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Lev Landau and the problem of singularities in cosmology

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1. Introduction

We consider different aspects of the problem of cosmological singularity such as the BKL oscillatory approach to singularity, the new features of cosmological dynamics in the neighborhood of the singularity in multidimensional and superstring cosmological models, and their connections with such a modern branch of mathematics as infinite-dimensional Lie algebras. In addition, we consider some new types of cosmological singularities that have been widely discussed during the last decade, after the discovery of the phenomenon of cosmic acceleration.

Many years ago, in conversations with his students, Lev Davidovich Landau used to say that three problems were most important for theoretical physics: the problem of cosmological singularity, the problem of phase transitions, and the problem of superconductivity [1]. We now know that the great breakthrough was achieved in the explanation of phenomena of superconductivity [2] and phase transitions [3]. The problem of cosmological singularity has been widely studied during the last 50 years and many important results have been obtained, but it still preserves some intriguing aspects. Moreover, some quite unexpected facets of the problem of cosmological singularity have been discovered.

In our review published 10 years ago [4] in the issue of this journal dedicated to the 90th anniversary of Landau's birth, we discussed some questions connected with the problem of singularity in cosmology. In the present paper, we dwell on relations between well-known old results of these studies and new developments in this area.

To begin, we recall that Penrose and Hawking [5] proved the impossibility of indefinite continuation of geodesics under certain conditions. This was interpreted as pointing to the existence of a singularity in the general solution of the Einstein equations. These theorems, however, did not allow finding the particular analytic structure of the singularity. The analytic behavior of the general solutions of the Einstein equations in the neighborhood of a singularity was investigated in [6–11]. These papers revealed the enigmatic phenomenon of an oscillatory approach to the singularity, which has become known as the *Mixmaster Universe* [12]. The model of a closed homogeneous but anisotropic universe with three degrees of freedom (Bianchi IX cosmological model) was used to demonstrate that the universe approaches the singularity such that its contraction along two axes is accompanied by an expansion along the third axis, and the axes change their roles according to a rather complicated law revealing chaotic behavior [10, 11, 13, 14].

The study of the dynamics of the universe in the vicinity of the cosmological singularity has become an explodingly developing field of modern theoretical and mathematical physics. We first note the generalization of the study of the oscillatory approach to the cosmological singularity in multidimensional cosmological models. It has been noticed [15] that the approach to the cosmological singularity in the multidimensional (Kaluza–Klein type) cosmological models has a chaotic character in space–times whose dimension is not higher than ten, while in space–times of higher dimensions, the universe enters a monotonic Kasner-type contracting regime after undergoing a finite number of oscillations.

The development of cosmological studies based on superstring models has revealed some new aspects of the dynamics in the vicinity of the singularity [17]. First, it was shown that these models involve mechanisms for changing Kasner epochs provoked not by the gravitational interactions but by the influence of other fields present in these theories. Second, it was proved that the cosmological models based on the six main superstring models plus the $D = 11$ supergravity model exhibit a chaotic oscillatory approach towards the singularity. Third, the connection between cosmological models manifesting the oscillatory approach towards singularity and a special subclass of infinite-dimensional Lie algebras [18], the so called hyperbolic Kac–Moody algebras, was discovered.

Another confirmation of the importance of the problem of singularity in general relativity has come from observational cosmology. At the end of the 1990s, the study of the relation between the luminosity and redshift of type-Ia supernovae revealed that the modern Universe is expanding with an acceleration [19]. To provide such an acceleration, it is necessary to have a particular substance, which was named 'dark energy' [20]. The main feature of this kind of matter is that it should have a negative pressure p such that $\rho + 3p < 0$, where ρ is the energy density. The simplest kind of this matter is the cosmological constant, for which $p = -\rho$. The so-called standard or ΛCDM cosmological model is based on the cosmological constant, whose energy density is responsible for roughly 70 percents of the general energy density of the Universe, while the rest is occupied by dust-like matter, both the baryonic one (approximately 4 percent) and a dark one. This model is in good agreement with observations, but other candidates for the role of dark energy are being intensively studied and new observations can give some surprises already in the nearest future. First of all, we note that some observations [21] suggest the possible existence of the so-called superacceleration, which is connected with the presence of phantom dark energy [22], characterized by the inequality $p < -\rho$. Under certain conditions, a universe filled with this type of dark energy can encounter a very particular cosmological singularity, the Big Rip [23]. When an expanding universe encounters this singularity, it has an infinite cosmological radius and an infinite value of the Hubble variable. Earlier, the possibility of this type of singularity was discussed in [24].

Study of different possible candidates for the role of dark energy has stimulated the elaboration of the general theory of possible cosmological singularities [25–29]. It is remarkable that while the 'traditional' Big Bang or Big Crunch singularities are associated with the vanishing size of the universe, i.e., with a universe squeezed to a point, these new singularities occur at finite or infinite value of the cosmological radius. The physical processes occurring in the vicinity of such singularities can have rather exotic features and their study is of great interest. Thus, we see that the development of both the theoretical and the observational branches of cosmology has confirmed the importance of the problem of singularity in general relativity mentioned by Landau many years ago.

