

METHODOLOGICAL NOTES

PACS numbers: **05.40.** – **a**, **05.45.** – **a**, 42.25.Dd, **46.65.** + **g**, 47.27.eb

Statistical topography and Lyapunov exponents in stochastic dynamical systems

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DOI: 10.1070/PU2008v051n04ABEH006450

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Abstract. This article discusses the relationship between the statistical description of stochastic dynamical systems based on the ideas of statistical topography and the traditional analysis of Lyapunov stability of dynamical systems with the use of the Lyapunov characteristic indices (Lyapunov exponents). As an illustration, some coherent phenomena are considered that occur with a probability of unity, i.e., in almost all realizations of the stochastic systems. Among such phenomena are the diffusion and clustering of a passive tracer in random hydrodynamic flows, the dynamic localization of plane waves in layered random media, and the emergence of caustic patterns of the wave field in multidimensional random media.

1. Introduction

In recent years, the attention of both theoreticians and experimenters has been commanded by the issue of the relationship between the dynamics of averaged characteristics of a solution to a problem and the behavior of the solution in particular realizations. This point is of particular importance in geophysical problems concerning the properties of atmosphere and ocean, where no appropriate averaging ensemble is generally present and, as a rule, experimenters deal with particular realizations.

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Received 2 August 2007, revised 19 November 2007
Uspekhi Fizicheskikh Nauk **178** (4) 419 – 431 (2008)
DOI: 10.3367/UFNr.0178.200804e.0419
Translated by A V Getling; edited by A Radzig

Dynamical problems for particular realizations of the parameters of a medium are extremely sophisticated in terms of mathematics, so that it is virtually hopeless to try to solve them. At the same time, investigators are interested in the basic features of the phenomena, without going into details. For this reason, the idea of using the well-elaborated mathematical technique of random processes and fields, i.e., considering statistical averages over a whole ensemble of possible realizations instead of particular realizations of the processes under study, proved to be highly attractive. For example, nearly all problems of atmospheric and oceanic physics are currently based, to a certain extent, on statistical analyses.

The introduction of randomness in the parameters of the medium entails the stochastic behavior of the physical fields themselves. The normally employed techniques of statistical averaging (i.e., the calculation of mean quantities such as the mean values of processes and fields, $\langle x(t)\rangle$ and $\langle \rho(\mathbf{r}, t)\rangle$, spatiotemporal correlation functions $\langle x(t)x(t')\rangle$, $\langle \rho(\mathbf{r}, t)\,\rho(\mathbf{r}', t')\rangle$, etc., where $\langle\ldots\rangle$ denotes averaging over the ensemble of all realizations of random parameters) ensure smoothing of the qualitative particular features of individual realizations. As a result, the obtained statistical characteristics in many cases not only have nothing in common with the behavior of individual realizations but even, at first glance, contradict them.

In a number of cases, however, certain physical processes and phenomena occur with a probability of unity (i.e., in almost all their realizations); they are termed *coherent* (see Refs [1, 2] and monographs [3 – 5], where this point is discussed in detail).

Definitely, the complete statistics contains full information about the dynamical system. However, only some simplest statistical characteristics related to one-time and one-point probability distributions can be studied in practice. Therefore, the following question arises: how can we describe the basic qualitative and quantitative characteristics of the behavior of individual realizations of the system knowing its simplest statistical characteristics and particular features?

This question can be fielded using *statistical topography techniques* (see, e.g., Ref. [6] and the above-mentioned monographs). On the one hand, we will consider below applications of these techniques to simplest physical problems; on the other hand, we will reveal relationships between these techniques and the traditional approach to a stability analysis based on considering Lyapunov exponents.

The approach based on the analysis of Lyapunov stability of the solutions of deterministic, linear, ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{x}(t) = A(t)\,\mathbf{x}(t)$$

has received much attention from numerous investigators. It implies analyzing the upper bound for the solution of the problem, namely

$$\lambda_{\mathbf{x}(t)} = \overline{\lim_{t\to+\infty}}\;\frac{1}{t}\,\ln\left|\mathbf{x}(t)\right|,$$

which is called the characteristic index of this solution [7]. In applying this approach to stochastic dynamical systems, a statistical analysis is frequently invoked at a final stage of the treatment to interpret and simplify the results obtained; in this case, statistical averages are calculated, such as

$$\left\langle \lambda_{\mathbf{x}(t)} \right\rangle = \overline{\lim_{t\to+\infty}}\;\frac{1}{t}\left\langle \ln\left|\mathbf{x}(t)\right|\right\rangle. \qquad (1)$$

2. Examples of dynamical systems

2.1 Ordinary differential equations

As a first example, let us mention papers [8, 9] where the problem of relative diffusion of weakly inertial particles in a random hydrodynamic flow with a velocity field $\mathbf{u}(\mathbf{r},t)$ (which has zero mean value) was considered in the framework of the Newton equation:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) = \mathbf{v}(t)\,, \qquad \mathbf{r}(0) = \mathbf{r}_0\,,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{v}(t) = -\lambda\big[\mathbf{v}(t) - \mathbf{u}\big(\mathbf{r}(t),t\big)\big]\,, \qquad \mathbf{v}(0) = \mathbf{v}_0(\mathbf{r}_0)\,. \end{aligned} \qquad (2)$$

In these studies, Eqns (2) were linearized with respect to the initial positions of the particles, and Lyapunov characteristic indices (1) were calculated. In the same papers, the results of numerical simulations of the stochastic system of equations (2) were also presented.

For inertialess particles, the parameter $\lambda\to\infty$. As follows from the system of equations (2), one has

$$\mathbf{v}(t) = \mathbf{u}\big(\mathbf{r}(t),t\big)$$

and the trajectory of a particle in a hydrodynamic flow with a velocity field $\mathbf{u}(\mathbf{r},t)$ is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t|\mathbf{r}_0) = \mathbf{u}\big(\mathbf{r}(t|\mathbf{r}_0),t\big)\,, \qquad \mathbf{r}(0|\mathbf{r}_0) = \mathbf{r}_0\,, \qquad (3)$$

so that the problem of determining the trajectories of inertialess particles in a hydrodynamic flow reduces to purely kinematic one. Here, the vertical bar denotes the dependence of the solution to the problem on the initial condition.

It should be noted that the quantity $j(t|\mathbf{r}_0) = \det\left\|\partial r_i(t|\mathbf{r}_0)/\partial r_{j0}\right\|$, called the *divergence*, is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,j(t|\mathbf{r}_0) = \frac{\partial\mathbf{u}(\mathbf{r},t)}{\partial\mathbf{r}}\,j(t|\mathbf{r}_0)\,, \qquad j(0|\mathbf{r}_0) = 1\,, \qquad (4)$$

and for a divergence-free hydrodynamic flow ($\operatorname{div}\mathbf{u}(\mathbf{r},t) = 0$) one has $j(t|\mathbf{r}_0)\equiv 1$.

Let us dwell on the stochastic peculiarities of the solution of problem (3) for a system of particles. Formally, each particle moves independently according to Eqn (3). However, if the random field $\mathbf{u}(\mathbf{r},t)$ has a finite spatial-correlation radius l_{cor}, particles separated by distances shorter than l_{cor} reside in a common zone of influence of the random field $\mathbf{u}(\mathbf{r},t)$, so that new collective features can emerge in the dynamics of such a system of particles.

For a time-independent velocity field, $\mathbf{u}(\mathbf{r},t)\equiv\mathbf{u}(\mathbf{r})$, equation (3) assumes the simpler form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) = \mathbf{u}(\mathbf{r})\,, \qquad \mathbf{r}(0) = \mathbf{r}_0\,.$$

In this case, the stationary points $\tilde{\mathbf{r}}$ at which $\mathbf{u}(\tilde{\mathbf{r}}) = 0$ remain immobile. Depending on whether they are stable or unstable, they will either attract or repel particles located in their neighborhood. Since the function $\mathbf{u}(\mathbf{r})$ is random, the positions of the points $\tilde{\mathbf{r}}$ are also random. A similar situation should also be present in the general case of a spatiotemporal random velocity field $\mathbf{u}(\mathbf{r},t)$.

If some points $\tilde{\mathbf{r}}$ remain stable over a sufficiently long time, particle-cluster regions (i.e., compact regions of enhanced particle concentrations mainly located in rarefied zones) should form in their neighborhoods in some particular realizations of the random field $\mathbf{u}(\mathbf{r},t)$. If, however, the stability of these points changes into instability sufficiently rapidly, so that the particles have no time for their substantial rearrangement, then cluster regions will not form.

Numerical simulations [10, 11] show that there are considerable differences in the dynamics of a system of particles between the cases of divergence-free and divergent random velocity fields. For a particular realization of a divergence-free time-independent velocity field $\mathbf{u}(\mathbf{r})$, Fig. 1a schematically represents an interval of the evolutionary history of a particles' system (in a two-dimensional case) in dimensionless time related to the statistical parameters of the field $\mathbf{u}(\mathbf{r})$. Initially, the particles were uniformly distributed inside a circle. In this case, the area bounded by the contour is preserved, and particles relatively uniformly fill the area enclosed by the deformed contour. Only a strong fractal indentation of this contour arises. This phenomenon, which came to be known as *chaotic advection*, is now actively being studied (see, e.g., Ref. [12]).

