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# The properties of isofrequency dependences and the laws of geometrical optics

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## Contents



**Abstract. The laws of geometrical optics for two-dimensional geometries of anisotropic materials and structures are presented through the analysis of the geometrical and mathematical properties of various isofrequency dependences which are also called sections of the wavevector surface. Relations are analyzed between certain peculiarities of these dependences, such as the presence of asymptotes, points of inflection, central or axial symmetry, single- or multiple valuedness, and the existence of certain phenomena, such as nonreciprocal propagation, unidirectional propagation, the emergence of two (or several) reflected or refracted beams, the absence of reflection, and the irreversibility of reflection or refraction. It is shown that simple rules based on seeking the extrema of the isofrequency dependences enable one to find out, for each given geometry, which angles of incidence correspond to positive reflection or refraction of the wave, and which to negative ones.**

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## 1. Introduction

The laws of geometrical optics were formulated many centuries ago by Euclid, V Snell, R Descartes, and P Fermat and are now known worldwide. They constitute a collection of several simple rules, confirmed in experiment, which describe propagation, reflection, and refraction of light (electromagnetic waves) in an isotropic medium. Although postulates of geometrical optics matter only first-order approximations and agree with the observed phenomena only in cases where interference and diffraction effects are negligible (i.e., where the wavelength is infinitely small), these postulates are always used and will be used for determining the imaging conditions in designing various optical and radiotechnical systems and devices.

Not long ago, V G Veselago introduced the laws of geometrical optics for isotropic materials with negative electrical permittivity $\varepsilon$ and negative magnetic permeability $\mu$ [1]. Although an isotropic medium with negative $\varepsilon$ and $\mu$ is a hypothetic one, the discovery of these laws, which differ from the analogous laws in usual isotropic media, has led to a considerable increase in research on realizing and employing these unusual laws in practice (see review papers [2, 3]).

The laws of propagation, reflection, and refraction of electromagnetic waves in anisotropic materials have been under study since rather long ago, in connection with research on well-known uniaxial crystals (see monographs [4 – 6] and the references cited therein). The wavevector surface of a uniaxial optical crystal consists of two surfaces, a sphere and an ellipsoid, which describe, respectively, ordinary and extraordinary waves. Sections of these surfaces, isofrequency dependences,[1] are circles and ellipses. There are more recent studies of wave processes in other anisotropic media: acoustic crystals [7–9], plasmas [10, 11], and magnetically ordered materials [12–14]. The isofrequency dependences in such media may take both the form of closed curves, with shapes more complicated than an ellipse (see, for instance, Refs [8, 15, 16]) and the form of open curves, resembling a hyperbola (see, for instance, Refs [11, Figs 104 and 138] and [12, Figs 4.13 and 5.6]). Propagation, reflection, and refraction laws for waves in the above-listed anisotropic media have proved not only to essentially differ from the corresponding laws in isotropic media (including those with negative $\varepsilon$ and $\mu$) but also to considerably extend and complete our knowledge on the behavior of waves in familiar optical crystals. At the same time, by analyzing the results of these studies, one can find some general laws or rules that, similarly to the postulates of geometrical optics for isotropic materials, provide the first-order approximation description of propagation, reflection, and refraction of electromagnetic and acoustic waves in anisotropic materials.

Differences in the behavior of waves in isotropic and anisotropic media originate for two main reasons. First, *in an anisotropic medium the wave itself*, as a rule, is 'unusual' and *differs principally from a wave in an isotropic medium by the fact that its wavevector* **k** *and group* (*ray*) *velocity vector* **V** *are noncollinear*. Second, the isofrequency dependences determining the laws of wave reflection and refraction in anisotropic materials can have rather complicated forms, which may lead to new physical effects. In fact, application of the momentum conservation law to noncollinear waves *has certain peculiarities*: momentum conservation 'is responsible' for the continuity of the tangential component of the **k** vector at the boundaries but the energy propagation direction is 'determined' by the **V** vector, connected with **k** through the dispersion relation. Hence, depending on the properties of isofrequency characteristics, applying the momentum conservation law to noncollinear waves may lead to such phenomena as negative reflection or refraction (when the incident, reflected, and refracted rays are found on the same side of the normal to the boundary), the emergence of two (or even several) reflected or refracted rays, the absence of reflection as such, unidirectional and nonreciprocal propagation, and some others.

Notice that by now, the most complete theoretical and experimental investigation of running noncollinear electromagnetic waves has been performed for magnetically ordered media, especially for ferrite films magnetized to saturation [12, 14, 17–25], whose magnetic properties are described by the second-rank tensor of magnetic permeability. Since such studies revealed a variety of physical effects and phenomena, it is mainly these investigations that will be discussed below. Nevertheless, the analysis carried out here is only based on the momentum conservation law, and hence all conclusions will be valid for any noncollinear waves in any anisotropic media. It should be noted that recently there has been much research on wave processes in spatially periodic media, often called photonic crystals, metamaterials, or superlattices. It is known that the isofrequency dependences of such media, in contrast to the isofrequency dependences of continuous anisotropic media, are periodic [26]. Therefore, the results presented in this work can be used for analyzing the isofrequency dependences of spatially periodic media within one period (for instance, in the first Brillouin zone); then the periodicity of the isofrequency dependences can be taken into account.

Below, the laws of geometrical optics for anisotropic media are formulated by analyzing the geometrical and mathematical properties of various isofrequency dependences. To give a more complete overview of the related subjects and to make the paper clearer for a broad audience, the narrative is made simple and concise, with a minimum of formulas and equations. At the same time, the treatment is sufficiently strict but does not go beyond the framework of geometrical optics.

## 2. Magnetostatic waves — electromagnetic waves propagating in magnetic films

First of all, for the paper not to look like a theoretical abstraction, let us briefly describe those waves whose isofrequency dependences will be analyzed for the most part.

It is known that thin layers of magnetically ordered materials, like ferrites, promote efficient excitation and low-loss propagation of electromagnetic waves whose wavevectors $k$ at microwave frequencies are on the order of $10-10^4$ cm$^{-1}$ (corresponding to wavelengths of 5 mm – 50 μm), which is much higher than in a vacuum, where $k \gg k_0 \equiv \omega/c \sim 1$ cm$^{-1}$ [12, 14]. Since the exchange energy variation at distances comparable to the wavelength is not noticeable for such waves, and their phase velocity is much less than the speed of light, their characteristics can be calculated within the framework of the magnetostatic approximation. In other words, one can neglect, on the one hand, the influence of the exchange field and, on the other hand, the terms $\sim \partial/\partial t$ in the Maxwell equations. Due of the magnetostatic approximation used for their description, these waves were termed in the literature magnetostatic waves (MSWs) [17]. Sometimes, due to neglecting the exchange energy, they are also called dipole spin waves, although this term is used much more seldom.

MSWs were considered in a large number of works (see Refs [12, 14] and the references cited therein), but only a few works among them deal with noncollinear MSWs, whose propagation in ferrite films gives rise to many unusual physical effects.

Most examples given in this paper describe the propagation, reflection, or refraction of MSWs in ferrite films, plates, or structures based on them. Note that magnetic films, being two-dimensional objects of investigations, have certain advantages over bulk (three-dimensional) anisotropic crystals. Indeed, on the flat surface of a film one can easily place both the excitation and the detection transducers; then, due to possibility of freely changing the positions and orientations of these transducers, one can investigate the propagation direction of the wave and the orientation of its phase front. (In a three-dimensional crystal, such a measurement is impossible.) Due to this opportunity, experimental studies carried out for magnetic films are much more advanced than the ones performed with other anisotropic materials. In addition, in the context of this paper, focused on the analysis of isofrequency dependences, MSWs are attractive not only because they provide a variety of such dependences (Figs 1 and 2), but also because, in their case, different isofrequency

[1] Since the terms 'wavevector surface' and 'isofrequency dependence' are absent in encyclopedias and reference books, their brief explanation is given in Section 4.

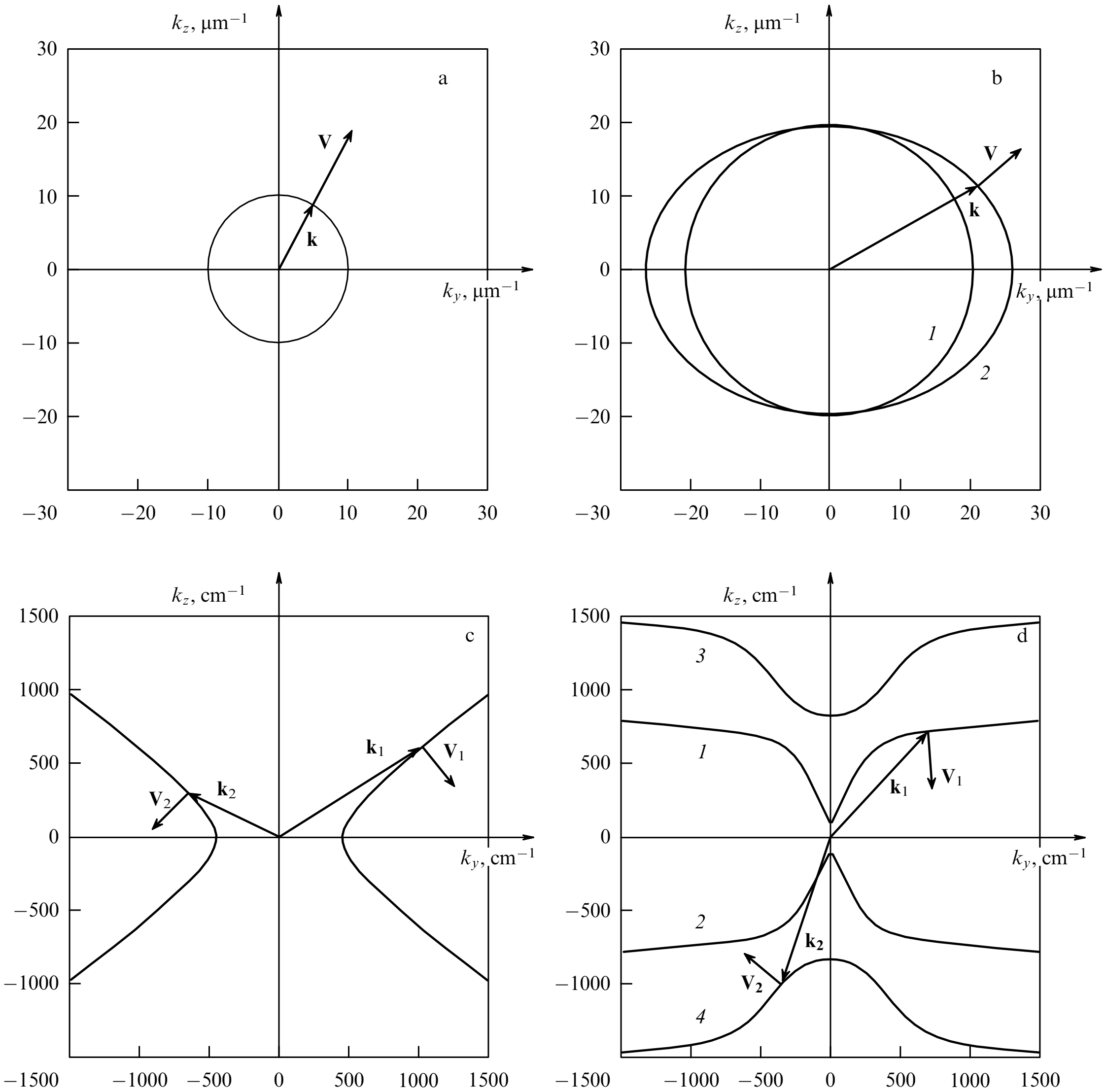


**Figure 1.** Centrally symmetric isofrequency dependences: (a) for red light in a vacuum ($\lambda = 0.6328$ μm, $f = 4.7375 \times 10^{14}$ Hz); (b) for red light in calomel (*1*, for an ordinary wave, $n_o = 1.96$; *2*, for an extraordinary wave, $n_e = 2.62$); (c, d) for a surface MSW with $f = 2900$ MHz and for a backward bulk MSW with $f = 2150$ MHz, respectively, in a free ferrite film with a thickness $d = 10$ μm and $4\pi M_0 = 1750$ G, in an external magnetic field $H_0 = 300$ Oe (*1*, *2* and *3*, *4* are the isofrequency curves for the first and second modes, respectively). The figure shows several arbitrary wavevectors **k** and the corresponding group velocity vectors **V**.

dependences correspond to waves of the same polarization. This considerably simplifies an analysis using MSWs as an example.

