{d}y}{y}\,w_{r\to i}\left(\frac{x}{y}\right) n_r(y)\,, \tag{7}$$

$$w_i = \sum_k \int_0^1 \mathrm{d}x\,x\,w_{i\to k}(x)\,, \tag{8}$$

where

$$\xi(Q^2) = \frac{N_{\mathrm{c}}}{2\pi} \int_{\mu^2}^{Q^2} \frac{\mathrm{d}\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2}\,\alpha(\mathbf{k}_\perp^2)\,. \tag{9}$$

Equation (7) has a clear probabilistic interpretation: the number of partons n_i decreases because of their decay into other partons in the opening phase space $\mathrm{d}\xi(Q^2)$ and increases because the decay products of other partons r can contain partons of the type i [3].

The momenta of parton distributions

$$n_i^j = \int_0^1 \mathrm{d}x\,x^{j-1}\,n_i(x) \tag{10}$$

satisfy the renormalization-group equations

$$\frac{\mathrm{d}}{\mathrm{d}\xi(Q^2)}\,n_i^j = \sum_r w_{r\to i}^j\,n_r^j\,, \tag{11}$$

and are related to the matrix elements of twist-2 operators

$$n_i(j) = \langle p\,|\,O_i^j\,|\,p\rangle\,. \tag{12}$$

The twist t is defined as the difference between their canonical dimension d measured in units of mass and the Lorentz spin j of the corresponding tensor. The quantities $w_{r\to i}^j$ are elements of the anomalous dimension matrix for the operators O_i^j.

3. High-energy interactions

Hadron–hadron scattering in the Regge kinematics (see Fig. 2)

$$s = (p_A + p_B)^2 = (2E)^2 \gg \mathbf{q}^2 = -(p_{A'} - p_A)^2 \sim m^2 \quad (13)$$

is usually described in terms of a t-channel exchange of the Reggeon (see Fig. 3),

$$A_p(s,t) = \xi_p(t)\, g(t)\, s^{j_p(t)}\, g(t)\,, \quad j_p(t) = j_0 + \alpha' t\,, \quad (14)$$

$$\xi_p = \frac{\exp\left(-\mathrm{i}\pi j_p(t)\right) + p}{\sin(\pi j_p)}\,, \quad (15)$$

where $j_p(t)$ is the Regge trajectory, assumed to be linear, and j_0 and α' are its intercept and slope. The signature factor ξ_p is a complex quantity depending on the Reggeon signature $p = \pm 1$. A special Reggeon–Pomeron is introduced to explain the approximately constant behavior of total cross sections at high energies and the fulfillment of the Pomeranchuk theorem $\sigma_{h\bar{h}}/\sigma_{hh} \to 1$. Its signature p is positive and its intercept is close to unity: $j_0^p = 1 + \Delta$, $\Delta \ll 1$. The field theory of Pomeron interactions was constructed by Gribov around 40 years ago.

Particle production at high energies can be investigated in the multi-Regge kinematics (see Fig. 4)

$$s \gg s_1\,, \quad s_2\,, \ldots\,, \quad s_{n+1} \gg t_1\,, \quad t_2\,, \ldots\,, \quad t_{n+1}\,, \quad (16)$$

where s_r are squares of the sums of neighboring particle momenta k_{r-1} and k_r, and $-t_r$ are squares of the momentum transfers $\mathbf{q}_r$. This amplitude can also be expressed in terms of the Reggeon exchanges in each of the t_r-channels:

$$A_{2\to 2+n} \sim \prod_{r=1}^{n+1} s_r^{j_p(t_r)}\,. \quad (17)$$

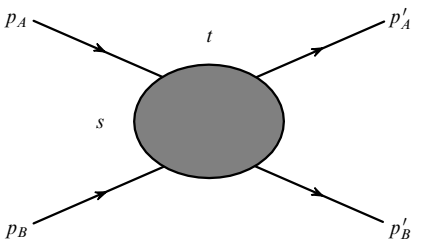

Figure 2.