The structure of this contribution is as follows: in Section 2, we briefly discuss the Landau theorem about the singularity, which was not published in a separate paper and was reported in book [30] and in review [6]; in Section 3, we recall the main features of the oscillatory approach to the singularity in relativistic cosmology; Section 4 is devoted to the modern development of the BKL ideas and methods, including the dynamics in the presence of a massless scalar field, multidimensional cosmology, superstring cosmology, and the correspondence between chaotic cosmological dynamics and hyperbolic Kac – Moody algebras; in Section 5, we describe some new types of cosmological singularities, and in Section 6 we present some concluding remarks.

2. Landau theorem about singularity

We consider the synchronous reference frame with the metric

$$ds^2 = dt^2 - \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta\,, \tag{1}$$

where $\gamma_{\alpha\beta}$ is the spatial metric. Landau pointed out that the determinant g of the metric tensor in a synchronous reference system must tend to zero at some finite time if the equation of state satisfies some simple conditions. To prove this statement, it is convenient to write the 0-0 component of the Ricci tensor as

$$R_0^0 = -\frac{1}{2}\,\frac{\partial K_\alpha^\alpha}{\partial t} - \frac{1}{4}\,K_\alpha^\beta K_\beta^\alpha\,, \tag{2}$$

where $K_{\alpha\beta}$ is the extrinsic curvature tensor defined as

$$K_{\alpha\beta} = \frac{\partial\gamma_{\alpha\beta}}{\partial t}\,, \tag{3}$$

and the spatial indices are raised and lowered by the spatial metric $\gamma_{\alpha\beta}$. The Einstein equation for R_0^0 is

$$R_0^0 = T_0^0 - \frac{1}{2}\,T, \tag{4}$$

where the energy – momentum tensor is

$$T_i^j = (\rho + p)\, u_i u^j - \delta_i^j p\,, \tag{5}$$

where ρ, p, and u_i are the energy density, the pressure, and the four-velocity, respectively. The quantity $T_0^0 - 1/2T$ in the right-hand side of Eqn (4) is

$$T_0^0 - \frac{1}{2}\,T = \frac{1}{2}\,(\rho + 3p) + (\rho + p)\, u_\alpha u^\alpha\,, \tag{6}$$

which is positive if

$$\rho + 3p > 0\,. \tag{7}$$

Thus, Eqn (4) implies that

$$\frac{1}{2}\,\frac{\partial K_\alpha^\alpha}{\partial t} + \frac{1}{4}\,K_\alpha^\beta K_\beta^\alpha \leqslant 0\,. \tag{8}$$

Because of the algebraic inequality

$$K_\alpha^\beta K_\beta^\alpha \geqslant \frac{1}{3}\,(K_\alpha^\alpha)^2\,, \tag{9}$$

we have

$$\frac{\partial K_\alpha^\alpha}{\partial t} + \frac{1}{6}\,(K_\alpha^\alpha)^2 \leqslant 0\,, \tag{10}$$

or

$$\frac{\partial}{\partial t}\,\frac{1}{K_\alpha^\alpha} \geqslant \frac{1}{6}\,. \tag{11}$$

If $K_\alpha^\alpha > 0$ at some instant of time, then if t decreases, the quantity $1/K_\alpha^\alpha$ decreases to zero within a finite time. Hence, K_α^α tends to $+\infty$; because of the identity

$$K_\alpha^\alpha = \gamma^{\alpha\beta}\,\frac{\partial\gamma_{\alpha\beta}}{\partial t} = \frac{\partial}{\partial t}\,\ln\gamma\,, \tag{12}$$

this means that the determinant g tends to zero [no faster than t^6 according to inequality (11)]. If $K_\alpha^\alpha < 0$ at the initial instant, then the same result is obtained for increasing time. A similar result was obtained in [31] in the case of dust-like matter and in [32].

This result does not prove that a true physical singularity inevitably exists that belongs to space – time itself and is not connected with the character of the chosen reference system. However, this result played an important role in stimulating the discussion about the existence and generality of singularities in cosmology. We note that energy dominance condition (7) used for the proof of the Landau theorem also appears in the proof of the Penrose and Hawking singularity theorem [5]. Moreover, the breakdown of this condition is necessary for an explanation of the phenomenon of cosmic acceleration.

The Landau theorem is deeply connected with the appearance of caustics studied in [33] and was discussed between those authors and Landau in 1961. In trying to geometrically construct the synchronous reference frame, one starts from the three-dimensional Cauchy surface and designs the family of geodesics orthogonal to this surface. The length along these geodesics serves as the time measure. It is known that these geodesics intersect on some two-dimensional caustic surface. This geometry constructed for the empty space is also valid in the presence of dust-like matter ($p = 0$). Such matter, moving along the geodesics, concentrates on the caustics, but the increase in density cannot be unbounded because the arising pressure destroys the caustics. [1] This question was studied in [34]. Later, caustics were used in [35] to explain the initial clustering of the dust, which, although not creating physical singularities, is nevertheless responsible for the creation of so-called pancakes. These pancakes represent the initial stage of the development of the large-scale structure of the universe.