Let us introduce a Cartesian frame of reference with the $yz$ plane parallel to the surface of the film (or the structure); then the $yz$ plane will correspond to the plane of wavenumbers $k_y k_z$. A uniform magnetic field $\mathbf{H}_0$, which magnetizes the film (structure) to saturation, is applied along the $z$-axis. Recall that a ferrite material is described by the magnetic permeability tensor $\overleftrightarrow{\mu}$ of the form [14]

$$\overleftrightarrow{\mu} = \begin{vmatrix} \mu & \mathrm{i}\nu & 0 \\ -\mathrm{i}\nu & \mu & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{1}$$

where

$$\mu = 1 + \frac{\omega_M \omega_H}{\omega_H^2 - \omega^2}, \tag{2}$$

$$\nu = \frac{\omega_M \omega}{\omega_H^2 - \omega^2}, \tag{3}$$

$\omega_H = \gamma H_0$, $\omega_M = 4\pi\gamma M_0$, $\omega = 2\pi f$, $\gamma$ is the gyromagnetic constant, $4\pi M_0$ is the saturation magnetization of the ferrite, and $f$ is the electromagnetic oscillation frequency.

Various studies have shown that in a free ferrite film with tangent magnetization, two types of MSWs can propagate: the surface MSW, and the backward bulk MSW, which differ not only by the propagation type (forward and backward) but also by the energy distribution across the film (exponential and trigonometric, respectively). Dispersion relations for these two types of waves can be found in Refs [12, 17].

It should be noted that in a ferrite plate, in addition to the MSW-type solution, there is another one, similar to a TH-wave (see, for instance, Ref. [27]). Since a TH-wave interacts with matter through the $\mu_{zz} = 1$ component of the magnetic permeability tensor, it propagates in a ferrite plate the same way as in a usual insulator with no magnetic properties, and its reflection and refraction laws are trivial.

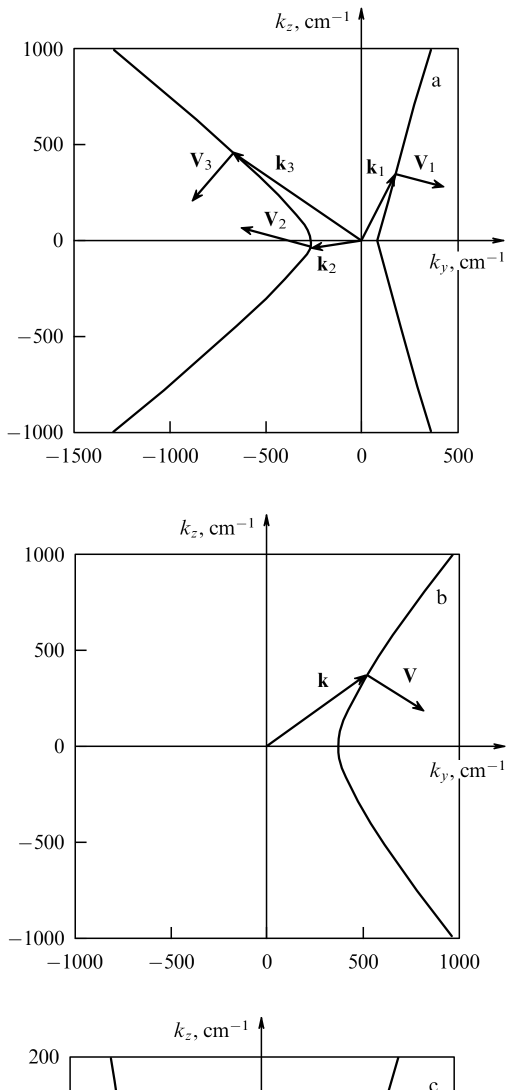


**Figure 2.** Noncentrosymmetric isofrequency dependences. They were calculated for a surface MSW in an FDM structure for various frequencies $f$ and distances $l$ between the ferrite film and the metal plane: (a) $f = 2800$ MHz, $l = 0$; (b) $f = 4200$ MHz, $l = 0$, and (c) $f = 3236$ MHz, $l = 17$ μm. The parameters $d$, $4\pi M_0$, and $H_0$ are the same as in Fig. 1. The figure shows several arbitrary wavevectors **k** and the corresponding group velocity vectors **V**. In the bottom figure, the scale along the $k_z$-axis is 10 times larger than along the $k_y$-axis, in order to show the oval more clearly. All **V** vectors of the oval are directed towards its center.

## 3. Preliminary remarks and definitions

In what follows, as in the geometrical optics of isotropic media, we will use the notion of a ray defined as the line along which wave energy is transferred. It should be emphasized that since waves propagating in anisotropic materials are, in the general case, noncollinear, the ray will be parallel to the group velocity vector **V**, while the wavevector **k** will be tilted from the ray direction by some angle $\chi$.

Let us generalize the known definitions of the forward and backward waves to the case of anisotropic media and define a forward wave as one for which the scalar product $\mathbf{Vk} > 0$, and a backward wave as one for which $\mathbf{Vk} < 0$. At $\mathbf{Vk} = 0$, which is possible only in theory, the wave does not propagate and has $|\mathbf{V}| = V = 0$. These definitions are valid for the classification of waves in any medium[2] with small losses.

In addition, we will define the axis of collinear propagation (or simply the collinear axis) for a certain wave as the direction in which the vectors **V** and **k** are collinear.

## 4. Isofrequency surfaces and dependences — characteristics determining the propagation, reflection, and refraction of waves

Since the terms 'isofrequency surface' and 'isofrequency dependence' are not commonly accepted, let us briefly explain their meaning.

Because the momentum conservation law is satisfied at the reflection and refraction of waves (this law requires that the tangential component of the wavevector **k** be preserved at the boundaries), it is convenient to analyze the reflection and refraction of a certain wave in the space of wavenumbers $\Sigma_k = \{0; k_x, k_y, k_z\}$ by introducing the *isofrequency surface* of the wave. This surface is directly described by the dispersion relation of the anisotropic medium at a fixed frequency $\omega$. Then, the group velocity **V** of the wave in an anisotropic medium can be found as the frequency gradient in the space of wavevectors [12, 29]:

$$\mathbf{V} = \frac{\partial \omega}{\partial \mathbf{k}} = \mathrm{grad}_{\mathbf{k}}\, \omega\,, \quad \text{or} \quad \mathbf{V} = \mathbf{x}_0\, \frac{\partial \omega}{\partial k_x} + \mathbf{y}_0\, \frac{\partial \omega}{\partial k_y} + \mathbf{z}_0\, \frac{\partial \omega}{\partial k_z}\,. \tag{4}$$

Therefore, the group velocity vector **V** is always directed along the normal to the isofrequency surface.

It is known that for an electromagnetic wave propagating in an isotropic medium with a fixed frequency $\omega$, the isofrequency surface (or the wavevector surface) represents a sphere. In this case, the wavevector **k** and the group velocity vector **V**, which determines the ray direction, are always parallel. However, the isofrequency surface is not spherical for anisotropic media and the vectors **V** and **k** are not parallel.

By analogy with the three-dimensional case, the propagation, reflection, and refraction of a wave in a two-dimensional structure can be described in terms of the *isofrequency dependence* which can be considered as the section of the dispersion surface $\omega(k_y, k_z)$ in the space of variables[3] $\{\omega, k_y, k_z\}$ by the plane corresponding to constant frequency. For a wave with a known character of propagation

[2] Note that these definitions are not yet included in reference books or encyclopedias. For instance, in Ref. [28, p. 384] it is mentioned that "in an anisotropic medium, the notions of forward and backward waves are strictly applicable only to certain fixed directions related to the principal axes of permittivity or deformation tensors."

[3] In the two-dimensional geometries considered below, the $x$ coordinate is orthogonal to the plane of the structure, and the wave propagates in the $yz$ plane.

(forward or backward), the isofrequency dependence provides a one-to-one correspondence between the wavevector **k** run out into a given point of this dependence and the direction of the group velocity vector **V** (orthogonal to the isofrequency dependence at that point). If it is not known whether the wave is forward or backward, one should find this out by calculating the **V** vector direction according to formula (4).

Isofrequency dependences are also convenient for the analysis of wave reflection and refraction in three-dimensional anisotropic media. For each medium, these dependences should be considered as sections of the isofrequency surfaces by the plane of incidence, since the wavevectors of the incident, reflected, and refracted waves lie in the plane of incidence.

Apparently, the isofrequency dependences for isotropic media are circles (Fig. 1a). In a uniaxial optical crystal, the isofrequency surface consists of two surfaces, a sphere and an ellipsoid, the first one describing an ordinary wave for which the **k** and **V** vectors are always parallel, as for a wave in an isotropic medium, and the second one corresponding to an extraordinary wave, for which **k** and **V** are, in general, not parallel. Correspondingly, the isofrequency dependence of a uniaxial crystal consists of a circle and an ellipse (Fig. 1b). In a biaxial optical crystal, where the isofrequency surface is a self-intersecting fourth-order surface, the isofrequency dependence consists of a circle and an oval, which intersect in some sections of the isofrequency surface [4, p. 747, Fig. 14.8]. The isofrequency dependence of an acoustic wave in tetragonal crystals (paratellurite, rutile, barium titanate, mercury bromide, etc.) looks like a truncated epicycloid or hypocycloid [4] [8, 15, 16]. For a surface MSW in a free ferrite film tangentially magnetized to saturation,[5] the isofrequency dependence resembles a hyperbola [6] (Fig. 1c).

The analysis of isofrequency dependences is most efficient in the studies of two-dimensional geometries, especially in solving problems where only orientations of the **k** and **V** vectors of the incident, reflected, and refracted waves are of interest, and not the amplitudes of the reflected and refracted rays. The isofrequency dependence has a simple physical meaning for the analysis of two-dimensional geometries: since this dependence describes all possible waves with the given frequency $\omega$ and various wavevectors, the directions of the reflected and the refracted rays can be determined by simply finding the *points* in isofrequency dependences of media that satisfy the momentum conservation law at a known orientation of the boundary and a given angle of incidence of the wave. From this it follows that *reflection and refraction of waves in any particular material will be fully determined by the following geometrical characteristics of the isofrequency dependences: the presence and number of asymptotes, axes of symmetry and points of inflection, the existence of central symmetry and single-valuedness or multivaluedness of the dependence*.

[4] The general form and description of epicycloids and hypocycloids can be found in Ref. [30, pp. 127, 128].

[5] Note that an MSW-type solution also exists in a normally magnetized ferrite plate, but in this case the isofrequency dependences of MSWs are circles and, hence, such MSWs obey the laws of geometrical optics for isotropic media.

[6] The isofrequency dependence of a surface MSW consists of two curves which, strictly speaking, are not hyperbolas but represent more complicated curves without special names. This is why one can only say they resemble hyperbolas.

For this reason, isofrequency dependences were most extensively used for studying wave propagation in two-dimensional structures (see, for instance, Ref. [31, p. 209, Fig. 92a] and Ref. [32]). They were first applied to the analysis of wave processes in magnetically ordered materials, probably, in Refs [33, 34], and to the analysis of the MSW propagation in magnetic films in Refs [35, 36].