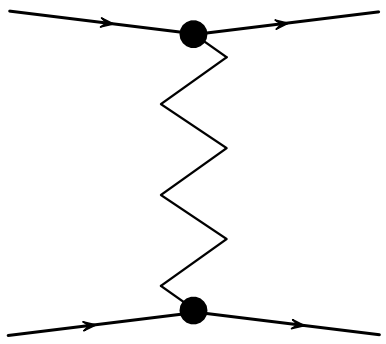

Figure 3.

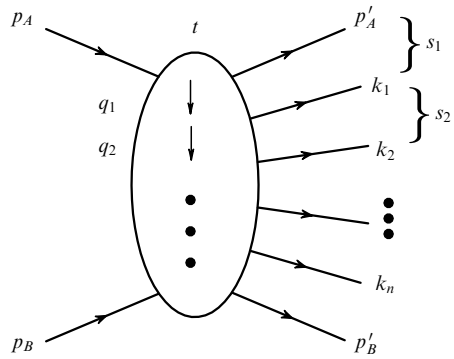

Figure 4.

4. Gluon Reggeization in QCD

In the Born approximation in QCD, the scattering amplitude for two-colored particle scattering is factored (see Fig. 2),

$$M_{AB}^{A'B'}(s,t)\big|_{\text{Born}} = \Gamma^{\text{c}}_{A'A}\,\frac{2s}{t}\,\Gamma^{\text{c}}_{B'B}\,, \quad \Gamma^{\text{c}}_{A'A} = g\,T^{\text{c}}_{A'A}\delta_{\lambda_{A'}\lambda_A}\,, \quad (18)$$

where T^{c} are the generators of the color group $\mathrm{SU}(N_{\text{c}})$ in the corresponding representation and λ_r are helicities of the colliding and final-state particles. In the LLA, the scattering amplitude in QCD can be written as [6]

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)\big|_{\text{Born}}\, s^{\omega(t)}, \quad \alpha_s \ln s \sim 1\,, \quad (19)$$

where the gluon Regge trajectory is

$$\omega\left(-|q|^2\right) = -\int \frac{\mathrm{d}^2 k}{4\pi^2}\,\frac{\alpha_s N_{\text{c}}\,|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_{\text{c}}}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}\,. \quad (20)$$

The fictitious gluon mass λ is introduced here to regularize the infrared divergence. This trajectory was also calculated in the two-loop approximation in QCD [7] and in supersymmetric gauge theories [8].

Further, the gluon production amplitude in the multi-Regge kinematics can be written in the factored form [6]

$$M_{2\to 1+n} = 2s\,\Gamma^{\text{c}_1}_{A'A}\,\frac{s_1^{\omega_1}}{|q_1|^2}\,g T^{d_1}_{\text{c}_2\text{c}_1}\,C(q_2,q_1) \times \frac{s_2^{\omega_2}}{|q_2|^2}\ldots C(q_n,q_{n-1})\,\frac{s_n^{\omega_n}}{|q_n|^2}\,\Gamma^{\text{c}_n}_{B'B}\,. \quad (21)$$

The Reggeon–Reggeon–gluon vertex for the produced gluon with a definite helicity is

$$C(q_2,q_1) = \frac{q_2\,q_1^*}{q_2^* - q_1^*}\,, \quad (22)$$

where we use the complex notation for the transverse components of particle momenta. This allows calculating the total cross section [6]

$$\sigma_{\text{t}} = \sum_n \int \mathrm{d}\Gamma_n \left|M_{2\to 1+n}\right|^2, \quad (23)$$

where Γ_n is the phase space for the produced particle momenta in the multi-Regge kinematics.

