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Impedance and parametric excitation of oscillators

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Contents

Abstract. This article deals with the linear ordinary differential equations for one or several coupled oscillators. Emphasis is placed on the separate notions of the frequency matrix (as a kinematic entity) and the impedance matrix. The latter matrix is explicitly introduced in this article, and its time dependence is shown to be responsible for parametric excitation and for non-conservation of adiabatic invariants.

1. Introduction

One of the favorite examples of parametric resonance in everyday life is a swing. Everybody believes that changing the length from the pivot axis of the swing to the mass center of the swinger is a simple and efficient way to excite the oscillations of the swing. We temporarily put aside the question of whether this belief describes physical reality correctly. What is more important to us is the following. In accordance with folklore tradition, the parametric excitation of the swing is attributed to the modulation of frequency, $\omega(t) = \sqrt{g/L}$, where g [m s^{-2}] is the gravity acceleration and L [m] is the length from the mass to the pivot axis. The most appealing part to connoisseurs is the Mathieu equation, which we write here in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_0^2 \left[1 + \frac{2\omega_1}{\omega_0} \cos(pt)\right] x(t) = 0\,. \tag{1.1}$$

It is common knowledge that if the 'instantaneous frequency' $\omega(t) \approx \omega_0 + \omega_1 \cos(pt)$ is modulated with the period $T = 2\pi/p$ that is close enough to half of the period $T_0 = 2\pi/\omega_0$ of the unmodulated motion, then parametric instability of the solutions of Mathieu equation (1.1) occurs. But in a laboratory environment, it is possible to modulate the length $L(t)$, i.e., the distance from the pivot axis to the swinging mass, and the effective gravity acceleration $g(t)$ separately. The latter can be done by moving the pivot axis up and down with an acceleration $a(t)$, with the result that $g_{\mathrm{eff}}(t) = g_0 + a(t)$ (Fig. 1).

Somewhat oversimplifying the main point of this paper, we try to answer the following question here: ***Is it true that just the modulation of the instantaneous frequency*** $\omega(t) = \sqrt{g(t)/L(t)}$ ***leads to parametric excitation*****?**

Surprisingly, most experts in parametric processes give the answer 'yes' to this question, while the correct one is 'no'! Indeed, in this article, we introduce the notion of *impedance for lumped systems*, including mechanical ones, and show that it is the modulation of the pendulum impedance $Z(t) = \sqrt{m^2 L^3(t)\, g_{\mathrm{eff}}(t)}$ that results in parametric excitation. In other words, if the instantaneous frequency $\omega(t) = \sqrt{g(t)/L(t)}$ does not change in time, but the impedance $Z(t)$ does, then parametric excitation is possible. On the contrary, if the impedance $Z(t)$ is constant in time, but the instantaneous frequency $\omega(t) = \sqrt{g(t)/L(t)}$ is time-modulated, then there is absolutely no parametric excitation of the oscillator.

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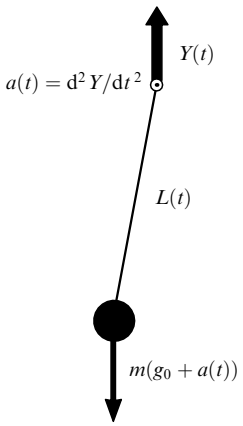

Figure 1. A pendulum (or a swing), where the pivot point height $Y(t)$ and the length $L(t)$ from the mass to the pivot point are modulated. The vertical acceleration of the pivot point $a(t) = \mathrm{d}^2 Y/\mathrm{d}t^2$ results in an effective change of the gravity force, $mg_{\mathrm{eff}}(t) = m\big(g_0 + a(t)\big)$.

The apparent contradiction with the well-known mathematical facts about Mathieu equation (1.1) is resolved in a rather simple manner. Namely, one should start with a system of two first-order ordinary differential equations (ODEs). It is in the process of the reduction of that system to the second order ODE in (1.1) where additional assumptions (sometimes correct, sometimes incorrect) are usually made.

This situation has its analogs in electric LC-circuits with inductance L and capacitance C, where the frequency equals $\omega = 1/\sqrt{LC}$ and the impedance is $Z = \sqrt{L/C}$. And again, most experts assume (incorrectly) that the frequency modulation is the reason for parametric excitation, while actually only the modulation of the impedance leads to parametric resonance.

A similar situation occurs in classical electrodynamics. Most people working in optics think that Fresnel reflection at normal incidence to the boundary of two media is governed by the step of the propagation speed $v = c/n$. Actually, it is the step of the impedance $Z = \sqrt{\mu/\varepsilon}$ that leads to the reflection at normal incidence, while the step of the propagation speed $v = 1/\sqrt{\varepsilon\mu}$ may be arbitrary. The 'stealth technology' of magneto-dielectric covers is based on matching the cover impedance to 377 Ω, i.e., to the impedance of the vacuum. In the television industry, a 75 Ω coaxial cable must be matched with another 75 Ω one, while the propagation speeds in the cables to be connected do not matter. The reason for people in optics to be confused is that $\mu = \mu_{\mathrm{vac}}$ in the optical spectral range, and therefore the impedance and propagation speed are rigidly connected there: $v = c/n$ and $Z = [377\ \Omega]/n$ in optics.

A considerable part of the material below is presented using the terminology of Lagrangian or Hamiltonian equations, with canonical (symplectic) transformations and generating functions (see, e.g., Refs [1–6]). *But the statements we make about the separate roles of impedance and frequency may be well understood directly from the systems of the ODE in question, without references to Hamiltonian or Lagrangian mechanics. In particular, the reader is advised to start from Eqns* (3.5)*, read to the end of Section 3, and proceed with Section 4*. After that, if the reader is still interested, he or she may return to the discussions of Lagrangians and Hamiltonians in Section 2.

2. The Lagrangian, momenta, and the Hamiltonian: basic definitions and equations

We consider a system of linear ordinary differential equations. It is not the most general system of linear ODEs, but one that may be produced as Euler–Lagrange equations of the variational principle with a bilinear Lagrangian. It is no surprise that the corresponding Hamiltonian is then also bilinear.

We use the 'vectors' and 'transposed vectors' of n coordinates and corresponding velocities:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \cdots \\ x_n(t) \end{pmatrix}, \qquad \dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \cdots \\ \dot{x}_n(t) \end{pmatrix}, \tag{2.1}$$

$$\mathbf{x}^{\mathrm{T}}(t) = \big(x_1(t), \ldots, x_n(t)\big), \qquad \dot{\mathbf{x}}^{\mathrm{T}}(t) = \big(\dot{x}_1(t), \ldots, \dot{x}_n(t)\big).$$

We assume the general Lagrange function to be bilinear in the coordinates and velocities:

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = -\frac{1}{2}\,\mathbf{x}^{\mathrm{T}}\hat{K}\mathbf{x} + \frac{1}{2}\,\dot{\mathbf{x}}^{\mathrm{T}}\hat{M}\dot{\mathbf{x}} + \mathbf{x}^{\mathrm{T}}\hat{\beta}\dot{\mathbf{x}}$$
$$\equiv -\frac{1}{2}\,x_i K_{ij} x_j + \frac{1}{2}\,\dot{x}_i M_{ij}\dot{x}_j + x_i\beta_{ij}\dot{x}_j\,. \tag{2.2}$$

Here, $K_{ij}(t)$, $M_{ij}(t)$, and $\beta_{ij}(t)$ are $n \times n$ matrices; $\hat{K}$ and $\hat{M}$ are symmetric, while $\hat{\beta}$ may or may not be symmetric in general; all three matrices may be time dependent. As usual, summation over repeated indices is assumed, and the hat denotes a matrix. The choice of signs and of the letters K, M, and β is aimed at bringing to mind the elasticity constant K of an oscillator, the mass M, and the magnetic field B. According to Faraday's law of electromagnetic induction, the time dependence of the antisymmetric part of $\hat{\beta}$ leads to the curly electric field, i.e., to an electromotive force.

Using the standard notation for the vectors of momenta $\mathbf{p}$ and of forces $\mathbf{f}$, we have

$$p_i = p_i(t) \equiv \frac{\partial L}{\partial \dot{x}_i} = M_{ij}\dot{x}_j + \hat{\beta}^{\mathrm{T}}_{ij}x_j\,,$$

$$f_i = f_i(t) \equiv \frac{\partial L}{\partial x_i} = -K_{ij}x_j + \hat{\beta}_{ij}\dot{x}_j\,, \tag{2.3}$$

$$\mathbf{p} = \hat{M}\dot{\mathbf{x}} + \hat{\beta}^{\mathrm{T}}\mathbf{x}\,, \qquad \mathbf{f} = \hat{\beta}\dot{\mathbf{x}} - \hat{K}\mathbf{x}\,.$$

We can express the velocities in terms of the momenta:

$$\dot{\mathbf{x}} = \hat{M}^{-1}(\mathbf{p} - \hat{\beta}^{\mathrm{T}}\mathbf{x})\,. \tag{2.4}$$

This expression is also needed for the transition to the Hamiltonian:

$$H(\mathbf{p}, \mathbf{x}, t) = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{x}}(\mathbf{p}) - L\big(\mathbf{x}, \dot{\mathbf{x}}(\mathbf{p}), t\big)$$
$$= \frac{1}{2}\,(\mathbf{p}^{\mathrm{T}} - \mathbf{x}^{\mathrm{T}}\hat{\beta})\,\hat{M}^{-1}(\mathbf{p} - \hat{\beta}^{\mathrm{T}}\mathbf{x}) + \frac{1}{2}\,\mathbf{x}^{\mathrm{T}}\hat{K}\mathbf{x}\,. \tag{2.5}$$

The standard Euler–Lagrange variational equations are written as $\mathrm{d}\mathbf{p}/\mathrm{d}t = \mathbf{f}$. Another way to express the same idea is to write the canonical Hamiltonian equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}^{\mathrm{T}}}\,, \qquad \dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}^{\mathrm{T}}}\,. \tag{2.6}$$

For our Hamiltonian (2.5) or, equivalently, for our Lagrangian (2.2), these equations are

$$\dot{p}_i(t) = \beta_{ij}\dot{x}_j - K_{ij}x_j \equiv (\hat{\beta}\hat{M}^{-1})_{ij}p_j - (K + \hat{\beta}\hat{M}^{-1}\hat{\beta}^{\mathrm{T}})_{ij}x_j\,,$$
$$\dot{x}_i(t) = M^{-1}_{ij}\,p_j - (\hat{M}^{-1}\hat{\beta}^{\mathrm{T}})_{ij}x_j\,. \tag{2.7}$$

The same equations can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{p}(t)\\ \mathbf{x}(t)\end{pmatrix} = V\begin{pmatrix}\mathbf{p}(t)\\ \mathbf{x}(t)\end{pmatrix},$$
$$V = \begin{pmatrix}\hat{\beta}\hat{M}^{-1} & -(\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^{\mathrm{T}})\\ \hat{M}^{-1} & -\hat{M}^{-1}\hat{\beta}^{\mathrm{T}}\end{pmatrix}, \tag{2.8}$$

where V is a $2n \times 2n$ matrix. If the properties of our system are stationary, i.e., if the three matrices $\hat{M}$, $\hat{K}$, and $\hat{\beta}$ are time-independent, then we can also write these equations in the form of 'Newton's second law':

$$\hat{M}\,\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = (\hat{\beta} - \hat{\beta}^{\mathrm{T}})\,\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \hat{K}\mathbf{x} \quad \text{(stationary case)}\,. \tag{2.9}$$

This means that in the stationary case, only the antisymmetric part of the $\hat{\beta}$ matrix is important, in accordance with the expression $\mathbf{B} = \mathrm{curl}\,(\mathbf{A}(\mathbf{r}))$ in the familiar case of the motion of a charged particle in the presence of a magnetic field B in 3-dimensional space. By itself, Eqn (2.9) is valid in the stationary case in any number of dimensions. Below, we prefer to work with system (2.8) for momenta and coordinates.

3. One-dimensional oscillator

For a one-dimensional oscillator (one coordinate x and one momentum p), the equations of motion become

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\beta(t)}{m(t)}\,p - \left[K(t) + \frac{\beta^2}{m}\right]x\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{m(t)}\,p - \frac{\beta(t)}{m(t)}\,x\,. \tag{3.1}$$

In this one-dimensional case, we can pass to the new canonical variables via a symplectic transformation, and to a new elasticity constant:

$$p_{\mathrm{new}} = p_{\mathrm{old}} - \beta(t)x_{\mathrm{old}}\,, \qquad x_{\mathrm{new}} = x_{\mathrm{old}}\,,$$
$$K_{\mathrm{new}}(t) = K_{\mathrm{old}}(t) - \frac{\mathrm{d}\beta}{\mathrm{d}t}\,. \tag{3.2}$$

This transformation is based on the generating function $\Gamma(p_{\mathrm{new}}, x_{\mathrm{old}})$ (see, e.g., Refs [1–6]),

$$\Gamma(p_{\mathrm{new}}, x_{\mathrm{old}}, t) = p_{\mathrm{new}}x_{\mathrm{old}} - \frac{1}{2}\,\beta(t)x^2_{\mathrm{old}}\,,$$
$$x_{\mathrm{new}} = \frac{\partial\Gamma}{\partial p_{\mathrm{new}}}\,, \qquad p_{\mathrm{old}} = \frac{\partial\Gamma}{\partial x_{\mathrm{old}}}\,, \qquad H_{\mathrm{new}} = H_{\mathrm{old}} + \frac{\partial\Gamma}{\partial t}\,. \tag{3.3}$$

It is worth noting that a similar transformation $\mathbf{x}_{\mathrm{new}} = \mathbf{x}$, $\mathbf{p}_{\mathrm{new}} = \mathbf{p} - \hat{\beta}^{\mathrm{T}}\mathbf{x}$ is not symplectic (i.e., not canonical) already for a two-dimensional $\mathbf{x}$. (It is rather ironic that exactly this nonsymplectic transformation is suggested to the readers of book [7] entitled *An Introduction to Symplectic Geometry*.)

Returning to the one-dimensional case, we have the resulting Hamiltonian

$$H_{\mathrm{new}}(p_{\mathrm{new}}, x_{\mathrm{new}}, t) = 0.5\left[\frac{p^2_{\mathrm{new}}}{m(t)} + K_{\mathrm{new}}(t)x^2_{\mathrm{new}}\right], \tag{3.4}$$

and the equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -K(t)x\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{m(t)}\,p\,, \tag{3.5}$$

where we dropped the subscript 'new' in Eqns (3.5) for brevity. *Actually, in the one-dimensional case, the reader may start directly with system* (3.5), *completely ignoring all the previous 'high-brow matter' of β-terms.*

Now we can introduce two different quantities: the instantaneous frequency $\omega(t)$ and the instantaneous impedance $Z(t)$ by the definitions

$$\omega(t) = \sqrt{\frac{K(t)}{m(t)}}\,, \qquad Z(t) = \sqrt{K(t)m(t)}\,,$$
$$K(t) = \omega Z\,, \qquad \frac{1}{m(t)} = \frac{\omega}{Z}\,. \tag{3.6}$$

Using the frequency and the impedance, we can rewrite Eqns (3.5) as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\omega(t)Z(t)x(t)\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega(t)}{Z(t)}\,p(t)\,. \tag{3.7}$$

A possible reason to call the quantity $Z(t)$ 'impedance' may be the following. The verb 'to impede' means "to hinder; to stop in progress; to obstruct; as, to impede the advance of troops." If the viscous force term $f_{\mathrm{visc}} = -Z_{\mathrm{visc}}\,\mathrm{d}x/\mathrm{d}t$ is added (see Appendix C), then the dimension of Z_{visc} is the same as the dimension of the impedance $Z(t)$. The critical value of the damping corresponds to $Z_{\mathrm{visc}} = 2Z$, when both eigenvalues of time evolution switch from the damped-oscillatory type to the purely damped type. The so-called quality factor Q of an oscillatory system (e.g., of an *LRC*-circuit in electronics) is defined as $Q = Z/Z_{\mathrm{visc}}$; it equals the ratio $Q = \omega_0/\gamma$ of the frequency ω_0 to the damping constant γ [s^{-1}] (for power), $\gamma = Z_{\mathrm{visc}}/m$. Incidentally, the quantity $\beta(t)$ also has the dimension of impedance. The dimension of impedance depends on the units for the coordinate x. Another physical interpretation of the impedance in the one-dimensional case is discussed below in terms of the *aspect ratio of the phase-space cell*.

We can perform one more canonical (symplectic) transformation to new variables: a new coordinate X and a new momentum P:

$$p(t) = \sqrt{Z(t)}\;P(t)\,, \qquad x(t) = \frac{1}{\sqrt{Z(t)}}\;X(t)\,. \tag{3.8}$$

The corresponding system of ODEs for these new variables X and P is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}P(t)\\ X(t)\end{pmatrix} = \begin{pmatrix}-g(t) & -\omega(t)\\ \omega(t) & g(t)\end{pmatrix}\begin{pmatrix}P(t)\\ X(t)\end{pmatrix},$$
$$g(t) = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(Z(t)\right). \tag{3.9}$$

It is easier to obtain Eqns (3.9) directly from (3.7) and (3.8); it can also be verified that the corresponding generating

function and the new Hamiltonian are

$$\Gamma(x,P,t) = xP\sqrt{Z(t)}\,, \tag{3.10}$$
$$H_{\rm new}(P,X,t) = H_{\rm old} + \frac{\partial\Gamma}{\partial t} = g(t)PX + \frac{\omega(t)}{2}\left(X^2+P^2\right),$$

with Eqns (3.9) being a direct consequence of Hamiltonian (3.10).