As for a potential velocity field $\mathbf{u}(\mathbf{r})$, the particles uniformly distributed at the initial time over a square form cluster regions in the process of time evolution. Figure 1b illustrates a numerically simulated fragment of such an evolutionary scenario. We emphasize again that the formation of *clusters* in this case is a purely kinematic effect. This feature of the particle dynamics completely disappears with averaging over the ensemble of realizations of the random velocity field.

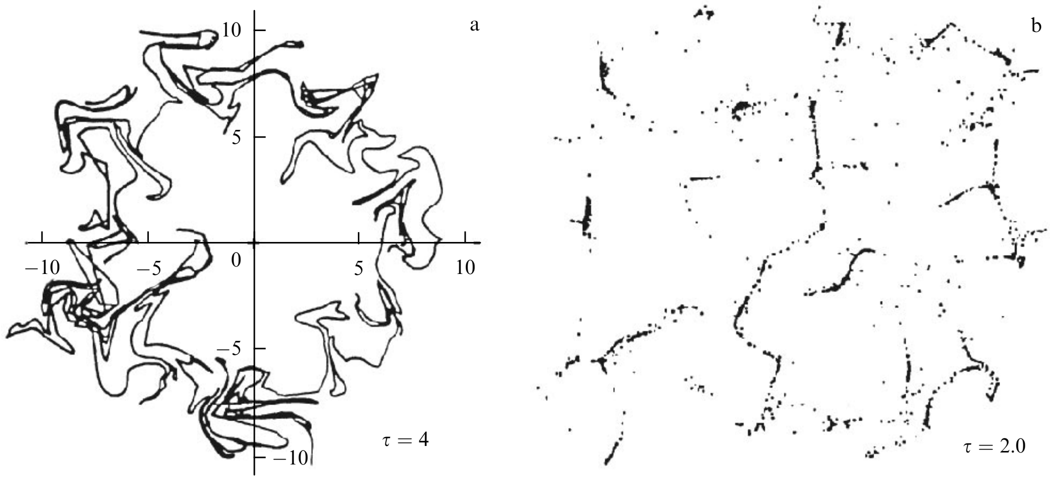

Figure 1. Simulated diffusion of a system of particles in a solenoidal (a) and potential (b) random velocity field.

A simple model of the tracer diffusion is known [13], which makes it possible to observe the basic difference between the diffusion processes in divergent and divergence-free velocity fields. In divergence-free (incompressible) velocity fields, the particles (and, therefore, the density field) have no time to be attracted by stable attracting centers within their lifetimes, thus slightly fluctuating about their initial positions. Conversely, in a divergent (compressible) velocity field (within the same lifetime of the stable attracting centers), the particles have time to be attracted by these centers, since this attraction process is exponentially speeded up.

To present a second example, we mention a monograph by Lifshits et al. [14], in which a one-dimensional problem of the overbarrier penetration of particles through a layer of a disordered medium was considered based on the one-dimensional, time-independent Schrödinger equation with a random potential. In the same monograph, Lyapunov characteristic indices (1) were calculated. This problem is similar to the problem of wave propagation in a one-dimensional random medium.

Let a layer of a chaotically inhomogeneous medium occupy a spatial region $L_0 < x < L$ and let an incident plane wave $u_0(x) = \exp\left[-\mathrm{i}k(x-L)\right]$ come to this layer from the region $x > L$. Due to the inhomogeneities, a wave reflected from the layer with a reflection coefficient $R_L = u(L) - 1$ appears, and another wave is transmitted by the layer with a transmission coefficient $T_L = u(L_0)$.

Inside the layer, the wave field is governed by the Helmholtz equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,u(x) + k^2\left[1+\varepsilon(x)\right]u(x) = 0\,, \qquad (5)$$

where the function $\varepsilon(x)$ is assumed to be random and describes the inhomogeneities of the medium. We also assume that $\varepsilon(x) = 0$ outside the layer, whereas $\varepsilon(x) = \varepsilon_1(x) + \mathrm{i}\gamma$ inside the layer, with the real part $\varepsilon_1(x)$ being responsible for wave-scattering processes in the medium, and the imaginary part $\gamma \ll 1$ describing the attenuation of the wave in the medium.

This equation must be complemented with boundary conditions, viz. the continuity conditions for the wave field $u(x)$ and its derivative $\mathrm{d}u(x)/\mathrm{d}x$ at the layer boundaries, and the radiation emission conditions at these boundaries; they can be written out as

$$u(L) + \frac{\mathrm{i}}{k}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\bigg|_{x=L} = 2\,, \qquad u(L_0) - \frac{\mathrm{i}}{k}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\bigg|_{x=L_0} = 0\,. \qquad (6)$$

Under the assumption that the statistical properties of the function $\varepsilon(x)$ are known, the statistical problem reduces to determining the statistical characterization of both the wave reflection and transmission coefficients related to the values of the field at the layer boundaries and the wave-field intensity $I(x) = \left|u(x)\right|^2$ inside the inhomogeneous medium.

The wave equation considered coincides in its form with the equation of an oscillator with a varying eigenfrequency (if the spatial variable x is substituted with the time variable t); as is well known, such an oscillator exhibits the phenomenon of *parametric resonance* at the frequencies $2k/n$ ($n = 1, 2, \ldots$). Since components with all frequencies, including these, are in general present in the function $\varepsilon(x)$, a similar phenomenon should obviously occur also in the problem under consideration and can be termed the phenomenon of *stochastic parametric wave resonance*. In this case, the boundary conditions fix the wave-field values at the boundaries of the inhomogeneous-medium layer, so that the statistical exponential growth of the field is possible only in the bulk of the layer, far from its boundaries.

However, this phenomenon does not occur in some particular realizations. Figure 2 illustrates two realizations of the wave-field intensity in a sufficiently deep layer of the medium, which correspond to two realizations of numerically simulated inhomogeneities of the medium (see, e.g., Refs [1, 15] and the monographs [3, 4]). Without going into details of the parameters of the problem, we only note that this figure clearly demonstrates a tendency to an abrupt exponential decline (with large spikes to either larger or nearly zero intensities), which is due to multiple reflections of the wave in the chaotically inhomogeneous medium (*dynamic localization*). In this case, the parameter $\gamma \ll 1$ and, therefore, the effect of weak absorption on the dynamic localization is not significant.

It should be noted that, with a passage to continual generalizations of the considered problem in the mechanics and electrodynamics of continuous media — i.e., to fields

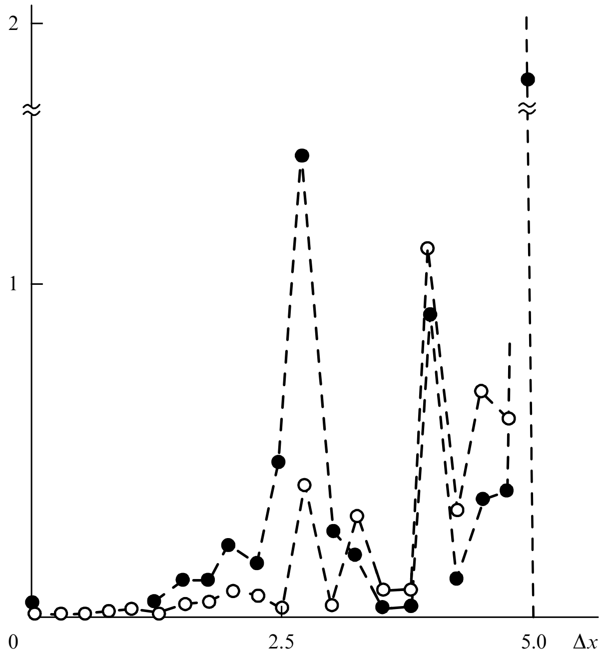

Figure 2. Numerical simulation of the dynamic localization for two realizations of inhomogeneities in a medium.

described by partial differential equations — an analysis of Lyapunov stability becomes possible only using series expansions of solutions in terms of a complete system of orthogonal functions, i.e., with a passage to an infinite-dimensional system of ordinary differential equations. Applying such a technique to stochastic problems raises the question of commutativity for the series-expansion and statistical-averaging procedures. In particular, these operations are, as a rule, not commutative if the statistical characteristics of the random processes and fields are approximated by singular (generalized) functions (as, for example, in the approximation of delta-correlated fluctuations in the parameters of the system).