Notice that although in this paper we use the term 'isofrequency dependence' (surface, curve), in other works the same dependence can be termed 'isofrequency' [22, 26, 32, 37], 'section of the isoenergy surface [34], or 'equifrequency line' [12]. In works concerned with the propagation of light in anisotropic optical crystals, electromagnetic waves in plasma, and acoustic waves in crystals, one often uses the term 'wavevector surface' [7–9, 11, 29, 38]. In the first and second cases, the wavevector surface is usually plotted in the coordinate axes of wavenumbers normalized to the wavenumber in a vacuum, $k_x/k_0$, $k_y/k_0$, and $k_z/k_0$ (i.e., the axes correspond to the refractive indices of the electromagnetic wave along the respective Cartesian coordinate axes). For acoustic waves, the wavevector surface is also called the acoustic slowness surface and plotted in the coordinate axes $1/V_x^{\rm ph}$, $1/V_y^{\rm ph}$, and $1/V_z^{\rm ph}$. Since the dispersion is linear in acoustic crystals, the phase velocity $\mathbf{V}^{\rm ph}$ of the wave in each direction is equal (in absolute value) for all frequencies, and the surface plotted versus $k_x/\omega$, $k_y/\omega$, and $k_z/\omega$ is, in fact, the characteristic of a certain anisotropic acoustic crystal. In media with frequency dispersion (for instance, in ferrite films), it is more convenient to use dependences (surfaces) plotted versus wavenumbers. However, regardless of the normalization method, the isofrequency dependence, section of the wavevector surface, section of the slowness surface, and other similar characteristics obviously possess the same physical meaning described above.

## 5. Types of two-dimensional geometries

Since this paper is aimed at describing and comparing the laws of geometrical optics in various anisotropic media and structures, which differ by the geometric characteristics of their isofrequency dependences, our consideration can be restricted to the analysis of two-dimensional geometries.

We will use the term 'two-dimensional geometry' for studying various two-dimensional plane-parallel structures and those particular cases of three-dimensional geometries of anisotropic media, where the plane of wave incidence coincides in each medium with one of the symmetry planes of the isofrequency surface of the medium. In these cases, all wavevectors **k** and all group velocity vectors **V** corresponding to the isofrequency dependence lie in one plane, the plane of incidence (as in the case of two-dimensional structures). Therefore, for a two-dimensional geometry all the analysis is performed in the plane of incidence, which considerably simplifies the study of wave reflection and refraction and makes it more explicit.

Among the two-dimensional geometries, there is a special class of two-dimensional plane-parallel structures which cannot be considered as a particular case of three-dimensional geometries. Indeed, if a two-dimensional structure is not symmetric (for instance, a ferrite plate with one surface covered by metal), its isofrequency dependence usually possesses no center of symmetry (see Fig. 2), which leads to unusual laws of wave propagation, reflection, and refraction.

## 6. Laws of wave propagation

### 6.1 Rectilinear propagation of a wave

*In a homogeneous anisotropic medium, similarly to an isotropic medium, a wave propagates from one point to another rectilinearly, i.e., a ray represents a straight line.* For an isotropic medium with a known refractive index, this formulation fully determines the parameters of the wave and its propagation direction. In an anisotropic medium, this law is not sufficient, since waves propagating in *various* directions have *different* parameters. Moreover, in an anisotropic medium there may be directions along which waves cannot propagate or the parameters of the wave are substantially different for opposite directions of propagation. For the description of these phenomena, new laws or rules should be formulated. Evidently, for discussing these laws in an anisotropic medium it is convenient to introduce a frame of reference connected with one of the symmetry axes of the isofrequency dependence.[7]

### 6.2 Rule for the wavevector orientation

*For different traveling directions $\psi$ of a ray, the tilt angle of the wavevector* **k** *with respect to the line of the ray is given by the expression*

$$\chi = \varphi - \psi\,, \tag{5}$$

where $\varphi$ and $\psi$ are, respectively, the tilt angles of the wavevector **k** and the group velocity vector **V** with respect to the chosen reference axis.[8] Apparently, if the values of $\chi$ fall within the interval $-90° < \chi < 90°$, the wave is forward, and if $90° < |\chi| \leqslant 180°$, it is backward.

The $\psi(\varphi)$ and $\chi(\psi)$ dependences (Fig. 3) are very important for the analysis of wave propagation in anisotropic media. It is the $\psi(\varphi)$ dependence that is usually measured in experiment. In the study of MSWs, for instance, the wave is excited by fixing the orientation of the input transducer with respect to the chosen axis, i.e., by setting the $\varphi$ angle, and then the ray propagation direction $\psi$ is measured. From the $\psi(\varphi)$ dependence, using Eqn (5), one can easily obtain the $\chi(\psi)$ dependence which gives the tilt of the **k** vector with respect to the line of the ray at different directions of the ray. As one can see from Fig. 3b, in media with elliptical isofrequency dependences $\chi$ never exceeds some maximal value $\chi_{\max}$ (for calomel, described by curve *2* in Fig. 3b, $\chi_{\max} \approx 16.4°$), whereas in media with hyperbola-like isofrequency dependences, $\chi$ can take all possible values: for a forward wave, it varies within $0 \leqslant |\chi| < 90°$ (Fig. 3b, curve *3*), and for a backward wave within $90° < |\chi| \leqslant 180°$ (Fig. 3b, curve *4*).

It should be emphasized that often several rays with different wavevectors can propagate in the certain direction in an anisotropic medium. For instance, as one can see from Fig. 3b, in calomel the propagation direction with $\psi = 30°$ corresponds to the ray of an ordinary wave with $\chi = 0$ (described by straight line *1*) and the ray of the extraordinary wave with $\chi = 15.9°$ (described by curve *2*), these two rays also differing by their polarizations. In a ferrite film, three rays of the first mode of the backward bulk MSW can propagate in the direction with $\psi = 80°$; the tilts of their wavevectors with respect to the line of the ray are $\chi = -97°$, $\chi = -131°$, and $\chi = -169°$ (Fig. 3b, curve *4*).

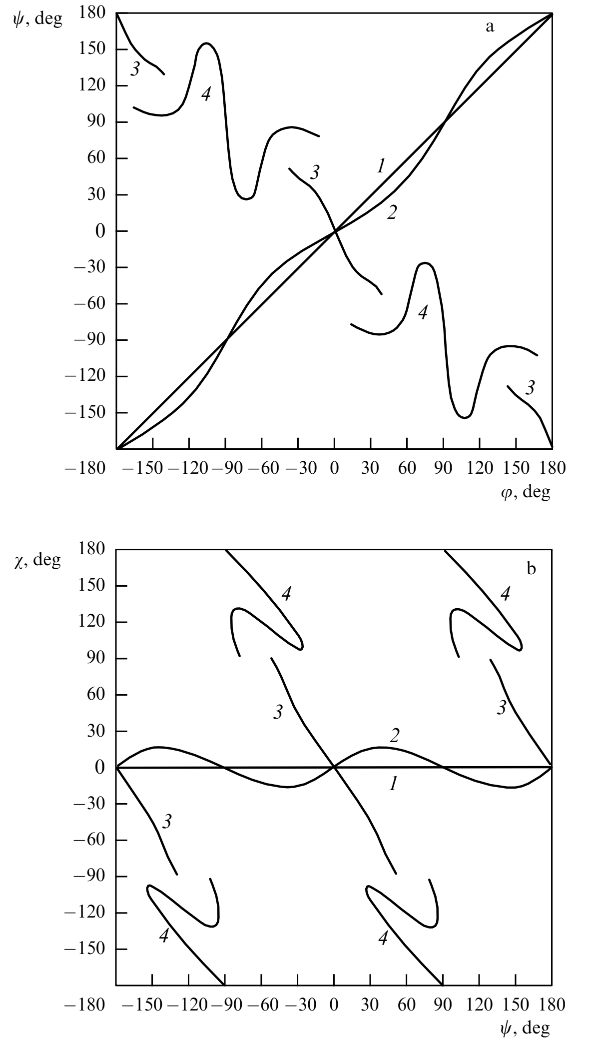


**Figure 3.** Direction $\psi(\varphi)$ of the group velocity vector **V** as a function of the angle between the wavevector **k** and the $y$-axis (a), and the angle $\chi(\psi)$ between the wavevector **k** and the line of the ray as a function of the ray direction $\psi$ (b) for isofrequency dependences plotted in Fig. 1: *1* and *2*, for ordinary and extraordinary waves of red light in calomel; *3* and *4*, for a surface MSW and the first mode of a backward bulk MSW in a ferrite film. The angles $\varphi$ and $\psi$ in Fig. 1 for a calomel crystal are reckoned from the optical $k_z$-axis, while for a ferrite film, from the $k_y$-axis, which is orthogonal to the magnetic field $\mathbf{H}_0$.

[7] In all anisotropic media and structures, the dispersion relation is derived in a reference frame connected with the symmetry axes of the isofrequency dependence (surface). The angle $\varphi$, giving the orientation of the wavevector **k** with respect to the chosen axis of symmetry, is contained, as a rule, directly in the dispersion relation written out in polar coordinates (see, for instance, equations for MSWs in a free film [12, p. 76, Eqn (4.14)] and in various structures [39]). It is not always true that every symmetry axis of the isofrequency dependence corresponds to the axis of collinear propagation of the wave (as, for instance, the $k_z$-axis in Fig. 1c or the $k_y$-axis in Fig. 1d). Below, propagation of MSWs in all structures is described with the angles reckoned from the $k_y$-axis, while the propagation of light in calomel crystals is described with the angles reckoned from the optical $k_z$-axis.

[8] Hereinafter, the angles $\varphi$ and $\psi$ will be reckoned from the chosen axis of symmetry in such a way that their values are within the interval from $-180°$ to $+180°$, and the counterclockwise direction will be assumed as positive. However, one should take into account that sometimes simple arithmetic operations may take the calculated angles outside of the chosen interval. In such cases, one should add or subtract 360°. For instance, calculating the angle $\chi$ between the vectors $\mathbf{V}_2$ and $\mathbf{k}_2$ in Fig. 1c, according to Eqn (5), we obtain $\chi_2 = \varphi_2 - \psi_2 = 154° - (-135°) = 289° = 289° - 360° = -71°$. Evidently, the calculated angles $\chi$ should be reckoned in the direction from the **V** vector to the **k** vector (as shown below in Fig. 4).

### 6.3 Possible propagation directions of a wave

As one can see from Figs 1 and 2, the isofrequency dependence of an anisotropic medium or structure can consist of one, two, three, or even infinitely many curves (if more than one mode is excited in the structure). Sometimes this dependence represents a set of similar curves (Fig. 1d), and sometimes it includes curves of different types (Fig. 2c). Since curves of various types possess different geometrical properties, one should analyze the properties of all the curves in order to find the possible propagation directions for a wave in the medium. However, sometimes the existence or absence of curves of some particular type may determine the properties of the medium in general.

**Rule 1.** *If at least one curve of the isofrequency dependence is closed, the wave can propagate in the plane of the isofrequency dependence in any direction, with possible exceptions for some particular directions.*

Examples of closed curves are a circle, an ellipse, various ovals and other, more complicated curves (for instance, those describing the isofrequency dependences of acoustic waves in the crystals of paratellurite, rutile, etc. [8, 15, 16]). It should be mentioned that there are two possible cases of the position of a closed isofrequency curve: the one where the 0 point of the $k_y k_z$ plane is outside the closed curve, and the other where it is inside the curve. In the first case (see, for instance, the oval in Fig. 2c), the **V** vector can have any orientation but the **k** vector can only be within some angular sector bounded by the tangential lines to the given isofrequency curve; moreover, the wave cannot propagate in the directions corresponding to the tangency points, where the **V** and **k** vectors are orthogonal.[9] In the second case, both the group velocity vector **V** and the wavevector **k** can have any orientation (Fig. 1a, b), and the wave can propagate in any direction along the plane.[10]

**Rule 2.** *If all curves of the isofrequency dependence are hyperbola-like and have asymptotes, then, as a rule, there is an angular sector in the plane of the isofrequency dependence where a wave cannot propagate.*

Let us consider the case (Fig. 1c) where the isofrequency dependence consists of two similar hyperbola-like curves with no inflection points (i.e., points where the second derivative is zero). If the asymptotes of the curves are directed at the angles $\varphi_1$, $\varphi_2$, $\varphi_3$, and $\varphi_4$ (in Fig. 1c: $\varphi_1 = 34°$, $\varphi_2 = 146°$, $\varphi_3 = -146°$, and $\varphi_4 = -34°$), then the possible directions of the wavevector **k** will lie within the angular interval $\varphi_4 < \varphi < \varphi_1$ and within the interval $\varphi_2 < |\varphi| \leqslant 180°$ which is symmetric to the first one with respect to the $k_z$-axis. Possible directions $\psi$ of the ray propagation will lie within the angular interval $\varphi_1 - 90° < \psi < \varphi_4 + 90°$ and the symmetric interval. The angles

$$\begin{gathered}\psi_1 = \varphi_1 - 90°\,, \qquad \psi_2 = \varphi_2 + 90°\,, \\ \psi_3 = \varphi_3 - 90°\,, \quad \text{and} \quad \psi_4 = \varphi_4 + 90°\end{gathered} \tag{6}$$

are usually called the *cutoff angles*.[11] They correspond to the limiting directions of the ray, for which the angle between the **k** and **V** vectors is $|\chi| \to 90°$ [formulas (6) can also be obtained from Eqn (5) by assuming $\chi = \pm 90°$]. Correspondingly, within the angular sectors $\psi_2 < \psi < \psi_1$ and $\psi_4 < \psi < \psi_3$ there can be no transfer of the wave energy. For instance, in Fig. 1c these sectors are $-124° < \psi < -56°$ and $56° < \psi < 124°$.