It is convenient to introduce complex amplitudes $a(t)$ and $a^*(t)$ and the corresponding slowly varying complex amplitudes $b(t)$ and $b^*(t)$ by the definitions

$$a(t) = \frac{X(t)+{\rm i}P(t)}{\sqrt{2\hbar}}\,, \qquad a^*(t) = \frac{X(t)-{\rm i}P(t)}{\sqrt{2\hbar}}\,, \tag{3.11}$$

$$b(t) = a(t)\exp\left({\rm i}\int_0^t \omega(t')\,{\rm d}t'\right), \tag{3.12}$$
$$b^*(t) = a^*(t)\exp\left(-{\rm i}\int_0^t \omega(t')\,{\rm d}t'\right).$$

From the standpoint of classical mechanics, the constant $2\hbar$ may be arbitrary, e.g., even 2. The convenience of interpreting $\hbar$ as the Planck constant can be seen in quantum mechanics, since the energy (Hamiltonian) for static ω and Z is $H = \hbar\omega(aa^* + a^*a)/2$, with a interpreted as the annihilation operator of one quantum $\hbar\omega$. Remaining within classical mechanics, we obtain the *exact* linear ODE for 'fast' amplitudes $a(t)$ and $a^*(t)$ and for 'slowly varying' amplitudes $b(t)$ and $b^*(t)$:

$$\frac{{\rm d}a}{{\rm d}t} = -{\rm i}\omega a(t) + g(t)\,a^*(t)\,, \qquad \frac{{\rm d}a^*}{{\rm d}t} = g(t)a(t) + {\rm i}\omega a^*(t)\,, \tag{3.13}$$

$$g(t) = \frac{1}{2}\frac{\rm d}{{\rm d}t}\ln\left(Z(t)\right), \tag{3.14}$$

$$\frac{{\rm d}b}{{\rm d}t} = g(t)b^*(t)\exp\left[2{\rm i}\int_0^t \omega(t')\,{\rm d}t'\right], \tag{3.15}$$
$$\frac{{\rm d}b^*}{{\rm d}t} = g(t)b(t)\exp\left[-2{\rm i}\int_0^t \omega(t')\,{\rm d}t'\right].$$

These systems, (3.13) or (3.15), allow coming to an important conclusion: *even in the case of a time-dependent frequency $\omega(t)$, the adiabatic invariant $aa^* \equiv bb^* = H/\hbar\omega$ is strictly conserved if the impedance $Z(t)$ is constant in time and therefore $g(t) \equiv 0$.*

Following the ideas expressed in [8] by P Paradoksov, we may introduce a quantum-mechanical interpretation of the above result of classical theory. Indeed, the wave functions $\psi_n(x)$ of the nth state of a stationary quantum oscillator are the well-known Hermit – Gauss functions, with the expectation values easily calculated in terms of $a(t)$ and $a^*(t)$ as $\left\langle\left(a(t)+a^*(t)\right)^2\right\rangle = 2n+1$, and hence

$$\psi_n(x) = {\rm const}\,H_n\left(x\sqrt{\frac{Z}{\hbar}}\right)\exp\left(-\frac{\left(x\sqrt{Z/\hbar}\right)^2}{2}\right), \tag{3.16}$$

$$\langle x^2\rangle = \left(n+\frac{1}{2}\right)\frac{\hbar}{Z}\,, \qquad \langle p^2\rangle = \left(n+\frac{1}{2}\right)\hbar Z\,, \tag{3.17}$$
$$\langle x^2\rangle\langle p^2\rangle = \left(n+\frac{1}{2}\right)^2\hbar^2\,,$$

with the minimum value of dispersions realized in the Gaussian ground state, $n = 0$.

We know from the Heisenberg commutation relations that the elementary area of the phase-space cell $\delta p\,\delta x$ is about $\hbar$ [J s] [actually, $2\pi\hbar(n+1/2)$ for the nth state of an oscillator]. Therefore, the elementary quantum of energy, according to Planck, is $\hbar\omega$ [J] [see Eqn (D.8) below], with *the frequency ω* [rad s^{-1}] *being the* '*kinematic parameter*.' *The shape* of a trajectory in the phase space for the oscillator *is an ellipse*.[1] ***The physical meaning of the impedance is the aspect ratio $\delta p/\delta x$ of that trajectory:***

$$\delta p\,\delta x \approx \hbar\left(n+\frac{1}{2}\right), \qquad \frac{\delta p}{\delta x} = Z\,. \tag{3.18}$$

It is important that this aspect ratio $\delta p/\delta x$ characterizes the orbit of oscillatory motion in any stationary state, including a state of motion in classical mechanics. The impedance has the dimensions $[Z] = [{\rm J\ s\ } x^{-2}]$ for any dimensions $[x]$ of the coordinate x.

If the impedance $Z(t)$ does not change with time, then each of the Hermit – Gauss wave functions keeps its shape, size ($\Delta x \approx \sqrt{\hbar(n+1/2)/Z}$), and the number n. The only changes are in the temporal phase factor of the corresponding wave function. This can also be interpreted somewhat differently. Liouville's theorem in classical mechanics informs us that the area of phase space is always conserved. If the aspect ratio (i.e., the impedance) is also conserved, then the adiabatic invariant is conserved.

Another way of understanding the exceptional case of constant impedance is to write the expression for Hamiltonian (3.4) in the old coordinate x and momentum p with the use of Eqns (3.6) for m and K via frequency and impedance:

$$H(x,p) = p\dot{x}(p) - L\left(x,\dot{x}(p)\right) = \frac{1}{2}\,\omega(t)\left(x^2Z(t) + \frac{p^2}{Z(t)}\right). \tag{3.19}$$

This form shows that for a constant (i.e., time-independent) impedance Z of our one-dimensional problem, time dependence of the frequency $\omega(t)$ amounts to a time-dependent rescaling of the Hamiltonian [also cf. (3.10)]. This time dependence of H corresponds to a change of the time variable,

$$\tau(t) = \int_0^t \omega(t')\,{\rm d}t'\,. \tag{3.20}$$

Therefore, time dependence of the frequency at a constant impedance results in no changes in actual trajectories in the phase space (x,p), although they change in the extended space (x,p,t). However, for multi-dimensional coupled oscillators, we show in what follows that the assumption of a time-independent impedance matrix is not sufficient for preserving the trajectories in the phase space $(\mathbf{x},\mathbf{p})$.

Figure 2 shows different actions on the phase plane (x,p) of two kinds of perturbation of a Hamiltonian that is initially stationary. Figure 2a depicts the vector field of frequency perturbation, and Fig. 2b does the same for impedance perturbation. It can be seen that the perturbation of impedance tends to introduce nontrivial changes to the original stationary ellipse.

The adiabatic invariant $E/\hbar\omega$ is conserved with reasonably good accuracy if $Z(t)$ changes adiabatically, i.e., slowly

[1] Here is the definition of an ellipse by a military person: "Ellipse is a circle inscribed into a square 3 by 4" (I learned this from the late Alexandr 'Sania' Ovchinnikov). In a sense, the transition from amplitudes a and a^* to the regular momentum p and coordinate x is exactly changing the aspect ratio of a circle to $\delta p/\delta x = Z^{1/2}/Z^{-1/2}$ instead of (3 by 4).

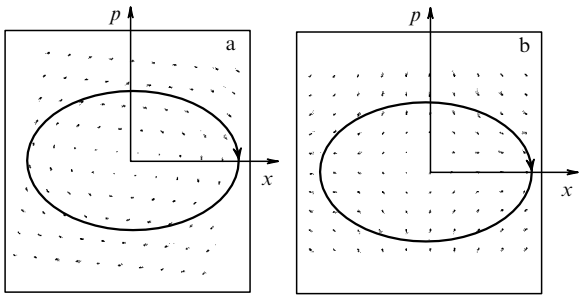

Figure 2. (a) Vector field corresponding to a frequency perturbation, and a stationary trajectory. (b) Vector field corresponding to an impedance perturbation, and a stationary trajectory.

at the time scale $\tau \approx 1/\omega$, since in that case the coupling between slow amplitudes $b(t)$ and $b^*(t)$ oscillates very fast and does not yield an accumulating effect. Following [8], we can say that the parametric transitions between a state n and states $(n \pm 2)$ are out of resonance with the external perturbations. This interpretation shows that approximate conservation of the adiabatic invariant is mostly due to the absence of parametric resonances of the first order ($\omega_{\text{parameter}} = 2\omega_0$) or of some higher order of the perturbation theory, and not necessarily due to slowness of the parameter change (see 'Arnold's tongues' [2]).

In our notation, parametric excitation is a change in the adiabatic invariant bb^* (i.e., a change in the number of quanta); it may occur only when the impedance $Z(t)$ is time dependent, and hence $g(t) = 1/2\,\mathrm{d}\ln\big(Z(t)\big)/\mathrm{d}t \neq 0$. Parametric excitation is especially effective when the impedance parameter of the system, $Z(t)$, changes in time at the doubled frequency of oscillations: this results in a slowly varying coupling between $b(t)$ and $b^*(t)$. For example, if

$$\begin{aligned} &\omega(t) \approx \omega_0 = \text{const}\,, \\ &Z(t) = Z_0 + Z_1 \cos(2\omega_0 t)\,, \qquad |Z_1| \ll Z_0\,, \end{aligned} \tag{3.21}$$

then

$$g(t) \approx g_1 \sin(2\omega_0 t)\,, \qquad g_1 = -\frac{Z_1}{Z_0}\,\omega_0\,. \tag{3.22}$$

For a small modulation Z_1/Z_0 in the vicinity of a parametric resonance, the fast-oscillating terms can be neglected in the equations for $b(t)$ and $b^*(t)$. Then the equations and their solutions become

$$\begin{aligned} &\frac{\mathrm{d}b}{\mathrm{d}t} = \mathrm{i}\,\frac{g_1}{2}\,b^*(t)\,, \qquad \frac{\mathrm{d}b^*}{\mathrm{d}t} = -\mathrm{i}\,\frac{g_1}{2}\,b(t)\,, \\ &b(t) = \cosh\frac{|g_1|t}{2}\,b(0) + \mathrm{i}\,\frac{g_1}{|g_1|}\sinh\frac{|g_1|t}{2}\,b^*(0)\,, \end{aligned} \tag{3.23}$$

clearly demonstrating parametric amplification with the amplitude growth rate $|g_1|/2\ [\mathrm{s}^{-1}]$. This is quite in agreement with the picture of resonant transitions between quantum states n and states $(n \pm 2)$, whose energy differences are equal to $\pm 2\hbar\omega_0$.

We compare (3.23) with solutions of the Mathieu equation. Representing the unknown variable $x(t)$ in the form of a complex amplitude plus its conjugate,

$$x(t) = \frac{1}{2}\left[c(t)\exp(-\mathrm{i}\omega_0 t) + c^*(t)\exp(\mathrm{i}\omega_0 t)\right], \tag{3.24}$$

we can reduce Mathieu equation (1.1) at $p = 2\omega_0$, i.e., at the exact parametric resonance and in the same approximation, to the following coupled equations and their solutions:

$$\begin{aligned} &\frac{\mathrm{d}c}{\mathrm{d}t} = -\mathrm{i}\,\frac{\omega_1}{2}\,c^*(t)\,, \qquad \frac{\mathrm{d}c^*}{\mathrm{d}t} = \mathrm{i}\,\frac{\omega_1}{2}\,c(t)\,, \\ &c(t) = \cosh\frac{|\omega_1|t}{2}\,c(0) - \mathrm{i}\,\frac{\omega_1}{|\omega_1|}\sinh\frac{|\omega_1|t}{2}\,c^*(0)\,. \end{aligned} \tag{3.25}$$

In this case, we see parametric amplification with the amplitude growth rate $|\omega_1|/2\ [\mathrm{s}^{-1}]$.

We now consider two cases for a pendulum, where either the length or the gravity is modulated (see Section 4). In the first case, $g_{\text{eff}} = \text{const}$, and it follows from (4.4) that

$$\begin{aligned} &L(t) = L_0 + L_1\cos(2\omega_0 t)\,, \\ &\omega(t) \approx \omega_0\left[1 - \frac{L_1}{2L_0}\cos(2\omega_0 t)\right], \\ &Z(t) \approx Z_0\left[1 + \frac{3L_1}{2L_0}\cos(2\omega_0 t)\right]. \end{aligned} \tag{3.26}$$

In the second case, $L(t) = \text{const}$ and

$$\begin{aligned} &g_{\text{eff}}(t) = g_0 + a_1\cos(2\omega_0 t)\,, \\ &\omega(t) \approx \omega_0 + \omega_0\left(\frac{a_1}{2g_0}\right)\cos(2\omega_0 t)\,, \\ &Z(t) \approx Z_0 + Z_0\left(\frac{a_1}{2g_0}\right)\cos(2\omega_0 t)\,. \end{aligned} \tag{3.27}$$

Comparing solutions (3.23) and (3.25), we conclude that Mathieu equation (1.1) for the length modulation at a constant effective gravity and for $x(t)$ interpreted as the angle $\varphi(t)$ yields a *wrong* result for the parametric amplitude gain rate (3 times less than the correct one, equal to $3\omega_0|L_1|/4L_0$) and a *wrong* phase for the amplified component. The same Mathieu equation (1.1) gives the correct amplitude gain rate $\omega_0|a_1|/4g_0$ and the correct phase for the gravity modulation at $L = \text{const}$.

This can be formulated in a slightly different manner. If we interpret $x(t)$ in Mathieu equation (1.1) as the angular momentum M_z of a pendulum,, then the length modulation, according to (1.1), would give a gain rate 3 times smaller than the correct one, but the correct phase of the amplified component, while the modulation of the effective gravity at constant length leads to the correct result for gain, but a wrong result for the phase of the amplified component.[2]

[2] A possible source of this confusion is the traditional use of the term 'hyperbolic rotation' for a vector field with a time-dependent impedance (Fig. 2b). The symplectic group $\mathrm{Sp}_{2n}(R)$ has the dimension $n(2n+1)$, which is 3 for $n = 1$ (one momentum and one coordinate; real 2×2 matrices with unit determinant). One of the generators of this group is given by the infinitesimal rotation around the origin in the (P, X) plane; only one type of such rotation exists in a proper sense. But there are two linearly independent generators for 'hyperbolic rotations,' one of which contracts the P-direction and dilates the X-direction by the same factor (as is shown in Fig. 2b), and the other contracts the direction at $+45^\circ$ and dilates the one at -45° by the same factor. The phase of an external perturbation (as we have seen, the impedance modulation phase) involves a certain linear combination of these two generators; on the other hand, although the time modulation of a kinematic parameter (i.e., the frequency) also makes the system nonautonomous, it does not select any preferred phase of motion because it corresponds just to a rotation in the (P, X) plane in the usual sense.

4. Particular examples: pendulum, *LC*-circuit in electronics

We consider the pendulum shown in Fig. 1. We let $\varphi(t)$ denote the angle of the pendulum from the vertical. The angular momentum of the pendulum with respect to the pivot axis z is $M_z = I(t)\dot{\varphi} = mL^2(t)\dot{\varphi}$, where $I(t) = mL^2(t)$ is the moment of inertia with respect to the z axis. The quantities φ and M_z are the generalized canonical coordinate and momentum, respectively. The vertical component of the force F_y acting on the mass and the corresponding torque T in the coordinate system of the pivot are given by

$$|F_y| = m\bigl(g_0 + a(t)\bigr) \equiv mg_{\text{eff}}(t)\,, \qquad a(t) = \frac{\mathrm{d}^2 Y(t)}{\mathrm{d}t^2}\,, \quad (4.1)$$

$$T(t) = -L(t)F_y(t)\sin\varphi(t) \approx -L(t)mg_{\text{eff}}(t)\varphi(t)\,. \quad (4.2)$$

The last expression for the torque T corresponds to linearization in the small amplitude of oscillations $\varphi(t)$ [rad]. In (4.1), we took the inertia force $ma(t)$ into account, where $a(t)$ is the vertical acceleration of the pivot axis. The equations of motion for the canonical variables φ and M_z are

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{mL^2(t)}\,M_z\,, \qquad \frac{\mathrm{d}M_z}{\mathrm{d}t} = -mL(t)g_{\text{eff}}(t)\varphi\,. \quad (4.3)$$

Introducing the instantaneous values of frequency $\omega(t)$ and of impedance $Z(t)$ (with respect to the coordinate φ) as

$$\omega(t) = \sqrt{\frac{g_{\text{eff}}(t)}{L(t)}}\,, \qquad Z(t) = \sqrt{m^2L^3(t)g_{\text{eff}}(t)}\,,$$
$$\frac{1}{mL^2(t)} = \frac{\omega(t)}{Z(t)}\,, \qquad mL(t)g_{\text{eff}}(t) = \omega(t)Z(t)\,, \quad (4.4)$$

we reduce our system to the form

$$\frac{\mathrm{d}M_z}{\mathrm{d}t} = -\omega(t)Z(t)\varphi(t)\,, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\omega(t)}{Z(t)}\,M_z(t)\,, \quad (4.5)$$

identical to Eqn (3.7) up to the change of notation $\varphi \to x$, $M_z \to p$. We have seen in Section 3 that just the modulation of impedance [$Z(t) = \sqrt{m^2L^3(t)g_{\text{eff}}(t)}$ for the pendulum] is responsible for the parametric excitation. It is worth noting that even the linearized equations for the coordinate $x(t) = L(t)\sin\varphi(t)$ are rather complicated in the case of a time-dependent length $L(t)$; therefore, we use the angle φ and the angular momentum M_z here.