2.2 Partial differential equations

An example of such a field in fluid mechanics is the field of the passive-tracer density $\rho(\mathbf{r}, t)$, which is governed by the continuity equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\, \mathbf{v}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t) = 0\,, \qquad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,. \tag{7}$$

Here, $\mathbf{v}(\mathbf{r}, t)$ is the tracer-velocity field in the hydrodynamic flow $\mathbf{u}(\mathbf{r}, t)$, and the total mass of the tracer is conserved during the evolution, namely

$$M = M(t) = \int \mathrm{d}\mathbf{r}\; \rho(\mathbf{r}, t) = \int \mathrm{d}\mathbf{r}\; \rho_0(\mathbf{r}) = \text{const}\,.$$

For weakly inertial particles, the field of the tracer velocity $\mathbf{v}(\mathbf{r}, t)$ itself in the hydrodynamic flow $\mathbf{u}(\mathbf{r}, t)$ can be described by the quasilinear partial differential equation (see, e.g., Ref. [16])

$$\left(\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{r}, t)\, \frac{\partial}{\partial \mathbf{r}}\right) \mathbf{v}(\mathbf{r}, t) = -\lambda\big[\mathbf{v}(\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)\big] \tag{8}$$

which should be considered to be phenomenological. In the general case, a solution to Eqn (8) may not be unique, discontinuities can be present, etc. However, in the asymptotic case of a weakly inertial tracer (with $\lambda \to \infty$, which is the limit that we are interested in), the solution of the problem will be unique over a reasonably long time interval. Notice that, on the right-hand side of Eqn (8), the term $\mathbf{F}(\mathbf{r}, t) = \lambda \mathbf{v}(\mathbf{r}, t)$ which is linear with respect to the velocity field $\mathbf{v}(\mathbf{r}, t)$ corresponds to the well-known *Stokes formula* for the drag force acting on a slowly moving particle and is brought about by the hydrodynamic flow $\mathbf{u}(\mathbf{r}, t)$. If the particle is approximated by a sphere of radius a, we have $\lambda = 6\pi a \eta / m_{\mathrm{p}}$, where η is the dynamic viscosity, and m_{p} is the mass of the particle (see, e.g., Refs [17, 18]).

Equations (7) and (8) provide an *Euler description* of the evolution of the passive-tracer-density field. They are first-order partial differential equations and can be solved using the method of characteristics. Then, the characteristic curves $\mathbf{r}(t), \mathbf{v}(t)$ for Eqn (8) coincide with equations (2) describing the motion of the particle. Equation (7) is written in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \rho(t|\mathbf{r}_0) = -\frac{\partial \mathbf{v}\big(\mathbf{r}(t|\mathbf{r}_0), t\big)}{\partial \mathbf{r}}\, \rho(t|\mathbf{r}_0)\,, \qquad \rho(0|\mathbf{r}_0) = \rho_0(\mathbf{r}_0)\,. \tag{9}$$

Such a passage leads to a *Lagrangian description* of the tracer dynamics.

For an inertialess tracer ($\lambda \to \infty$), we have $\mathbf{v}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}, t)$, and Eqns (7), (9) can be written in the simplified form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\, \mathbf{u}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t) = 0\,, \qquad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \rho(t|\mathbf{r}_0) = -\frac{\partial \mathbf{u}\big(\mathbf{r}(t|\mathbf{r}_0), t\big)}{\partial \mathbf{r}}\, \rho(t|\mathbf{r}_0)\,, \qquad \rho(0|\mathbf{r}_0) = \rho_0(\mathbf{r}_0)\,. \tag{11}$$

In this case, the clustering of the density field $\rho(\mathbf{r}, t)$ in the divergent hydrodynamics flow with the velocity field $\mathbf{u}(\mathbf{r}, t)$ [for $\operatorname{div} \mathbf{u}(\mathbf{r}, t) \neq 0$] occurs with a probability of unity.

A comparison between Eqns (11) and (4) leads to the following relationship between the Lagrangian particle density and the divergence:

$$\rho(t|\mathbf{r}_0) = \frac{\rho_0(\mathbf{r}_0)}{j(t|\mathbf{r}_0)}\,. \tag{12}$$

As a second example, let us consider wave propagation in a randomly inhomogeneous three-dimensional medium using a scalar parabolic equation valid for the description of wave propagation in a medium with large-scale three-dimensional inhomogeneities and describing the scattering of a wave by small angles (see, e.g., Refs [19–21] and the monograph [3]):

$$\begin{gathered} \frac{\partial}{\partial x}\, U(x, \mathbf{R}) = \frac{\mathrm{i}}{2k}\, \Delta_{\mathbf{R}} U(x, \mathbf{R}) + \frac{\mathrm{i}k}{2}\, \varepsilon(x, \mathbf{R}) U(x, \mathbf{R})\,, \\ U(0, \mathbf{R}) = U_0(\mathbf{R})\,. \end{gathered} \tag{13}$$

Here, the following notation was introduced: x is the coordinate in the direction of propagation of the wave, $\mathbf{R}$ are the coordinates in the lateral plane, and $\varepsilon(x, \mathbf{R})$ is the departure of the permittivity from unity. Clearly, this equation is approximate.

If we introduce the amplitude and phase of the wave field according to the formula

$$U(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\big[\mathrm{i} S(x, \mathbf{R})\big]\,,$$

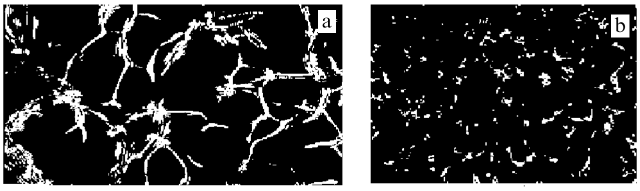

Figure 3. Cross section of a laser beam propagating in a turbulent medium (under laboratory conditions) in the region of strong focusing (a) and in the region of strong (saturated) fluctuations (b).

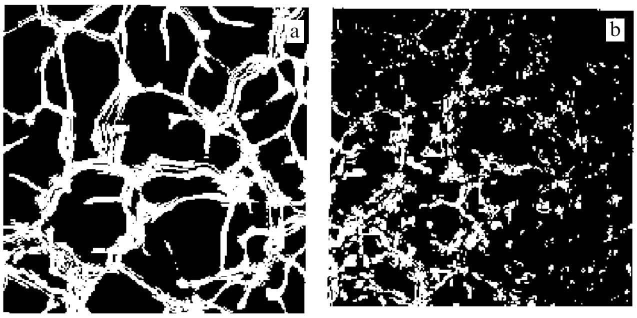

Figure 4. Cross section of a laser beam propagating in a turbulent medium (numerical simulation) in the region of strong focusing (a) and in the region of strong (saturated) fluctuations (b).

the transport equation for the wave-field intensity $I(x, \mathbf{R}) = |u(x, \mathbf{R})|^2$ can be written out as

$$\frac{\partial}{\partial x} I(x, \mathbf{R}) + \frac{1}{k} \nabla_{\mathbf{R}} \{ \nabla_{\mathbf{R}} S(x, \mathbf{R}) I(x, \mathbf{R}) \} = 0 \,, \qquad I(0, \mathbf{R}) = I_0(\mathbf{R}) \,. \tag{14}$$

Therefore, in the general case of an arbitrarily directed incoming wave beam, the power of the wave in the plane $x = \text{const}$, viz.

$$E_0 = \int I(x, \mathbf{R}) \, \mathrm{d}\mathbf{R} = \int I_0(\mathbf{R}) \, \mathrm{d}\mathbf{R} \,,$$

is conserved.

Equation (14) coincides in form with Eqn (7) and can thus be treated as the transport equation for a conservative tracer in a potential velocity field.

As is well known, the realizations of the intensity field are of a cluster nature, and the clustering manifests itself in *caustic patterns* due to the effects of random focusing and defocusing of the wave field in a random medium. As an example, Fig. 3 presents photographs of the cross section of a laser beam propagating in a turbulent medium in laboratory studies [22] for various intensities of the permittivity fluctuations. Similar patterns taken from Ref. [23] are presented in Fig. 4; they were obtained by numerical simulations [24, 25] using the representation of the solution of Eqn (13) as a continual integral. The development of the caustic wave-field pattern is clearly seen in these figures.

3. Statistical topography of random processes and fields

We will now discuss the essentials of the statistical topography method. Let us first introduce the concept of typical realization of the random process $z(t)$ to characterize the basic peculiarities of behavior for a particular realization of the process as a whole, over the entire time interval $t \in (0, \infty)$.

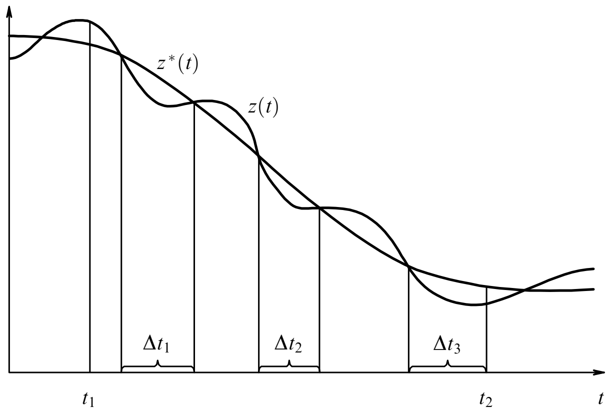

Figure 5. Toward a definition of a typical-realization curve for the random process.

3.1 The typical-realization curve for a random process

Let $z(t)$ be a random process. The statistical characteristics of the process $z(t)$ at a fixed time t are completely described by either the probability density $P(z, t)$, which parametrically depends on time, or the integral distribution function

$$F(z, t) = \text{Prob}\,\big(z(t) < z\big) = \int_{-\infty}^{z} \mathrm{d}z' \, P(z', t)$$

which gives the probability of the process value at a time t, satisfying the inequality $z(t) < z$.