3. Oscillatory approach to the singularity in relativistic cosmology

One of the first exact solutions found in the framework of general relativity was the Kasner solution [16] for the Bianchi-I cosmological model representing the gravitational field in an empty space with a Euclidean metric depending on time according to the formula

$$ds^2 = dt^2 - t^{2p_1}\, dx^2 - t^{2p_2}\, dy^2 - t^{2p_3}\, dz^2\,, \tag{13}$$

where the exponents p_1, p_2, and p_3 satisfy the relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1\,. \tag{14}$$

[1] In an empty space, the caustic is a mathematical but not a physical singularity. This follows simply from the fact that its location can always be shifted by changing the initial Cauchy surface.

Choosing the ordering of exponents as

$$p_1 < p_2 < p_3\,, \tag{15}$$

we can parameterize them as [6]

$$p_1 = \frac{-u}{1+u+u^2}\,,\; p_2 = \frac{1+u}{1+u+u^2}\,,\; p_3 = \frac{u(1+u)}{1+u+u^2}\,. \tag{16}$$

As the parameter u varies in the range $u \geqslant 1$, p_1, p_2, and p_3 take all their allowed values:

$$-\frac{1}{3} \leqslant p_1 \leqslant 0\,, \quad 0 \leqslant p_2 \leqslant \frac{2}{3}\,, \;\; \frac{2}{3} \leqslant p_3 \leqslant 1\,. \tag{17}$$

The values $u < 1$ lead to the same range of values of p_1, p_2, and p_3 because

$$p_1\left(\frac{1}{u}\right) = p_1(u)\,,\; p_2\left(\frac{1}{u}\right) = p_3(u)\,,\; p_3\left(\frac{1}{u}\right) = p_2(u)\,. \tag{18}$$

The parameter u introduced in the early 1960s turned out to be very useful and its properties are attracting the attention of researchers in different contexts. For example, in recent paper [36], a connection was established between the Lifshitz–Khalatnikov parameter u and invariants arising in the context of Petrov's classification of the Einstein spaces [37].

In the case of Bianchi-VIII or Bianchi-IX cosmological models, the Kasner regime described by (13) and (14) is not an exact solution of the Einstein equations; however, a generalized Kasner solutions can be constructed [7–11]. It is possible to construct some kind of perturbation theory where the exact Kasner solution in (13), (14) plays the role of the zeroth-order approximation, with the role of perturbations played by the terms in the Einstein equations that depend on spatial curvature tensors (apparently, such terms are absent in Bianchi-I cosmology). This perturbation theory is effective in the vicinity of a singularity or, in other words, at $t \to 0$. The remarkable feature of these perturbations is that they imply a transition from the Kasner regime with one set of parameters to the Kasner regime with another one.

The metric of the generalized Kasner solution in a synchronous reference system can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\,, \tag{19}$$

where

$$a = t^{p_l},\; b = t^{p_m},\; c = t^{p_n}\,. \tag{20}$$

The three-dimensional vectors **l**, **m**, and **n** define the directions along which the spatial distances vary with time according to power laws (20). Let $p_l = p_1$, $p_m = p_2$, and $p_n = p_3$, with

$$a \sim t^{p_1}\,,\; b \sim t^{p_2}\,,\; c \sim t^{p_3}\,, \tag{21}$$

which means that the Universe is contracting in directions given by the vectors **m** and **n** and is expanding along **l**. It was shown in [10] that the perturbations caused by spatial curvature terms make the variables a, b, and c undergo transition to another Kasner regime characterized by the formulas

$$a \sim t^{p'_l}\,,\; b \sim t^{p'_m}\,,\; c \sim t^{p'_n}\,, \tag{22}$$

where

$$p'_l = \frac{|p_1|}{1-2|p_1|}\,,\; p'_m = -\frac{2|p_1| - p_2}{1-2|p_1|}\,,\; p'_n = -\frac{p_3 - 2|p_1|}{1-2|p_1|}\,. \tag{23}$$

Thus, the effect of the perturbation is to replace one 'Kasner epoch' by another such that the negative power of t is transformed from the l to the m direction. During the transition, the function $a(t)$ reaches a maximum and $b(t)$ a minimum. Hence, the previously decreasing quantity b now increases, the quantity a decreases, and $c(t)$ remains a decreasing function. The previously increasing perturbation caused the transition from regime (21) to regime (22), and is therefore damped and eventually vanishes. Then another perturbation begins to grow, which leads to a new replacement of one Kasner epoch by another, etc.

We emphasize that just the fact that perturbation implies a change of dynamics extinguishing it allows using the perturbation theory so successfully. We note that the effect of changing the Kasner regime already exists in cosmological models that are simpler than those of Bianchi IX and Bianchi VIII. As a matter of fact, in the Bianchi II universe, there exists only one type of perturbation connected with spatial curvature and this perturbation makes one change of Kasner regime (one bounce). This fact was known to Lifshitz and Khalatnikov in the early 1960s, and they discussed this topic with Landau (just before the tragic accident), who highly appreciated it. The results describing the dynamics of the Bianchi IX model were reported by Khalatnikov in his talk given in January 1968 at the Henri Poincaré Seminar in Paris. Wheeler, who was present there, pointed out that the dynamics of the Bianchi IX universe represent a nontrivial example of the chaotic dynamic system. Later, Thorn distributed a preprint with the text of this talk.