We now consider another important example of a lumped element system: the *LC*-circuit in electronics, with the notation presented in Fig. 3. A voltage $V(t)$ [V] in the capacitor results in the charge $Q(t) = -C(t)V(t)$ (in the lower electrode of the capacitor in Fig. 3). Here, $C(t)$ [F] is the time-dependent capacitance, and the dimension of the charge Q is [Coulomb]. The voltage $V(t)$ tends to increase the magnetic flux $\Phi(t)$ in the inductor. This magnetic flux (summed over all the coils of the inductor) equals $\Phi(t) = L(t)I(t)$, where $I(t)$ [A] is the current and $L(t)$ [H] is the time-dependent value of inductance.

The following equations describe the system:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = V(t)\,, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = I(t)\,,$$
$$\Phi(t) = L(t)I(t)\,, \qquad Q(t) = -C(t)V(t)\,. \quad (4.6)$$

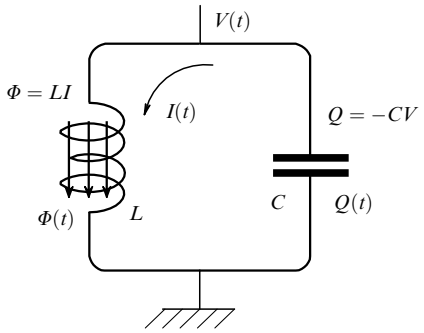

Figure 3. Notations for *LC*-circuit.

The first equation in (4.6) is Faraday's law of electromagnetic induction (an analog of $\dot{p} = f$), the second expresses the idea of continuity of charge (an analog of $v = \dot{x}$), and the third and forth equations are the material relations (analogs of $p = mv$ and $f = -kx$). Eliminating the intermediate quantities V and I, we can write the system in the form of two coupled linear equations:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{Q(t)}{C(t)}\,, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\Phi(t)}{L(t)}\,. \quad (4.7)$$

We introduce the frequency ω [rad s^{-1}] and impedance Z [Ω] by the formulas

$$\omega = \frac{1}{\sqrt{LC}}\,, \qquad Z = \sqrt{\frac{L}{C}}\,, \qquad \frac{1}{L} = \frac{\omega}{Z}\,, \qquad \frac{1}{C} = \omega Z\,. \quad (4.8)$$

Then our system of equations (4.7) becomes

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\omega(t)Z(t)Q(t)\,, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\omega(t)}{Z(t)}\,\Phi(t)\,, \quad (4.9)$$

and therefore, up to the substitution $\Phi \to p$, $Q \to x$, it is identical to Eqns (3.5), (3.7). This proves that the parametric excitation in *LC*-circuits is governed by time modulation of the impedance $Z(t)$, not of the frequency $\omega(t)$.

5. Multi-dimensional case, no 'magnetic' forces

In this section, we consider the case where $\hat{\beta} = 0$, i.e., the case of absent 'magnetic-type' forces. Then system of equations (2.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{p}\\ \mathbf{x}\end{pmatrix} = V\begin{pmatrix}\mathbf{p}\\ \mathbf{x}\end{pmatrix};$$
$$V = \begin{pmatrix}\hat{0} & -\hat{K}\\ \hat{M}^{-1} & \hat{0}\end{pmatrix}, \quad \text{or} \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\hat{K}\mathbf{x}\,, \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \hat{M}^{-1}\mathbf{p}\,. \quad (5.1)$$

We wish to introduce tensor analogs of the frequency and the impedance. Here is our approach to the system of equations (5.1) where both matrices $\hat{M}$ and $\hat{K}$ may be time dependent. We define the instantaneous symmetric real matrix $\hat{Z}(t)$ as a solution of the matrix equation

$$\hat{Z}\hat{M}^{-1}\hat{Z} = \hat{K}\,, \quad \hat{M} = \hat{M}^{\mathrm{T}}\,, \quad \hat{K} = \hat{K}^{\mathrm{T}}\,, \quad \hat{Z} = \hat{Z}^{\mathrm{T}} \quad (5.2)$$

(we recall that the superscript T denotes the transposition of a matrix, and hence the three matrices $\hat{M}$, $\hat{K}$, and $\hat{Z}$ are symmetric). An explicit (albeit somewhat awkward) expres-

sion for the solution of this equation is

$$\hat{Z} = \hat{M}^{1/2}(\hat{M}^{-1/2}\hat{K}\hat{M}^{-1/2})^{1/2}\hat{M}^{1/2}\,. \tag{5.3}$$

Another expression, which yields an identical result, is

$$\hat{Z} = \hat{K}^{1/2}(\hat{K}^{1/2}\hat{M}^{-1}\hat{K}^{1/2})^{-1/2}\hat{K}^{1/2}\,. \tag{5.4}$$

Equation (5.2) *defining the* $\hat{Z}$ *matrix, together with its formal solution* (5.3) *or* (5.4)*, constitutes one of the main results in this paper*: *an extension of the notion of impedance to the multi-dimensional case without magnetic-type forces*. In Appendix A, we give the explicit solution $\hat{Z}$ of Eqn (5.2) for 2×2 matrices $\hat{M}$ and $\hat{K}$; for a larger number of dimensions, expressions (5.3) and (5.4) require using eigenvalues of $\hat{M}$ and $\hat{K}$.

That somewhat unwieldy formula (5.3) is derived as follows. We first perform a symplectic transformation to new coordinates and momenta such that the mass becomes the unit matrix:

$$\mathbf{p}_{\rm new} = \frac{1}{\hat{M}^{1/2}}\,\mathbf{p}\,, \qquad \mathbf{x}_{\rm new} = \hat{M}^{1/2}\mathbf{x}\,. \tag{5.5}$$

For a stationary system (i.e., for time-independent matrices $\hat{M}$ and $\hat{K}$), the equations of motion then become

$$\begin{aligned} &\frac{{\rm d}\mathbf{p}_{\rm new}}{{\rm d}t} = -\hat{M}^{-1/2}\hat{K}\hat{M}^{-1/2}\mathbf{x}_{\rm new}\,,\\ &\frac{{\rm d}\mathbf{x}_{\rm new}}{{\rm d}t} = \mathbf{p}_{\rm new} \qquad \text{(stationary case)}\,. \end{aligned} \tag{5.6}$$

We may interpret $\hat{K}_{\rm new} = \hat{M}^{-1/2}\hat{K}\hat{M}^{-1/2}$ as the new elasticity matrix. After that, the solution of an equation like (4.2), $\hat{Z}_{\rm new}\hat{Z}_{\rm new} = \hat{K}_{\rm new}$, is evident: $\hat{Z}_{\rm new} = \hat{K}_{\rm new}^{1/2}$. Returning to the original coordinates $(\mathbf{p},\mathbf{x})$, we obtain the result in (5.3). Certainly, after that 'derivation,' result (5.3) was verified by direct substitution in Eqn (5.2). A similar procedure, where a new elasticity matrix becomes a unit matrix by a symplectic transformation, yields formula (5.4), which may also be verified by direct substitution.

Returning to the general time-dependent case, we can use this $n\times n$ matrix $\hat{Z}$ of impedance to define new 'vectors' of canonical momenta $\mathbf{P}$ and coordinates $\mathbf{X}$ by

$$\begin{aligned} &\begin{pmatrix}\mathbf{P}\\ \mathbf{X}\end{pmatrix} = \hat{Z}_{2n}\begin{pmatrix}\mathbf{p}\\ \mathbf{x}\end{pmatrix}, \qquad \hat{Z}_{2n}(t) = \begin{pmatrix}\hat{Z}^{-1/2} & \hat{0}\\ \hat{0} & \hat{Z}^{1/2}\end{pmatrix};\\ &\begin{pmatrix}\mathbf{p}\\ \mathbf{x}\end{pmatrix} = \hat{Z}_{2n}^{-1}\begin{pmatrix}\mathbf{P}\\ \mathbf{X}\end{pmatrix}, \qquad \hat{Z}_{2n}^{-1} = \begin{pmatrix}\hat{Z}^{1/2} & \hat{0}\\ \hat{0} & \hat{Z}^{-1/2}\end{pmatrix}. \end{aligned} \tag{5.7}$$

Indeed, the transformation matrix $\hat{Z}_{2n}$ and its inverse $\hat{Z}_{2n}^{-1}$ are symplectic $2n\times 2n$ matrices. (The definition of symplectic matrices is recalled in Appendix B.) Now the equations for $\mathbf{P}$ and $\mathbf{X}$ become

$$\frac{\rm d}{{\rm d}t}\begin{pmatrix}\mathbf{P}(t)\\ \mathbf{X}(t)\end{pmatrix} = \begin{pmatrix}-\hat{G}_1 & -\hat{\Omega}_2\\ \hat{\Omega}_1 & \hat{G}_2\end{pmatrix}\begin{pmatrix}\mathbf{P}(t)\\ \mathbf{X}(t)\end{pmatrix}, \tag{5.8}$$

where we have introduced four $n\times n$ matrices

$$\begin{aligned} &\hat{\Omega}_1 = \hat{Z}^{1/2}\hat{M}^{-1}\hat{Z}^{1/2} \equiv \hat{\Omega}_1^{\rm T}\,, \qquad \hat{\Omega}_2 = \hat{Z}^{-1/2}\hat{K}\hat{Z}^{-1/2} \equiv \hat{\Omega}_2^{\rm T}\,,\\ &\hat{G}_1 = \hat{Z}^{-1/2}\,\frac{{\rm d}\hat{Z}^{1/2}}{{\rm d}t}\,,\\ &\hat{G}_2 = -\hat{Z}^{1/2}\,\frac{{\rm d}\hat{Z}^{-1/2}}{{\rm d}t} \equiv \frac{{\rm d}\hat{Z}^{1/2}}{{\rm d}t}\,\hat{Z}^{-1/2} \equiv \hat{G}_1^{\rm T}\,. \end{aligned} \tag{5.9}$$

Transformation of the formula for $\hat{G}_2$ was facilitated by the use of identities

$$\begin{aligned} &\frac{\rm d}{{\rm d}t}\,\hat{A}^{-1} = -\hat{A}^{-1}\left(\frac{\rm d}{{\rm d}t}\,\hat{A}\right)\hat{A}^{-1}\,,\\ &\frac{\rm d}{{\rm d}t}\,\hat{Z}^{-1/2} = -\hat{Z}^{-1/2}\left(\frac{\rm d}{{\rm d}t}\,\hat{Z}^{1/2}\right)\hat{Z}^{-1/2}\,, \end{aligned} \tag{5.10}$$

which, in turn, may be easily derived by differentiating the identity $\hat{B}\hat{B}^{-1} = \hat{1}$. The Hamiltonian corresponding to Eqns (5.8) is

$$H(\mathbf{P},\mathbf{X},t) = \mathbf{P}^{\rm T}\hat{G}_2\mathbf{X} + \frac{1}{2}(\mathbf{P}^{\rm T}\hat{\Omega}_1\mathbf{P} + \mathbf{X}^{\rm T}\hat{\Omega}_2\mathbf{X})\,, \tag{5.11}$$

which may be found by fitting $H(\mathbf{P},\mathbf{X},t)$ to Eqns (5.8). The same Hamiltonian can also be found differently, with a proof of the canonical (symplectic) character of transformation (5.7) as a spin-off, if we use a generating function $\Gamma(\mathbf{x},\mathbf{P},t)$:

$$\begin{aligned} &\mathbf{X} = \frac{\partial\Gamma}{\partial\mathbf{P}^{\rm T}}\,, \qquad \mathbf{p} = \frac{\partial\Gamma}{\partial\mathbf{x}^{\rm T}}\,,\\ &H_{\rm new}(\mathbf{P},\mathbf{X},t) = H_{\rm old}\big(\mathbf{p}(\mathbf{P}),\mathbf{x}(\mathbf{X}),t\big) + \frac{\partial\Gamma(\mathbf{P},\mathbf{x},t)}{\partial t}\,,\\ &\Gamma(\mathbf{P},\mathbf{x},t) = \mathbf{x}^{\rm T}\hat{Z}^{1/2}\mathbf{P}\,. \end{aligned} \tag{5.12}$$

It can be seen that because the impedance matrix $\hat{Z}$ satisfies Eqn (5.2), the two symmetric real 'frequency matrices' $\hat{\Omega}_1$ and $\hat{\Omega}_2$ introduced in (5.9) are actually equal to each other and are denoted just by $\hat{\Omega}$ in what follows:

$$\hat{\Omega} = \hat{\Omega}_1 = \hat{\Omega}_2 = \hat{Z}^{1/2}\hat{M}^{-1}\hat{Z}^{1/2} = \hat{Z}^{-1/2}\hat{K}\hat{Z}^{-1/2}\,. \tag{5.13}$$

Similarly to the single-oscillator case, we introduce a 'complex amplitude' vector $\mathbf{a}(t)$ and its complex conjugate vector $\mathbf{a}^*(t)$ by the definition

$$\mathbf{a}(t) = \frac{\mathbf{X}(t) + {\rm i}\mathbf{P}(t)}{\sqrt{2\hbar}}\,, \qquad \mathbf{a}^*(t) = \frac{\mathbf{X}(t) - {\rm i}\mathbf{P}(t)}{\sqrt{2\hbar}}\,. \tag{5.14}$$

The coupled equations for the complex vectors $\mathbf{a}(t)$ and $\mathbf{a}^*(t)$ then become

$$\begin{aligned} &\frac{{\rm d}\mathbf{a}(t)}{{\rm d}t} = -{\rm i}\hat{\omega}(t)\mathbf{a}(t) + \hat{g}(t)\mathbf{a}^*(t)\,,\\ &\frac{{\rm d}\mathbf{a}^*(t)}{{\rm d}t} = {\rm i}\hat{\omega}(t)\mathbf{a}^*(t) + \hat{g}(t)\mathbf{a}(t)\,, \end{aligned} \tag{5.15}$$

where $\hat{\omega}(t)$ is the Hermitian matrix

$$\hat{\omega} = \hat{\Omega} + \frac{\hat{G}_1 - \hat{G}_1^{\rm T}}{2{\rm i}} \equiv \hat{\omega}^+ = \hat{\Omega} - \frac{\hat{G}_1^{\rm T} - \hat{G}_1}{2{\rm i}}\,, \tag{5.16}$$

and a real Hermitian matrix $\hat{g}(t)$ is defined as

$$\hat{g}(t) = \frac{\hat{G}_1 + \hat{G}_1^{\rm T}}{2}\,. \tag{5.17}$$

We first consider the case where the impedance matrix is time independent. Then $\hat{g}(t)\equiv 0$, and the Hermitian character of $\hat{\omega}(t)$ results in the evolution of the vector $\mathbf{a}(t)$ under a

unitary transformation:

$$\mathbf{a}(t) = \hat{U}(t,t_0)\mathbf{a}(t_0)\,, \qquad \frac{\mathrm{d}\hat{U}(t,t_0)}{\mathrm{d}t} = -\mathrm{i}\hat{\omega}(t)\hat{U}(t,t_0)\,, \qquad \hat{U}^{+}(t,t_0) = \hat{U}^{-1}(t,t_0)\,. \tag{5.18}$$

Generally, the matrices $\hat{\omega}(t_1)$ and $\hat{\omega}(t_2)$ do not commute with each other at different instants t_1 and t_2; for that reason, a simple exponential formula for $\hat{U}(t,t_0)$ [as we had in Eqn (3.12)] is not valid here. It must be substituted by a time-ordered exponential (see, e.g., Ref. [9]); we do not go into these details.

We next consider the equations for the 'slowly varying' complex vectors $\mathbf{b}(t)$ and $\mathbf{b}^{*}(t)$:

$$\mathbf{a}(t) = \hat{U}(t)\,\mathbf{b}(t)\,, \qquad \mathbf{a}^{*}(t) = \hat{U}^{*}(t)\,\mathbf{b}^{*}(t)\,, \qquad \frac{\mathrm{d}\hat{U}(t)}{\mathrm{d}t} = -\mathrm{i}\hat{\omega}(t)\hat{U}(t)\,, \tag{5.19}$$

where we allow $\hat{g}(t) \neq 0$ and assume expression (5.16) for the Hermitian matrix $\hat{\omega}(t)$. For definiteness, we have chosen the initial instant $t_0 = 0$. Then the equations for $\mathbf{b}(t)$ and $\mathbf{b}^{*}(t)$ are

$$\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = \hat{U}^{-1}(t)\hat{g}(t)\hat{U}^{*}(t)\mathbf{b}^{*}(t)\,, \qquad \frac{\mathrm{d}\mathbf{b}^{*}(t)}{\mathrm{d}t} = \left[\hat{U}^{-1}(t)\right]^{*}\hat{g}(t)\hat{U}(t)\mathbf{b}(t)\,. \tag{5.20}$$

Just as in the case of one oscillator, we come to an important conclusion here; we repeat it. *Even in the case of a time-dependent frequency matrix $\hat{\omega}(t)$, the adiabatic invariant $\mathbf{a}\mathbf{a}^{*} \equiv \mathbf{b}\mathbf{b}^{*} = \sum H/\hbar\omega$ is strictly conserved if the impedance matrix $\hat{Z}$ is constant in time and hence the matrix $\hat{g}(t) \equiv 0$.* More familiar is the statement that the adiabatic invariant is conserved with reasonably good accuracy if the impedance $\hat{Z}(t)$ changes adiabatically, i.e., slowly at the time scale $\tau \approx 1/\omega$, since in that case the coupling between slow amplitudes $\mathbf{b}(t)$ and $\mathbf{b}^{*}(t)$ oscillates very fast and does not yield an accumulating effect. The expression for the conserved adiabatic invariant $\mathbf{a}\mathbf{a}^{*} \equiv \mathbf{b}\mathbf{b}^{*} = \sum H/\hbar\omega$ is a particular case of Manley – Row conserved quantities, in the case where any possible transitions may only substitute one quantum of a mode by one quantum of another mode. Another description of the conservation of the adiabatic invariant is the statement that parametric excitation of our system of coupled oscillators is absent for a time-independent impedance matrix.