When the isofrequency dependence has inflection points (curves in Fig. 1d and the right-hand curve in Fig. 2a), possible directions of the ray propagation will be restricted by the values of $\psi$ at the inflection points but not by the corresponding cutoff angles $\psi_1 - \psi_4$. For instance, although for curves *1* and *2* in Fig. 1d the asymptotes are directed at the angles $\varphi_1 = 14°$, $\varphi_2 = 166°$, $\varphi_3 = -166°$, and $\varphi_4 = -14°$, and the corresponding cutoff angles are $\psi_1 = -76°$, $\psi_2 = -104°$, $\psi_3 = 104°$, and $\psi_4 = 76°$, the wave energy can be transferred within a broader angular sector $-154° < \psi < -26°$ and $26° < \psi < 154°$ (and not for $-104° < \psi < -76°$ and $76° < \psi < 104°$) because the $\psi$ values at the inflection points differ considerably from the cutoff angles (compare the maximal and end $\psi$ values for curves *4* in Fig. 3a). Thus, the existence of inflection points always leads to a considerable widening of the angular sector of allowed ray directions and to a considerable narrowing of the angular sector of forbidden ray propagation directions. Sometimes the angular sector where ray is impossible can completely disappear, which would be the case for a complex hyperbola-like curve. For instance, this situation would be realized if the oval in Fig. 2c were tangent to the neighboring curve, so that there would be a narrow 'exit' from the inside of the oval.

I should be noted that if the isofrequency dependence looks like conjugate hyperbolas,[12] then there will be only selected directions in such a medium [specified by the cutoff angles (6)] in which the wave cannot propagate.

### 6.4 Nonreciprocal and unidirectional propagation of a wave

**Rule 1** (for the nonreciprocal propagation of a wave). *If the isofrequency dependence of the medium is not represented by the centrally symmetric figure, then nonreciprocal propagation of a wave takes place, which means that the parameters of waves propagating in two opposite directions differ.*

Indeed, if the isofrequency dependence is not centrally symmetric and one can find in it two points corresponding to oppositely directed group velocity vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ (Fig. 2a), we see that the two rays characterized by these vectors will have different wavelengths, values of the $\chi$ angle, and group velocities, since $k_1 \neq k_2$, $\chi_1 \neq \chi_2$, and $V_1 \neq V_2$ (Fig. 4).

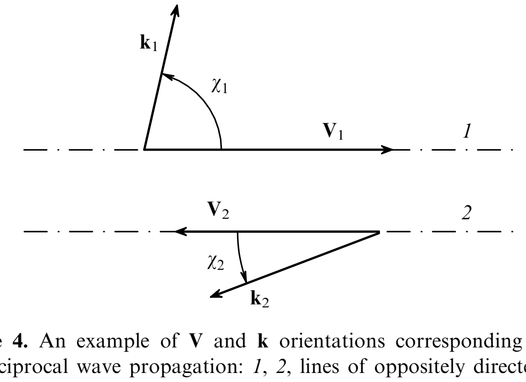


**Figure 4.** An example of **V** and **k** orientations corresponding to the nonreciprocal wave propagation: *1*, *2*, lines of oppositely directed rays (the rays shown here are given by the vectors $\mathbf{V}_1$, $\mathbf{k}_1$ and $\mathbf{V}_2$, $\mathbf{k}_2$ in Fig. 2a).

[9] It should be noted that if the isofrequency dependence includes other curves, waves with other wavevectors can propagate in these directions.

[10] If the isofrequency curve is rather complicated and has points where the **V** and **k** vectors are orthogonal to each other, then the wave cannot propagate in the directions of the **V** vectors given by these points.

[11] The angles $\psi_1 - \psi_4$ are often called the cutoff angles of the group velocity **V**, and the angles $\varphi_1 - \varphi_4$ are called the cutoff angles of the wavevector **k**.

[12] For conjugate hyperbolas, see Ref. [30, p. 203].

Moreover, it may happen that for a given group velocity vector $\mathbf{V}$ (for instance, for the $\mathbf{V}_3$ vector in Fig. 2a) there is no group velocity vector with the opposite direction! Conditions under which such an unusual situation arises are discussed below.

If the isofrequency dependence is centrally symmetric (Fig. 1a–d), the group velocity vectors corresponding to any two wavevectors with opposite directions are also oppositely directed and equal in the absolute value. Hence, any two rays propagating in opposite directions have the same parameters (wavelength $\lambda$, angle $\chi$, group velocity $V$). For a multimode wave, the isofrequency dependence consists of an infinite number of curves (Fig. 1d), and an infinite number of rays with different wavevectors can propagate in both forward and backward directions; still, each pair of centrally symmetric curves will correspond to oppositely directed rays with the same parameters.

However, it should be noted that in some two-dimensional structures, oppositely directed rays with the same parameters $\lambda$, $\chi$, $V$ may still differ. For instance, calculations and experiments carried out for a surface MSW in a free ferrite film (the isofrequency dependence is given in Fig. 1c) have shown [12, 17] that, for two rays propagating in opposite directions, the wave energy is localized near the opposite film surfaces. Therefore, when a surface MSW is excited on one side of the film surface, a ray localized near this surface is excited much more efficiently than a ray localized near the opposite surface and propagating in the opposite direction. This phenomenon is also usually called the nonreciprocal propagation of a wave. Later on, it was discovered that oppositely directed rays of a noncollinear backward volume MSW in a free film also have nonsymmetric energy distribution throughout the film thickness and exhibit nonreciprocal wave propagation [25]. Thus, in some two-dimensional structures,[13] oppositely directed rays with the same parameters $\lambda$, $\chi$, $V$ may differ by the wave energy distribution throughout the thickness of the structure.

**Rule 2** (for the unidirectional propagation of the wave). *If the isofrequency dependence of the medium consists of a single hyperbola-like curve, has no inflection points, and is fully contained in the positive or negative semiplane of the wavenumber $k_y$ (or $k_z$), then all possible rays given by this dependence are unidirected, i.e., none of them has a counterpropagating ray.*

The above-described isofrequency dependence is typical for an MSW in a ferrite plate with one surface covered by metal, for the frequency range $\omega_{\mathrm{H}} + 0.5\omega_{\mathrm{M}} < \omega < \omega_{\mathrm{H}} + \omega_{\mathrm{M}}$ (Fig. 2b). In addition, a similar isofrequency dependence arises in ferrite plate–'magnetic wall' [40] and 'magnetic wall'–ferrite plate–metal structures [39], providing unidirectional wave propagation within a broader frequency range. Analysis of the properties of metallized ferrite plates showed that as the frequency decreases, the angles between the asymptotes of the isofrequency curve and the $k_y$-axis increase, and the sector of angles $\psi$ where the wave can propagate becomes narrower. However, unidirectional wave propagation in a metallized ferrite plate disappears at frequencies $\omega$ below the value of $\omega_{\mathrm{H}} + 0.5\omega_{\mathrm{M}}$ (another isofrequency curve appears in the left-hand semiplane, as in Fig. 2a), whereas in a 'magnetic wall'–ferrite plate–metal structure, even at lower frequencies there remains only one isofrequency curve which as $\omega \to \sqrt{\omega_{\mathrm{H}}(\omega_{\mathrm{H}} + \omega_{\mathrm{M}})}$ tends to a straight line coinciding with the $k_z$-axis. At frequencies slightly higher than this value, the isofrequency curve of the last structure is almost indistinguishable from a straight line and the wave can propagate only in the vicinity of a single direction — the positive direction of the $y$-axis [39]!

Note that in some structures with nonsymmetric isofrequency dependence, unidirectional wave propagation can take place only for a certain interval of directions of the group velocity vector $\mathbf{V}$, even if the isofrequency dependence is not localized in one semiplane. Indeed, from Fig. 2a one can see that vectors $\mathbf{V}$ corresponding to distant parts of the left-hand isofrequency curve (for instance, vector $\mathbf{V}_3$) do not have their oppositely directed counterparts at any point of the isofrequency dependence.

Note that, from the physical viewpoint, the phenomenon of unidirectional wave propagation should be understood as the impossibility of propagating in opposite directions for waves of a *given* type (in these cases, MSWs), but not for any waves in the medium considered. Indeed, it is only for MSWs (in other words, for waves with the wavenumber $k$ falling within the range $10-10^4$ cm$^{-1}$) that the isofrequency dependences look like hyperbolas shown, for instance, in Fig. 2b. At the same time, in the vicinities of the asymptotes of the MSW isofrequency dependence (where $k$ does not lie in the indicated range since $k \to \infty$), calculation of the dispersion dependences requires, already for $k \sim 10^5$ cm$^{-1}$, taking into account exchange interaction, which means that the isofrequency dependence starts to deviate from its asymptotes and becomes similar to the isofrequency dependence of an exchange spin wave.[14] In addition, one should keep in mind that, besides MSWs, ferrite plates can also support TH-type waves with $k \sim k_0 \sim 1$ cm$^{-1}$ (see Section 2 and Ref. [27]).

## 7. Basic relations for the study of wave reflection and refraction

Reflection and refraction of electromagnetic waves preserve the tangential component of the wave momentum. Let us find out what special features and differences arise when this law is applied to waves propagating in isotropic and anisotropic media.

Figures 5 and 6 outline, both in the wavenumber plane and in the real plane, the geometries of reflection and refraction of an electromagnetic wave in an isotropic and anisotropic media, respectively. Figure 5 illustrates reflection and refraction of red light incident from a vacuum on the surface of glass with the refractive index $n = 1.5$, while Fig. 6 depicts reflection and refraction of a surface MSW propagating from a free ferrite film into a ferrite–dielectric–metal (FDM) structure. An FDM structure [20–23, 41–47], which is a ferrite film with a metal plane placed at a distance $l$ over one of its surfaces, possesses an asymmetric isofrequency dependence [47], which allows one to use this example for considering most features of reflection and refraction of waves in anisotropic media.

[13] No matter whether the isofrequency dependence of the structure is centrally symmetric or not.

[14] If the isofrequency dependence of some other anisotropic medium with negligible exchange interaction looked like a hyperbola, then in the vicinity of the asymptotes, where $k \to \infty$, the continuous-medium approximation would soon be violated (as soon as $k$ became comparable to the intermolecular distance) and the asymptotic behavior of the isofrequency dependence would disappear again. Thus, the isofrequency dependence in a real medium may resemble a hyperbola only conventionally, within a certain (sometimes rather large) interval of $k$ values.