5. The BFKL equation

Using the fact that the production amplitudes in QCD are factored, we can write a Bethe – Salpeter-type equation for the total cross section $\sigma_{\rm t}$. Also using the optical theorem, we can represent this equation as the Balitsky – Fadin – Kuraev – Lipatov (BFKL) equation for the Pomeron wave function [6]:

$$E\,\Psi(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = H_{12}\Psi(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\,, \quad \Delta = -\frac{\alpha_s N_{\rm c}}{2\pi}\,E\,, \tag{24}$$

where $\sigma_{\rm t} \sim s^{\Delta}$ and the BFKL Hamiltonian in the coordinate representation is

$$H_{12} = \ln|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}\left(\ln|\rho_{12}|^2\right) p_1 p_2^* + \frac{1}{p_1^* p_2}\left(\ln|\rho_{12}|^2\right) p_1^* p_2 - 4\psi(1)\,, \quad \rho_{12} = \rho_1 - \rho_2\,. \tag{25}$$

It is invariant under the Möbius transformations [9, 10]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,. \tag{26}$$

We use the complex notation for transverse coordinates and their canonically conjugate momenta. The conformal weights for the principal series of unitary representations of the Möbius group are

$$m = \gamma + \frac{n}{2}\,, \quad \widetilde{m} = \gamma - \frac{n}{2}\,, \quad \gamma = \frac{1}{2} + {\rm i}\nu\,, \tag{27}$$

where γ is the anomalous dimension of the twist-2 operators and n is the conformal spin.

The Bartels – Kwiecinski – Praszalowicz equation for colorless composite states of several Reggeized gluons has the form [11]

$$E\,\Psi(\boldsymbol{\rho}_1,\ldots) = H\,\Psi(\boldsymbol{\rho}_1,\ldots)\,, \quad H = \sum_{k<l} \frac{\mathbf{T}_k\mathbf{T}_l}{-N_{\rm c}}\,H_{kl}\,, \tag{28}$$

where H_{kl} is the BFKL Hamiltonian. In addition to the Möbius invariance, its wave function in the multi-color QCD ($N_{\rm c} \to \infty$) has the holomorphic factorization property [12]

$$\Psi(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1,\ldots,\rho_n)\,\Psi_s(\rho_1^*,\ldots,\rho_n^*)\,, \tag{29}$$

where the sum is taken over the degenerate set of solutions of the corresponding holomorphic and antiholomorphic BFKL equations. These equations have the duality symmetry $p_k \to \rho_{k,k+1} \to p_{k+1}$ $(k = 1, 2, ..., n)$ [13] and n integrals of motion q_r, q_r^* [14]. The corresponding Hamiltonians h and h^* are local Hamiltonians of an integrable Heisenberg spin model in which spins are generators of the Möbius group [15]. We can introduce the transfer (T) and monodromy (t) matrices according to the definitions [14]

$$T(u) = {\rm tr}\,t(u)\,, \quad t(u) = L_1 L_2 \ldots L_n = \sum_{r=0}^{n} u^{n-r}\,q_r\,, \tag{30}$$

$$L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix}. \tag{31}$$

Then the monodromy matrix $t(u)$ satisfies the Yang – Baxter equation [14]

$$t_{r_1'}^{s_1}(u)\,t_{r_2'}^{s_2}(v)\,l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\,t_{r_2}^{s_2'}(v)\,t_{r_1}^{s_1'}(u)\,,$$

$$\hat{l}(u) = u\hat{1} + {\rm i}\hat{P}\,, \tag{32}$$

where $\hat{l}(u)$ is the monodromy matrix for the usual Heisenberg spin model and $\hat{P}$ is the permutation operator. This equation can be solved with the use of the Bethe ansatz and the Baxter – Sklyanin approach.