Returning to the rules governing the bouncing of the negative power of time from one direction to another, it can be shown that they can be conveniently expressed in terms of parameterization (16),

$$p_l = p_1(u)\,,\; p_m = p_2(u)\,,\; p_n = p_3(u)\,, \tag{24}$$

and then

$$p'_l = p_2(u-1)\,,\; p'_m = p_1(u-1)\,,\; p'_n = p_3(u-1)\,. \tag{25}$$

The greater of the two positive powers remains positive.

The successive changes as in (25), accompanied by a bouncing of the negative power between the directions **l** and **m**, continue until the integral part of u is exhausted, i.e., until u becomes less than one. Then, according to Eqn (18), the value $u < 1$ transforms into $u > 1$; at this instant, either the exponent p_l or p_m is negative and p_n becomes the smaller of the two positive numbers ($p_n = p_2$). The next sequence of changes bounces the negative power between the directions **n** and **l** or **n** and **m**. We emphasize that the Landau–Khalatnikov parameter u is useful because it allows encoding rather complicated laws of transitions between different Kasner regimes (23) by simple rules such as $u \to u-1$ and $u \to 1/u$.

Consequently, the evolution of our model towards a singular point consists of successive periods (called eras) in which distances oscillate along two axes and decrease monotonically along the third axis, and the volume decreases according to a law that is near $\sim t$. In the transition from one era to another, the axes along which the distances decrease monotonically are interchanged. The order in which the pairs of axes are interchanged and the order in which eras of different lengths follow each other acquire a stochastic character.

To every (sth) era, there corresponds a decreasing sequence of values of the parameter u. This sequence has the form $u_{\max}^{(s)}, u_{\max}^{(s)} - 1, \ldots, u_{\min}^{(s)}$, where $u_{\min}^{(s)} < 1$. We introduce the notation

$$u_{\min}^{(s)} = x^{(s)}, \quad u_{\max}^{(s)} = k^{(s)} + x^{(s)}, \tag{26}$$

i.e., $k^{(s)} \equiv [u_{\max}^{(s)}]$ (the square brackets denote the greatest integer $\leqslant u_{\max}^{(s)}$). The number $k^{(s)}$ defines the era length. For the next era, we obtain

$$u_{\max}^{(s+1)} = \frac{1}{x^{(s)}}, \quad k^{(s+1)} = \left[\frac{1}{x^{(s)}}\right]. \tag{27}$$

The ordering with respect to the length of $k^{(s)}$ of the successive eras (measured by the number of Kasner epochs contained in them) acquires an asymptotically stochastic character. The random nature of this process arises because of rules (26) and (27) that define the transitions from one era to another in the infinite sequence of values of u. If all this infinite sequence begins from some initial value $u_{\max}^{(0)} = k^{(0)} + x^{(0)}$, then the lengths of series $k^{(0)}, k^{(1)}, \ldots$ are the numbers occurring in the expansion into a continued fraction:

$$k^{(0)} + x^{(0)} = k^{(0)} + \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \cdots}}. \tag{28}$$

We can statistically describe this sequence of eras if, instead of a given initial value $u_{\max}^{(0)} = k^{(0)} + x^{(0)}$, we consider a distribution of $x^{(0)}$ over the interval $(0, 1)$ governed by some probability law. Then we also obtain some distributions of the values of $x^{(s)}$ that terminate every sth series of numbers. It can be shown that with increasing s, these distributions tend to a stationary (independent of s) probability distribution $w(x)$ in which the initial value $x^{(s)}$ is completely 'forgotten':

$$w(x) = \frac{1}{(1+x)\ln 2}. \tag{29}$$

It follows from Eqn (29) that the probability distribution of the lengths of series k is given by

$$W(k) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}. \tag{30}$$

Moreover, probability distributions for other parameters describing successive eras, such as the parameter δ, can be calculated exactly, giving a relation between the amplitudes of logarithms of the functions a, b, and c and the logarithmic time [14].

Thus, we have seen from the results of statistical analysis of evolution in the neighborhood of a singularity [13] that the stochasticity and probability distributions of parameters already arise in classical general relativity.

At the end of this section, a historical remark is in order. Continued fraction (28) was shown in 1968 to I M Lifshitz (Landau had already passed away) and he immediately noticed that the formula for a stationary distribution of the value of x in (29) can be derived. Later, it became known that this formula was derived in the nineteenth century by Gauss, who had not published it but had described it in a letter to a colleague.