Calculations similar to those in the one-dimensional case lead to the mean values $\langle x_i x_k\rangle$ and $\langle p_i p_k\rangle$ for the ground state of the system:

$$\langle x_i x_k\rangle = \frac{1}{2}\,\hbar\hat{Z}^{-1}_{ik}\,, \qquad \langle p_i p_k\rangle = \frac{1}{2}\,\hbar\hat{Z}_{ik}\,, \qquad \langle x_i p_k\rangle = -\frac{1}{2}\,\mathrm{i}\hbar\delta_{ik}\,, \qquad \langle p_i x_k\rangle = \frac{1}{2}\,\mathrm{i}\hbar\delta_{ik}\,, \qquad \langle p_i x_k + x_k p_i\rangle = 0\,, \qquad \langle p_i x_k - x_k p_i\rangle = -\mathrm{i}\hbar\delta_{ik}\,. \tag{5.21}$$

The last formula expresses a direct consequence of the canonical commutation relation. The wave function of the ground state in the coordinate representation is multidimensional Gaussian:

$$\psi(\mathbf{x}) = \text{const}\,\exp\left(-\frac{1}{2\hbar}\sum_{i,k} Z_{ik}x_i x_k\right). \tag{5.22}$$

Wave functions of higher states can be found by application of an appropriate number of the relevant creation operators $\mathbf{a}^{*}$ to ground-state wave function (5.22), and coherent states can be produced in a similar manner; we do not go into these details (see, e.g., Ref. [10]). Again, it is almost evident that if simultaneous variations of mass and elasticity do not change the impedance matrix, the wave function of the ground state is preserved and, easily generalizing to excited states, we see that the adiabatic invariant $\mathbf{a}\mathbf{a}^{*}$ is also preserved.

This discussion may be broadened to a more general $\mathbf{p},\mathbf{x}$-dependence of the Hamiltonian $H(\mathbf{p},\mathbf{r},t)$ than the previously considered bilinear Lagrangians and Hamiltonians. Specifically, let

$$H(\mathbf{p},\mathbf{r},t) = H_0(\mathbf{p},\mathbf{r})\,s(t)\,. \tag{5.23}$$

Evidently, the Hamilton equations for the Hamiltonian $H(\mathbf{p},\mathbf{r},t)$ differ from similar equations for $H_0(\mathbf{p},\mathbf{r})$ by a simple time rescaling $t \to \tau$, such that $\mathrm{d}\tau/\mathrm{d}t = s(t)$. In other words, trajectories in the phase space $(\mathbf{p},\mathbf{r})$ for these two Hamiltonians are identical, while trajectories in the extended phase space $(\mathbf{p},\mathbf{r},t)$ are different for $s(t) \neq 1$. Because the Hamiltonian $H_0(\mathbf{p},\mathbf{r})$ is time-independent, the motion of the corresponding system is limited to a constant-energy hypersurface $H_0(\mathbf{p},\mathbf{r}) = E_0$. However, as we have seen above for a multidimensional system of coupled oscillators, the system with a constant impedance matrix may still have a nontrivial time dependence of the Hamiltonian, which cannot be reduced just to a nonlinear change of the time variable (or, which is equivalent, of the energy unit).

6. Two-dimensional motion in the presence of a magnetic field

We now consider a particular case of motion in the (x,y) plane in the presence of a magnetic field $\mathbf{B} = B\mathbf{e}_z$, with the action of this field described by the antisymmetric tensor $\hat{\beta}$. For definiteness, we consider the mass tensor $\hat{M}$ to be isotropic. The tensor $\hat{\beta}$ can then be conveniently expressed in terms of the so-called Larmor frequency $L = qB/2m$ [rad s^{-1}], where q is the electric charge of the particle and m is its mass. We also allow a certain anisotropy of the elasticity tensor $\hat{K}$, which by an appropriate choice of (x,y) coordinates can be diagonalized in the (x,y) axes. In other words, we assume that

$$\beta_{xy} = -\beta_{yx} = mL\,, \qquad \beta_{xx} = \beta_{yy} = 0\,, \qquad \hat{\beta} = \begin{pmatrix} 0 & mL \\ -mL & 0 \end{pmatrix}, \qquad \hat{K} = \begin{pmatrix} m\omega_x^2 & 0 \\ 0 & m\omega_y^2 \end{pmatrix}, \qquad \hat{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. \tag{6.1}$$

The explicit expression for the Hamiltonian in this notation is

$$H(\mathbf{p},\mathbf{x},t) = (2m)^{-1}\left[(p_x + ymL)^2 + (p_y - xmL)^2\right] + \frac{1}{2}\,(mx^2\omega_x^2 + my^2\omega_y^2)\,. \tag{6.2}$$

The Lagrange – Hamilton equations then become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix} & \begin{pmatrix} -m(\omega_x^2+L^2) & 0 \\ 0 & -m(\omega_y^2+L^2) \end{pmatrix} \\ \begin{pmatrix} 1/m & 0 \\ 0 & 1/m \end{pmatrix} & \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix}. \tag{6.3}$$

These equations are also valid for time-dependent parameters m, L, ω_x, and ω_y. According to Eqn (5.3), the actual precession (for $q > 0$) occurs in the direction opposite to the magnetic field (diamagnetism). We introduce new momenta P_+ and P_-, as well as new coordinates X_+ and X_-, via the transformation with a symplectic matrix $\hat{Z}_4$:

$$\begin{pmatrix} P_+ \\ P_- \\ X_+ \\ X_- \end{pmatrix} = \hat{Z}_4 \begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix}, \qquad \hat{Z}_4 = \begin{pmatrix} \hat{S} & \hat{U} \\ \hat{V} & \hat{W} \end{pmatrix}, \tag{6.4}$$

$$\hat{S} = \begin{pmatrix} \dfrac{1}{\sqrt{2Z_1}} & 0 \\ \dfrac{1}{\sqrt{2Z_1}} & 0 \end{pmatrix}, \qquad \hat{U} = \begin{pmatrix} 0 & \dfrac{\sqrt{Z_1}}{\sqrt{2}} \\ 0 & -\dfrac{\sqrt{Z_1}}{\sqrt{2}} \end{pmatrix}, \tag{6.5}$$

$$\hat{V} = \begin{pmatrix} 0 & -\dfrac{1}{\sqrt{2Z_1}} \\ 0 & \dfrac{1}{\sqrt{2Z_1}} \end{pmatrix}, \qquad \hat{W} = \begin{pmatrix} \dfrac{\sqrt{Z_1}}{\sqrt{2}} & 0 \\ \dfrac{\sqrt{Z_1}}{\sqrt{2}} & 0 \end{pmatrix}.$$

(Unfortunately, none of the matrices S, U, V, and W is invertible, because each has a vanishing determinant, and therefore the elegant formulas for block matrices are inapplicable here.) In other words, we introduce new momenta P_+ and P_- and new coordinates X_+ and X_- as

$$\begin{aligned} P_+ &= \frac{p_x}{\sqrt{2Z_1}} + y\sqrt{\frac{Z_1}{2}}, \qquad P_- = \frac{p_x}{\sqrt{2Z_1}} - y\sqrt{\frac{Z_1}{2}}, \\ X_+ &= \frac{-p_y}{\sqrt{2Z_1}} + x\sqrt{\frac{Z_1}{2}}, \qquad X_- = \frac{p_y}{\sqrt{2Z_1}} + x\sqrt{\frac{Z_1}{2}}. \end{aligned} \tag{6.6}$$

The inverse transformation is also sufficiently simple:

$$\begin{aligned} p_x &= \sqrt{\frac{Z_1}{2}}(P_+ + P_-), \qquad p_y = \sqrt{\frac{Z_1}{2}}(-X_+ + X_-), \\ x &= \frac{1}{\sqrt{2Z_1}}(X_+ + X_-), \qquad y = \frac{1}{\sqrt{2Z_1}}(P_+ - P_-). \end{aligned} \tag{6.7}$$

In the general case, when the quantity $Z_1(t)$ is time dependent, the equations of motion are given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_+ \\ P_- \end{pmatrix} &= -g_1(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P_+ \\ P_- \end{pmatrix} - \hat{\Omega}_2\begin{pmatrix} X_+ \\ X_- \end{pmatrix}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X_+ \\ X_- \end{pmatrix} &= g_1(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_+ \\ X_- \end{pmatrix} + \hat{\Omega}_1\begin{pmatrix} P_+ \\ P_- \end{pmatrix}, \\ g_1(t) &= \frac{1}{2}\frac{\mathrm{d}\ln Z_1(t)}{\mathrm{d}t} \end{aligned} \tag{6.8}$$

(to be compared also to Fig. 2b). Specific expressions for the symmetric matrices $\hat{\Omega}_1$ and $\hat{\Omega}_2$ are given below [see Eqns (6.11) and (6.12)]. Equations (6.8) are the canonical equations for the Hamiltonian

$$H(P_+, P_-, X_+, X_-, t) = g_1(t)(X_+P_- + X_-P_+) + \frac{1}{2}\,\mathbf{P}^{\mathrm{T}}\hat{\Omega}_1\mathbf{P} + \frac{1}{2}\,\mathbf{X}^{\mathrm{T}}\hat{\Omega}_2\mathbf{X}. \tag{6.9}$$

If this 'intermediate' impedance Z_1 is independent of time, then the equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_+ \\ P_- \end{pmatrix} = -\hat{\Omega}_2\begin{pmatrix} X_+ \\ X_- \end{pmatrix}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X_+ \\ X_- \end{pmatrix} = \hat{\Omega}_1\begin{pmatrix} P_+ \\ P_- \end{pmatrix}. \tag{6.10}$$

Remarkably, the transformation with $Z_1(t) = \text{const}$ produces equations where the new momenta $\mathbf{P}$ are 'pulled back by a restoring force' proportional to $-\hat{\Omega}_2\mathbf{X}$, while the new coordinates $\mathbf{X}$ increase with time proportionally to $+\hat{\Omega}_1\mathbf{P}$. In other words, this transformation allowed eliminating the terms relating $\mathrm{d}\mathbf{P}/\mathrm{d}t$ to $\mathbf{P}$, and the terms relating $\mathrm{d}\mathbf{X}/\mathrm{d}t$ to $\mathbf{X}$. As a result, the matrix $\hat{\Omega}_2$ plays the role of a new elasticity matrix and the matrix $\hat{\Omega}_1$ plays the role of a new inverse mass matrix. In some sense, this transformation resembles the transition to the new coordinate frame revolving with the Larmor precession rate L. The transformation to a rotating frame would, however, make the anisotropic restoring force (at $\omega_x \neq \omega_y$) explicitly time dependent. On the contrary, our symplectic transformation is time independent, and hence does not result in a violation of the energy conservation law, even for $\omega_x \neq \omega_y$.

The symmetric real matrices $\hat{\Omega}_1$ and $\hat{\Omega}_2$ have a rather complicated structure in the case $\omega_x \neq \omega_y$:

$$\hat{\Omega}_1 = \begin{pmatrix} L + \dfrac{m(\omega_y^2+L^2)}{2Z_1^2} + \dfrac{Z_1^2}{2m} & -\dfrac{m(\omega_y^2+L^2)}{2Z_1^2} + \dfrac{Z_1^2}{2m} \\ -\dfrac{m(\omega_y^2+L^2)}{2Z_1^2} + \dfrac{Z_1^2}{2m} & -L + \dfrac{m(\omega_y^2+L^2)}{2Z_1^2} + \dfrac{Z_1^2}{2m} \end{pmatrix}, \tag{6.11}$$

$$\hat{\Omega}_2 = \begin{pmatrix} L + \dfrac{m(\omega_x^2+L^2)}{2Z_1^2} + \dfrac{Z_1^2}{2m} & \dfrac{m(\omega_x^2+L^2)}{2Z_1^2} - \dfrac{Z_1^2}{2m} \\ \dfrac{m(\omega_x^2+L^2)}{2Z_1^2} - \dfrac{Z_1^2}{2m} & -L + \dfrac{m(\omega_x^2+L^2)}{2Z_1^2} + \dfrac{Z_1^2}{2m} \end{pmatrix}. \tag{6.12}$$

Their further transformation via an extra dimensionless matrix $\hat{z}_2$ satisfying the equation

$$\hat{z}_2\hat{\Omega}_1^{-1}\hat{z}_2 = \hat{\Omega}_2 \tag{6.13}$$

allows making the new $\hat{\Omega}_1$ and $\hat{\Omega}_2$ identical. Explicit expressions for the matrix $\hat{z}_2$ satisfying Eqn (6.13) are given in Eqns (5.3), (5.4), and (A.7).

Eigenfrequencies of two linearly independent modes in the stationary case are

$$\omega_{+,-} = \sqrt{2L^2 + \frac{1}{2}(\omega_x^2+\omega_y^2) \pm \sqrt{\left[2L^2 + \frac{1}{2}(\omega_x^2+\omega_y^2)\right]^2 - \omega_x^2\omega_y^2}}. \tag{6.14}$$

Below, we limit ourself to the isotropic potential well, $\omega_x = \omega_y = \omega_{xy}$; then choosing

$$Z_1 = m\sqrt{L^2 + \omega_{xy}^2} \tag{6.15}$$

results in a simple diagonal form of the matrix $\hat{\Omega}_1 = \hat{\Omega}_2 = \hat{\Omega}$:

$$\hat{\Omega}_1 = \hat{\Omega}_2 = \hat{\Omega} = \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix}, \qquad \omega_+ = \sqrt{\omega_{xy}^2 + L^2} + L\,, \quad \omega_- = \sqrt{\omega_{xy}^2 + L^2} - L\,. \tag{6.16}$$

Indeed, an honest solution of the characteristic equation for the matrices in Eqns (6.11) and (6.12) in the isotropic case $\omega_x = \omega_y = \omega_{xy}$ exactly yields these two eigenfrequencies, ω_+ and ω_-, even without the special choice of $Z_1 = \text{const}$.

Just as in the previous nonmagnetic case, we can introduce complex amplitudes, to become annihilation and creation operators in quantum mechanics:

$$a_+(t) = \frac{X_+(t) + \mathrm{i}P_+(t)}{\sqrt{2\hbar}}\,, \quad a_+^*(t) = \frac{X_+(t) - \mathrm{i}P_+(t)}{\sqrt{2\hbar}}\,, \qquad a_-(t) = \frac{X_-(t) + \mathrm{i}P_-(t)}{\sqrt{2\hbar}}\,, \quad a_-^*(t) = \frac{X_-(t) - \mathrm{i}P_-(t)}{\sqrt{2\hbar}}\,. \tag{6.17}$$

Quantum mechanical energy levels of this oscillatory system are characterized by the excitation numbers n_+ and n_- of the corresponding quanta $\hbar\omega_+$ and $\hbar\omega_-$:

$$E(n_+, n_-) = \hbar\omega_+\left(n_+ + \frac{1}{2}\right) + \hbar\omega_-\left(n_- + \frac{1}{2}\right). \tag{6.18}$$

If only the magnetic field is present and the potential well is absent, $\omega_{xy} = 0$, we obtain the famous Landau levels, with $\omega_+ = 2L$ being equal to twice the Larmor frequency, i.e., to the cyclotron frequency, while ω_- vanishes, and the centers of Landau orbits become degenerate.

Expressions like Eqns (6.17) allow the calculation of transition matrix elements, fluctuations, etc. Corresponding expressions for the regular coordinates (x, y) and momenta (p_x, p_y) are

$$x(t) = \sqrt{\frac{\hbar}{4Z_1}}\,(a_+ + a_+^* + a_- + a_-^*)\,, \tag{6.19}$$

$$y(t) = \sqrt{\frac{\hbar}{4Z_1}}\,\frac{a_+ - a_+^* - a_- + a_-^*}{\mathrm{i}}\,,$$

$$p_x(t) = \sqrt{\hbar Z_1}\,\frac{a_+ - a_+^* + a_- - a_-^*}{2\mathrm{i}}\,, \tag{6.20}$$

$$p_y(t) = \sqrt{\hbar Z_1}\,\frac{-a_+ - a_+^* + a_- + a_-^*}{2}\,.$$

The expressions for velocities v_x and v_y may also be useful:

$$v_x(t) = Ly + \frac{p_x}{m}\,, \qquad v_y(t) = -Lx + \frac{p_y}{m}\,. \tag{6.21}$$

We consider the state of the system with definite values of the number of quanta n_+ and n_-. Then

$$\langle x\rangle = \langle y\rangle = \langle p_x\rangle = \langle p_y\rangle = \langle v_x\rangle = \langle v_y\rangle = 0\,, \tag{6.22}$$

$$Z_1\langle x^2\rangle = Z_1\langle y^2\rangle = \frac{\langle p_x^2\rangle}{Z_1} = \frac{\langle p_y^2\rangle}{Z_1} = \frac{\hbar}{2}\left(n_+ + \frac{1}{2} + n_- + \frac{1}{2}\right), \tag{6.23}$$

$$\langle xp_x + p_x x\rangle = \langle yp_y + p_y y\rangle = 0\,, \tag{6.24}$$

$$\langle xp_x - p_x x\rangle = \langle yp_y - p_y y\rangle = \mathrm{i}\hbar\,,$$

$$\langle xp_y - p_y x\rangle = \langle yp_x - p_x y\rangle = 0\,, \tag{6.25}$$

$$-\langle xp_y + p_y x\rangle = \langle yp_x + p_x y\rangle = \hbar(n_+ - n_-)\,,$$

$$\left\langle \frac{m(v_x^2 + v_y^2)}{2} + \frac{m\omega_{xy}^2(x^2 + y^2)}{2} \right\rangle = \hbar\omega_+\left(n_+ + \frac{1}{2}\right) + \hbar\omega_-\left(n_- + \frac{1}{2}\right). \tag{6.26}$$

These formulas may be useful in describing the system in statistical physics.