6. Pomeron in the $N = 4$ SUSY

We can also calculate the integral kernel for the BFKL equation in two loops [16]. Its eigenvalue can be written as

$$\omega = 4\hat{a}\,\chi(n,\gamma) + 4\,\hat{a}^2\Delta(n,\gamma)\,, \quad \hat{a} = \frac{g^2 N_{\rm c}}{16\pi^2}\,, \tag{33}$$

where

$$\chi(n,\gamma) = 2\psi(1) - \psi\left(\gamma + \frac{|n|}{2}\right) - \psi\left(1 - \gamma + \frac{|n|}{2}\right) \tag{34}$$

and $\psi(x) = \Gamma'(x)/\Gamma(x)$. The one-loop correction $\Delta(n,\gamma)$ in QCD contains nonanalytic terms, the Kronecker symbols $\delta_{|n|,0}$ and $\delta_{|n|,2}$ [8]. But in the $N = 4$ SUSY, they cancel and we obtain the following result for $\Delta(n,\gamma)$ in the Hermitian separable form [8, 17]:

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,$$

$$M = \gamma + \frac{|n|}{2}\,, \tag{35}$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\,\zeta(2)\,,$$

$$\beta'(z) = \frac{1}{4}\left[\psi'\left(\frac{z+1}{2}\right) - \psi'\left(\frac{z}{2}\right)\right]. \tag{36}$$

It is interesting that all functions entering these expressions have the maximal transcendentality property [17]. Moreover, $\phi(M)$ can be written as

$$\phi(M) = 3\zeta(3) + \psi''(M) - 2\Phi(M) + 2\beta'(M)\big(\psi(1) - \psi(M)\big)\,, \tag{37}$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\psi'(k+1) - \frac{\psi(k+1) - \psi(1)}{k+M}\right), \tag{38}$$

where $\psi(M)$ has the transcedentality equal to 1, its derivatives $\psi^{(n)}$ have transcedentalities $n+1$, and the additional poles in the sum over k increase the transcedentality of $\Phi(M)$ up to 3, which is also the transcendentality of $\zeta(3)$. The maximal transcendentality hypothesis is also valid for the anomalous dimensions of twist-2 operators in the $N = 4$ SUSY [18, 19], in contrast to the case of QCD [20]. This result is discussed in the next section.

Generally, the BFKL equation in the diffusion approximation can be written in the simple form [6]

$$j = 2 - \Delta - Dv^2 , \tag{39}$$

where v is related to the anomalous dimension of the twist-2 operators as [16]

$$\gamma = 1 + \frac{j-2}{2} + \mathrm{i}v . \tag{40}$$

The parameters Δ and D are functions of the coupling constant $\hat{a}$ and are known up to two loops. Higher-order perturbative corrections can be obtained with the use of the effective action [21, 22]. For large coupling constants, we can expect that the leading Pomeron singularity in the $N = 4$ SUSY is moved to the point $j = 2$ and the Pomeron asymptotically coincides with the graviton Regge pole. This assumption is related to the AdS/CFT correspondence, formulated in the framework of the Maldacena hypothesis that the $N = 4$ SUSY is equivalent to a superstring model living on the 10-dimensional anti-de Sitter space [23–25]. For the BFKL equation in the diffusion approximation, it is therefore natural to impose the physical condition that γ is zero for the conserved energy–momentum tensor $\vartheta_{\mu\nu}(x)$ having the Lorents spin $j = 2$. As a result, we obtain that the parameters Δ and D coincide [19]. In this case, we can solve the above BFKL equation for γ:

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1 + \sqrt{1 + (j-2)/\Delta}}\right) . \tag{41}$$

Using the dictionary developed in the framework of the AdS/CFT correspondence [24], we can rewrite the BFKL equation in the form of the graviton Regge trajectory [19]

$$j = 2 + \frac{\alpha'}{2}\, t , \quad t = \frac{E^2}{R^2} , \quad \alpha' = \frac{R^2}{2}\, \Delta . \tag{42}$$

On the other hand, Gubser, Klebanov, and Polyakov predicted the following asymptotic form of the anomalous dimension at large $\hat{a}$ and j [26]:

$$\gamma_{|\,\hat{a}, j\to\infty} = -\sqrt{j-2}\, \Delta^{-1/2}_{|\,j\to\infty} = \sqrt{2\pi j}\, \hat{a}^{\,1/4} . \tag{43}$$