4. Oscillatory approach to the singularity: modern development

The oscillatory approach to the cosmological singularity described in the preceding section was developed for an empty space – time. It is not difficult to understand that if the universe is filled with a perfect fluid with the equation of state $p = w\rho$, where p is the pressure, ρ is the energy density, and $w < 1$, then the presence of this matter cannot change the dynamics in the vicinity of the singularity. Indeed, using the energy conservation equation, we can show that

$$\rho = \frac{\rho_0}{(abc)^{w+1}} = \frac{\rho_0}{t^{w+1}}, \tag{31}$$

where ρ_0 is a positive constant. Thus, the term representing matter in the Einstein equations behaves as $\sim 1/t^{1+w}$ and is weaker as $t \to 0$ than the terms of geometric origin coming from the time derivatives of the metric, which behave as $1/t^2$, let alone the perturbations due to the presence of spatial curvature, responsible for changes in the Kasner regime, which behave as $1/t^{2+4|p_1|}$. But the situation changes drastically if the parameter w is equal to unity, i.e., the pressure is equal to the energy density. This matter is called 'stiff matter' and can be represented by a massless scalar field. In this case, $\rho \sim 1/t^2$ and the contribution of matter is of the same order as the leading terms of geometrical origin. Hence, it is necessary to find a Kasner-type solution, with the presence of terms connected with the presence of stiff matter (a massless scalar field) taken into account. Such a study was carried out in [38]. It was shown there that the scale factors a, b, and c can again be represented as t^{2p_1}, t^{2p_2}, and t^{2p_3}, where the Kasner indices satisfy the relations

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1 - q^2, \tag{32}$$

where the number q^2 reflects the presence of stiff matter and is bounded by

$$q^2 \leqslant \frac{2}{3}. \tag{33}$$

It can be seen that if $q^2 > 0$, then there exist combinations of the positive Kasner indices satisfying relations (32). Moreover, if $q^2 \geqslant 1/2$, only triples of positive Kasner indices can satisfy relations (32). If the universe finds itself in a Kasner regime with three positive indices, the perturbative terms existing due to the spatial curvatures are too weak to change this Kasner regime, and it therefore becomes stable. This means that in the presence of stiff matter, after a finite number of changes of Kasner regimes, the universe finds itself in a stable regime and the oscillations stop. Thus, the massless scalar field plays an 'antichaotizing' role in the process of cosmological evolution [38]. The Lifshitz – Khalatnikov parameter can also be used in this case. The Kasner indices satisfying relations (32) are conveniently represented as [38]

$$p_1 = \frac{-u}{1+u+u^2}\,,$$
$$p_2 = \frac{1+u}{1+u+u^2}\left[u - \frac{u-1}{2}\left(1-(1-\beta^2)^{1/2}\right)\right],$$
$$p_3 = \frac{1+u}{1+u+u^2}\left[1 + \frac{u-1}{2}\left(1-(1-\beta^2)^{1/2}\right)\right],$$
$$\beta^2 = \frac{2(1+u+u^2)^2}{(u^2-1)^2}\,. \tag{34}$$

The range of the parameter u is now $-1 \leqslant u \leqslant 1$, while the admissible values of q at a fixed u are

$$q^2 \leqslant \frac{(u^2-1)^2}{2(1+u+u^2)^2}\,. \tag{35}$$

It is easy to show that after one bounce, the value of q^2 changes according to the rule

$$q^2 \to q'^2 = q^2\,\frac{1}{(1+2p_1)^2} > q^2\,. \tag{36}$$

Thus, the parameter q^2 increases and, hence, the probability of finding all three Kasner indices to be positive increases. This again confirms the statement that after a finite number of bounces, the universe in the presence of the massless scalar field finds itself in the Kasner regime with three positive indices and the oscillations stop.

In the second half of the 1980s, a series of papers was published [15] where solutions of the Einstein equations in the vicinity of a singularity for $(d+1)$-dimensional space – times were studied. The multidimensional analog of a Bianchi-I universe was considered with the generalized Kasner metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{i=1}^{d} t^{2p_i}\,\mathrm{d}x^{i2}\,, \tag{37}$$

where the Kasner indices p_i satisfy the conditions

$$\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} p_i^2 = 1\,. \tag{38}$$

In the presence of spatial curvature terms, the transition from one Kasner epoch to another occurs and this transition is described by the following rule: the new Kasner exponents are equal to

$$p'_1, p'_2, \ldots, p'_d = (q_1, q_2, \ldots, q_d)\,, \tag{39}$$

where

$$q_1 = \frac{-p_1 - P}{1+2p_1+P}\,,\quad q_2 = \frac{p_2}{1+2p_1+P}\,, \ldots,$$
$$q_{d-2} = \frac{p_{d-2}}{1+2p_1+P}\,,\quad q_{d-1} = \frac{2p_1+P+p_{d-1}}{1+2p_1+P}\,,$$
$$q_d = \frac{2p_1+P+p_d}{1+2p_1+P}\,, \tag{40}$$

with

$$P = \sum_{i=2}^{d-2} p_i\,. \tag{41}$$

However, such a transition from one Kasner epoch to another occurs if at least one of the numbers α_{ijk} is negative. These numbers are defined as

$$\alpha_{ijk} \equiv 2p_i + \sum_{l\neq j,k,i} p_l\,,\quad (i\neq j\,,\ i\neq k\,,\ j\neq k)\,. \tag{42}$$

For space – times with $d < 10$, one of the factors α is always negative and, hence, one change of Kasner regime is followed by another, implying the oscillatory behavior of the universe in the neighborhood of the cosmological singularity. But for space – times with $d \geqslant 10$, combinations of Kasner indices satisfying Eqn (38) with all the numbers α_{ijk} positive exist. If the universe enters the Kasner regime with such indices (so-called 'Kasner stability region'), its chaotic behavior disappears and this Kasner regime is preserved. Thus, the hypothesis was put forward that in space – times with $d \geqslant 10$, after a finite number of oscillations, the universe under consideration finds itself in the Kasner stability region and the oscillating regime is replaced by the monotonic Kasner behavior.