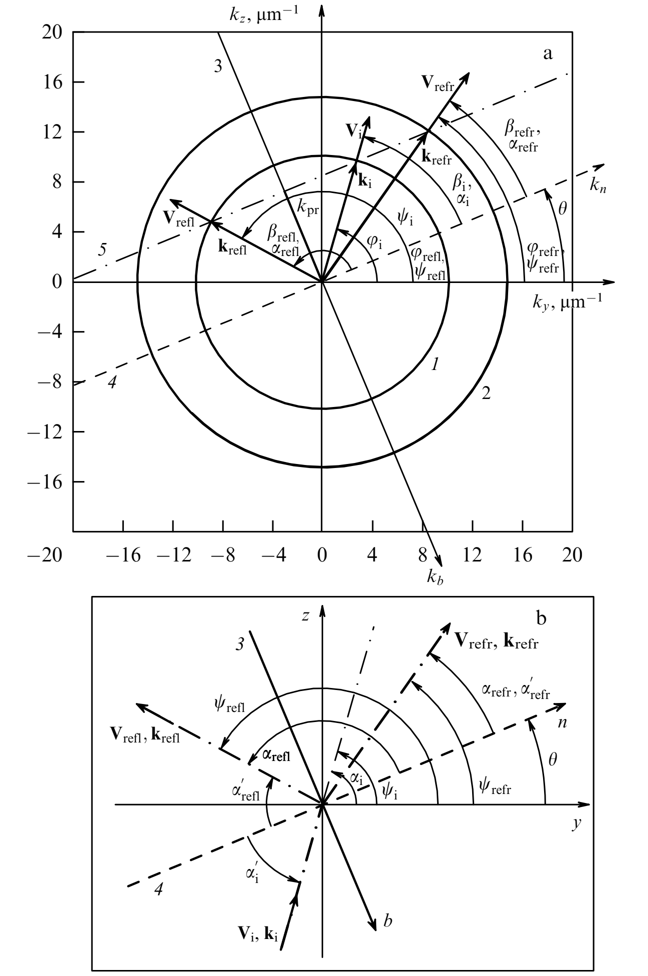


**Figure 5.** Geometry of reflection and refraction of red light with $f = 4.7375 \times 10^{14}$ Hz in the wavenumber plane $k_y k_z$ (a), and in the $yz$ plane of incidence (b) (in explanation for the conservation of the tangential component of momentum): *1*, the isofrequency dependence in a vacuum; *2*, the isofrequency dependence in glass; *3*, the interface; *4*, the normal to the interface; *5*, the projecting line.

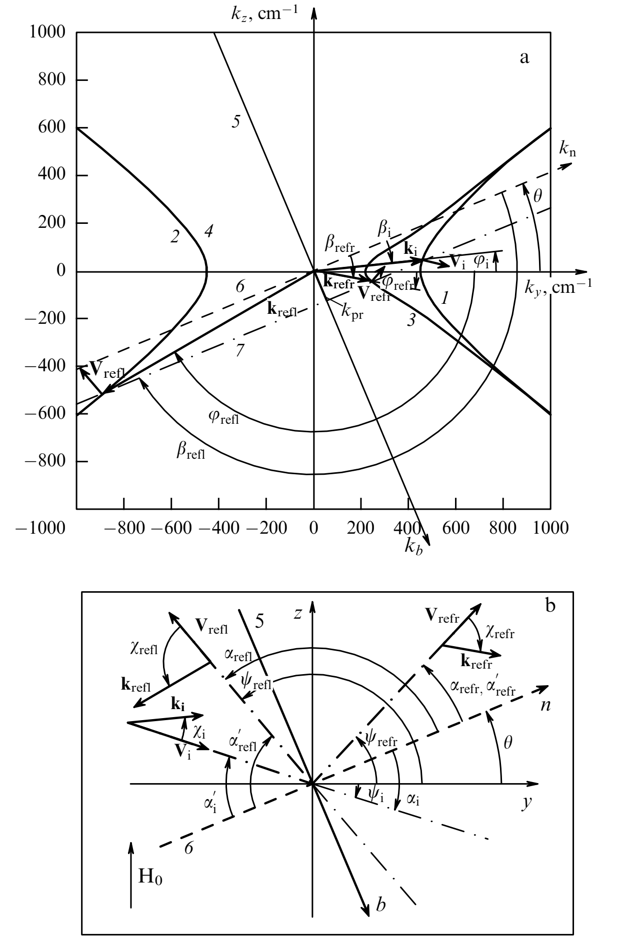


**Figure 6.** Geometry of reflection and refraction of a surface MSW with $f = 2900$ MHz in the wavenumber plane $k_y k_z$ (a), and in the $yz$ plane of a ferrite film (b) (in explanation for the conservation of the tangential component of momentum): *1*, *2*, isofrequency curves of a surface MSW in a free ferrite film; *3*, *4*, isofrequency curves of a surface MSW in an FDM structure with the gap of 30 μm between ferrite and metal layers; *5*, interface; *6*, the normal to the interface; *7*, the projecting line.

For describing waves in anisotropic media, let us employ the traditional frame of reference $\Sigma = \{0; y, z\}$ related to the common axis of symmetry and the attendant frame of reference $\Sigma_k = \{0; k_y, k_z\}$ in the plane of wavenumbers. In order to accentuate a more complete analogy between wave reflection and refraction in isotropic and anisotropic media, we will introduce similar reference frames $\Sigma$ and $\Sigma_k$ in an isotropic medium.

Let the normal $\mathbf{n}$ to the plane interface (surface) make an angle $\theta$ with the $y$-axis in each medium. Let the incident wave be described by a wavevector $\mathbf{k}_\mathrm{i}$ oriented at an angle $\varphi_\mathrm{i}$ to the $k_y$-axis. Drawing a straight line parallel to the normal through the end of the $\mathbf{k}_\mathrm{i}$ vector and finding its points of intersection with the isofrequency dependences of the two media, we can determine the wavevectors $\mathbf{k}_\mathrm{refl}$ and $\mathbf{k}_\mathrm{refr}$ of the reflected and refracted waves and their appropriate angles $\varphi_\mathrm{refl}$ and $\varphi_\mathrm{refr}$ with respect to the $k_y$-axis, predicted by the momentum conservation law (see Figs 5 and 6).

Reflection and refraction processes in isotropic media are usually described in the reference frame $\Sigma' = \{0; b, n\}$ wherein the abscissa axis $b$ is tied up with the interface and the ordinate axis $n$ coincides with the normal to the interface. Let us additionally introduce, both for isotropic and anisotropic media, the reference frame $\Sigma'$ and the appropriate reference frame $\Sigma'_k = \{0; k_b, k_n\}$ in the wavenumber plane, the origin of the $\Sigma'_k$ frame coinciding with the origin of the $\Sigma_k = \{0; k_y, k_z\}$ frame. It is convenient to choose the positive directions for the $n$ and $k_n$ axes so that the unit vectors $\mathbf{n}_0$ and $\mathbf{k}_{n0}$ are directed towards the interface, and positive directions for the $b$ and $k_b$ axes so that, together with the ordinate axes $n$ and $k_n$, the abscissa axes $b$ and $k_b$ form the right-hand reference frame.

The angles in the reference frames $\Sigma$, $\Sigma_k$ and $\Sigma'$, $\Sigma'_k$ will be laid off so that they lie within the intervals from $-180°$ to $180°$, and will be reckoned, respectively, from the $y$ (or $k_y$)-axis and from the normal $n$ (or $k_n$), and the counter-clockwise direction will be assumed to be positive.

As will be seen from what follows, simultaneously using reference frames $\Sigma$, $\Sigma_k$ and $\Sigma'$, $\Sigma'_k$ is convenient and allows one to observe some special features of wave reflection and refraction.

By considering right triangles with hypotenuses formed by the vectors $\mathbf{k}_\mathrm{i}$, $\mathbf{k}_\mathrm{refl}$, and $\mathbf{k}_\mathrm{refr}$, one can easily see that, for both isotropic and anisotropic media, projection $k_\mathrm{pr}$ of the $\mathbf{k}_\mathrm{i}$ vector onto the interface is given by[15]

$$k_{\rm pr} = k_{\rm i} \sin\beta_{\rm i} = k_{\rm i} \sin(\varphi_{\rm i} - \theta)\,, \tag{7}$$

projection $k_{\rm pr}$ of the $\mathbf{k}_{\rm refl}$ vector onto the interface is equal to

$$k_{\rm pr} = k_{\rm refl} \sin\beta'_{\rm refl} = k_{\rm refl} \sin\beta_{\rm refl} = k_{\rm refl} \sin(\varphi_{\rm refl} - \theta)\,, \tag{8}$$

and projection $k_{\rm pr}$ of the $\mathbf{k}_{\rm refr}$ vector onto the interface is

$$k_{\rm pr} = k_{\rm refr} \sin\beta_{\rm refr} = k_{\rm refr} \sin(\varphi_{\rm refr} - \theta)\,. \tag{9}$$

By equating the two expressions (7) and (8) according to the momentum conservation law, we get

$$\frac{k_{\rm i}}{k_{\rm refl}} = \frac{\sin\beta_{\rm refl}}{\sin\beta_{\rm i}} \quad \text{or} \quad \frac{k_{\rm i}}{k_{\rm refl}} = \frac{\sin(\varphi_{\rm refl} - \theta)}{\sin(\varphi_{\rm i} - \theta)}\,, \tag{10}$$

and by equating the two expressions (8) and (9), we arrive at

$$\frac{k_{\rm i}}{k_{\rm refr}} = \frac{\sin\beta_{\rm refr}}{\sin\beta_{\rm i}} \quad \text{or} \quad \frac{k_{\rm i}}{k_{\rm refr}} = \frac{\sin(\varphi_{\rm refr} - \theta)}{\sin(\varphi_{\rm i} - \theta)}\,. \tag{11}$$

In Eqns (7)–(11), $\beta_{\rm i}$, $\beta_{\rm refl}$, and $\beta_{\rm refr}$ denote the angles between the wavevectors $\mathbf{k}_{\rm i}$, $\mathbf{k}_{\rm refl}$, and $\mathbf{k}_{\rm refr}$, respectively, and the unit vector $\mathbf{k}_{n0}$ (or $\mathbf{n}_0$). The angles $\beta_{\rm i}$, $\beta_{\rm refl}$, and $\beta_{\rm refr}$ in the reference frame $\Sigma'_k$ are connected with the orientation $\theta$ of the normal and the corresponding angles $\varphi_{\rm i}$, $\varphi_{\rm refl}$, and $\varphi_{\rm refr}$ in the $\Sigma_k$ reference frame via relations (see Figs 5 and 6)

$$\beta_{\rm i} = \varphi_{\rm i} - \theta\,, \tag{12}$$

$$\beta_{\rm refl} = \varphi_{\rm refl} - \theta\,, \tag{13}$$

$$\beta_{\rm refr} = \varphi_{\rm refr} - \theta\,. \tag{14}$$

Note that if the angles $\beta_{\rm i}$, $\beta_{\rm refl}$, and $\beta_{\rm refr}$ are reckoned from the wavevector $\mathbf{k}_{n0}$ as defined above, their values are either all positive or all negative (see Figs 5 and 6). Therefore, the sines in Eqns (7)–(11) are either all positive or all negative.