As a result, we can obtain the explicit expression for the Pomeron intercept at large coupling constants [19, 27],

$$j = 2 - \Delta , \quad \Delta = \frac{1}{2\pi}\, \hat{a}^{\,-1/2} . \tag{44}$$

7. Maximal transcedentality

According to the hypothesis discussed above, the anomalous dimension

$$\gamma(j) = \hat{a}\gamma_1(j) + \hat{a}^2\gamma_2(j) + \hat{a}^3\gamma_3(j) + \ldots \tag{45}$$

should contain the maximally transcendental functions [17]. Indeed, we have

$$\gamma_1(j+2) = -4S_1(j) , \tag{46}$$

$$\frac{\gamma_2(j+2)}{8} = 2S_1\left(S_2 + S_{-2}\right) - 2S_{-2,1} + S_3 + S_{-3} \tag{47}$$

in two loops [17, 18], and

$$\begin{aligned}
\frac{\gamma_3(j+2)}{32} &= -12\big(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\big) \\
&+ 6\big(S_{-4,1} + S_{-3,2} + S_{-2,3}\big) - 3S_{-5} - 2S_3\, S_{-2} - S_5 \\
&- 2S_1^2\big(3\, S_{-3} + S_3 - 2\, S_{-2,1}\big) - S_2\big(S_{-3} + S_3 - 2\, S_{-2,1}\big) \\
&+ 24S_{-2,1,1,1} - S_1\big(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2\big) \\
&- S_1\big(3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1}\big)
\end{aligned} \tag{48}$$

in three loops [19], where the harmonic sums are defined as

$$\begin{aligned}
&S_a(j) = \sum_{m=1}^{j} \frac{1}{m^a} , \quad S_{a,b,c,\ldots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\, S_{b,c,\ldots}(m) , \\
&S_{-a}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a} , \quad S_{-a,b,\ldots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\, S_{b,\ldots}(m) , \\
&\overline{S}_{-a,b,c,\ldots}(j) = (-1)^j S_{-a,b,\ldots}(j) + S_{-a,b,\ldots}(\infty)\big(1 - (-1)^j\big) .
\end{aligned} \tag{49}$$

It was argued in Ref. [28] that for the $N = 4$ SUSY, the evolution equations for anomalous dimensions of quasi-partonic operators are integrable in the LLA. Later, such an integrability was generalized to other operators [29] and to higher loops [30]. With the additional use of the maximal transcendentality hypothesis, the integral equation for the so-called casp anomalous dimension was constructed in all orders of the perturbation theory [31, 32].

To calculate the anomalous dimension of the twist-2 operators in 4 loops, we can apply the integrability approach based on the asymptotic Bethe ansatz [30]. The corresponding equations for the Bethe roots u_k are

$$\left(\frac{x_k^+}{x_k^-}\right)^2 = \prod_{r=1}^{j-2} \frac{x_k^- - x_r^+}{x_k^+ - x_r^-}\, \frac{1 - g^2/x_k^+x_r^-}{1 - g^2/x_k^-x_r^+} \exp\big(2\mathrm{i}\theta(u_k, u_r)\big) , \tag{50}$$

where we use the notation

$$x_k^\pm = \frac{u_k^\pm}{2} + \sqrt{\frac{(u_k^\pm)^2}{4} - g^2} , \quad u^\pm = u \pm \frac{\mathrm{i}}{2} \tag{51}$$

and the dressing phase expansion [32]

$$\theta(u_k, u_j) = 4\zeta(3)\, g^6\, \big(q_2(u_k)\, q_3(u_j) - q_3(u_k)\, q_2(u_j)\big) + \ldots . \tag{52}$$