We define *the typical-realization curve* for the random process $z(t)$ as the deterministic curve $z^*(t)$ that is the *median of the integral distribution function*. This function can be determined as the solution of the algebraic equation

$$F\big(z^*(t), t\big) = \frac{1}{2} \,.$$

This definition is based on the property of the median that, for any time interval (t_1, t_2), the random process $z(t)$ appears to 'wind' around the curve $z^*(t)$, so that the mean time over which the inequality $z(t) > z^*(t)$ holds true coincides with the mean time over which the opposite inequality $z(t) < z^*(t)$ is valid (Fig. 5), namely

$$\big\langle T_{z(t) > z^*(t)} \big\rangle = \big\langle T_{z(t) < z^*(t)} \big\rangle = \frac{1}{2} (t_2 - t_1) \,.$$

Naturally, the curve $z^*(t)$ can substantially differ from any particular realization of the process $z(t)$ and it does not describe the magnitude of possible spikes. Thus, the typical-realization curve $z^*(t)$ of the random process $z(t)$, which was obtained relying on the one-time probability density, is nevertheless defined over the whole time interval $t \in (0, \infty)$.

For specific types of random processes, additional information characterizing the spikes against this curve can obviously be obtained, as well.

3.1.1 Very simple random processes. For a Gaussian random process $z(t)$ with a mean value $\langle z(t) \rangle$ and a variance $\sigma^2(t)$, the one-time probability density is given by

$$P(z, t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left\{ -\frac{\big[z - \langle z(t) \rangle\big]^2}{2\sigma^2(t)} \right\},$$

and the typical-realization curve coincides with the mean value of the process $z(t)$:

$$z^*(t) = \langle z(t) \rangle \,. \tag{15}$$

The generating (or characteristic) function for this process has the form

$$\langle \exp[\lambda z(t)] \rangle = \int_{-\infty}^{\infty} \mathrm{d}z \exp[\lambda z(t)] P(z,t)$$
$$= \exp\left\{ \lambda \langle z(t) \rangle + \frac{\lambda^2}{2}\, \sigma^2(t) \right\}. \tag{16}$$

For a logarithmically normal random process $y(t)$ whose logarithm is a Gaussian random process $z(t)$, viz.

$$y(t) = \exp z(t) \,,$$

the one-time probability density $P(y,t)$ can be written out as

$$P(y,t) = \frac{1}{y\sqrt{2\pi\sigma^2(t)}} \exp\left\{ -\frac{\ln^2\left[\exp\left[-\langle z(t)\rangle\right] y\right]}{2\sigma^2(t)} \right\},$$

and the typical-realization curve is determined by the equality

$$y^*(t) = \exp\langle z(t)\rangle = \exp\langle \ln y(t)\rangle \,. \tag{17}$$

If we know the behavior of the moment functions of the logarithmically normal random process $y(t)$, i.e., the functions $\langle y^n(t)\rangle$ $(n = 1, 2, \ldots)$, the statistical characteristics of the random process $z(t) = \ln y(t)$ are also known. Indeed, according to formula (16), we have at $\lambda = n$ the following relationship

$$\langle y^n(t)\rangle = \langle \exp[n \ln y(t)]\rangle = \exp\left\{ n\langle \ln y(t)\rangle + \frac{n^2}{2}\,\sigma^2_{\ln y}(t) \right\},$$

and hence

$$\langle \ln y(t)\rangle = \lim_{n\to 0} \frac{1}{n} \ln\langle y^n(t)\rangle \,, \qquad \sigma^2_{\ln y}(t) = \lim_{n\to\infty} \frac{2}{n^2} \ln\langle y^n(t)\rangle \,. \tag{18}$$

3.1.2 The simplest Markovian random processes

Wiener random process. The Wiener random process is defined as the solution of the stochastic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, w(t) = z(t) \,, \qquad w(0) = 0 \,,$$

where $z(t)$ is a Gaussian process delta-correlated in time (a 'white noise' process) with the parameters

$$\langle z(t)\rangle = 0 \,, \qquad \langle z(t) z(t')\rangle = 2\sigma^2\tau_0\, \delta(t - t') \,. \tag{19}$$

For a discussion of the property of delta correlation of processes in time and the physical meaning of the parameters σ^2 and τ_0, see, for instance, the monographs [3, 4].

The solution of this equation,

$$w(t) = \int_0^t \mathrm{d}\tau\, z(\tau) \,,$$

is a continuous, time-dependent Gaussian random process with the parameters

$$\langle w(t)\rangle = 0 \,, \qquad \langle w(t) w(t')\rangle = 2\sigma^2\tau_0 \min(t, t') \,.$$

Wiener random process with a drift. Let us discuss a more general process with a drift, depending on the parameter α according to the formula

$$w(t;\alpha) = -\alpha t + w(t) \,, \qquad \alpha > 0 \,.$$

The process $w(t;\alpha)$ is Markovian; its probability density $P(w,t;\alpha)$ is described by the expression

$$P(w,t;\alpha) = \frac{1}{2\sqrt{\pi D t}} \exp\left\{ -\frac{(w + \alpha t)^2}{4Dt} \right\}, \tag{20}$$

where the quantity $D = \sigma^2\tau_0$ is the diffusion coefficient. The typical-realization curve for a Wiener random process with a drift is the linear function of time:

$$w^*(t;\alpha) = -\alpha t \,.$$

Wiener random processes can be used to construct various other processes convenient for the simulation of various physical phenomena. For positive quantities, a very simple approximation is a logarithmically normal (lognormal) process. We will consider it in more detail.

Logarithmically normal process. We define a lognormal random process by the formula

$$y(t;\alpha) = \exp w(t;\alpha) = \exp\left\{ -\alpha t + \int_0^t \mathrm{d}\tau\, z(\tau) \right\}, \tag{21}$$

where $z(t)$ is a white-noise Gaussian process with the parameters specified by formulas (19). It can be governed by the stochastic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y(t;\alpha) = \{-\alpha + z(t)\}\, y(t;\alpha) \,, \qquad y(0;\alpha) = 1 \,.$$

The one-time probability density of the lognormal process is given by

$$P(y,t;\alpha) = \frac{1}{2y\sqrt{\pi D t}} \exp\left\{ -\frac{\ln^2\left[y \exp(\alpha t)\right]}{4Dt} \right\}. \tag{22}$$

A feature typical of distribution (22) is the appearance of a long flat '*tail*' for $Dt \gg 1$, indicative of the increasing role of large spikes of the process $y(t;\alpha)$ in the formation of one-time statistics.

If we know only one-point statistical characteristics of the process $y(t;\alpha)$, we can obtain important information on the behavior of the realizations of the process $y(t;\alpha)$ over the entire time interval $(0,\infty)$.

In particular, the lognormal process $y(t;\alpha)$ is Markovian and its one-time probability density (22) obeys the Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial y}\, y \right) P(y,t;\alpha) = D \frac{\partial}{\partial y}\, y \frac{\partial}{\partial y}\, y P(y,t;\alpha) \,,$$
$$P(y,0;\alpha) = \delta(y-1) \,. \tag{23}$$

Based on Eqn (23), we can easily derive equations for the moment functions of the process $y(t;\alpha)$, whose solutions are determined by the equalities

$$\left\langle y^n(t;\alpha)\right\rangle = \exp\left[n\left(n-\frac{\alpha}{D}\right)Dt\right],$$
$$\left\langle \frac{1}{y^n(t;\alpha)}\right\rangle = \exp\left[n\left(n+\frac{\alpha}{D}\right)Dt\right], \qquad n=1,2,\ldots \tag{24}$$

and grow exponentially with time.

From Eqn (23), we can also easily obtain the equality

$$\left\langle \ln y(t)\right\rangle = -\alpha t,$$

therefore, the parameter α can be written in the form

$$-\alpha = \frac{1}{t}\left\langle \ln y(t)\right\rangle. \tag{25}$$

This means that, according to equality (1), it is the *Lyapunov characteristic index* (Lyapunov exponent) for the lognormal random process $y(t)$. Next, the typical-realization curve for the process $y(t;\alpha)$ proves to be the exponentially declining curve

$$y^*(t) = \exp\left\langle \ln y(t)\right\rangle = \exp(-\alpha t), \tag{26}$$

in agreement with formula (17).

The exponential growth of the moments is due to the spikes of the process $y(t;\alpha)$ against the background of the typical-realization curve $y^*(t;\alpha)$ in the direction of both larger and smaller y values; therefore, it is a purely statistical effect that results from averaging over the whole ensemble of realizations.

Thus, we see a distinct contradiction between the behavior of the statistical characteristics of the process $y(t;\alpha)$ and the behavior of the process in concrete realizations.

All these properties of the lognormal process manifest themselves in the dynamics of specific physical systems in the form of coherent phenomena such as *clustering* and *localization*.

3.2 Random fields

The principal subject of investigation in the statistical topography of random fields is, as in the normal topography of mountains, the system of contours — level lines (in the two-dimensional case) or surfaces of constant values (in the three-dimensional case) defined by the equality $f(\mathbf{r},t)=f=\text{const}$.

To analyze the system of contours (in the two-dimensional case), it is convenient to introduce the singular indicator function

$$\varphi(\mathbf{R},t;f) = \delta\big(f(\mathbf{R},t)-f\big)$$

localized at the contours.