7. Statistical physics of an oscillator in a magnetic field

For a state with definite values of n_+ and n_-, it is instructive to calculate $\langle M_z\rangle$, the expectation value of the magnetic moment $M_z = q/2(xv_y - yv_x)$ [A m^2]. Substitution of expressions (6.22)–(6.26) in M_z yields (for $q > 0$)

$$\langle M_z\rangle = \frac{q}{2}\langle xv_y - yv_x\rangle = \frac{q\hbar}{2m\sqrt{L^2 + \omega_{xy}^2}} \times \left[-L + n_-\left(\sqrt{L^2 + \omega_{xy}^2} - L\right) - n_+\left(\sqrt{L^2 + \omega_{xy}^2} + L\right)\right]. \tag{7.1}$$

The well-known Planck formula for the thermodynamically equilibrium expectation value of the number of quanta takes a particular limit form in the case of high temperatures:

$$\langle n\rangle = \left[\exp\left(\frac{\hbar\omega}{k_\mathrm{B}T}\right) - 1\right]^{-1} \approx \frac{k_\mathrm{B}T}{\hbar\omega} - \frac{1}{2} + \frac{1}{12}\,\frac{\hbar\omega}{k_\mathrm{B}T} + O\left(\frac{\hbar\omega}{k_\mathrm{B}T}\right)^2. \tag{7.2}$$

At the same time, $\langle n\rangle \to 0$ at a very low temperature. As a result, the expectation value of the magnetic moment at very low temperatures is

$$\langle M_z\rangle = -\frac{|q|\hbar}{2m}\,\frac{L}{\sqrt{L^2 + \omega_{xy}^2}} \equiv -\frac{q^2\hbar}{4m}\,\frac{B}{\sqrt{\omega_{xy}^2 + (qB/2m)^2}} \tag{7.3}$$

(diamagnetic response), with $|q|\hbar/2m$ being the Bohr magneton. As $B \to 0$, this result coincides with the Langevin formula. Equations (7.1) and (7.2) also allow illustrating the validity of the theorem by Niels Bohr and Van Leuven (see, e.g., Ref. [11]):

$$\langle M_z\rangle = \frac{q}{2}\langle xv_y - yv_x\rangle \approx -\frac{|q|\hbar}{2m}\,\frac{\hbar L}{3k_\mathrm{B}T} \quad \text{as} \quad k_\mathrm{B}T \to \infty\,, \tag{7.4}$$

and hence the induced magnetic moment vanishes in the classical limit $\hbar \to 0$.

If the magnetic field is strong, i.e., if $L \gg \omega_x, \omega_y$, then the frequency ω_- is even much smaller than ω_x and ω_y. For example, in the isotropic case ($L \gg \omega_x = \omega_y = \omega_{xy}$), the lower eigenfrequency ω_- can be approximately calculated without accounting for inertia: the Lorentz force $qBv = qBr\omega_-$ (in the circular motion of radius r and yet unknown angular velocity ω_-) is to be set equal to the elastic restoring force $m(\omega_{xy})^2 r$. The higher eigenfrequency ω_+ may be approximately calculated without accounting for the elastic restoring force, but just by setting the mass times the centripetal acceleration, $m(\omega_+)^2 r$, equal to the Lorentz force $qBv \equiv qB\omega_+ r$. In that approximation,

$$\omega_- \approx \frac{\omega_{xy}^2}{2L}\,, \qquad \omega_+ \approx 2L\,, \tag{7.5}$$

where $2L = qB/m$ is the well-known cyclotron frequency.

From the methodological standpoint, there is an interesting intermediate case, where

$$\begin{gathered} L \gg \omega_{xy}\,, \qquad \omega_+ \approx 2L\,, \qquad \omega_- \approx \frac{\omega_{xy}^2}{2L} \ll \omega_{xy} \ll L\,, \\ \hbar\omega_- \ll k_{\mathrm{B}}T \ll \hbar\omega_+\,, \qquad \langle n_+\rangle \cong 0\,, \qquad \langle n_-\rangle \approx \frac{k_{\mathrm{B}}T}{\hbar\omega_-} - \frac{1}{2}\,. \end{gathered} \tag{7.6}$$

In this case, a nonzero diamagnetic response occurs

$$\langle M_z\rangle = \frac{q}{2}\langle xv_y - yv_x\rangle \approx \frac{-|q|\hbar}{2m} + \frac{k_{\mathrm{B}}T}{B_z}\,, \tag{7.7}$$

where the second 'classical' paramagnetic term is a small correction to the leading quantum diamagnetic term. The reason for the small paramagnetic correction is that the magnetic field was assumed to be very strong: the cyclotron quantum $2\hbar L$ is much larger than the thermal energy $k_{\mathrm{B}}T$, while the low-frequency quantum $\hbar\omega_-$ is much smaller than the thermal energy.

8. General case. Discussion

We now consider the most general case, where the tensor $\hat{\beta}$ may have both symmetric and antisymmetric parts. In the stationary case, the symmetric part $\hat{\beta}^{\mathrm{s}}$ of $\hat{\beta}$ influences the trajectories in the $(\mathbf{p}, \mathbf{x})$ space but does not influence trajectories in the $(\mathbf{x}, \dot{\mathbf{x}})$ space. Indeed, we can perform a gauge transformation of the Lagrangian with respect to the symmetric part $\hat{\beta}^{\mathrm{s}}$ of $\hat{\beta}$:

$$x_i\,\beta_{ij}^{\mathrm{s}}\dot{x}_j = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,x_i\,\beta_{ij}^{\mathrm{s}}x_j\right) - x_i x_j\,\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\beta_{ij}^{\mathrm{s}}\,. \tag{8.1}$$

It is well known that adding a total time derivative of a function of time and coordinates (but not velocities) does not change the Euler–Lagrange equations for $(\mathbf{x}, \dot{\mathbf{x}})$. The time-dependent symmetric part $\hat{\beta}^{\mathrm{s}}$ of the tensor $\hat{\beta}$ yields an extra contribution to the elastic coefficient [cf. formula (3.2)]. The antisymmetric part $\hat{\beta}^{\mathrm{a}}$ of $\hat{\beta}$ gives rise to magnetic-type forces, while the time dependence of $\hat{\beta}^{\mathrm{a}}$ (of the magnetic field) leads, according to Faraday's law of electromagnetic induction, to curly forces with nonzero work along a closed loop. That work is called the 'electromotive force (EMF) of induction' by physicists.

We suppose that a symplectic $2n \times 2n$ matrix $\hat{Z}_{2n}$ (i.e., a matrix satisfying symplecticity conditions (B.5)–(B.8) in Appendix B) has been found such that

$$\begin{gathered} \hat{Z}_{2n} = \begin{pmatrix} \hat{S} & \hat{U} \\ \hat{V} & \hat{W} \end{pmatrix}, \qquad \hat{Z}_{2n}^{-1} = \begin{pmatrix} \hat{W}^{\mathrm{T}} & -\hat{U}^{\mathrm{T}} \\ -\hat{V}^{\mathrm{T}} & \hat{S}^{\mathrm{T}} \end{pmatrix}, \\ \hat{S}\hat{U}^{\mathrm{T}} - \hat{U}\hat{S}^{\mathrm{T}} = 0\,, \quad \hat{V}\hat{W}^{\mathrm{T}} - \hat{W}\hat{V}^{\mathrm{T}} = 0\,, \quad \hat{S}\hat{W}^{\mathrm{T}} - \hat{U}\hat{V}^{\mathrm{T}} = \hat{1}\,, \end{gathered} \tag{8.2}$$

$$\hat{Z}_{2n}\begin{pmatrix} \hat{\beta}\hat{M}^{-1} & -(\hat{K} + \hat{\beta}\hat{M}^{-1}\hat{\beta}^{\mathrm{T}}) \\ \hat{M}^{-1} & -\hat{M}^{-1}\hat{\beta}^{\mathrm{T}} \end{pmatrix}\hat{Z}_{2n}^{-1} = \begin{pmatrix} \hat{0} & -\hat{\Omega} \\ \hat{\Omega} & \hat{0} \end{pmatrix}. \tag{8.3}$$

At the time of writing, the author does not have a mathematical proof of the existence (or uniqueness) of such a matrix $\hat{Z}_{2n}$. But the intuition of a physicist suggests that such a solution must exist. Moreover, in the absence of degeneracy, the matrix $\hat{Z}_{2n}$ that diagonalizes the yet unknown $n \times n$ 'frequency matrix' $\hat{\Omega}$ should be unique up to a relabeling of n modes.

The expression for the 'old' momenta and coordinates are then given by

$$\begin{aligned} \mathbf{p} &= \sqrt{\frac{\hbar}{2}}\left[(-\hat{U}^{\mathrm{T}} - \mathrm{i}\hat{W}^{\mathrm{T}})\mathbf{a} + (-\hat{U}^{\mathrm{T}} - \mathrm{i}\hat{W}^{\mathrm{T}})\mathbf{a}^*\right], \\ \mathbf{x} &= \sqrt{\frac{\hbar}{2}}\left[(\mathrm{i}\hat{V}^{\mathrm{T}} + \hat{S}^{\mathrm{T}})\mathbf{a} + (-\mathrm{i}\hat{V}^{\mathrm{T}} + \hat{S}^{\mathrm{T}})\mathbf{a}^*\right]. \end{aligned} \tag{8.4}$$

Furthermore, if the state of the system in the Hilbert space can be presented as a product of states with definite numbers n_j of quanta in the jth oscillator mode, then

$$\begin{aligned} &\langle a_k\rangle = \langle a_k^*\rangle = 0\,, \qquad \langle a_k a_m\rangle = \langle a_k^* a_m^*\rangle = 0\,, \\ &\langle a_k a_m^*\rangle - \delta_{km} = \langle a_k^* a_m\rangle = n_k\delta_{km}\,. \end{aligned} \tag{8.5}$$

The corresponding formulas for the expectation values $\langle\mathbf{x}\rangle$, $\langle\mathbf{p}\rangle$, $\langle\mathbf{xp}\rangle$, $\langle\mathbf{xv}\rangle$, etc., are logically simple, but cumbersome in writing, and therefore we do not present them here (cf. the rather detailed expressions in the case of a magnetic field in Section 6).

A possible approach to multidimensional systems with antisymmetric components of the $\hat{\beta}$ tensor was already mentioned in Section 6. Namely, the transition to the coordinate frame revolving with the Larmor angular velocity L can be performed. For two momenta and two coordinates, we can use the symplectic transformation

$$\begin{gathered} \begin{pmatrix} \tilde{p}_x \\ \tilde{p}_y \\ \tilde{x} \\ \tilde{y} \end{pmatrix} = \hat{Z}_4\begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix}, \qquad \hat{Z}_4(t) = \begin{pmatrix} \hat{R}(t) & \hat{0} \\ \hat{0} & \hat{R}(t) \end{pmatrix}, \\ \hat{R}(t) = \begin{pmatrix} \cos(Lt) & -\sin(Lt) \\ \sin(Lt) & \cos(Lt) \end{pmatrix}, \qquad \hat{R}^{\mathrm{T}}(t) = \hat{R}^{-1}(t)\,. \end{gathered} \tag{8.6}$$

Equations (6.3) can then be transformed with the help of the identity

$$\begin{aligned} \left[\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{R}(t)\right]\hat{R}^{-1}(t) &= \begin{pmatrix} -L\sin(Lt) & -L\cos(Lt) \\ L\cos(Lt) & -L\sin(Lt) \end{pmatrix} \\ &\times \begin{pmatrix} \cos(Lt) & \sin(Lt) \\ -\sin(Lt) & \cos(Lt) \end{pmatrix} = \begin{pmatrix} 0 & -L \\ L & 0 \end{pmatrix}. \end{aligned} \tag{8.7}$$

As a result, Eqns (6.3) become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\tilde{\mathbf{p}}\\ \tilde{\mathbf{x}}\end{pmatrix}=\frac{\mathrm{d}\hat{Z}_4(t)}{\mathrm{d}t}\,\hat{Z}_4^{-1}(t)\begin{pmatrix}\tilde{\mathbf{p}}\\ \tilde{\mathbf{x}}\end{pmatrix}+\hat{Z}_4(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{p}\\ \mathbf{x}\end{pmatrix}$$
$$=\begin{pmatrix}\hat{0} & -\hat{R}(t)(\hat{K}-\hat{\beta}^{\mathrm{a}}\hat{M}^{-1}\hat{\beta}^{\mathrm{a}})\hat{R}^{-1}(t)\\ \hat{R}(t)\hat{M}^{-1}\hat{R}^{-1}(t) & \hat{0}\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{p}}\\ \tilde{\mathbf{x}}\end{pmatrix}. \tag{8.8}$$

We recall that the antisymmetric matrix $\hat{\beta}$ is here assumed to have form (6.1). In a truly remarkable way, the terms relating momenta to themselves, and radius vectors to themselves in Eqn (6.3) have disappeared! We can say that these terms were compensated (in the Larmor-revolving frame) by the Coriolis force. We can now apply the procedure of finding the impedance matrix with respect to the radius vector $\tilde{\mathbf{x}}$. After that, we should make the transformation to the *complex* amplitudes $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}^*$. Then, still using these complex amplitudes, we can make a transformation back to the laboratory (i.e., nonrotating) frame. Then we can pass from laboratory *complex* amplitudes to the laboratory *real* coordinates and momenta P_+, P_-, X_+, X_-. Because this procedure is valid at any time instant, the limit as $t \to 0$ can be taken by using something like l'Hôpital's rule. The procedure of going from real momenta and coordinates to a new basis, then going to complex amplitudes, then going to complex amplitudes in the old basis, and finally going to new real momenta and coordinates is similar to the procedure of finding real 'momenta' and 'coordinates' for the running-wave oscillators in classical electrodynamics, starting with the standing-wave oscillators (see, e.g., § 52 in Ref. [12][3]).

Returning to the problem with the antisymmetric real matrix β of arbitrary size, we can use the well-known theorem (see, e.g., Ref. [13]) about the reduction of a general antisymmetric matrix by rotations (which are known to preserve symplecticity) to a block form, with each block being either

$$\begin{pmatrix}0 & L_j\\ -L_j & 0\end{pmatrix} \tag{8.9}$$

or a zero matrix of size 1. Subsequently, we can pass to the coordinate frame that revolves in the plane of the corresponding matrix (7.9) with the appropriate 'Larmor' angular velocity L_j, individual for each jth plane, and perform all the tricks with *complex*/*real* amplitudes described above. This program might possibly be carried out, but this has not yet been done by the present author,[4] except in the case with two momenta and two coordinates (see Section 6).

Returning to the general ideology of impedance, it is worth making the following statement, which does not seem entirely evident. Common wisdom and the experience of working with quantum oscillators of mass m and frequency ω suggest the following estimate:

$$\delta y_+ \approx \sqrt{\frac{\hbar}{2Z_+}} = (?) = \sqrt{\frac{\hbar}{2m\omega_+}}\,, \tag{8.10}$$
$$\delta y_- \approx \sqrt{\frac{\hbar}{2Z_-}} = (?) = \sqrt{\frac{\hbar}{2m\omega_-}}$$

(cf. the example involving the magnetic field in Section 6). The results of our consideration in Section 6 show that an intuitive estimation like (8.10) for the impedance Z_- is radically wrong there. Indeed, in the isotropic case, both high- and low-frequency modes of motion have exactly the same impedances, $Z_1 = m(L^2 + \omega_{xy}^2)^{1/2}$.

A typical objection to the use of a separate notion of impedance Z (as opposed to the product $m\omega$) is that the mass m is usually time-independent. One argument against that objection is that the relativistic 'transverse' mass depends on speed.

The motion of electrons near the bottom of the conductance band in a solid state with an anisotropic mass tensor is another example. The tensor of inertia, which plays the role of mass for the rotational motion, may evidently be time-dependent; this is especially clear for the pendulum with variable length. In electrical circuits (see Section 4), the frequency $\omega = 1/\sqrt{LC}$ and the impedance $Z = \sqrt{L/C}$ are evidently independent parameters, because both the inductance L and the capacitance C may change independently in time.

The case of motion in the presence of a magnetic field is especially dramatic in this respect: even for a constant mass, the frequency and the impedance can be changed independently. Even more importantly, the impedance Z_1 in Eqn (6.15) can be kept time-independent even for $m(t) = \text{const}$ by simultaneously changing the magnetic field (or the cyclotron frequency $2L$) and the elasticity of the mechanical restoring 'spring' (isotropic frequency ω_{xy}) in opposite directions: increasing one and decreasing the other in appropriate proportions.

Apparently, gyroscopic forces in the classical dynamics of rigid bodies and Coriolis forces in a rotating coordinate frame may give rise to antisymmetric components of the $\hat{\beta}$ tensor. However, this problem requires further study.