The solution for $u_k^\pm$ allows finding the anomalous dimensions

$$\gamma(g, M) = 2g^2 \sum_{k=1}^{M} \left(\frac{\mathrm{i}}{x_k^+} - \frac{\mathrm{i}}{x_k^-}\right) . \tag{53}$$

In four loops, in particular, we can obtain [33]

$$\begin{aligned}
\frac{\gamma_4}{256} &= 4S_{-7} + 6S_7 \\
&+ 2(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) + \ldots \\
&- 80S_{1,1,-4,1} - \zeta(3)S_1(S_3 - S_{-3} + 2S_{-2,1}) ,
\end{aligned} \tag{54}$$

where the harmonic sums depend on $j-2$ and the dots denote the omitted terms (their number exceeds 200). All these terms satisfy the maximal transcendentality property. The last term appears from the dressing phase.

It turns out that after the analytic continuation of this expression in the complex j-plane, the first two terms give rise to the pole $1/\omega^7$ for $\omega = j-1 \to 0$, which contradicts the singularity at this point predicted in 4 loops from the BFKL equation,

$$\lim_{j\to 1} \gamma_4(j) = -\frac{32}{\omega^4}\left(32\zeta_{(3)} + \frac{\pi^4}{9}\,\omega\right) + \dots . \qquad (55)$$

This means that the asymptotic Bethe ansatz should be modified starting from 4 loops. Specifically, wrapping effects should be taken into account [33].

Interesting results were also obtained for the scattering amplitudes in the $N=4$ SUSY for particles on the mass shell [34]. These amplitudes were used in Ref. [35] for the construction of higher-loop corrections to the BFKL kernel in this model. But it was shown in [35] that the BDS ansatz in [34] does not satisfy the correct factorization properties in the multi-Regge kinematics.

8. Discussion of the obtained results

It was demonstrated that the Pomeron in QCD is a composite state of reggeized gluons. The BFKL dynamics is integrable in the LLA. In the next-to-leading approximation in the $N=4$ SUSY, the equation for the Pomeron wave function has remarkable properties, including analyticity in the conformal spin n and maximal transcendentality. In this model, the BFKL Pomeron coincides with the Reggeized graviton. The anomalous dimension for twist-2 operators has the maximal transcendentality property, which allows calculating it analytically in 2 and 3 loops. The integrability based on the asymptotic Bethe ansatz reproduces these results, but fails to reproduce the BFKL prediction in 4 loops due to the presence of wrapping effects. The BDS ansatz for scattering amplitudes in the $N=4$ SUSY does not agree with the BFKL approach in the multi-Regge kinematics.