Quantities such as the total area of regions that are bounded by contours and where $f(\mathbf{R},t)>f$, viz.

$$S(t;f) = \int \theta\big(f(\mathbf{R},t)-f\big)\,\mathrm{d}\mathbf{R} = \int_f^\infty \mathrm{d}f' \int \mathrm{d}\mathbf{R}\,\varphi(\mathbf{R},t;f'),$$

and the total 'mass' of the field in these regions,

$$M(t;f) = \int_f^\infty f'\,\mathrm{d}f' \int \mathrm{d}\mathbf{R}\,\varphi(\mathbf{R},t;f'),$$

can be expressed in terms of this function.

The indicator function averaged over the ensemble of realizations determines the one-time, one-point probability density $P(\mathbf{R},t;f)=\left\langle\varphi(\mathbf{R},t;f)\right\rangle$; therefore, the mean values of all expressions are determined by this probability density.

Additional information on the structure of the field $f(\mathbf{R},t)$ can be gained if its spatial gradient, $\mathbf{p}(\mathbf{R},t)=\nabla f(\mathbf{R},t)$, is included in the consideration. For example, the total length of the contours is described by the expression

$$l(t;f) = \int \mathrm{d}\mathbf{R}\,\left|\mathbf{p}(\mathbf{R},t)\right|\,\delta\big(f(\mathbf{R},t)-f\big) = \oint \mathrm{d}l.$$

The inclusion of the second-order spatial derivatives makes it possible to estimate the total number of the contours $f(\mathbf{R},t)=f=\text{const}$ using the approximate (correct up to counting unclosed curves) formula

$$N(t;f) \sim \frac{1}{2\pi}\int \mathrm{d}\mathbf{R}\,\kappa(\mathbf{R},t;f)\left|\mathbf{p}(\mathbf{R},t)\right|\,\delta\big(f(\mathbf{R},t)-f\big),$$

where $\kappa(\mathbf{R},t;f)$ is the curvature of the level line.

It should be noted that, for a spatially homogeneous field $f(\mathbf{R},t)$ with $\mathbf{R}$-independent appropriate one-point probability densities, the statistical averages of all quantities will characterize the corresponding specific (calculated per unit area) values of these quantities. In this case, for one-point statistical characteristics, random fields are statistically equivalent to random processes for which the typical-realization curve characterizes the time behavior of the random field at any fixed point in space. This constitutes a fundamental difference of the statistical description from the traditional Lyapunov approach.

We now illustrate the applications of the statistical topography ideology using the problems considered in Section 2 as examples.

4. An inertialess tracer in random hydrodynamic flows

In the general case, we assume the random velocity field to be divergent ($\operatorname{div}\mathbf{u}(\mathbf{r},t)\neq 0$) and, simultaneously, a (statistically homogeneous and isotropic in space and stationary in time) Gaussian random field with a correlation tensor and a spectral tensor $\big(\left\langle\mathbf{u}(\mathbf{r},t)\right\rangle=0\big)$:

$$B_{ij}(\mathbf{r}-\mathbf{r}',t-t') = \left\langle u_i(\mathbf{r},t)u_j(\mathbf{r}',t')\right\rangle$$
$$= \int \mathrm{d}\mathbf{k}\,E_{ij}(\mathbf{k},t-t')\exp\left[\mathrm{i}\mathbf{k}(\mathbf{r}-\mathbf{r}')\right],$$
$$E_{ij}(\mathbf{k},t) = \frac{1}{(2\pi)^d}\int \mathrm{d}\mathbf{r}\,B_{ij}(\mathbf{r},t)\exp(-\mathrm{i}\mathbf{k}\mathbf{r}), \tag{27}$$
$$E_{ij}(\mathbf{k},t) = E^{\mathrm{s}}_{ij}(\mathbf{k},t)+E^{\mathrm{p}}_{ij}(\mathbf{k},t),$$

where d is the dimension of space, and the spectral components of the velocity field tensor have the structure

$$E^{\mathrm{s}}_{ij}(\mathbf{k},t) = E^{\mathrm{s}}(k,t)\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right),$$
$$E^{\mathrm{p}}_{ij}(\mathbf{k},t) = E^{\mathrm{p}}(k,t)\,\frac{k_ik_j}{k^2}. \tag{28}$$

Here, $E^{\mathrm{s}}(k,t)$ and $E^{\mathrm{p}}(k,t)$ are the solenoidal and the potential components of the spectral density of the velocity field, respectively.

We define the time-correlation radius of the field $\mathbf{u}(\mathbf{r}, t)$ via the equality

$$\tau_0 \sigma_{\mathbf{u}}^2 = \int_0^\infty \mathrm{d}\tau \, B_{ii}(0, \tau) = \int_0^\infty \mathrm{d}\tau \int \mathrm{d}\mathbf{k} \left[(d-1) E^{\mathrm{s}}(k, \tau) + E^{\mathrm{p}}(k, \tau) \right],$$

where the variance of the velocity field is $\sigma_{\mathbf{u}}^2 = B_{ii}(0,0) = \langle \mathbf{u}^2(\mathbf{r}, t) \rangle$.

In our statistical analysis, we will utilize the approximation of a random field $\mathbf{u}(\mathbf{r}, t)$ delta-correlated in time, so that the correlation tensor of the field $\mathbf{u}(\mathbf{r}, t)$ is approximated by the expression (see, e.g., the monographs [3, 4])

$$B_{ij}(\mathbf{r}, t) = 2 B_{ij}^{\mathrm{eff}}(\mathbf{r}) \delta(t), \qquad B_{ij}^{\mathrm{eff}}(\mathbf{r}) = \int_0^\infty \mathrm{d}t \, B_{ij}(\mathbf{r}, t).$$

4.1 Lagrangian description (diffusion of particles)

4.1.1 One-point statistical characteristics. From the system of equations (3) for the one-time Lagrangian probability density $P(\mathbf{r}, j, t|\mathbf{r}_0)$ for the particle coordinate $\mathbf{r}(t|\mathbf{r}_0)$ and its divergence $j(t|\mathbf{r}_0)$, in the approximation of a Gaussian velocity field $\mathbf{u}(\mathbf{r}, t)$ delta-correlated in time, we arrive at the Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} - D_0 \Delta \right) P(\mathbf{r}, j, t|\mathbf{r}_0) = D^{\mathrm{p}} \frac{\partial^2}{\partial j^2} j^2 P(\mathbf{r}, j, t|\mathbf{r}_0), \qquad (29)$$
$$P(\mathbf{r}, j, 0|\mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)\, \delta(j-1).$$

The solution of equation (29) has the form

$$P(\mathbf{r}, j, t|\mathbf{r}_0) = P(\mathbf{r}, t|\mathbf{r}_0)\, P(j, t|\mathbf{r}_0), \qquad (30)$$

where $P(\mathbf{r}, t|\mathbf{r}_0)$ is the probabilistic distribution of particle coordinates that satisfies the equation

$$\frac{\partial}{\partial t} P(\mathbf{r}, t|\mathbf{r}_0) = D_0 \frac{\partial^2}{\partial \mathbf{r}^2} P(\mathbf{r}, t|\mathbf{r}_0), \qquad P(\mathbf{r}, 0|\mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0).$$

Consequently, equation (30) represents a Gaussian distribution

$$P(\mathbf{r}, t|\mathbf{r}_0) = \frac{1}{(4\pi D_0 dt)^{d/2}} \exp\left\{ -\frac{(\mathbf{r} - \mathbf{r}_0)^2}{4 D_0 t} \right\}, \qquad (31)$$

where d is the dimension of the space.

We emphasize that the obtained solution (30) implies that the coordinates $\mathbf{r}(t|\mathbf{r}_0)$ and divergences $j(t|\mathbf{r}_0)$ are statistically independent in the neighborhood of the particle with the Lagrangian coordinate $\mathbf{r}_0$.

The probability density for the divergence satisfies the Fokker–Planck equation that follows from Eqn (29):

$$\frac{\partial}{\partial \tau} P(j, \tau) = \frac{\partial^2}{\partial j^2} j^2 P(j, \tau), \qquad P(j, 0|\mathbf{r}_0) = \delta(j-1). \quad (32)$$

Hereinafter, we use the dimensionless time $\tau = D^{\mathrm{p}} t$. The random process $j(t|\mathbf{r}_0)$ is then lognormal, and its probability density does not depend on the parameter $\mathbf{r}_0$:

$$P(j, t|\mathbf{r}_0) = \frac{1}{2j\sqrt{\pi\tau}} \exp\left\{ -\frac{\ln^2 (j \exp \tau)}{4\tau} \right\}. \qquad (33)$$

Therefore, we obtain the following expression for the Lagrangian moments of divergence:

$$\langle j^n(t|\mathbf{r}_0) \rangle = \exp\left[n(n-1)\tau \right];$$

this implies, in particular, an exponential growth of its moments (for $n > 1$) in the Lagrangian representation.

For the realizations of divergence, we have an exponentially declining typical-realization curve of the form

$$j^*(t) = \exp(-\tau),$$

which is precisely the Lyapunov exponent.

We emphasize that the above-discussed Lagrangian statistical properties of the particle in the flows containing a random potential component qualitatively differ from the statistical properties of the particle resided in divergence-free flows, where the divergence is conserved, viz. $j(t) = 1$. The above-presented statistical estimates of the random process $j(t)$ indicate that they are formed by the spikes of their realizations against the background of the typical-realization curves.