The discovery of the fact that the chaotic character of the approach to the cosmological singularity disappears in space – times with $d \geqslant 10$ was unexpected and looked like an accidental result of an interplay of real numbers satisfying generalized Kasner relations (40). Later, it became clear that underlying this fact is a deep mathematical structure, the hyperbolic Kac – Moody algebras. Indeed, in a series of works by Damour, Henneaux, Nicolai, and some other authors (see, e.g., Refs [17]) on the cosmological dynamics in models based on superstring theories living in 10-dimensional space – time and in the $d+1 = 11$ supergravity model, it was shown that in the vicinity of the singularity, these models reveal oscillating behavior of the BKL type. The important new feature of the dynamics in these models is the role played by nongravitational bosonic fields (p-forms), which are also responsible for transitions from one Kasner regime to another. For a description of these transitions, the Hamiltonian formalism [12] becomes very convenient. In the framework of this formalism, the configuration space of the Kasner parameters describing the dynamics of the universe can be treated as a billiard system, while the curvature terms in the Einstein theory and p-form potentials in superstring theories play the role of the walls of these billiards. The transition from one Kasner epoch to another is a reflection from one of the walls. Thus, there is a correspondence between the rather complicated dynamics of a universe in the vicinity of the cosmological singularity and the motion of a ball on a billiard table.

However, there exists a more striking and unexpected correspondence between the chaotic behavior of the universe in the vicinity of the singularity and such an abstract mathematical object as the hyperbolic Kac – Moody algebras [17]. We briefly explain what it means. Every Lie algebra is defined by its generators h_i, e_i, f_i, $i = 1, \ldots, r$, where r is the rank of the Lie algebra, i.e., the maximal number of its generators h_i that commute with each other (these generators constitute the Cartan subalgebra). The commutation relations between the generators are

$$[e_i, f_j] = \delta_{ij}\,h_i\,,$$
$$[h_i, e_j] = A_{ij}\,e_j\,,$$
$$[h_i, f_j] = -A_{ij} f_j\,,$$
$$[h_i, h_j] = 0\,. \tag{43}$$

The coefficients $A_{i,j}$ constitute the generalized Cartan $r \times r$ matrix such that $A_{ii} = 2$, its off-diagonal elements are nonpositive integers, and $A_{ij} = 0$ for $i \neq j$ implies $A_{ji} = 0$. We can say that the e_i are rising operators, similar to the well-known operator $L_+ = L_x + iL_y$ in the theory of angular momentum, while the f_i are lowering operators like $L_- = L_x - iL_y$. The generators h_i of the Cartan subalgebra can be compared with the operator L_z. The generators must also satisfy the Serre relations

$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0\,,$$
$$(\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0\,, \tag{44}$$

where $(\mathrm{ad}\, A)B \equiv [A, B]$.

The Lie algebras $\mathcal{G}(A)$ built on a symmetrizable Cartan matrix A have been classified according to the properties of their eigenvalues:

if A is positive definite, $\mathcal{G}(A)$ is a finite-dimensional Lie algebra;

if A allows one null eigenvalue and all the others are strictly positive, $\mathcal{G}(A)$ is an affine Kac – Moody algebra;

if A allows one negative eigenvalue and all the others are strictly positive, $\mathcal{G}(A)$ is a Lorentz Kac – Moody algebra.

A correspondence exists between the structure of a Lie algebra and a certain system of vectors in the r-dimensional Euclidean space, which essentially simplifies the task of classification of the Lie algebras. These vectors, called roots, represent the rising and lowering operators of the Lie algebra. The vectors corresponding to the generators e_i and f_i are called simple roots. The system of positive simple roots (i.e., roots corresponding to the rising generators e_i) can be represented by nodes of their Dynkin diagrams, while the edges connecting (or not connecting) the nodes give information about the angles between simple positive root vectors.

An important subclass of Lorentz Kac – Moody algebras can be defined as follows. A Kac – Moody algebra such that deleting one node from its Dynkin diagram gives a sum of finite or affine algebras is called a hyperbolic Kac – Moody algebra. These algebras are all known. In particular, there exists no hyperbolic algebras with the rank higher than 10.

We recall some more definitions from the theory of Lie algebras. Reflections with respect to hyperplanes orthogonal to simple roots leave the root system invariant. The corresponding finite-dimensional group is called the Weyl group. Finally, the hyperplanes mentioned above divide the r-dimensional Euclidean space into regions called Weyl chambers. The Weyl group transforms one Weyl chamber into another.

We can now briefly formulate the results of the approach in [17] following paper [39]: the links between the billiards describing the evolution of the universe in the neighborhood of singularity and its corresponding Kac – Moody algebra can be described as follows:

the Kasner indices describing the 'free' motion of the universe between the reflections from the walls correspond to the elements of the Cartan subalgebra of the Kac – Moody algebra;

the dominant walls, i.e., the terms in the equations of motion responsible for the transition from one Kasner epoch to another, correspond to the simple roots of the Kac – Moody algebra;

the group of reflections in the cosmological billiard system is the Weyl group of the Kac – Moody algebra;

the billiard table can be identified with the Weyl chamber of the Kac – Moody algebra.