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References

1. Bjorken J D, in *Selected Topics in Particle Physics*: *Proc. of Intern. School of Physics "Enrico Fermi", Course 41* (Ed. J Steinberger) (New York: Academic Press, 1968); Bjorken J D, Paschos E A *Phys. Rev.* **185** 1975 (1969)
2. Feynman R P *Phys. Rev. Lett.* **23** 1415 (1969)
3. Lipatov L N *Yad. Fiz.* **20** 181 (1975) [*Sov. J. Nucl. Phys.* **20** 94 (1975)]
4. Gross D J, Wilczek F *Phys. Rev. Lett.* **30** 1343 (1973); Politzer H D *Phys. Rev. Lett.* **30** 1346 (1973)
5. Gribov V N, Lipatov L N *Yad. Fiz.* **15** 781 (1972) [*Sov. J. Nucl. Phys.* **15** 438 (1972)]; Altarelli G, Parisi G *Nucl. Phys. B* **126** 298 (1977); Dokshitzer Yu L *Zh. Eksp. Teor. Fiz.* **73** 1216 (1977) [*Sov. Phys. JETP* **46** 641 (1977)]
6. Lipatov L N *Yad. Fiz.* **23** 642 (1976) [*Sov. J. Nucl. Phys.* **23** 338 (1976)]; Fadin V S, Kuraev E A, Lipatov L N *Phys. Lett. B* **60** 50 (1975); Kuraev E A, Lipatov L N, Fadin V S *Zh. Eksp. Teor. Fiz.* **71** 840 (1976); **72** 377 (1977) [*Sov. Phys. JETP* **44** 443 (1976); **45** 199 (1977)]; Balitsky I I, Lipatov L N *Yad. Fiz.* **28** 1597 (1978) [*Sov. J. Nucl. Phys.* **28** 822 (1978)]
7. Fadin V S, Fiore R, Kotsky M I *Phys. Lett. B* **387** 593 (1996)
8. Kotikov A V, Lipatov L N *Nucl. Phys. B* **582** 19 (2000)
9. Lipatov L N *Phys. Lett. B* **309** 394 (1993)
10. Lipatov L N *Zh. Eksp. Teor. Fiz.* **90** 1536 (1986) [*Sov. Phys. JETP* **63** 904 (1986)]
11. Bartels J *Nucl. Phys. B* **175** 365 (1980); Kwieciński J, Praszałowicz M *Phys. Lett. B* **94** 413 (1980)
12. Lipatov L N *Phys. Lett. B* **251** 284 (1990)
13. Lipatov L N *Nucl. Phys. B* **548** 328 (1999)
14. Lipatov L N "High energy asymptotics of multi-colour QCD and exactly solvable lattice models", Preprint DFPD/93/TH/70 (Padua: Inst. Naz. Fis. Nucleare, 1993); hep-th/9311037
15. Lipatov L N *Pis'ma Zh. Eksp. Teor. Fiz.* **59** 571 (1994) [*JETP Lett.* **59** 596 (1994)]; Faddeev L D, Korchemsky G P *Phys. Lett. B* **342** 311 (1995)
16. Fadin V S, Lipatov L N *Phys. Lett. B* **429** 127 (1998); Ciafaloni M, Camici G *Phys. Lett. B* **430** 349 (1998)
17. Kotikov A V, Lipatov L N *Nucl. Phys. B* **661** 19 (2003)
18. Kotikov A V, Lipatov L N, Velizhanin V N *Phys. Lett. B* **557** 114 (2003)
19. Kotikov A V et al. *Phys. Lett. B* **595** 521 (2004); 'Erratum' **632** 754 (2006)
20. Moch S, Vermaseren J A M, Vogt A *Nucl. Phys. B* **688** 101 (2004)
21. Lipatov L N *Nucl. Phys. B* **452** 369 (1995); *Phys. Rep.* **286** 131 (1997)
22. Antonov E N et al. *Nucl. Phys. B* **721** 111 (2005)
23. Maldacena J M *Adv. Theor. Math. Phys.* **2** 231 (1998)
24. Gubser S S, Klebanov I R, Polyakov A M *Phys. Lett. B* **428** 105 (1998)
25. Witten E *Adv. Theor. Math. Phys.* **2** 253 (1998)
26. Gubser S S, Klebanov I R, Polyakov A M *Nucl. Phys. B* **636** 99 (2002)
27. Brower R C et al. *JHEP* (12) 005 (2007)
28. Lipatov L N, in *Perspectives in Hadronic Physics*: *Proc. of Conf. ICTP, Triest, Italy, May 1997* (Eds S Boffi et al.) (Singapore: World Scientific, 1998)
29. Minahan J A, Zarembo K *JHEP* (03) 013 (2003)
30. Beisert N, Staudacher M *Nucl. Phys. B* **670** 439 (2003)
31. Eden B, Staudacher M *J. Stat. Mech.* P11014 (2006)
32. Beisert N, Eden B, Staudacher M *J. Stat. Mech.* P01021 (2007)
33. Kotikov A V et al. *J. Stat. Mech.* P10003 (2007)
34. Bern Z, Dixon L J, Smirnov V A *Phys. Rev. D* **72** 085001 (2005)
35. Bartels J, Lipatov L N, Sabio Vera A, arXiv:0802.2065