However, the ray propagation directions are specified not by the angles $\varphi_{\rm i}$, $\varphi_{\rm refl}$, $\varphi_{\rm refr}$ or $\beta_{\rm i}$, $\beta_{\rm refl}$, $\beta_{\rm refr}$ entering into relations (7)–(11), but by the angles $\psi_{\rm i}$, $\psi_{\rm refl}$, $\psi_{\rm refr}$ and $\alpha_{\rm i}$, $\alpha_{\rm refl}$, $\alpha_{\rm refr}$ determining the directions of group velocity vectors $\mathbf{V}_{\rm i}$, $\mathbf{V}_{\rm refl}$, and $\mathbf{V}_{\rm refr}$ of the incident, reflected, and refracted rays, respectively, in the reference frames $\Sigma$, $\Sigma_k$ and $\Sigma'$, $\Sigma'_k$. The relation between the angles $\psi_{\rm i}$, $\psi_{\rm refl}$, $\psi_{\rm refr}$ and $\alpha_{\rm i}$, $\alpha_{\rm refl}$, $\alpha_{\rm refr}$ is given by the formulas (see Figs 5 and 6)

$$\alpha_{\rm i} = \psi_{\rm i} - \theta\,, \tag{15}$$

$$\alpha_{\rm refl} = \psi_{\rm refl} - \theta\,, \tag{16}$$

$$\alpha_{\rm refr} = \psi_{\rm refr} - \theta\,. \tag{17}$$

The angles $\psi_{\rm i}$, $\psi_{\rm refl}$, and $\psi_{\rm refr}$ [and then, using Eqns (15)–(17), also the angles $\alpha_{\rm i}$, $\alpha_{\rm refl}$, and $\alpha_{\rm refr}$] can be found from the known values of the angles $\varphi_{\rm i}$, $\varphi_{\rm refl}$, $\varphi_{\rm refr}$ and the values of $k_{\rm i}$, $k_{\rm refl}$, $k_{\rm refr}$ wavenumbers using the dispersion relation for a given wave.[16]

Let us also introduce the traditionally utilized angles of incidence, reflection, and refraction, $\alpha'_{\rm i}$, $\alpha'_{\rm refl}$, and $\alpha'_{\rm refr}$, related to the angles $\alpha_{\rm i}$, $\alpha_{\rm refl}$, and $\alpha_{\rm refr}$, reckoned from the $\mathbf{n}_0$ vector (or $\mathbf{k}_{n0}$ vector), in the following way:[17]

$$\alpha'_{\rm i} = \alpha_{\rm i}\,, \tag{18}$$

$$\alpha'_{\rm refl} = 180^\circ - \alpha_{\rm refl}\,, \tag{19}$$

$$\alpha'_{\rm refr} = \alpha_{\rm refr}\,. \tag{20}$$

*For the reflection angle $\alpha'_{\rm refl}$ calculated according to formula* (19), in contrast to all other angles, *the clockwise direction of angle reckoning should be considered as positive*. Only if the angle $\alpha'_{\rm refl}$ is defined in this way can one speak about the equality between the angles of incidence and reflection in isotropic media, taking into account the sign.

Thus, if the angles $\alpha'_{\rm refl}$ and $\alpha'_{\rm refr}$ found from Eqns (18)–(20) have the same sign as the $\alpha'_{\rm i}$ angle (as in Fig. 5), we are looking at positive wave reflection and refraction, and if the signs of the $\alpha'_{\rm i}$ angle and of $\alpha'_{\rm refl}$, $\alpha'_{\rm refr}$ angles are different (as in Fig. 6), we are looking at negative wave reflection and refraction. As one can see from Figs 5 and 6, the above consideration is valid for both isotropic and anisotropic media. Special features of reflection and refraction in various geometries are determined by the geometric properties of isofrequency dependences, which will be considered below.

It is worth noting that for given positions of the interface and the normal $\mathbf{n}_0$, the incident wave cannot be described by any wavevector of the isofrequency dependence but only by a vector for which the appropriate group velocity vector satisfies the incidence condition (the incident ray should be directed towards the interface):

$$(\mathbf{V}_{\rm i}\,\mathbf{n}) > 0\,, \quad \text{or} \quad (\mathbf{V}_{\rm i}\,\mathbf{k}_{n0}) > 0\,, \quad \text{or} \quad |\alpha'_{\rm i}| < 90^\circ\,. \tag{21}$$

Moreover, not all points of intersection between the isofrequency curves of both media and the projecting line[18] will correspond to reflected or refracted waves. Evidently, reflection will correspond only to those points of intersection with the original isofrequency dependence at which the group velocity vector satisfies the reflection condition (the reflected ray is directed from the interface towards the medium with a wave source):

$$(\mathbf{V}_{\rm refl}\,\mathbf{n}) < 0\,, \quad \text{or} \quad (\mathbf{V}_{\rm refl}\,\mathbf{k}_{n0}) < 0\,, \quad \text{or} \quad |\alpha'_{\rm refl}| < 90^\circ\,. \tag{22}$$

Refraction, in turn, will correspond only to those intersection points of the projecting line with the isofrequency dependence of the second medium where the group velocity vector satisfies the refraction condition (the refracted ray is directed from the medium with a wave source towards the medium behind the interface):

$$(\mathbf{V}_{\rm refr}\,\mathbf{n}) > 0\,, \quad \text{or} \quad (\mathbf{V}_{\rm refr}\,\mathbf{k}_{n0}) > 0\,, \quad \text{or} \quad |\alpha'_{\rm refr}| < 90^\circ\,. \tag{23}$$

*Meeting conditions* (22) *and* (23) *is necessary and sufficient for the existence of reflected and refracted rays only in those cases where all curves of the isofrequency dependences for both media describe waves with the same polarization*. Otherwise,

[15] In Refs [12, 18, and 23], instead of relations (7)–(9), a more complicated expression is given for an anisotropic medium, but it can be reduced to formulas (7)–(9) after some simplifications.

[16] For instance, dispersion relations for MSWs in a free film and in various structures can be found in Refs [12–14, 17, 25, 39, 41–43, 47]. A method of obtaining the $\psi(\varphi)$ dependence is described, using an MSW as an example, in Ref. [12, pp. 232–236] and in Ref. [25].

[17] Recall that since according to our assumptions we always used the values of angles falling within the interval from $-180^\circ$ to $180^\circ$, so in determining the value of $\alpha'_{\rm refl}$ at negative values of $\alpha_{\rm refl}$ in formula (19) one should additionally subtract $360^\circ$ from the result.

[18] In what follows, we will briefly call this way the straight line drawn parallel to the normal and passing through the end of the wavevector $\mathbf{k}_{\rm i}$ of the incident wave (line *7* in Fig. 6).

inequalities (22) and (23) are only necessary conditions. Therefore, if the isofrequency curves of one medium or both media describe waves of different polarizations, then, after finding the points in isofrequency dependences satisfying conditions (22) and (23), one should check whether the reflected and refracted waves corresponding to these points will emerge if one takes into account the incident ray polarization with respect to the plane of incidence and to the characteristic directions in each anisotropic medium (i.e., directions that determine the anisotropy of the medium, such as crystallographic axes or the directions of applied constant magnetic or electric field). For instance, when linearly polarized light is normally incident on an optical crystal [5, p. 383], then, depending on its polarization direction with respect to the optical axis, the intensity of the excited ordinary and extraordinary rays in the crystal will change, in turn, from zero to the maximal value. Since taking polarization into account is described in detail in the literature and is not the subject of the present paper, which is devoted to the analysis of the mathematical properties of isofrequency dependences, in what follows we will assume that the polarization of the incident ray is such that all intersection points of the projecting line with the isofrequency dependences correspond to waves with nonzero amplitudes. Notice that this assumption is consistent with the most general case of wave reflection and refraction. Moreover, since in the framework of this paper we are not interested in the polarization and intensities of the reflected and refracted rays, this assumption will enable us to concentrate on analyzing the properties of isofrequency dependences without frequent distractions caused by taking polarization into account. Therefore, in what follows we will mention polarization only if it is necessary.

Let us note at this point that, in contrast to isotropic media where it does not matter whether the $\alpha'_{\mathrm{refr}}(\alpha'_{\mathrm{i}})$ and $\alpha'_{\mathrm{refl}}(\alpha'_{\mathrm{i}})$ dependences [19] are measured by changing the orientation of the interface [20] or by rotating the excitation transducer (the transmitting antenna), in an anisotropic medium these two ways will lead to completely different results. In the first method, when the orientation of the excitation transducer with respect to the chosen axis does not change, *the parameters of the incident wave* (the $\mathbf{k}_{\mathrm{i}}$ and $\mathbf{V}_{\mathrm{i}}$ vectors and the angles $\varphi_{\mathrm{i}}$ and $\psi_{\mathrm{i}}$ related to them) *remain constant*. At the same time, the parameters of the incident wave in the second method are different for each new value of the angle of incidence $\alpha'_{\mathrm{i}}$ (since the orientations of the excitation transducer with respect to the chosen axis are changed). This feature should be taken into account in both theoretical and experimental studies.

## 8. Laws of wave reflection

### 8.1 General remarks

In isotropic media, where the isofrequency curve is a circle, the wavevectors of the incident and reflected waves, $\mathbf{k}_{\mathrm{i}}$ and $\mathbf{k}_{\mathrm{refl}}$, are the radii of the same circle, and the equalities

$$k_{\mathrm{refl}} = k_{\mathrm{i}} = \mathrm{const} \tag{24}$$

always hold true. Therefore, expression (10) takes the form $\sin\beta_{\mathrm{i}} = \sin\beta_{\mathrm{refl}}$, whence one finds

$$\beta_{\mathrm{refl}} = \beta_{\mathrm{i}} \quad \text{or} \quad \beta_{\mathrm{refl}} = 180^\circ - \beta_{\mathrm{i}}\,. \tag{25}$$

Evidently, the first solution in Eqn (25) corresponds to the initial wave, and the second one to the reflected wave. Furthermore, a circular isofrequency curve has another unique property: any wavevector $\mathbf{k}$ connecting the center with a point of the circle and the group velocity vector $\mathbf{V}$ orthogonal to the circle at the same point are collinear for all points of the circle. For forward waves propagating in an ordinary isotropic material, this property leads to the fact that always $\psi_{\mathrm{i}} = \varphi_{\mathrm{i}}$, $\psi_{\mathrm{refl}} = \varphi_{\mathrm{refl}}$ and $\beta_{\mathrm{i}} = \alpha_{\mathrm{i}}$, $\beta_{\mathrm{refl}} = \alpha_{\mathrm{refl}}$. Hence, taking into account the second expression in Eqn (25) we obtain the relation for the group velocity orientations:

$$\alpha_{\mathrm{refl}} = 180^\circ - \alpha_{\mathrm{i}}\,, \tag{26}$$

which, after substituting it into formula (19), leads to the well-known Euclidian law — the angle of reflection is equal to the angle of incidence:

$$\alpha'_{\mathrm{refl}} = \alpha'_{\mathrm{i}}\,. \tag{27}$$

For a backward wave propagating in a hypothetical isotropic medium with negative $\varepsilon$ and $\mu$, the second equality in Eqn (25) also leads to relations (26) and (27) [it is sufficient to apply the relations $\psi_{\mathrm{i}} = \varphi_{\mathrm{i}} - 180^\circ$, $\psi_{\mathrm{refl}} = \varphi_{\mathrm{refl}} - 180^\circ$, $\beta_{\mathrm{i}} = \alpha_{\mathrm{i}} - 180^\circ$, and $\beta_{\mathrm{refl}} = \alpha_{\mathrm{refl}} - 180^\circ$].

Similar speculation applied to an anisotropic medium leads to the fact that, in the general case, $k_{\mathrm{i}} \neq k_{\mathrm{refl}}$; therefore, reflection in general should be discussed separately for each given anisotropic medium.

However, let us pose a question: can we find some general features typical for wave reflection in anisotropic media? The answer turns out to be positive. These features are formulated below and followed by short comments.

### 8.2 Euclid's rule

*If the boundary of the medium is parallel to one of the symmetry axes of the isofrequency dependence, and reflection of a wave with the wavevector* $\mathbf{k}_{\mathrm{i}}$ *and the group velocity vector* $\mathbf{V}_{\mathrm{i}}$ *results in the appearance of a single reflected ray with the appropriate vectors* $\mathbf{k}_{\mathrm{refl}}$ *and* $\mathbf{V}_{\mathrm{refl}}$*, then the angle of reflection* $\alpha'_{\mathrm{refl}}$ *will be equal to the angle of incidence* $\alpha'_{\mathrm{i}}$ *and the lengths of the respective vectors* $\mathbf{k}_{\mathrm{i}}$, $\mathbf{V}_{\mathrm{i}}$ *and* $\mathbf{k}_{\mathrm{refl}}$, $\mathbf{V}_{\mathrm{refl}}$ *will be equal, viz.*

$$k_{\mathrm{refl}} = k_{\mathrm{i}}\,, \qquad V_{\mathrm{refl}} = V_{\mathrm{i}}\,, \qquad \alpha'_{\mathrm{refl}} = \alpha'_{\mathrm{i}}\,. \tag{28}$$

*If reflection of the wave under the described geometry results in the appearance of the multitude of reflected rays, there will necessarily be one ray among them, for which relations* (28) *hold true.*

Indeed, if the boundary in Fig. 6a is oriented parallel to a symmetry axis, for instance, the $k_y$-axis, so that the normal is directed oppositely to $k_z$, then, by plotting a straight line parallel to $k_z$ through the end of the $\mathbf{k}_{\mathrm{i}}$ vector, we will easily find the vectors $\mathbf{k}_{\mathrm{refl}}$ and $\mathbf{V}_{\mathrm{refl}}$ shown in Fig. 7. A similar reflection scheme can be obtained for any other anisotropic medium. It should be kept in mind that the isofrequency dependence of an anisotropic medium (structure) may have

[19] In isotropic media, the $\alpha'_{\mathrm{refl}}(\alpha'_{\mathrm{i}})$ dependence is represented by a straight line $\alpha'_{\mathrm{refl}} = \alpha'_{\mathrm{i}}$.