We can imagine two types of billiard tables: infinite, where the linear motion without collisions with the walls is possible (nonchaotic regime), and those where reflections from the walls are inevitable and the regime can be only chaotic. Remarkably, the Weyl chambers of the hyperbolic Kac – Moody algebras are such that infinitely repeating collisions with the walls occur. It was shown that all the theories with the oscillating approach to the singularity such as the Einstein theory in dimensions $d < 10$ and superstring cosmological models correspond to hyperbolic Kac – Moody algebras.

The existence of links between the BKL approach to the singularities and the structure of some infinite-dimensional Lie algebras has inspired some authors to declare a new program of development of quantum gravity and cosmology [40]. They propose "to take seriously the idea that near the singularity (i.e., when the curvature gets larger than the Planck scale) the description of a spatial continuum and space – time based (quantum) field theory breaks down, and should be replaced by a much more abstract Lie algebraic description."

5. New types of cosmological singularities

As mentioned in the Introduction, the development of the theoretical and observational cosmology and, in particular, the discovery of the cosmic acceleration have stimulated the elaboration of cosmological models where new types of singularities are described. In contrast to the 'traditional' Big Bang and Big Crunch singularities, these singularities occur not at zero but at finite or even infinite values of the cosmological radius. The most famous of these singularities is, perhaps, the Big Rip singularity [23, 24] arising if the absolute value of the negative pressure p of dark energy is larger than the energy density ρ. Indeed, we consider a flat Friedmann universe with the metric

$$\mathrm{d}s^2 - a^2(t)\,\mathrm{d}l^2\,, \tag{45}$$

filled with a perfect fluid with the equation of state

$$p = w\rho\,, \quad w = \mathrm{const} < -1\,. \tag{46}$$

The dependence of the energy density ρ on the cosmological radius a is, as usual,

$$\rho = \frac{C}{a^{3(1+w)}}\,, \tag{47}$$

and the Friedmann equation in this case has the form

$$\frac{\dot{a}^2}{a} = \frac{C}{a^{3(1+w)}}\,, \tag{48}$$

where C is a positive constant. Integrating Eqn (48), we obtain

$$a(t) = \left(a_0^{3(1+w)/2} + \frac{2\sqrt{C}\,(t - t_0)}{3(1+w)}\right)^{2/3(1+w)}. \tag{49}$$

It is easy to see that at the finite time $t_R > t_0$ equal to

$$t_R = t_0 - \frac{3(1+w)}{2\sqrt{C}}\, a_0^{3(1+w)/2}\,, \tag{50}$$

the cosmological radius becomes infinite and the same also occurs with the Hubble variable $\dot{a}/a$ and, hence, with the scalar curvature. Thus, we encounter a new type of cosmological singularity, characterized by infinite values of the cosmological radius, its time derivative, the Hubble variable, and the scalar curvature. It is usually called the 'Big Rip' singularity. Its properties have attracted a considerable attention of researchers because some observational data indicate that the actual value of the equation of state parameter w is indeed smaller than -1.

There are also other types of cosmological singularities that can be encountered at finite values of the cosmological radius (see, e.g., [25–29]). For illustration, we here consider one type of singularity, the Big Brake singularity [25]. This singularity can be achieved in a finite lapse of cosmic time and is characterized by a finite value of the cosmological radius, by the vanishing first time derivative of the radius, and by the second time derivative of the cosmological radius tending to minus infinity (an infinite deceleration). We consider a perfect fluid with the equation of state

$$p = \frac{A}{\rho}\,, \tag{51}$$

where A is a positive constant. This fluid could be called an 'anti-Chaplygin' gas because the widely used Chaplygin gas cosmological model [41] is based on the equation of state $p = -A/\rho$. The dependence of the energy density on the cosmological radius for equation of state (51) is

$$\rho = \sqrt{\frac{B}{a^6} - A}\,, \tag{52}$$

where B is a positive constant. When a is small, $\rho \sim 1/a^3$ and behaves like dust. Then, as $a \to a_B$,

$$a_B = \left(\frac{B}{A}\right)^{1/6}, \tag{53}$$

and the energy density tends to zero. The solution of the Friedmann equation in this limit gives

$$a(t) = a_B - C_0(t_B - t)^{4/3}, \quad C_0 = 2^{-7/3}3^{5/3}(AB)^{1/6}\,. \tag{54}$$

Now, it can be easily verified that as $t \to t_B$, $\dot{a} \to 0$ and $\ddot{a} \to -\infty$. Thus, we indeed encounter the Big Brake cosmological singularity.

6. Conclusions

We have shown in this short review that the opinion expressed by Landau many years ago concerning the importance of the problem of singularity in cosmology has proved to be prophetic. The study of the cosmological singularity has revealed the existence of an oscillatory behavior of the universe as the curvature of space–time increases, which in turn has a deep connection with quite new branches of modern mathematics. On the other hand, the latest successes of observational cosmology have stimulated the development of various cosmological models, which reveal new types of cosmological singularities whose investigation from both the physical and mathematical standpoints can be very promising.