9. Conclusion

In this paper, we have emphasized separate notions of impedance as a 'dynamic parameter,' and of frequency as a 'kinematic parameter.' The modulation of not the frequency but the impedance is responsible for the parametric excitation. *The shape* of a trajectory in the phase space for the oscillator is an ellipse. The physical meaning of the *impedance Z is the aspect ratio of the phase-space trajectory*: $Z = \delta p/\delta x$, of the dimension $[Z] = [\mathrm{J\,s}\,x^{-2}]$ for the appropriate dimension of the coordinate x.

We derived the equation for the impedance matrix in the multi-dimensional case. The conservation of the adiabatic invariant is discussed in the example of the Manley – Row relations. It seems that the generation of particle – antiparticle pairs by gravitational fields may also be discussed in terms of some kind of impedance, but this question deserves further study.

The author is grateful to the numerous friends, colleagues, and relatives with whom he discussed the results of this paper: N B Baranova, G I Barenblatt, V B Braginskii, S M Voronin, L P Grishchuk, J Goodman, P V Elyutin, V S Liberman, M A Liberman, V I Zalyapin, K B Zeldovich, A E Kaplan, D Kaup, E I Kats, D Christodoulides, K K Likharev, S Mokhov, J Neife, S P Novikov, M A Ovchinnikov, M Ya Ovchinnikova, L P Pitaevskii, O V Rudenko, and A E Siegman. S M Voronin found an essential misprint in the expression for the impedance of a pendulum. M A Ovchinni-

[3] Section 52 in Landau and Lifshitz's *Field Theory* bears the humble title "Characteristic vibrations of the field," suggesting that its contents should be almost trivial. In reality, however, the procedure described there is not very simple, for the author of this paper at least.

[4] "He who loves stronger than me writes longer than me."

kov suggested the idea underlying Figs 2a, b. M Ya Ovchinnikova suggested the term 'aspect ratio of a phase-space trajectory.' The author expresses his deep gratitude to G I Barenblatt, V I Zalyapin, L B Glebov, A E Kaplan, and E W van Stryland for moral and organizational support of his work on the subject of impedance.

10. Appendices

A. Solution of the impedance equation for 2×2 matrices

We consider the task of solving the equation for the unknown impedance matrix $\hat{Z}$:

$$\hat{Z}\hat{M}^{-1}\hat{Z} = \hat{K}, \tag{A.1}$$

where $\hat{M}$ is a symmetric real matrix of mass, $\hat{K}$ is a symmetric real matrix of elasticity, and both these matrices are assumed to be positive definite. For 2×2 matrices, we were able to find an explicit solution of impedance equation (A.1); below is the description of that solution.

It is convenient to reduce the problem to one where all the three matrices have unit determinant. Namely, for general 2×2 matrices, we can set

$$\hat{\mu} = (\det \hat{M})^{-1/2}\,\hat{M}, \qquad \hat{\kappa} = (\det \hat{K})^{-1/2}\,\hat{K}, \qquad \hat{\zeta} = (\det \hat{Z})^{-1/2}\hat{Z}. \tag{A.2}$$

Each of the new matrices $\hat{\mu}$, $\hat{\kappa}$, and $\hat{\zeta}$ has unit determinant, and Eqn (A.1) implies that

$$\det \hat{Z} = \sqrt{\det \hat{M} \det \hat{K}}, \qquad \hat{\zeta}\hat{\mu}^{-1} = \hat{\kappa}\hat{\zeta}^{-1}. \tag{A.3}$$

The most general symmetric real positive definite 2×2 matrices with unit determinant are

$$\hat{\mu} = \begin{pmatrix} m & t \\ t & \dfrac{1+t^2}{m} \end{pmatrix}, \qquad \hat{\kappa} = \begin{pmatrix} k & u \\ u & \dfrac{1+u^2}{k} \end{pmatrix}, \tag{A.4}$$

where m and k are arbitrary real positive numbers, and t and u are any real numbers. The inverse matrix to $\hat{\mu}$ looks simple because the determinant of $\hat{\mu}$ is 1:

$$\hat{\mu}^{-1} = \begin{pmatrix} \dfrac{1+t^2}{m} & -t \\ -t & m \end{pmatrix}. \tag{A.5}$$

We introduce the notation for entries of a matrix $\hat{\zeta}$ with unit determinant:

$$\hat{\zeta} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1, \qquad \hat{\zeta}^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}. \tag{A.6}$$

With this notation, Eqn (A.3) becomes linear (for the 2×2 matrices $\hat{\zeta}$ and $\hat{\zeta}^{-1}$) in the coefficients α, β, γ, and δ, and therefore yields a relatively simple solution:

$$\beta = \gamma = (u+t)\sqrt{\frac{mk}{2km(1-ut) + m^2(1+u^2) + k^2(1+t^2)}}, \qquad \alpha = \frac{\beta}{u+t}\,(m+k), \qquad \delta = \frac{\beta}{u+t}\left(\frac{1+u^2}{k} + \frac{1+t^2}{m}\right). \tag{A.7}$$

Thus, the impedance matrix $\hat{Z}$ turns out to be symmetric, as expected, and exists at any values of the real parameters t and u, and real positive m and k. Formulas (A.7) were verified by direct substitution. An extra test is applicable in the especially simple diagonal case, where $t = u = 0$. Then

$$\beta = \gamma = 0, \qquad \alpha = \sqrt{km}, \qquad \delta = \frac{1}{\sqrt{km}}, \tag{A.8}$$

just as it should be for the diagonal matrices $\hat{\mu}$ and $\hat{\kappa}$ of type (A.4) with $t = u = 0$.

B. Linear transformations of coordinates and momenta: when are they canonical? Symplectic matrices

We consider 'vectors' and 'transposed vectors' of n original coordinates and n original momenta:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \cdots \\ x_N(t) \end{pmatrix}, \qquad \mathbf{p}(t) = \begin{pmatrix} p_1(t) \\ \cdots \\ p_N(t) \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{T}}(t) = \big(x_1(t), \ldots, x_N(t)\big), \quad \mathbf{p}^{\mathrm{T}}(t) = \big(p_1(t), \ldots, p_N(t)\big). \tag{B.1}$$

We consider linear transformations to vectors $\mathbf{X}$ and $\mathbf{P}$ of 'new' coordinates and momenta:

$$\mathbf{P} = \hat{S}\mathbf{p} + \hat{U}\mathbf{x}, \qquad \mathbf{X} = \hat{V}\mathbf{p} + \hat{W}\mathbf{x}, \qquad \begin{pmatrix} \mathbf{P} \\ \mathbf{X} \end{pmatrix} = \hat{Z}_{2n}\begin{pmatrix} \mathbf{p} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \hat{S} & \hat{U} \\ \hat{V} & \hat{W} \end{pmatrix}\begin{pmatrix} \mathbf{p} \\ \mathbf{x} \end{pmatrix}, \tag{B.2}$$

where $\hat{S}$, $\hat{U}$, $\hat{V}$, and $\hat{W}$ are $n \times n$ real matrices, and $\hat{Z}_{2n}$ is a real $2n \times 2n$ matrix. Here is the way to memorize the order of matrices in our notation: they appear in alphabetical order from left to right in the top row of $\hat{Z}_{2n}$ and then in the bottom row. (We skip the letter 'T' because it is needed to denote the transposition.) These transformations are 'canonical' if the Poisson brackets of the new variables with respect to the old variables satisfy the relations

$$\{X_i, X_j\} = 0, \qquad \{P_i, P_j\} = 0, \qquad \{P_i, X_j\} = \delta_{ij}. \tag{B.3}$$

Here, the Poisson bracket of two functions $f(x,p)$ and $g(x,p)$ is defined by

$$\{f,g\} = \sum_{k=1}^{N}\left(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x_k} - \frac{\partial f}{\partial x_k}\frac{\partial g}{\partial p_k}\right). \tag{B.4}$$

In quantum mechanics, these Poisson brackets are replaced by commutators of operators of physical quantities f and g. Direct substitution yields the conditions on the matrices $\hat{S}$, $\hat{U}$, $\hat{V}$, and $\hat{W}$ for the transformation to be canonical:

$$\hat{S}\hat{U}^{\mathrm{T}} = (\hat{S}\hat{U}^{\mathrm{T}})^{\mathrm{T}} \equiv \hat{U}\hat{S}^{\mathrm{T}}, \qquad \hat{V}\hat{W}^{\mathrm{T}} = (\hat{V}\hat{W}^{\mathrm{T}})^{\mathrm{T}} \equiv \hat{W}\hat{V}^{\mathrm{T}}, \qquad \hat{S}\hat{W}^{\mathrm{T}} - \hat{U}\hat{V}^{\mathrm{T}} = \hat{1}_n, \qquad \hat{W}\hat{S}^{\mathrm{T}} - \hat{V}\hat{U}^{\mathrm{T}} = \hat{1}_n. \tag{B.5}$$

Actually, the last equality in (B.5) is a transposition of the previous equality, whereas the transposition of the first and second equalities in (B.5) just reproduces them. Conditions (B.5) mean that the $2n \times 2n$ matrix $\hat{Z}_{2n}$ is symplectic, i.e., has the properties

$$\hat{Z}_{2n}\hat{R}(\hat{Z}_{2n})^{\mathrm{T}} = \hat{R}, \qquad \hat{R} = \begin{pmatrix} \hat{0} & \hat{1}_n \\ -\hat{1}_n & \hat{0} \end{pmatrix}; \qquad \hat{Z}_{2n} = \begin{pmatrix} \hat{S} & \hat{U} \\ \hat{V} & \hat{W} \end{pmatrix} \Rightarrow (\hat{Z}_{2n})^{-1} = \begin{pmatrix} \hat{W}^{\mathrm{T}} & -\hat{U}^{\mathrm{T}} \\ -\hat{V}^{\mathrm{T}} & \hat{S}^{\mathrm{T}} \end{pmatrix}. \tag{B.6}$$

It can be verified that the symplecticity condition can also be written as

$$(\hat{Z}_{2n})^{\mathrm{T}}\hat{R}\hat{Z}_{2n} = \hat{R}\,, \tag{B.7}$$

i.e., with interchanged $\hat{Z}_{2n}$ and $(\hat{Z}_{2n})^{\mathrm{T}}$ in comparison with (B.6). Then, in addition to (B.5), we also have the relations

$$\begin{aligned} &\hat{S}^{\mathrm{T}}\hat{V} = (\hat{S}^{\mathrm{T}}\hat{V})^{\mathrm{T}} \equiv \hat{V}^{\mathrm{T}}\hat{S}\,, \qquad \hat{U}^{\mathrm{T}}\hat{W} = (\hat{U}^{\mathrm{T}}\hat{W})^{\mathrm{T}} \equiv \hat{W}^{\mathrm{T}}\hat{U}\,,\\ &\hat{S}^{\mathrm{T}}\hat{W} - \hat{V}^{\mathrm{T}}\hat{U} = \hat{1}_n\,, \qquad \hat{W}^{\mathrm{T}}\hat{S} - \hat{U}^{\mathrm{T}}\hat{V} = \hat{1}_n\,. \end{aligned} \tag{B.8}$$

And again, the last equality in (B.8) is a transposition of the previous equality, whereas the transposition of the first and second equalities in (B.8) just reproduces them.

C. Oscillators with two types of dissipation

The systems considered in this appendix are described by oscillatory equations with two types of friction, i.e., of dissipation. Both examples are actually rather simple; however, considerable effort is needed to choose the coordinates and notation such that the equations for these systems take a simple form and reduce to the standard form in (C.6) and (C.9).[5] The first system is a mass m on a spring (Fig. 4). One end of the spring is rigidly attached to the mass, whose coordinate is denoted as $x(t)$. The other end of the spring is attached to a weightless pivot, whose coordinate is $x(t) - y(t) - L_0$; here, L_0 is the equilibrium length of the spring. The pivot is at a rail and exhibits a 'viscous' type of friction. The equation of motion for the weightless pivot is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(x(t) - y(t) - L_0\bigr) = \eta f_{\mathrm{pivot}}\,. \tag{C.1}$$

Here, the parameter η [s kg^{-1}] is the viscous mobility of the pivot, i.e., the proportionality coefficient between the applied force and the velocity of the pivot relative to the motionless rail. This force, acting on the pivot (with the + sign) and on the mass (with the − sign), is proportional to the deformation $y(t)$ of the spring:

$$f_{\mathrm{pivot}} = ky(t)\,. \tag{C.2}$$

The mass is subject to two forces: from the deformation of the spring and from its own viscous friction, and therefore the relation between its velocity $\mathrm{d}x/\mathrm{d}t$ and momentum p, and Newton's second law are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p(t)}{m}\,, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = f_x = -ky(t) - Z_{\mathrm{visc}}\frac{\mathrm{d}x}{\mathrm{d}t}\,, \tag{C.3}$$

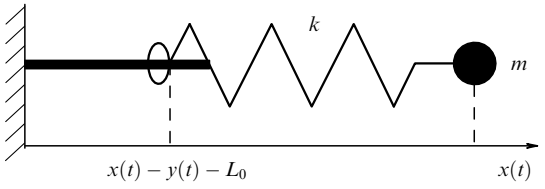

Figure 4. A mass on a spring with two sources of dissipation.

[5] The reader who tends to assume that this is very easy is invited to read Appendix C once and then rederive Eqns (C.9) with the correct expressions for Γ_1 and Γ_2.

where Z_{visc} [kg s^{-1}] is the coefficient of viscous friction of the mass. We can eliminate the variables $x(t)$ and $\mathrm{d}x/\mathrm{d}t$ to obtain a closed system for $y(t)$ and $p(t)$:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{Z_{\mathrm{visc}}}{m}\,p(t) - ky(t)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{m}\,p(t) - \eta k y(t)\,. \tag{C.4}$$

We now introduce the undamped frequency ω_0 and the impedance Z_y (with respect to the coordinate y):

$$\omega_0 = \sqrt{\frac{k}{m}}\,, \qquad Z_y = \sqrt{km}\,, \qquad k = \omega_0 Z_y\,, \qquad \frac{1}{m} = \frac{\omega_0}{Z_y}\,. \tag{C.5}$$

System (C.4) then becomes

$$\begin{aligned} &\frac{\mathrm{d}p}{\mathrm{d}t} = -\omega_0\,\frac{Z_{\mathrm{visc}}}{Z_y}\,p - \omega_0 Z_y y \equiv -\Gamma_1\,p - \omega_0 Z_y y\,,\\ &\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\omega_0}{Z_y}\,p - \omega_0(\eta Z_y)y \equiv \frac{\omega_0}{Z_y}\,p - \Gamma_2 y\,. \end{aligned} \tag{C.6}$$

This system has one 'dynamic parameter,' the impedance Z_y [kg s^{-1}], one 'kinematic parameter,' the frequency ω_0 [rad s^{-1}], and two dimensionless parameters characterizing dissipation, $\alpha = Z_{\mathrm{visc}}/Z_y = \Gamma_1/\omega_0$ and $\beta = \eta Z_y = \Gamma_2/\omega_0$:

$$\Gamma_1 = \alpha\omega_0 \equiv \omega_0\,\frac{Z_{\mathrm{visc}}}{Z_y}\,, \qquad \Gamma_2 = \beta\omega_0 \equiv \omega_0(\eta Z_y)\,. \tag{C.7}$$

If all the parameters are independent of time, the impedance Z_y can be eliminated by the transformation

$$P(t) = \frac{p(t)}{\sqrt{Z_y}}\,, \qquad Y(t) = y(t)\sqrt{Z_y}\,, \tag{C.8}$$

and hence the system acquires a purely 'kinematic' form:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\Gamma_1 P - \omega_0 Y\,, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \omega_0 P - \Gamma_2 Y\,. \tag{C.9}$$

The physical meaning of the individual relaxation constants Γ_1 and Γ_2 is best revealed in the respective cases where $\omega_0 \ll \Gamma_1$ and $\omega_0 \ll \Gamma_2$ or, alternatively, in the cases where $Y \equiv 0$ and $P \equiv 0$. In the first case, the momentum P relaxes as $\exp(-\Gamma_1 t)$, and in the second case, the coordinate Y relaxes as $\exp(-\Gamma_2 t)$. Seeking a solution of the standardized system (C.9) in the form $\propto \exp(-\mathrm{i}\omega t)$, we obtain two complex eigenvalues of the frequency ω:

$$\omega_{1,2} = -\mathrm{i}\,\frac{1}{2}(\Gamma_1 + \Gamma_2) \pm \sqrt{\omega_0^2 - \frac{1}{4}(\Gamma_1 - \Gamma_2)^2}\,. \tag{C.10}$$

We emphasize that our two sources of dissipation contribute to the amplitude damping $\Gamma = -\mathrm{Im}\,\omega_{1,2}$ by their *arithmetic mean* $(\Gamma_1 + \Gamma_2)/2$. The factor $1/2$ in this arithmetic mean is easy to interpret. Indeed, an oscillator 'spends half of the time' having mostly kinetic energy (which is dissipated via Γ_1), and the other 'half of the time' having mostly potential energy (which is dissipated via Γ_2). But the correction to the real part of the eigenfrequency is determined by the *difference* of the two damping constants: $(1/4)(\Gamma_1 - \Gamma_2)^2$. In particular, the two sources of dissipation can be adjusted such that $\Gamma_1 = \Gamma_2$. Then the real part of the frequency equals ω_0 at any value of the Q factor [defined as $Q = \omega_0/(\Gamma_1 + \Gamma_2)$], including the case $Q \ll 1$.