At the same time, the probability distributions for particle coordinates are virtually the same in the cases of both divergent and divergence-free flows.

4.1.2 Two-point statistical characteristics. Now we consider the combined dynamics of two particles in the absence of a mean flow. In this case, the combined probability density for the relative diffusion of the two particles is defined as

$$P(\mathbf{l}, t) = \left\langle \delta\big(\mathbf{r}_1(t) - \mathbf{r}_2(t) - \mathbf{l} \big) \right\rangle,$$

and it satisfies the Fokker–Planck equation

$$\frac{\partial}{\partial t} P(\mathbf{l}, t) = \frac{\partial^2}{\partial l_\alpha \partial l_\beta} D_{\alpha\beta}(\mathbf{l})\, P(\mathbf{l}, t), \qquad P(\mathbf{l}, 0) = \delta(\mathbf{l} - \mathbf{l}_0), \quad (34)$$

where

$$D_{\alpha\beta}(\mathbf{l}) = 2\left[B_{\alpha\beta}^{\mathrm{eff}}(0) - B_{\alpha\beta}^{\mathrm{eff}}(\mathbf{l}) \right]$$

is the structural matrix of the vector field $\mathbf{u}(\mathbf{r}, t)$, and $\mathbf{l}_0$ is the initial distance between the particles.

Equation (34) has not been solved in the general case. If, however, for the initial distance between the particles we have $l_0 \ll l_{\mathrm{cor}}$, where l_{cor} is the spatial correlation radius of the velocity field $\mathbf{u}(\mathbf{r}, t)$, the functions $D_{\alpha\beta}(\mathbf{l})$ can be expanded into a Taylor series and, to a first approximation, we will obtain

$$D_{\alpha\beta}(\mathbf{l}) = -\frac{\partial^2 B_{\alpha\beta}^{\mathrm{eff}}(\mathbf{l})}{\partial l_i \partial l_j} \bigg|_{\mathbf{l}=0} l_i l_j.$$

The diffusion tensor can be simplified using representation (28) and written out as

$$D_{\alpha\beta}(\mathbf{l}) = \frac{1}{d(d+2)} \Big\{ \left[D^{\mathrm{s}}(d+1) + D^{\mathrm{p}} \right] \delta_{\alpha\beta} l^2 - 2(D^{\mathrm{s}} - D^{\mathrm{p}}) l_\alpha l_\beta \Big\}, \qquad (35)$$

where d is the dimension of the space.

Now we substitute Eqn (35) into Eqn (34), multiply both parts of the resulting equation by $|\mathbf{l}|^n = l^n$, and integrate with respect to $\mathbf{l}$ to obtain the closed equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \left\langle l^n(t) \right\rangle = \frac{1}{d(d+2)} \Big\{ \left[D^{\mathrm{s}}(d+1) + D^{\mathrm{p}} \right] n(d+n-2) - 2(D^{\mathrm{s}} - D^{\mathrm{p}}) n(n-1) \Big\} ,$$

whose solution corresponds to functions exponentially growing in time for all moments ($n = 1, 2, \ldots$). The probability distribution for the random process $l(t)/l_0$ will then be logarithmically normal.

Therefore, in accordance with formulas (17) and (18), the typical-realization curve for the distance between the two particles will be the exponential function of time in the form

$$l^*(t) = l_0 \exp\left\{ \frac{1}{d(d+2)} \left[D^{\mathrm{s}} d(d-1) - D^{\mathrm{p}}(4-d) \right] t \right\} , \quad (36)$$

which is related to the *Lyapunov exponent*.

From this it follows that, in the two-dimensional case ($d = 2$), the expression

$$l^*(t) = l_0 \exp\left[\frac{1}{4} (D^{\mathrm{s}} - D^{\mathrm{p}}) t \right]$$

is essentially dependent on the sign of the difference $D^{\mathrm{s}} - D^{\mathrm{p}}$. In particular, for a divergence-free velocity field ($D^{\mathrm{p}} = 0$), we have an exponentially growing typical realization, which corresponds to an exponentially rapid recession of particles at small distances between them. This result is valid for times

$$\frac{1}{4} D^{\mathrm{s}} t \ll \ln \frac{l_{\mathrm{cor}}}{l_0} ,$$

for which expansion (35) is valid. In the other limiting case of a potential velocity field ($D^{\mathrm{s}} = 0$), the typical realization will be an exponentially declining curve, i.e., the particles will obviously tend to 'merge'. This means that the *clusters* — zones of compact concentration of particles, mainly located in rarefied zones — should form, which agrees with the results of evolutionary numerical simulations of a realization of an initially uniform particle distribution in a random potential velocity field (such a distribution is shown in Fig. 1b, albeit for a completely different statistical model of the velocity field). Seemingly, the clustering phenomenon itself does not depend on the model of the random velocity field, although the statistical parameters characterizing this phenomenon can certainly be sensitive to the model. Thus, the following inequality must be satisfied for the clustering of particles:

$$D^{\mathrm{s}} < D^{\mathrm{p}} . \quad (37)$$

In contrast, for the three-dimensional case ($d = 3$), it follows from Eqn (36) that

$$l^*(t) = l_0 \exp\left[\frac{1}{15} (6D^{\mathrm{s}} - D^{\mathrm{p}}) t \right] ,$$

and the typical-realization curve will exponentially decline with time, provided a more restrictive condition (compared to the two-dimensional case) is satisfied:

$$D^{\mathrm{p}} > 6D^{\mathrm{s}} .$$

In the one-dimensional case, one obtains

$$l^*(t) = l_0 \exp\left(-D^{\mathrm{p}} t \right) ,$$

and the typical-realization curve always declines with time, since the velocity field is always divergent in this case.

4.2 Eulerian description

To describe the local behavior of the realizations of the tracer spatial field in the random velocity field $\mathbf{u}(\mathbf{r}, t)$, it is necessary to know the probabilistic distribution of the tracer density. If we proceed from stochastic equation (7), the equation for the probability density of the tracer density (concentration) field can be written out as

$$\left(\frac{\partial}{\partial t} - D_0 \Delta \right) P(\mathbf{r}, t; \rho) = D_\rho \frac{\partial^2}{\partial \rho^2} \rho^2 P(\mathbf{r}, t; \rho) , \qquad P(\mathbf{r}, 0; \rho) = \delta\left(\rho - \rho_0(\mathbf{r}) \right) , \quad (38)$$

where the diffusion coefficient in the ρ-space is equal to $D_\rho = D^{\mathrm{p}}$.

In particular, it follows from Eqn (38) that the moment functions of the density field obey the equation

$$\left(\frac{\partial}{\partial t} - D_0 \Delta \right) \left\langle \rho^n(\mathbf{r}, t) \right\rangle = D_\rho n(n-1) \left\langle \rho^n(\mathbf{r}, t) \right\rangle , \qquad \left\langle \rho^n(\mathbf{r}, 0) \right\rangle = \rho_0^n(\mathbf{r}) . \quad (39)$$

Its solution has the following structure

$$\left\langle \rho^n(\mathbf{r}, t) \right\rangle = \exp\left[n(n-1)\tau \right] \int \mathrm{d}\mathbf{r}' \, P(\mathbf{r}, t | \mathbf{r}') \, \rho_0^n(\mathbf{r}') , \quad (40)$$

where the function $P(\mathbf{r}, t|\mathbf{r}')$ is described by equality (31), and the parameter $\tau = D_\rho t$.

If the initial tracer density is everywhere the same, $\rho_0(\mathbf{r}) = \rho_0 = \mathrm{const}$, the probabilistic density distribution does not depend on $\mathbf{r}$ and is governed by the equation

$$\frac{\partial}{\partial t} P(t; \rho) = D_\rho \frac{\partial^2}{\partial \rho^2} \rho^2 P(t; \rho) , \qquad P(0; \rho) = \delta(\rho - \rho_0) , \quad (41)$$

which coincides with Eqn (32) for the divergence, differing only in the initial condition.

Therefore, the Eulerian density field is lognormal in this case, its probability density and the corresponding integral distribution function being as follows:

$$P(t; \rho) = \frac{1}{2\rho\sqrt{\pi\tau}} \exp\left\{ -\frac{\ln^2\left(\rho \exp(\tau)/\rho_0 \right)}{4\tau} \right\} , \qquad F(t; \rho) = \Phi\left(\frac{\ln\left(\rho \exp(\tau)/\rho_0 \right)}{2\sqrt{\tau}} \right) , \quad (42)$$

where $\Phi(z)$ is the probability integral (error function)

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{d}y \, \exp\left(-\frac{y^2}{2} \right) .$$

In terms of the one-point characterization of the density field $\rho(\mathbf{r}, t)$, the problem in this case is statistically equivalent to the divergence random process in the Lagrangian description, $j(t)$, and all the moment functions, starting from the second one, exponentially grow with time:

$$\left\langle \rho(\mathbf{r}, t) \right\rangle = \rho_0 , \qquad \left\langle \rho^n(\mathbf{r}, t) \right\rangle = \rho_0^n \exp\left[n(n-1)\tau \right] . \quad (43)$$

In contrast, the typical-realization curve of the density field at any fixed point in space exponentially declines with time, according to formulas (17) and (18):

$$\rho^*(t) = \rho_0 \exp(-\tau), \tag{44}$$

which testifies to a clustered nature of medium-density fluctuations in arbitrary divergent flows. The formation of the Eulerian statistics of density at any fixed point in space is controlled by density fluctuations around this curve.