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Axial anomaly in quantum electro- and chromodynamics and the structure of the vacuum in quantum chromodynamics

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1. Introduction

In this report, I discuss the current state of the problem of the axial anomaly in quantum electrodynamics (QED) and quantum chromodynamics (QCD) and the relation of the axial anomaly to the structure of the vacuum in QCD. In QCD, the vacuum average of the axial anomaly is proportional to a new quantum number n, the winding number. There are an infinite number of vacuum states $|n\rangle$. The transition amplitudes between these states are amplitudes of tunnel transitions along certain paths in the space of gauge fields. I show that the axial anomaly condition implies that there are zero modes of the Dirac equation for a massless quark and that spontaneous chiral symmetry breaking occurs in QCD, which leads to the formation of a quark condensate. The axial anomaly can be represented in the form of a sum rule for the structure function in the dispersion representation of the axial – vector – vector (AVV) vertex. On the basis of this sum rule, we calculate the width of the $\pi^0 \to 2\gamma$ decay with an accuracy of 1.5%.

2. The definition of an anomaly

We suppose that the classical field-theory Lagrangian has a certain symmetry, i.e., is invariant under transformations of the fields corresponding to this symmetry. According to the Noether theorem, the symmetry corresponds to a conservation law. An anomaly is a phenomenon in which the given symmetry and the conservation law are violated as we pass to quantum theory. The reason for this violation lies in the singularity of quantum field operators at small distances, such that finding the physical quantities requires fixing not only the Lagrangian but also the renormalization procedure. (See reviews dealing with various anomalies in Refs [1 – 4].)

There are two types of anomalies, internal and external. In the first case, the gauge invariance of the classical Lagrangian is broken at the quantum level, the theory becomes unrenormalizable, and is not self-consistent. This problem can be resolved by a special choice of fields in the Lagrangian, for which all the internal anomalies cancel. (Such an approach is used in the standard model of electroweak interaction and is known as the Glashow – Illiopoulos – Maiani mechanism.) External anomalies emerge as a result of the interaction between the fields in the Lagrangian and external sources. It is these anomalies that appear in quantum electrodynamics and quantum chromodynamics; they are discussed in what follows. We show that anomalies play an important role in QED and especially in QCD. Hence, the term 'anomaly' should not mislead us — it is a normal and important ingredient of most quantum field theories.

3. Axial anomaly in QED

The Dirac equation for the electron in an external electromagnetic field $A_\mu(x)$ is

$$\mathrm{i}\gamma_\mu \frac{\partial\psi(x)}{\partial x_\mu} = m\psi(x) - e\gamma_\mu A_\mu(x)\,\psi(x)\,. \tag{1}$$

The axial current is defined as

$$j_{\mu5}(x) = \bar\psi(x)\,\gamma_\mu\gamma_5\psi(x)\,. \tag{2}$$

Its divergence calculated in classical theory, i.e., with the use of Eqn (1), is

$$\partial_\mu j_{\mu5}(x) = 2\mathrm{i}m\bar\psi(x)\,\gamma_5\psi(x) \tag{3}$$

and tends to zero as $m \to 0$. In quantum theory, the axial current must be redefined because $j_{\mu5}(x)$ is the product of two local fermionic fields, with the result that it is singular when both fields act at the same point. (Naturally, a similar statement is true for a vector current.) To achieve a meaningful approach, we split the points where the two fermionic fields act by a distance ε, such that

$$j_{\mu5}(x,\varepsilon) = \bar\psi\left(x+\frac{\varepsilon}{2}\right)\gamma_\mu\gamma_5 \times \exp\left[\mathrm{i}e\int_{x-\varepsilon/2}^{x+\varepsilon/2} \mathrm{d}y_\alpha A_\alpha(y)\right]\psi\left(x-\frac{\varepsilon}{2}\right), \tag{4}$$

and take $\varepsilon \to 0$ in the final result. The exponential factor in (4) is introduced to ensure the local gauge invariance of $j_{\mu5}(x,\varepsilon)$. The divergence of axial current (4) has the following form (we use Eqn (1) and keep only the terms that are linear in ε):

$$\partial_\mu j_{\mu5}(x,\varepsilon) = 2\mathrm{i}m\bar\psi\left(x+\frac{\varepsilon}{2}\right)\gamma_5\psi\left(x-\frac{\varepsilon}{2}\right) - \mathrm{i}e\varepsilon_\alpha\bar\psi\left(x+\frac{\varepsilon}{2}\right)\gamma_\mu\gamma_5\psi\left(x-\frac{\varepsilon}{2}\right)F_{\alpha\mu}\,, \tag{5}$$

where $F_{\alpha\mu}$ is the electromagnetic field strength. For simplicity, we assume that $F_{\mu\nu} = \mathrm{const}$ and use the fixed-point gauge (the Fock – Schwinger gauge) $x_\mu A_\mu(x) = 0$. Then $A_\mu(x) = (1/2)\,x_\nu F_{\nu\mu}$. We calculate the vacuum average of (5). To calculate the right-hand side of (5), we use the expression for the electron propagator in an external electromagnetic field