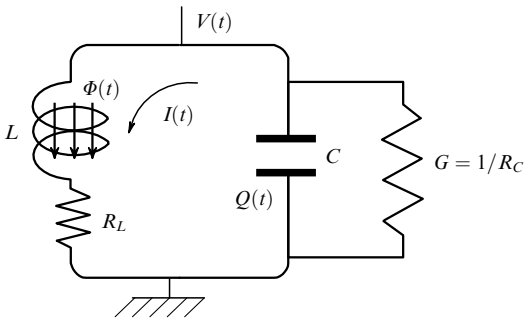

Figure 5. An *LC*-*RG* circuit.

The other system is a classical *LC*-circuit (Fig. 5), where we take two ways of energy dissipation into account. One of them is the loss due to a resistance R_L [Ω] put serially with the inductance L. The other is the conductivity G [Ω^{-1}] of the leakage of the charge of our capacitor C. The total magnetic flux $\Phi(t)$ (with the multiple turns of the coil taken into account) is a result of the passage of the current $I(t)$ through the inductor:

$$\Phi(t) = LI(t)\,, \tag{C.11}$$

where L [H] is the inductance. Faraday's law of electromagnetic induction states that the voltage $V_L(t) = V(t) - R_L I(t)$ results in an increase in the magnetic flux:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = V(t) - R_L I(t)\,. \tag{C.12}$$

The charge $Q(t)$ of the capacitor (at its lower electrode in Fig. 5) is

$$Q(t) = -CV(t)\,. \tag{C.13}$$

Finally, the current $I(t)$ is expressed as

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} - GV(t)\,. \tag{C.14}$$

Eliminating $V(t)$ and $I(t)$, we obtain a closed system of ODEs for $\Phi(t)$ and $Q(t)$:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{R_L}{L}\,\Phi(t) - \frac{1}{C}\,Q(t)\,, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{L}\,\Phi(t) - \frac{G}{C}\,Q(t)\,. \tag{C.15}$$

And again, introducing the system impedance Z, the undamped frequency ω_0, and two damping rates Γ_L for the RL circuit and Γ_C for the GC circuit as

$$Z = \sqrt{\frac{L}{C}}\,, \quad \omega_0 = \frac{1}{\sqrt{LC}}\,, \quad \Gamma_L = \frac{R_L}{L}\,, \quad \Gamma_C = \frac{G}{C} \equiv \frac{1}{R_C C} \tag{C.16}$$

allows reducing system (C.15) to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\Gamma_L \Phi(t) - \omega_0 Z Q(t)\,, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\omega_0}{Z}\,\Phi(t) - \Gamma_C Q(t)\,. \tag{C.17}$$

This is the standard system of form (C.6), (C.9), for which all the conclusions obtained above are valid.

The main body of this paper is devoted to the study of similarities and dissimilarities between the potential and kinetic energy parts of the Hamiltonian. Similarities are covered by the kinematic parameter, the frequency ω_0; the dissimilarity is emphasized by the notion of impedance. In Appendix C, to continue this line of reasoning, we discussed a certain symmetry and asymmetry between the dissipation of potential energy and that of kinetic energy. The topics discussed in this appendix are relevant to the asymptotic behavior of strongly damped oscillations.

D. What is the adiabatic invariant?

In discussing the contents of this paper with students, I noted that the young generation (which 'chooses Pepsi,' as I nearly wrote) is not necessarily familiar with adiabatic invariants — a notion from classical Lagrangian/Hamiltonian mechanics. Below is the formal definition of an adiabatic invariant, given by a simple example of one-dimensional motion $x(t)$ in a stationary potential field $U(x)$ (Fig. 6). In this problem, the energy conservation law can be written as

$$\frac{p^2}{2m} + U(x) = E = \mathrm{const}\,. \tag{D.1}$$

As a result, the momentum p depends on the position x as

$$p(x) = \pm\big|\,p(x)\big| = \pm\sqrt{2m\big(E - U(x)\big)}\,. \tag{D.2}$$

Here, the + and − signs correspond to the respective stages of motion in positive and negative directions of x. Stationary oscillatory motion occurs between two points x_1 and x_2 such that all the energy E becomes potential at these points:

$$U(x_1) = E\,, \qquad U(x_2) = E\,. \tag{D.3}$$

Because $\mathrm{d}x/\mathrm{d}t = p/m$, the period T of this motion is

$$T = T(E) = \oint \mathrm{d}t = 2\int_{x_1}^{x_2} \mathrm{d}t = 2m\int_{x_1}^{x_2} \frac{\mathrm{d}x'}{\big|\,p(x')\big|} \equiv \sqrt{2m}\int_{x_1(E)}^{x_2(E)} \frac{\mathrm{d}x'}{\sqrt{E - U(x')}}\,. \tag{D.4}$$

The adiabatic invariant is formally defined as the area inside the trajectory on the phase plane,

$$A = A(E) = \oint p\,\mathrm{d}x = 2\int_{x_1}^{x_2} p(x')\,\mathrm{d}x' = 2\sqrt{2m}\int_{x_1(E)}^{x_2(E)} \sqrt{E - U(x')}\;\mathrm{d}x'\,, \tag{D.5}$$

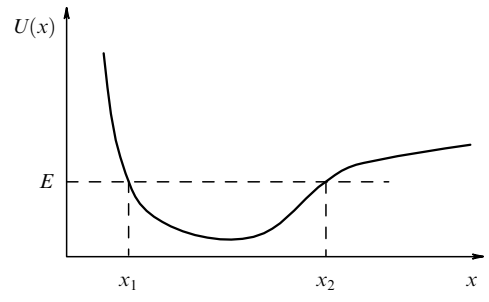

Figure 6. Motion of a particle between turning points x_1 and x_2 in a stationary potential $U(x)$.

with the evident relation

$$T(E) = \frac{\mathrm{d}A}{\mathrm{d}E}\,. \tag{D.6}$$

The quantum mechanical motivation underlying this definition (see [8]) is that in the semiclassical (WKB) approximation, the de Broglie relation $\Lambda = 2\pi\hbar/p$ holds between the momentum p and the 'wavelength' Λ. The stationary motion of a particle with wave-like properties requires that an integer number $\tilde{n}$ of wavelengths fit the trajectory:

$$\oint \frac{\mathrm{d}x'}{\Lambda(x')} = \frac{2}{2\pi\hbar}\int_{x_1}^{x_2} p(x')\,\mathrm{d}x' = \frac{A(E)}{2\pi\hbar} = \tilde{n}\,. \tag{D.7}$$

A more refined approach (the so-called Bohr – Sommerfeld version of the old quantum theory quantization) amounts to replacing [for smooth potentials $U(x)$] the number $\tilde{n}$ with $(n + 1/2)$, where n is an integer. As a result, the purely classical definition of the adiabatic invariant is related to this number n from the old quantum theory as

$$A(E) = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{D.8}$$

This equation, with the functional dependence $A(E)$ calculated for a particular potential curve $U(x)$ via Eqn (D.5), allows, at least approximately, finding the discrete energy level energies E_n. In the same WKB approximation, the frequency $\omega = 2\pi/T$ of classical motion approximately corresponds to the energy difference of adjacent energy levels divided by $\hbar$:

$$\omega = \frac{2\pi}{T} = \frac{E_{n+1} - E_n}{\hbar}\,. \tag{D.9}$$

Most textbooks on classical mechanics make the following correct and important statement: "*Slow changes of the parameters of the system leave adiabatic invariant almost unchanged.*" Here is P Paradoksov's explanation of the conservation of an adiabatic invariant. Slow changes in the parameters of the system do not provide temporal Fourier components of perturbations that could possibly induce transitions $n \to n'$ because the corresponding transitions have the frequencies

$$\omega(n \to n') \approx \frac{2\pi(n - n')}{T}\,. \tag{D.10}$$

As a result, the system stays in a quantum state with unchanged n, and, by virtue of Eqn (D.8), with the unchanged adiabatic invariant $A(E)$. This logic shows that the slow character of perturbation is a sufficient condition for good preservation of $A(E)$, but not a necessary one. Even fast changes may not affect $A(E)$ considerably if they do not contain appropriate resonant frequencies.

For the power-law potential

$$U(x) = \frac{k}{\alpha}\,|x|^{\alpha}\,, \qquad \alpha > 0\,, \tag{D.11}$$

simple calculation shows that

$$x_{2,1}(E) = \pm\left(\frac{\alpha E}{k}\right)^{1/\alpha}\,, \qquad A(E) = 4x_2(E)\sqrt{2mE}\,B(\alpha)\,,$$
$$B(\alpha) = \int_0^1 \sqrt{1 - y^{\alpha}}\;\mathrm{d}y\,, \tag{D.12}$$

$$T(E) = 2\sqrt{2m}\int_0^{x_2(E)} \frac{\mathrm{d}x}{\sqrt{E - (k/\alpha)x^{\alpha}}} = 2\sqrt{2m}\,\frac{x_2(E)}{\sqrt{E}}\,D(\alpha)\,,$$
$$D(\alpha) = \int_0^1 \frac{\mathrm{d}y}{\sqrt{1 - y^{\alpha}}}\,, \tag{D.13}$$

$$A(E) = \text{const}\,E^{1/\alpha + 0.5}\,, \qquad T(E) = \text{const}'\,E^{1/\alpha - 0.5}\,. \tag{D.14}$$

The case of a linear harmonic oscillator, where $\alpha = 2$, is especially instructive:

$$A(E) = \frac{2\pi E}{\omega_0}\,, \qquad T = \frac{2\pi}{\omega_0}\,, \qquad \omega_0 = \sqrt{\frac{k}{m}}\,, \tag{D.15}$$

and hence the period is independent of the amplitude (i.e., of the energy), and the Bohr – Sommerfeld quantization formula yields the exact result:

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right). \tag{D.16}$$

Conservation of the adiabatic invariant for a linear harmonic oscillator means that $E \propto \omega_0$, i.e., an adiabatic change in energy follows the change in frequency. Because the quantum operator is related to the complex amplitudes a^* and a by

$$\hat{n} + \frac{1}{2} = \frac{aa^* + a^*a}{2}\,, \tag{D.17}$$

the conservation of aa^* also implies that $E \propto \omega$, i.e., the conservation of the adiabatic invariant.

E. Classical mechanics of the adiabatic invariant conservation

The authors has always tended to feel some confusion when reading various books on classical mechanics. The adiabatic invariant $A(E)$ is the area on the phase plane inside a constant-energy trajectory under the assumption that the Hamiltonian is stationary. From the Liouville theorem, it is well known that the area (volume) in the phase space is strictly conserved in the course of evolution. Therefore, it would be natural to expect a certain relation between the adiabatic invariant conservation and the Liouville theorem. To the author's disappointment, virtually all the books that he has been able to look through do not mention the Liouville theorem in the relevant sections (however, see [6]). For this reason, in what follows, the author tries to explain the conservation of $A(E)$ in a way that would suit his personal tastes (which does not necessarily mean that it would suit others', unfortunately).

We consider the motion of a 1-dimensional mechanical system with the coordinate $x(t)$ and the momentum $p(t)$, which satisfy the standard Hamilton equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H\bigl(x, p, \lambda(t)\bigr)}{\partial x}\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H\bigl(x, p, \lambda(t)\bigr)}{\partial p}\,. \tag{E.1}$$

The parameter $\lambda(t)$ has a certain constant value λ_1 at infinite negative time and a constant value λ_2 at infinite positive time. Figures 7 and 8 depict possible changes with $\lambda_1 \neq \lambda_2$; however, these two limit values may also coincide. It is important that $\lambda(t)$ generally changes inside this time interval. The energy is conserved in both the initial state and the final state, but not in-between.

The conservation of energy E in the one-dimensional finite (oscillatory, but not necessarily harmonic) motion

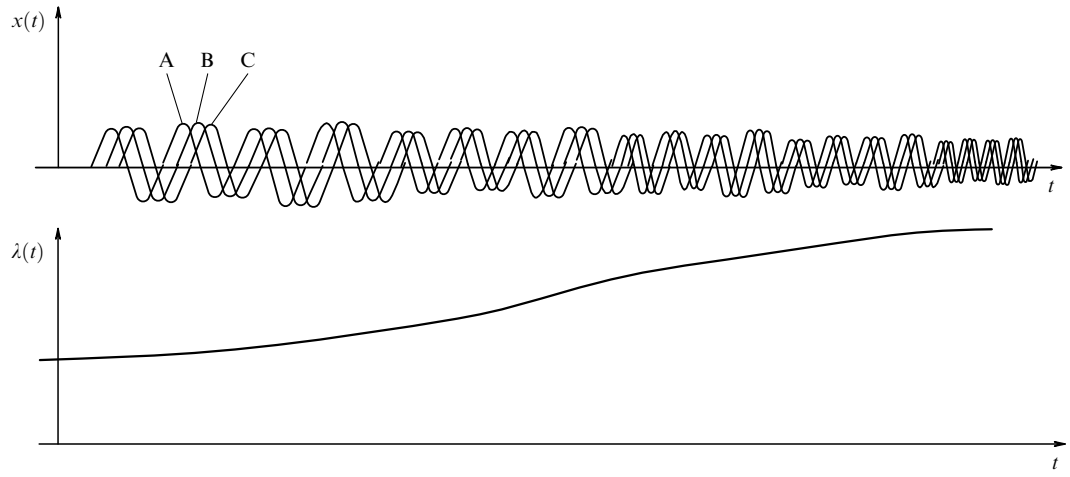

Figure 7. Oscillatory motion of a system. Graphs A, B, and C represent motions with the same energy as $t \to -\infty$. This means that as $t \to -\infty$, these graphs differ by a time shift (or, equivalently, a phase shift) only. The change in the parameter $\lambda(t)$ is slow in this case.

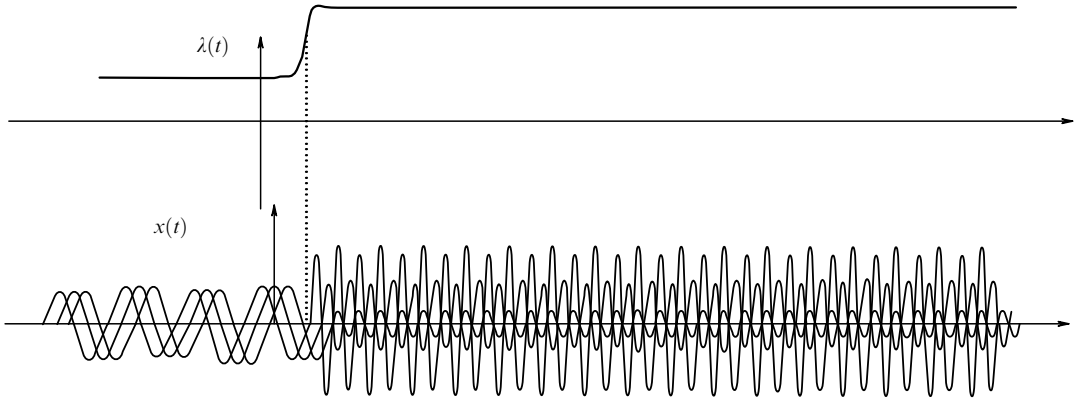

Figure 8. Oscillatory motion of a system. The three graphs represent motions with the same energy (as $t \to -\infty$). This means that as $t \to -\infty$, these graphs differ by a time shift (or, equivalently, a phase shift) only. The change in the parameter $\lambda(t)$ is fast in this case.

means that each trajectory, depicted for $x(t)$ in Figs 7 and 8, is a closed line in the phase plane (p, x), depicted in Fig. 9. Figure 9a shows trajectories in the phase space of a linear harmonic oscillator: they are ellipses with the area equal to the adiabatic invariant $A(E)$, and with the aspect ratio $\delta p/\delta x = Z$, where Z is the impedance. By passing to new variables $P = p/\sqrt{Z}$ and $X = x\sqrt{Z}$, we can make this ellipse into a circle (Fig. 9b). Then the time evolution at $\lambda(t) = \text{const}$ corresponds to the conservation of energy and hence to pure rotation in the P, X plane (Fig. 9c). We emphasize that the points A, B, C, D, etc. at fixed energy are mapped into one another under time evolution, and thus cover a closed trajectory: an ellipse or circle, depending on the p, x or P, X scales. Some of these phase points are shown in the graphs in Fig. 7 as A, B, and C.

The time dependence of the coordinate $x(t)$ and of the momentum $p(t)$ may be represented as a 'circulation' of the depicted points A–B–C–D–...–A along the closed trajectory, and the particular position of a point corresponds to a particular phase of such an oscillatory motion. Anharmonicity may lead to a peculiar shape of the closed trajectory (Fig. 10) and, even more importantly, to a dependence of the period of oscillations on the amplitude (on the energy).

We now consider two different ensembles of points in the phase plane, both of which correspond to a homogeneously

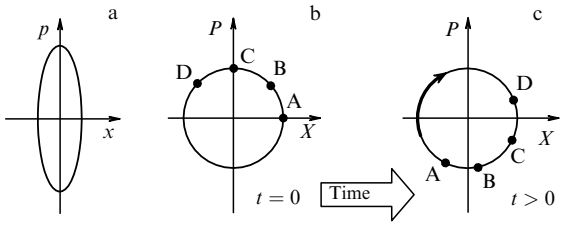

Figure 9. Phase plane of a linear harmonic oscillator with a constant parameter λ and hence with a constant energy. (a) Elliptic phase-space curve. (b) Change in (p, x) scales allows making a circle out of the original ellipse. (c) Time evolution corresponds to motion of the points along the circle, i.e., to rotation of points around the origin.

populated area of the phase plane. The homogeneous density stays invariant in the process of motion, as follows from the Liouville theorem.