We discussed above the one-point probability distribution for the tracer density in an Eulerian representation, which was even sufficient to make some inferences on the temporal behavior of the density-field realizations at fixed points in space. We will now demonstrate that this distribution also makes it possible to find out some characteristic features of the spatiotemporal structure of the density-field realizations.

For the sake of clarity, we restrict ourselves here to the two-dimensional case. As noted above, important information on the spatial behavior of realizations can be gained from an analysis of contours which are specified by the equality

$$\rho(\mathbf{r}, t) = \rho = \text{const}. \tag{45}$$

In particular, such functionals of the density field as the total area in the region where $\rho(\mathbf{r}, t) > \rho$ [we denote it as $S(t, \rho)$] and the total mass $M(t, \rho)$ of tracer in this region (their mean values are determined by the one-point probability density) are described by the expressions

$$\begin{aligned} \langle S(t,\rho)\rangle &= \int_\rho^\infty \mathrm{d}\tilde\rho \int \mathrm{d}\mathbf{r}\, P(\mathbf{r}, t; \tilde\rho), \\ \langle M(t,\rho)\rangle &= \int_\rho^\infty \tilde\rho\, \mathrm{d}\tilde\rho \int \mathrm{d}\mathbf{r}\, P(\mathbf{r}, t; \tilde\rho). \end{aligned} \tag{46}$$

If we substitute here the solution of Eqn (38), we will easily find, after simple manipulations, explicit expressions for these quantities:

$$\begin{aligned} \langle S(t,\rho)\rangle &= \int \mathrm{d}\mathbf{r}\, \Phi\left(\frac{1}{2\sqrt{\tau}} \ln \frac{\rho_0(\mathbf{r}) \exp(-\tau)}{\rho}\right), \\ \langle M(t,\rho)\rangle &= \int \mathrm{d}\mathbf{r}\, \rho_0(\mathbf{r})\, \Phi\left(\frac{1}{2\sqrt{\tau}} \ln \frac{\rho_0(\mathbf{r}) \exp \tau}{\rho}\right). \end{aligned} \tag{47}$$

It can be seen from here that, as $\tau \gg 1$, the mean area of the regions where the density exceeds a given value ρ decreases with time according to the law

$$\langle S(t,\rho)\rangle \approx \frac{1}{\sqrt{\pi\tau\rho}} \exp\left(-\frac{\tau}{4}\right) \int \mathrm{d}\mathbf{r} \sqrt{\rho_0(\mathbf{r})}, \tag{48}$$

while the mean mass of the tracer contained in these regions, viz.

$$\langle M(t,\rho)\rangle \approx M - \sqrt{\frac{\rho}{\pi\tau}} \exp\left(-\frac{\tau}{4}\right) \int \mathrm{d}\mathbf{r} \sqrt{\rho_0(\mathbf{r})}, \tag{49}$$

monotonically approaches the total mass $M = \int \mathrm{d}\mathbf{r}\, \rho_0(\mathbf{r})$. This confirms again our previously-made conclusion that, as time passes, the particles of the tracer tend to gather into clusters — compact enhanced-density regions surrounded by rarefied regions.

5. Localization of plane waves in layered random media

Normally, to compute the specific statistical characteristics of a solution to the wave problem, the model of fluctuations of the function $\varepsilon(x)$ is used in the form of a Gaussian delta-correlated random process (*white-noise* process) with the parameters

$$\langle \varepsilon_1(x)\rangle = 0, \quad \langle \varepsilon_1(x)\varepsilon_1(x')\rangle = B_\varepsilon(x - x') = 2\sigma_\varepsilon^2 l_0 \delta(x - x'), \tag{50}$$

where $B_\varepsilon(x - x')$ is the correlation function, $\sigma_\varepsilon^2 \ll 1$ is the variance, and l_0 is the correlation radius for the random function $\varepsilon_1(x)$. This approximation implies that a passage to the asymptotic case $l_0 \to 0$ in the exact solution of the problem with a finite correlation radius l_0 leads to results coinciding with the solution of the statistical problem with the parameters (50).

Notice that the principal fraction of the results weakly depends on the model of the medium. Using a particular model makes it possible to quantify the basic parameters of the problem.

A statistical analysis of the solution of the problem indicates that, for a sufficiently thick layer of the medium, viz. $D(L - L_0) \gg 1$ [where the quantity $D = k^2\sigma_\varepsilon^2 l_0/2$ is related to the statistical characteristics of the function $\varepsilon(x)$], with a probability of unity one finds $|T_L| \to 0$ and, therefore, $|R_L| \to 1$; in other words, the half-space ($L_0 \to -\infty$) of the randomly inhomogeneous medium totally reflects the incident wave due to multiple reflections in the medium, i.e., a *dynamic localization of the wave field* occurs in this layer.

However, the mean value of the wave-field intensity for the half-space of the random medium is $\langle I(L - x)\rangle = 2$, while higher moments normalized to their values at the layer boundary are described by the expression

$$\langle I^n(L - x)\rangle = \exp\left[Dn(n-1)(L - x)\right],$$

i.e., the intensity of the wave field exhibits a logarithmically normal probability distribution, while the moment functions grow exponentially to the medium's interior.

The typical realization of the wave intensity in the medium is described in this case by the exponentially declining curve

$$I^*(x) = 2 \exp\left[-D(L - x)\right]$$

and coincides with the Lyapunov exponent; the quantity $l_{\text{loc}} = 1/D$ is called the *localization length* (see, e.g., Ref. [14]) and determines the spatial scale of wave-field-intensity damping in the medium for particular realizations of this field.

Thus, we can see that the formation of the statistics is due to the large spikes against the background of the typical-realization curve, which means the absence of a *statistical localization of the wave field.*

6. The caustic wave-field structure in a randomly inhomogeneous medium

Let us introduce the amplitude and phase of a wave field and a complex wave phase according to the formula

$$u(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\left[\mathrm{i}S(x, \mathbf{R})\right] = \exp\left[\phi(x, \mathbf{R})\right],$$

where

$$\phi(x, \mathbf{R}) = \chi(x, \mathbf{R}) + \mathrm{i}S(x, \mathbf{R}),$$

$\chi(x, \mathbf{R}) = \ln A(x, \mathbf{R})$ is the amplitude level of the wave, and $S(x, \mathbf{R})$ is the wave-phase fluctuation relative to the phase kx of the incident wave. From parabolic equation (13), the following nonlinear equation of so-called Rytov's *method of smooth perturbations* (*MSP*) can be obtained for the complex phase:

$$\frac{\partial}{\partial x}\,\phi(x, \mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}}\phi(x, \mathbf{R}) + \frac{\mathrm{i}}{2k}\left[\boldsymbol{\nabla}_{\mathbf{R}}\,\phi(x, \mathbf{R})\right]^2 + \mathrm{i}\,\frac{k}{2}\,\varepsilon(x, \mathbf{R}). \tag{51}$$

For a plane incident wave, which we will consider in what follows, it can be assumed without any loss of generality that $u_0(\mathbf{R}) = 1$ and, therefore, $\phi(0, \mathbf{R}) = 0$.

We separate the real and the imaginary parts in Eqn (51) to arrive at the equations

$$\begin{aligned} &\chi(x, \mathbf{R}) + \frac{1}{2k}\,\Delta_{\mathbf{R}}S(x, \mathbf{R}) + \frac{1}{k}\left[\boldsymbol{\nabla}_{\mathbf{R}}\,\chi(x, \mathbf{R})\right]\left[\boldsymbol{\nabla}_{\mathbf{R}}S(x, \mathbf{R})\right] = 0\,, \\ &\frac{\partial}{\partial x}\,S(x, \mathbf{R}) - \frac{1}{2k}\,\Delta_{\mathbf{R}}\chi(x, \mathbf{R}) - \frac{1}{2k}\left[\boldsymbol{\nabla}_{\mathbf{R}}\chi(x, \mathbf{R})\right]^2 \\ &\qquad + \frac{1}{2k}\left[\boldsymbol{\nabla}_{\mathbf{R}}S(x, \mathbf{R})\right]^2 = \frac{k}{2}\,\varepsilon(x, \mathbf{R}). \end{aligned} \tag{52}$$

If the function $\varepsilon(x, \mathbf{R})$ is sufficiently small, iterative series in terms of the field $\varepsilon(x, \mathbf{R})$ can be constructed for solving equations (52). Once this is done, Gaussian fields $\chi(x, \mathbf{R})$ and $S(x, \mathbf{R})$ correspond to the so-called first approximation of Rytov's MSP; their statistical characteristics can be determined by statistically averaging the corresponding iterative series. In particular, the second moments (including variances) of these fields can be found from the linearized system of equations (52), i.e., the system

$$\begin{aligned} &\frac{\partial}{\partial x}\,\chi_0(x, \mathbf{R}) = -\frac{1}{2k}\,\Delta_{\mathbf{R}}S_0(x, \mathbf{R})\,, \\ &\frac{\partial}{\partial x}\,S_0(x, \mathbf{R}) = \frac{1}{2k}\,\Delta_{\mathbf{R}}\chi_0(x, \mathbf{R}) + \frac{k}{2}\,\varepsilon(x, \mathbf{R})\,, \end{aligned} \tag{53}$$

while the mean values can be obtained by the direct averaging of equations (52). The linear system of equations (53) can be solved using a Fourier transform with respect to the transverse coordinate.