[20] It should be noted that the orientation of the interface cannot be easily changed in any medium. For instance, in a tangentially magnetized ferrite film with a straight edge, this is not difficult to do by rotating the film around the normal to its surface (provided that the anisotropy in the film plane is small), while for optical or acoustic crystals one should manufacture a set of samples with different interface orientations.

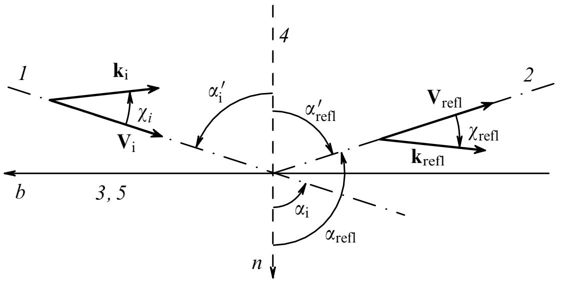


**Figure 7.** Reflection according to Euclid's rule when the interface is parallel to an axis of symmetry: *1*, *2*, lines of the incident and reflected rays; *3*, interface; *4*, normal to the interface; *5*, symmetry axis of the medium or structure.

various numbers of symmetry axes: an ellipse (Fig. 1b) and a dependence resembling symmetric hyperbolas (Fig. 1c, d) have two such axes, the dependences depicted in Fig. 2 have one, the isofrequency dependences of tetragonal acoustic crystals have four [8, 15, 16], and so on. The isofrequency dependence of an isotropic material (a circle) has an infinite number of symmetry axes, therefore in such materials Euclid's rule holds true for any orientation of the boundary.

If the isofrequency dependence of the medium consists of several curves or a family of curves[21] and the multitude of reflected rays appear (this is the case, for instance, for a backward MSW in a medium with the boundary parallel to the $k_y$-axis in Fig. 1d), then the reflected ray satisfying relations (28) corresponds to the point in the isofrequency dependence that is symmetric, with respect to the boundary, to the point depicting the incident ray.

### 8.3 Backward reflection

*If the isofrequency dependence is centrally symmetric and the interface is orthogonal to the wavevector* $\mathbf{k}_i$ *of the incident ray, then the wavevectors of all reflected rays will be collinear to the* $\mathbf{k}_i$ *vector, and for one of the reflected rays the angle of reflection* $\alpha'_{refl}$, *the wavevector* $\mathbf{k}_{refl}$, *and the group velocity vector* $\mathbf{V}_{refl}$ *will be related to the corresponding parameters of the incident ray* (*the angle of incidence* $\alpha_i'$, *and the vectors* $\mathbf{k}_i$ *and* $\mathbf{V}_i$) *through the equalities*

$$\mathbf{k}_{refl} = -\mathbf{k}_i\,, \qquad \mathbf{V}_{refl} = -\mathbf{V}_i\,, \qquad \alpha'_{refl} = -\alpha_i'\,. \tag{29}$$

Evidently, if reflection results in a single reflected ray, then it is for this ray that relations (29) are valid (Fig. 8).

Consider first the case where only one reflected ray emerges. In isotropic media, where the corresponding $\mathbf{k}$ and $\mathbf{V}$ vectors are always parallel, backward reflection occurs only at normal incidence of the wave on the interface, i.e., at $\alpha_i' = -\alpha'_{refl} = 0$. In anisotropic media, where the vectors $\mathbf{k}$ and $\mathbf{V}$ are, in the general case, not parallel, *backward reflection also takes place at slanting incidence of the wave on the interface*. For instance, if the orientation of the interface in Fig. 6 is chosen so that the normal $\mathbf{k}_{n0}$ is directed the same way as the incident ray wavevector $\mathbf{k}_i$, then the reflected ray will have a wavevector centrally symmetric to the $\mathbf{k}_i$ vector, and the reflection will take place as shown in Fig. 8. Evidently, if only one reflected ray emerges, then both the incident and reflected waves are the same and are either forward or backward. Moreover, if both waves are forward, as in Fig. 8, the angles of incidence and reflection are related to the $\chi_i$ angle between the vectors $\mathbf{V}_i$ and $\mathbf{k}_i$ through the equalities [taking into account Eqn (5)]

$$\alpha_i' = -\alpha'_{refl} = -\chi_i = \psi_i - \varphi_i\,, \tag{30}$$

and when both waves are backward, the following relations hold:

$$\alpha_i' = -\alpha'_{refl} = 180^\circ - \chi_i = 180^\circ - \varphi_i + \psi_i\,. \tag{31}$$

[21] Note that in such media the multitude of incident waves corresponding to different isofrequency curves can be excited simultaneously. In order to have only one incident wave in experiment (otherwise, each incident wave may create the multitude of reflected rays), one should take special precautions. For instance, an efficient way to excite only the first mode of a backward MSW is to use an excitation transducer whose width is larger than half of the second-mode wavelength.

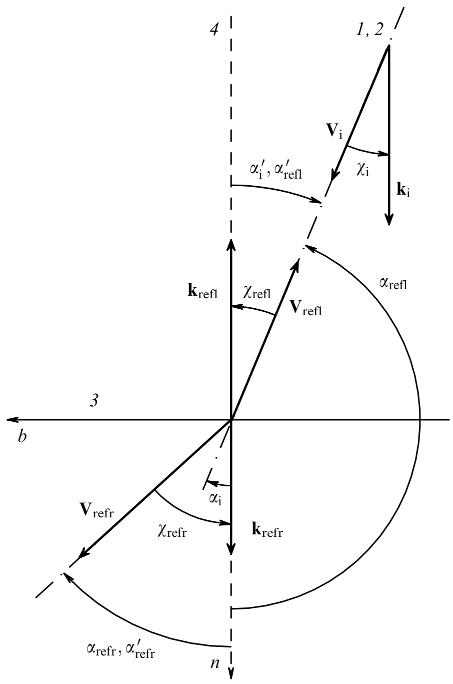


**Figure 8.** Reflection and refraction of a wave whose phase front is parallel to the interface: *1*, *2*, lines of the incident and reflected rays; *3*, interface, *4*, normal to the interface.

If the isofrequency dependence of the medium consists of several curves, it is possible that several reflected rays appear. (For instance, many reflected rays may emerge for a backward MSW, when the incident ray corresponds to the $\mathbf{k}_1$ vector in Fig. 1d, and the interface is orthogonal to the $\mathbf{k}_1$ vector.) In this case, the reflected ray, which satisfies relations (29) and (31), corresponds to a point in the isofrequency dependence that is centrally symmetric to the point depicting the incident ray. If all curves of the isofrequency dependence describe waves of the same type, i.e., either all forward or all backward, as in Fig. 1d, then the wavevectors of all reflected rays will be directed opposite to the wavevector $\mathbf{k}_i$ of the incident wave. In the other case, where one of the reflected waves is of a different type than the incident wave, then its wavevector is directed the same way as the $\mathbf{k}_i$ vector. The last case can be realized, for instance, in a structure with metal planes placed on both sides of a ferrite plate at equal distances $l$ from its surface. The isofrequency dependence of such a structure at certain values of $l$ consists of two centrally symmetric ovals (resembling the oval shown in Fig. 2c) for which parts of the arc close to the origin describe forward waves, and parts of the arc far from the origin describe backward waves.

It should be noted that for anisotropic media, where isofrequency curves are not centrally symmetric (see Fig. 2), the law of backward reflection (29) is not valid.

### 8.4 Positive and negative reflection

In order to formulate the conditions for the reflection to be positive or negative, let us study whether the isofrequency dependence of the medium has extrema or singular points in the reference frame $\Sigma_k' = \{0; k_b, k_n\}$. With this in mind we will first describe such a study for the case of an arbitrary isofrequency dependence and then consider it for the example of the isofrequency dependence of a surface MSW in a free film. Let the isofrequency dependence of some wave be described in the reference system $\Sigma_k = \{0; k_y, k_z\}$ by the dispersion relation $F(k_y, k_z) = 0$. To represent this dispersion relation in the reference system $\Sigma_k'$, let us change the variables according to the formulas [22]

$$k_y = k_n \cos\theta + k_b \sin\theta\,,\qquad k_z = k_n \sin\theta - k_b \cos\theta\,. \tag{32}$$

Then, the isofrequency dependence of the wave is described by the equation $F(k_b, k_n) = 0$ or, if the variable $k_n$ can be written explicitly, by the dependence $k_n = P(k_b)$. Further, let the dependence $P(k_b)$ be within the range [23] $k_{b\,\min} < k_b < k_{b\,\max}$. Then we will divide all the range $k_{b\,\min} < k_b < k_{b\,\max}$ into intervals separated by points $k_{b1}, k_{b2}, \ldots, k_{bm}$, where the derivative [24] $\partial P/\partial k_b$ is either equal to zero or tends to infinity, or the $P(k_b)$ dependence has a discontinuity. Recall that from the physics viewpoint, each $k_b$ value corresponds to a potential existence of a common projection for the wavevectors of the incident and reflected rays, $\mathbf{k}_{\mathrm{i}}$ and $\mathbf{k}_{\mathrm{refl}}$. Therefore, every interval $k_{b\,\min} < k_b < k_{b1}$, $k_{b1} < k_b < k_{b2}, \ldots, k_{bm} < k < k_{b\,\max}$ and every point $k_{b1}, k_{b2}, \ldots, k_{bm}$ obviously correspond to a certain type of wave reflection, since at each point [25] $k_{b1}, k_{b2}, \ldots, k_{bm}$ either the incident-wave group velocity vector $\mathbf{V}_{\mathrm{i}}$ or the reflected-wave group velocity vector $\mathbf{V}_{\mathrm{refl}}$ is either collinear or orthogonal to the $\mathbf{k}_{n0}$ vector (and the normal vector $\mathbf{n}_0$).

When analyzing the $P(k_b)$ dependence one should take into account that, as a rule, it is not a function, and each $k_b$ value belonging to one of the intervals given above may correspond to several values of $k_n$. Therefore, it is convenient to consider the $P(k_b)$ dependence in each of these intervals as consisting of separate sections $P_0(k_b), P_1(k_b), P_2(k_b), \ldots$, etc.

Apparently, for reflection to be enabled for some of the above-given intervals or values of $k_b$, it is necessary that, in the given interval of $k_b$ (or for the given value of $k_b$), the incidence condition (21) be satisfied for at least one section, and reflection condition (22) for another section (if only for one of the other sections). The section of the $P(k_b)$ dependence satisfying condition (21) and corresponding to the incident ray [26] will be denoted as $P_0(k_b)$, and the section satisfying condition (22) and corresponding to the $j$th reflected ray will be denoted by $P_j(k_b)$.[27] Now, one can formulate the rule determining the wave reflection character.