The ensemble of the first type, encapsulated by the curves A–B–C–...–A in Figs 9 and 10, is surrounded by a closed trajectory with a fixed energy E_0. The points A–B–C–...–H–A on that trajectory represent different phases of motion; they are mapped into each other's positions during the evolution with the time-independent Hamiltonian, i.e., at $\lambda = \text{const}$. Therefore, the ensemble of all the points,

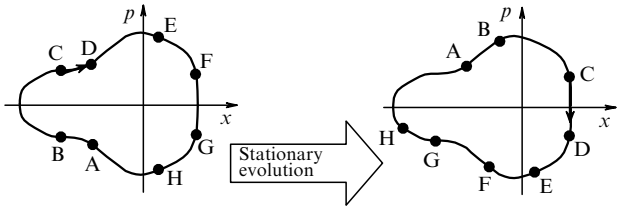

Figure 10. Nonlinear oscillator in stationary conditions. Points A–B–C– ... –H–A are distributed over the actual trajectory (on a constant-energy line, stationary ensemble). Therefore, the time evolution keeps them on the same line, with a phase shift.

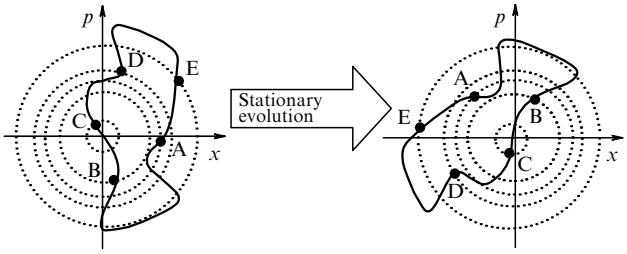

Figure 11. Linear harmonic oscillator in stationary conditions. Points A, B, C, D, E are distributed over the line containing *different* values of energy (a nonstationary ensemble). Therefore, in the process of evolution, this line changes its position in the phase plane, even for a stationary Hamiltonian. In this particular case of a harmonic oscillator, this stationary evolution is a rotation of the figure.

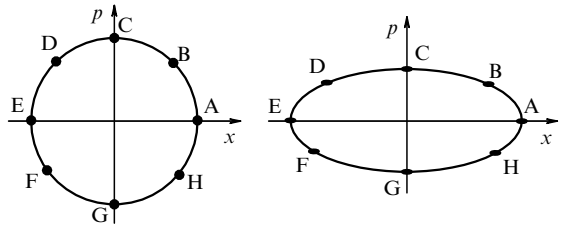

Figure 12. *Adiabatic* evolution of a linear harmonic oscillator. New positions of the points A–B– ... –H–A also constitute a fixed-energy trajectory. Because the frequency and, most of all, the impedance have changed, the second curve is an ellipse in the (p, x) canonical coordinates.

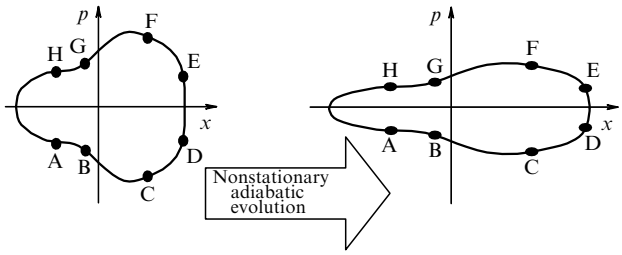

Figure 13. *Adiabatic* evolution of a *nonlinear* oscillator. New positions of the points A–B– ... –H–A also constitute a fixed-energy trajectory.

both inside and at the boundary, is stationary, i.e., it reproduces itself in the process of motion. We can also say that such an ensemble is phase invariant: none of the phases occurs with greater probability than any other.

The second type of ensemble also consists of points with a constant density at $t = t_0$, but not all phases are present in equal proportions (Fig. 11). For definiteness, Fig. 11 shows the case of the linear harmonic oscillator described in the coordinates such that the corresponding phase-space trajectories are circles. It is evident, for example, that after the 3/8 of the period, this ensemble moves into another position in the phase plane (see Fig. 11). The introduction of such an ensemble should be accompanied by explicitly indicating the instant $t = t_0$ at which we consider it.

We now discuss the evolution of the system between $t \to -\infty$ and $t \to +\infty$, as the parameter $\lambda(t)$ changes from a constant value λ_1 to a constant value λ_2 (see Figs 7 and 8). We also suppose that at $t \to -\infty$, we take the stationary (i.e., phase-invariant) ensemble of points on the phase plane, with the maximum value of energy E_1 at the boundary A–B–C– ... –A. We should then consider two limit cases of the possible behavior of the parameter $\lambda(t)$.

The first limit case is the adiabatic case of *slow change* of $\lambda(t)$, shown in Fig. 7. The main 'logical input' to the proof resides here. This slow change *cannot* (up to a reasonable accuracy) *select* any particular phase of the original motion. Therefore, all the points of the ensemble undergo the same evolution, and hence the final ensemble is also phase invariant, with the boundary points A–B–C– ... –A on a trajectory with some new common energy value E_2 (Figs 12 and 13). Then the conservation of the phase-space volume (the Liouville theorem) immediately yields the conservation of the adiabatic invariant $A(E)$, which here just coincides with the phase-space area, which had to be shown.

The other limit case is where the parameter $\lambda(t)$ changes rapidly enough such that we can pinpoint a particular phase of the oscillatory motion, for example, the one that corresponds to the maximum positive value of $\mathrm{d}\lambda(t)/\mathrm{d}t$ (see Fig. 8). The behavior of the trajectories on the phase plane in this limit case is shown in Figs 14 and 15. In other words, evolution of the points of our ensemble with different initial phases (e.g., A, B, C) yields final states with *different* energies: $E_2(\mathrm{A})$, $E_2(\mathrm{B})$, $E_2(\mathrm{C})$.

This is the point where the treatment of adiabatic invariants in this paper differs from the generally adopted ones (see, e.g., Refs [1–6]). Namely, the main statement to be made about the nonadiabatic case is *not* that a certain change in $A(E)$ occurs. The main point to be observed is that the nonadiabatic evolution *introduces a phase-dependent spread* of the trajectories, of the final energy values E_2, and of the final values of the adiabatic invariant $A(E_2)$.

Actually, in most cases of a weak nonadiabaticity, the *value* $\langle A(E_2) - A(E_1) \rangle$ *averaged over the initial phases* is zero in the first order in some small parameter that characterizes the change in $A(E)$. A *spread of* $A(E)$ occurs in the first order in this parameter. Therefore, statements like 'the adiabatic invariant changes by 3%' may be misleading: they detract attention *from the fact of the phase sensitivity*, from the notion of the spread of the final values of E_2 and $A(E_2)$, sensitive to the initial phase.

A nice example of an antiadiabatic perturbation is given by the action of a pulse $\lambda(t)$ of the resonant force $F(t) = \lambda(t) \cos(\omega t)$ on a harmonic oscillator; this example is taken from [3]. The equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p(t)}{m}\,, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -m\omega^2 x(t) + \lambda(t) \cos(\omega t)\,, \qquad \text{(E.2)}$$

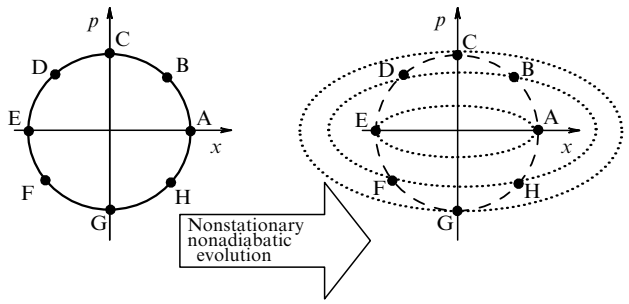

Figure 14. *Antiadiabatic* (fast) evolution of a linear harmonic oscillator. Different A–B– ... –H–A points acquire different change of energy, and the new A–B– ... –H–A sequence of points does not constitute a trajectory.

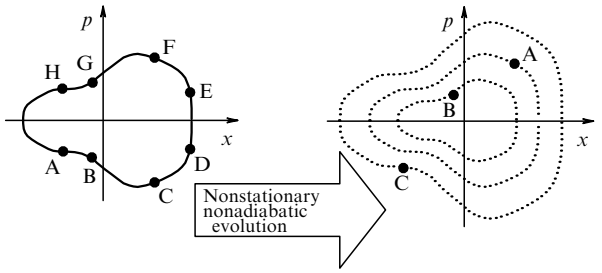

Figure 15. *Antiadiabatic* (fast) evolution of a *nonlinear* oscillator. Different A–B– ... –H–A points acquire different changes of energy, and the new A–B– ... –H–A sequence of points does not constitute a trajectory.

under the assumption that $\lambda(-\infty) = \lambda(+\infty) = 0$. The general solution of this equation as $t \to +\infty$ is

$$x(t) = a_0 \cos(\omega t - \varphi) + b \cos(\omega t - \beta)\,, \tag{E.3}$$

$$p(t) = -m\omega a_0 \sin(\omega t - \varphi) - m\omega b \sin(\omega t - \beta)\,, \tag{E.4}$$

where

$$b \exp(\mathrm{i}\beta) = (m\omega)^{-1} \int \lambda(\tau) \cos(\omega\tau) \exp(-\mathrm{i}\omega\tau)\, \mathrm{d}\tau\,, \tag{E.5}$$

and the integration ranges from $\tau = -\infty$ to $\tau = +\infty$. The final value of the adiabatic invariant $A(E_2)$ is easily calculated:

$$A(E_2) = A(E_1) + 2b\sqrt{2\pi m\omega A(E_1)}\, \cos(\varphi - \beta) + 2\pi m\omega b^2\,. \tag{E.6}$$

This shows that the first-order effect with respect to b is a phase-sensitive spread of the final values $A(E_2)$. Only the second-order term $\propto b^2$ gives the phase-insensitive systematic increase in the adiabatic invariant.

Another example of a nonadiabatic perturbation is the parametric resonance induced by the modulation of impedance at the frequency 2ω. Amplification or deamplification also occur here, in a manner extremely sensitive to the phase of initial oscillations, which is to be referenced to the phase of the modulation parameter (the impedance).

F. Energy transfer in resonant coupled oscillators

Most textbooks treat oscillators using the standard second-order ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0\,, \tag{F.1}$$

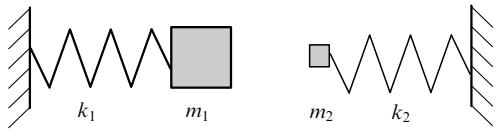

Figure 16. Two oscillators may have identical eigenfrequencies but be different physically: their impedances are not equal to each other.

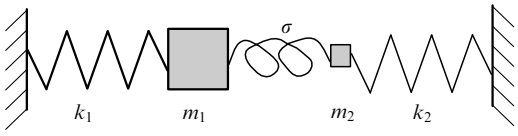

Figure 17. Two oscillators have identical eigenfrequencies and are coupled by a weak spring (constant σ). Do the beats result in complete energy transfer from one oscillator to the other? If the impedances are not equal, then the answer is 'no.'

where the only parameter of the oscillator is its eigenfrequency $\omega = \sqrt{k/m}$. What is missing here?[6] Most people do not realize that the information about the impedance $Z = \sqrt{km}$ is missing, with k being the elasticity constant and m being the mass. At first glance, with the given initial coordinate $x(t = 0)$ and the initial velocity $v_x(t = 0)$, the trajectory can be found using Eqn (F.1) only. This may produce the false impression that two oscillators, one with a large mass m_1 and a strong spring (large elasticity constant k_1) and the other with a small mass m_2 and a weak spring (small elasticity constant k_2), are physically equivalent if their eigenfrequencies coincide (Fig. 16). In the main body of this paper, we (hopefully) were able to persuade the reader that the notion of impedance is necessary for an oscillator with time-dependent parameters. Now, how about the stationary case? In this Appendix, we show that in a stationary system of two coupled oscillators with the same eigenfrequencies, the energy exchange depends on their impedances in an essential way.

The first step in understanding this is to write Eqn (F.1) in the presence of an external force $F(t)$:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = \frac{1}{m} F(t) \equiv \frac{\omega}{Z} F(t)\,. \tag{F.2}$$

It follows that the effect of the same force $F(t)$ is smaller for the oscillator with larger Z.

It is instructive to consider the system of two mutually resonant oscillators whose eigenfrequencies were the same before the coupling (Fig. 17). We consider the coupling of these oscillators by a weak spring, with an elasticity constant σ. Standard thinking is that the existence of a resonance between two original eigenfrequencies leads to the following beautiful phenomenon. For example, we excite a

[6] There is a well-known Russian joke about a person who had been trained to become a cook; at the exam, he was asked to taste some borshch and to tell what was missing (the question referred to the ingredients that may have been omitted in cooking the dish). The answer "Bread is missing" shows not only how notoriously dumb that person was but also the Russian tradition to serve lunch with bread. In Eqn (F.1), the information about the impedance is missing. This information is "outside Eqn (F.1)" much as bread is "outside the borshch," but both are actually important. We may say that the frequency and the impedance are the bare necessities, in terms of food, of the theory of oscillations.

finite displacement of the first oscillator, $x_1(t=0)=a$, and produce no deflection of the second one, $x_2(t=0)=0$. The commonly expected effect is that in a certain time, all the energy of oscillations is to be transferred to the second oscillator, then back to first one, etc. But direct calculations show that the displacement $x_1(t)$ behaves as

$$x_1(t)=\frac{a}{Z_1+Z_2}\left(Z_1\cos(\omega_- t)+Z_2\cos(\omega_+ t)\right), \qquad \text{(F.3)}$$

where ω_- and ω_+ are the frequencies of two eigenmodes in the presence of coupling:

$$\omega_-=\omega\,, \qquad \omega_+=\omega\sqrt{1+\frac{\sigma}{\omega}\left(\frac{1}{Z_1}+\frac{1}{Z_2}\right)}\,. \qquad \text{(F.4)}$$

It follows from Eqn (F.3) that the complete energy transfer occurs only if $Z_1=Z_2$.

A natural question is whether the coupling can be modified so as to achieve the complete energy transfer. The solution is to take the interaction in a modified form, with the total Hamiltonian given by

$$H(p_1,p_2,x_1,x_2)=\frac{1}{2}\left[\frac{1}{m_1}p_1^2+k_1x_1^2+\frac{1}{m_2}p_2^2 + k_2x_2^2+\sigma\left(\frac{x_1}{\alpha}-\frac{x_2}{\beta}\right)^2\right], \qquad \text{(F.5)}$$

where we introduced two dimensionless coefficients α and β. The assumption of equal original eigenfrequencies $\omega_1=\omega_2=\omega$ is easily described using Eqn (3.19) for the Hamiltonian:

$$H(p_1,p_2,x_1,x_2)=\frac{\omega}{2}\left[\frac{1}{Z_1}p_1^2+Z_1x_1^2+\frac{1}{Z_2}p_2^2+Z_2x_2^2\right] +\frac{1}{2}\,\sigma\left(\frac{x_1}{\alpha}-\frac{x_2}{\beta}\right)^2. \qquad \text{(F.6)}$$

The transformation to the new variables P_1, P_2, X_1, X_2, via

$$p_1=P_1\sqrt{Z_1}\,, \quad p_2=P_2\sqrt{Z_2}\,, \quad x_1=\frac{X_1}{\sqrt{Z_1}}\,, \quad x_2=\frac{X_2}{\sqrt{Z_2}} \qquad \text{(F.7)}$$

allows writing the Hamiltonian of the coupled system as

$$\tilde{H}(P_1,P_2,X_1,X_2)=\frac{\omega}{2}(P_1^2+X_1^2+P_2^2+X_2^2) +\frac{1}{2}\,\sigma\left(\frac{X_1}{\alpha\sqrt{Z_1}}-\frac{X_2}{\beta\sqrt{Z_2}}\right)^2. \qquad \text{(F.8)}$$

It is evident that our system becomes symmetric with respect to the new variables, and thus allows the complete energy transfer, only if

$$\left|\alpha\sqrt{Z_1}\right|=\left|\beta\sqrt{Z_2}\right|. \qquad \text{(F.9)}$$

The reader is welcome to practice in designing various 'levers' with the appropriate ratio α/β, which may be called 'impedance transformers.' A possible design is shown in Fig. 18, where a weightless stick is attached to an axis, to one mass via a spring, and to another mass directly. Evidently, the longer lever (proportional to β) must be attached to the oscillator with the smaller impedance Z_2.

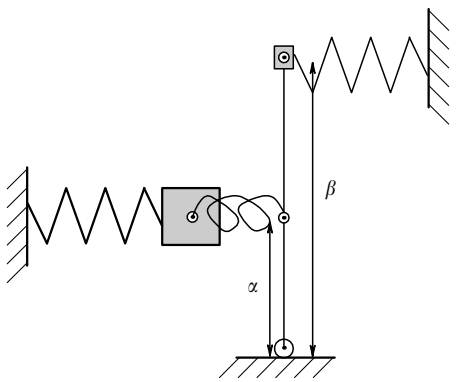

Figure 18. A possible design of an 'impedance transformer' for resonant coupled oscillators.

Oscillators coupled by a common contribution to the kinetic energy can also be imagined. Impedance matching can be achieved in that case as well, and formulas (F.7) are especially helpful for that task.

Impedance transformers (including the standard 220/127 transformers: see the novel by E Uspenskii about the little guarantee men) are widely used and quite well studied in electronics, and therefore the impedance matching of *LC* oscillators is not discussed here.

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