Now consider a statistical description of the wave field. We assume that the random field $\varepsilon(x, \mathbf{R})$ is a Gaussian uniform and isotropic field with the parameters

$$\left\langle \varepsilon(x, \mathbf{R}) \right\rangle = 0\,, \qquad B_\varepsilon(x - x', \mathbf{R} - \mathbf{R}') = \left\langle \varepsilon(x, \mathbf{R})\varepsilon(x', \mathbf{R}') \right\rangle.$$

In the approximation of delta-correlated fluctuations in the parameters of the medium, this correlation function can be approximated by the 'effective' function

$$B_\varepsilon(x, \mathbf{R}) = B_\varepsilon^{\mathrm{eff}}(x, \mathbf{R}) = \delta(x)A(\mathbf{R})\,,$$

$$A(\mathbf{R}) = \int_{-\infty}^{\infty} \mathrm{d}x\, B_\varepsilon(x, \mathbf{R})\,,$$

and the random field $\phi(x, \mathbf{R})$ will then be a statistically homogeneous field in the plane $\mathbf{R}$, with all its one-point statistical characteristics being independent of the parameter $\mathbf{R}$. The statistical properties of the amplitude fluctuations are described in the considered approximation by the variance of the amplitude level, i.e., by the parameter $\sigma_0^2(x) = \left\langle \chi_0^2(x, \mathbf{R}) \right\rangle$.

To find the mean value of the amplitude level, we make use of equation (14). For a plane incident wave, averaging this equation over the ensemble of realizations of the field $\varepsilon(x, \mathbf{R})$ yields the equality $\left\langle I(x, \mathbf{R}) \right\rangle = 1$, which can be rewritten as

$$\begin{aligned} \left\langle I(x, \mathbf{R}) \right\rangle &= \left\langle \exp\left[2\chi_0(x, \mathbf{R})\right] \right\rangle \\ &= \exp\left[2\left\langle \chi_0(x, \mathbf{R}) \right\rangle + 2\sigma_0^2(x)\right] = 1\,. \end{aligned}$$

Therefore, in the first MSP approximation one obtains $\left\langle \chi_0(x, \mathbf{R}) \right\rangle = -\sigma_0^2(x)$. As for the variance of the wave intensity, which is called the *scintillation index*, we find that, in the first MSP approximation, it is equal to

$$\beta_0^2(x) = \left\langle I^2(x, \mathbf{R}) \right\rangle - 1 = \left\langle \exp\left[4\chi_0(x, \mathbf{R})\right] \right\rangle - 1 \approx 4\sigma_0^2(x)\,. \tag{54}$$

Thus, the wave-field intensity is a logarithmically normal random field, and its one-point probability density is given by the expression

$$P(x; I) = \frac{1}{I\sqrt{2\pi\beta_0(x)}}\exp\left\{-\frac{1}{2\beta_0(x)}\ln^2\left(I\exp\left[\frac{1}{2}\,\beta_0(x)\right]\right)\right\}. \tag{55}$$

Now we can consider a statistically equivalent random process $I(x)$ with probability density (55). For this process, the typical-realization curve of the wave-field intensity is a curve exponentially declining with the distance, namely

$$I^*(x) = \exp\left[-\frac{1}{2}\,\beta_0(x)\right],$$

at any fixed point of space $\mathbf{R}$; this testifies to the emergence of a cluster (caustic) structure of the intensity field. The formation of the statistics (for example, the moment functions $\left\langle I^n(x, \mathbf{R}) \right\rangle$) is controlled by large spikes of the process $I(x)$ against the background of this curve.

The obtained description of intensity fluctuations based on the first MSP approximation is valid for $\beta_0(x) \leqslant 1$. As the parameter $\beta_0(x)$ is further increased, the method of smooth perturbations fails, and it becomes necessary to take into account the nonlinearity of the equation for the complex wave-field phase. This range of fluctuations, called the *range of strong focusing*, can hardly be treated analytically. As the parameter $\beta_0(x)$ is further increased ($\beta_0^2(x) \geqslant 10$), the statistical characteristics of field intensity reach a saturation regime, and this range of variations of the parameter $\beta_0(x)$ is called the *range of strong intensity fluctuations*.

In the range of strong focusing, the intensity moments can be approximated by the expression (see, e.g., Ref. [26])

$$\left\langle I^n(x, \mathbf{R}) \right\rangle = n!\exp\left\{n(n-1)\,\frac{\beta^2(x) - 1}{4}\right\},$$

with the corresponding probability density in the form

$$\begin{aligned} P(x, I) &= \frac{1}{\sqrt{\pi\left(\beta(x) - 1\right)}} \\ &\quad \times \int_0^\infty \mathrm{d}z\exp\left\{-zI - \frac{1}{\beta(x) - 1}\left[\ln z - \frac{\beta(x) - 1}{4}\right]^2\right\}, \end{aligned}$$

where the parameter $\beta^2(x)$ describes the variance of intensity fluctuations in this region; for example, in the case of a turbulent medium, one obtains

$$\beta^2(x) = 1 + 0.861\left(\beta_0^2(x)\right)^{-2/5}.$$

Here, $\beta_0^2(x)$ is the variance of the wave-field intensity calculated in the first MSP approximation.

These asymptotic formulas describe a passage to the range of *saturated* intensity fluctuations, where $\beta(x) \to 1$ as $\beta_0^2(x) \to \infty$. Accordingly, in this range we have

$$\left\langle I^n(x, \mathbf{R})\right\rangle = n!\,, \qquad P(x, I) = \exp\left(-I\right).$$

The exponential probability distribution of the wave-field intensity implies that the complex field $u(x, \mathbf{R})$ is a Gaussian random field. Consequently, in this region the mean specific area of the regions where $I(x, \mathbf{R}) > I$, and the specific averaged power localized in these regions are constant and do not reflect the behavior of the wave-field intensity in particular realizations. Similarly, the passage to a statistically equivalent random process is not informative in this case, since the typical-realization curve for such a process is also represented by a constant.

The structure of the wave field in this case can be understood by analyzing such quantities as the specific mean length of the contours and the mean specific number of contours (see, e.g., Ref. [21]). These quantities continue growing with the parameter $\beta_0(x)$, in contrast to the specific mean area. This results from the fact that a leading role in this regime is played by the interference of partial waves coming from various directions.

The dynamical pattern of behavior of the contours depends on the relationship between the processes of radiation focusing and defocusing by particular regions of the turbulent medium. The focusing on large-scale inhomogeneities gives rise to high peaks in the random relief of intensity. In the maximum-focusing regime $[\beta_0(x) \sim 1]$, about half the total power of the wave is concentrated in high, narrow peaks. As the parameter $\beta_0(x)$ is increased, the defocusing of radiation becomes predominant, which smears high peaks and produces a highly rugged (interferential) relief with a large number of peaks at the levels of $I \sim 1$, which was actually observed in both laboratory experiments (see Fig. 3) and numerical simulations (see Fig. 4).

7. Conclusions

Our approach to the analysis of stochastic dynamical problems is based on the ideas of statistical topography and makes it possible to infer a quantitative and qualitative characterization of particular realizations of the quantities of interest over the whole time interval (whole space) from one-point statistical characteristics of the random processes and fields; it was conceived in discussions with experimenters who mainly deal with individual realizations.

Many investigators still give much attention to the traditional approach of analyzing the Lyapunov stability of dynamical systems, based on Lyapunov characteristic indices. As we have shown above, the Lyapunov exponent for a random process can be identified with the typical-realization curve corresponding to a lognormal law for positive, time-dependent characteristics of the solutions for stochastic dynamical systems. This is natural, since both methods proceed in essence from the linearization of the original dynamical system. The only difference lies in the fact that, as the typical-realization curve is calculated, some properties of the random parameters are immediately used, such as their stationarity in time and homogeneity and isotropy in space. Furthermore, the analysis can virtually always be carried out using the approximation of the delta correlation of the fluctuating parameters in time, i.e., using the Fokker – Planck equation, so that cumbersome computations can be avoided and the procedure can be substantially simplified.

For stochastic dynamical systems homogeneous in space, which are described by partial differential equations, a passage can be made to statistically equivalent random processes and, therefore, the typical-realization curve for such a process can be studied. In contrast to Lyapunov's approach, this curve can be used to gain information on the formation of clustered structures in the random field.

If, however, linearization is not applicable or the one-time probability distribution for the solution of the problem is time-independent (for example, as in the case of wave propagation in a turbulent medium in the radiation-intensity range corresponding to 'saturated fluctuations'), the typical-realization curve ceases to be informative, and the complex pattern of the wave-field caustic structure can be accounted for only by studying the statistical topography of the wave field in full.

Acknowledgments
This work was carried out in the framework of the Regular and Chaotic Hydrodynamics European Research Network and was also supported by the Russian Foundation for Basic Research (project Nos 07-05-0006a, 05-05-64745a, and 07-05-92210-NTsNIL-a).

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