*In those intervals of $k_b$ values where within the section $P_j(k_b)$ of the isofrequency dependence corresponding to the jth reflected ray the derivative $\partial P_j/\partial k_b$ has the same sign as the derivative $\partial P_0/\partial k_b$ within the section $P_0(k_b)$ corresponding to the incident ray, the jth reflected ray undergoes negative reflection. Otherwise, if the signs of the derivatives $\partial P_j/\partial k_b$ and $\partial P_0/\partial k_b$ are opposite, then the jth reflected ray undergoes positive reflection. For those values of $k_b$ from the set $\{k_{b1}, k_{b2}, \ldots, k_{bm}\}$ for which the derivative $\partial P_0/\partial k_b = 0$, the incident ray is oriented normally to the interface, and if $\partial P_j/\partial k_b = 0$, then the jth reflected ray is normal to the interface.*

Let us briefly illustrate this rule using the example of the isofrequency dependence $P(k_b)$ for a surface MSW propagating in a free ferrite film (Fig. 9).[28] For the given orientation of the medium interface with respect to the symmetry axis $k_y$,

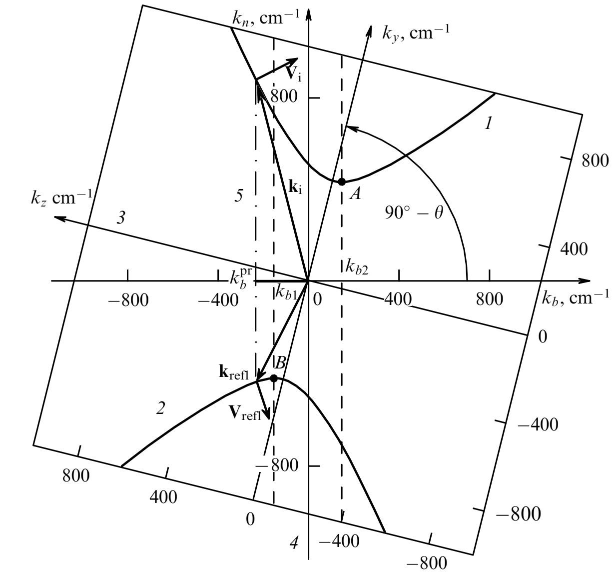


**Figure 9.** Example for determining the character of reflection: a surface MSW in a reference system connected with the interface and the normal: *1*, *2*, isofrequency curves; *3*, boundary of the film and the $k_b$-axis connected with it; *4*, normal to the boundary and the $k_n$-axis connected with it; *5*, projecting line; *A*, *B*, extremum points of the isofrequency dependence in the reference system $k_b k_n$.

[22] Expressions (32) can be derived from Ref. [30, p. 196] by assuming that the reference system $\Sigma_k$ is obtained from the reference system $\Sigma_k'$ through the rotation by an angle of $90° - \theta$. This is clear, for instance, from Fig. 6a.

[23] In some cases, for instance, where the isofrequency dependence resembles a hyperbola, the range of $k_b$ values may not be limited, i.e., $k_{b\,\min} = -\infty$ and $k_{b\,\max} = +\infty$.

[24] If the dependence $P(k_b)$ is given implicitly by means of equation $F(k_b, k_n) = 0$, this derivative can be found according to the rules of differentiating an implicit dependence [30].

[25] Excepting the discontinuity points of the $P(k_b)$ dependence.

[26] Note that in some cases there may be several sections satisfying condition (21). Then, the section corresponding to the incident ray should be chosen according to the specific experimental conditions or to the problem situation (for instance, if one is only interested in the incident ray with a given orientation or absolute value of the wavevector, and so on).

[27] For simplicity, hereinafter we assume that either all curves of the isofrequency dependence describe waves with the same polarization, or the incident ray is polarized in such a way that all intersection points of the projecting line with the isofrequency dependence correspond to waves with nonzero amplitudes. In a real situation, if the curves of the isofrequency dependence describe differently polarized waves, one should make sure that, given the polarization of the incident ray and the specific geometry of the wave incidence on the interface, the corresponding reflected waves have nonzero amplitudes.

[28] Both curves of the isofrequency dependence for a surface MSW describe waves with the same polarization.

there are two extremum points, $A$ and $B$, and in every interval of $k_b$ values there are always two sections of isofrequency dependence, $P_0(k_b)$ and $P_1(k_b)$, with the first one satisfying condition (21) and always belonging to curve *1*, and the second one satisfying condition (22) and always belonging to curve *2* (see Fig. 9). Let us denote the projection of the $\mathbf{k}_\mathrm{i}$ wavevector onto the interface (the $k_b$-axis) as $k_b^{\mathrm{pr}}$. Depending on which vector $\mathbf{k}_\mathrm{i}$ corresponds to the incident ray, the following cases of wave reflection arise:

(1) if the $k_b^{\mathrm{pr}}$ value falls within the interval $k_b^{\mathrm{pr}} < k_{b1}$ (as shown in Fig. 9), then the derivative $\partial P_0/\partial k_b < 0$ in the section $P_0(k_b)$ containing the end of the $\mathbf{k}_\mathrm{i}$ vector, while in the corresponding section $P_1(k_b)$ containing the end of the $\mathbf{k}_\mathrm{refl}$ vector, the derivative $\partial P_1/\partial k_b > 0$; since the derivatives have opposite signs, positive reflection takes place;

(2) if $k_b^{\mathrm{pr}} = k_{b1}$, then the projecting line (straight line *5* in Fig. 9) goes through the point $B$ wherein $\partial P_1/\partial k_b = 0$; in this case, the reflected ray is orthogonal to the interface, although the incident ray is directed at an angle to the interface;

(3) if $k_{b1} < k_b^{\mathrm{pr}} < k_{b2}$, then $\partial P_0/\partial k_b < 0$ and $\partial P_1/\partial k_b < 0$; since the derivatives have the same sign, negative reflection takes place; for $k_b^{\mathrm{pr}} = 0$ there is backward reflection with relations (29) satisfied;

(4) if $k_b^{\mathrm{pr}} = k_{b2}$, then the projecting line passes through point $A$ wherein $\partial P_0/\partial k_b = 0$; in this case, the incident wave is normal to the interface but the reflected ray is directed at an angle to the interface;

(5) if $k_b^{\mathrm{pr}} > k_{b2}$, then $\partial P_0/\partial k_b > 0$ and $\partial P_1/\partial k_b < 0$; since the derivatives have different signs, positive reflection takes place.

Notice that the rule formulated above is valid regardless of whether the wave propagating in the medium is forward or backward.[29]

Both backward reflection and negative reflection at slanted incidence of the wave on the medium interface can be observed for MSWs in ferrite films [25], for acoustic waves in acoustic crystals [48], and for light reflection into an optical crystal (although the latter effect is almost never mentioned in textbooks on crystal optics).

### 8.5 The absence of the reflected ray

Evidently, for the reflected ray to be absent, condition (22) must not be satisfied for any section of the isofrequency dependence. For a given orientation of the medium interface with respect to the symmetry axis, this is possible in the following cases.

*Reflection will be absent at any angles of incidence of the ray on the interface if*

(1) *the isofrequency dependence consists of only centrally symmetric curves similar to hyperbolas and having no inflection points, and the medium interface is orthogonal to one of their asymptotes*;

(2) *the isofrequency dependence is a function*[30] *in some frame of reference, and the interface is parallel to the abscissa axis of this frame of reference, and*

(3) *the isofrequency dependence comprises a family of curves, each of which is single-valued* (*like a function*) *in some frame of reference, all curves belong to the same semiplane, and the medium interface is parallel to the boundary of this semiplane*.

Notice that the term 'the absence of reflection' means the absence of the reflected wave described by the same isofrequency dependence. Apparently, in a real anisotropic medium the energy of the incident wave can be transformed into other types of waves; in the case of an MSW propagating in a ferrite film these can be, for instance, exchange spin waves or edge waves propagating along the interface.[31]

The isofrequency dependence of a surface MSW in a free ferrite film is appropriate to the first case from the list given above. Indeed, if the interface in Fig. 9 is arranged orthogonal to one of the asymptotes, then for any parameters of the incident ray the projecting line does not cross the isofrequency dependence at any other point, and no reflection emerges.

Examples illustrating the second case are given by the isofrequency dependence of an MSW in a ferrite film with one surface covered by metal for the frequency range $\omega_\mathrm{H} + 0.5\omega_\mathrm{M} < \omega < \omega_\mathrm{H} + \omega_\mathrm{M}$ (Fig. 2b), and by the isofrequency dependence of an MSW in a ferrite film with the 'magnetic wall'-type boundary conditions satisfied on one its surface for the frequency range $\sqrt{\omega_\mathrm{H}(\omega_\mathrm{H} + \omega_\mathrm{M})} < \omega < \omega_\mathrm{H} + 0.5\omega_\mathrm{M}$ (see Fig. 5 in Ref. [40]). In contrast to the case of a surface MSW in a free ferrite film, where reflection is absent only for a single orientation $\theta$ of the boundary with respect to the symmetry axis $k_y$, in these structures there is no reflection within a rather broad range of $\theta$. Let us also mention the structure metal – ferrite plate – 'magnetic wall', where the isofrequency dependence tends to a straight line coinciding with the $k_z$-axis as $\omega \to \sqrt{\omega_\mathrm{H}(\omega_\mathrm{H} + \omega_\mathrm{M})}$ (see Ref. [39] and Section 6.4). If the $\omega$ values are somewhat higher than this value, one can arrive at an isofrequency dependence that is very close to a straight line but does not coincide with the $k_z$-axis. Thus, in such a structure reflection is absent in a maximally broad range of $\theta$ values.

### 8.6 The emergence of two or more reflected rays

Apparently, for the appearance of two reflected rays it is necessary that the isofrequency dependence and the projecting line intersect at three points: one of which should satisfy condition (21), and the other two condition (22). The following rule can be formulated.

*For a given orientation of the medium interface with respect to the symmetry axis of the isofrequency dependence there are not less than two reflected rays if the isofrequency dependence possesses at least one of the following properties*:[32]

(1) *it consists of two curves, one of which is closed*;

(2) *it consists of three or more curves*;[33]

(3) *one of the curves of the isofrequency dependence has inflection points*.

Isofrequency dependences of optical crystals possess the first of these properties (see, for instance, Fig. 1b); this is why

[29] At first sight, it may seem that this rule should be formulated in the opposite way for a backward wave. However, if directions of the $\mathbf{V}$ vectors in Fig. 9 are changed to opposite ones, then, according to conditions (21) and (22), sections $P_0(k_b)$ and $P_1(k_b)$ of the isofrequency dependence will switch the roles: the first one will correspond to the reflected wave, and the second one to the incident wave. Formulation of the rule will remain the same.

[30] Recall that a function is defined as a dependence such that each one of its abscissa values corresponds to a *single* ordinate value.

[31] Unfortunately, there is still no detailed and experimentally confirmed answer to the question into which particular types of waves or into which forms of energy the MSW energy is transformed when the reflected MSW wave is absent.

[32] See footnote 27.

[33] There is one hypothetic exception to this rule — the case where the isofrequency dependence consists of parallel straight lines.

double reflection arises for a certain polarization of the incident ray [49–51]. A similar property is manifested by the isofrequency dependence of an MSW propagating in an FDM structure at certain values of its parameters (Fig. 2c).

The isofrequency dependence for an acoustic wave in tetragonal crystals and the isofrequency dependence of a metallized ferrite film (Fig. 2a) offer the third of these properties,[34] while the isofrequency dependence of a bulk backward MSW in a free ferrite film (Fig. 1d) has the second and the third. In both cases, theoretical predictions about the existence of two reflected rays were confirmed by experimental observations [48, 25].

If one of the curves of the isofrequency dependence has inflection points, then, for a certain location of the projecting line in the neighborhood of one of these points, it crosses this isofrequency curve at three or more points.[35] The orientation of the normal vector $\mathbf{n}_0$ (or the $\mathbf{k}_{n0}$ vector) can always be chosen in such a way that condition (21) is satisfied at one intersection point and condition (22) at two other points, which will then correspond to two reflected rays. Notice that since all three points lie in the same curve, polarization does not influence the occurrence of reflected rays in this case. The more inflection points the isofrequency curve has, the more reflected rays can appear for a certain orientation of the medium interface.

The isofrequency dependences shown in Figs 1a, c cannot have more than two intersection points with the projecting line, since these dependences satisfy neither of the conditions listed above. (One can see from Fig. 1c that two hyperbola-like curves are not sufficient for obtaining three intersection points.) Therefore, the second reflected ray never emerges in the media with such isofrequency dependences.

If the isofrequency dependence describes a multimode wave (Fig. 1d), then, regardless of the existence of inflection points, several reflected rays corresponding to higher modes will emerge at a certain orientation of the interface.

### 8.7 Irreversibility of the reflected ray path

*If the isofrequency dependence of a medium has no center of symmetry, irreversibility of the reflected ray path shows itself.* This rule follows from the law of nonreciprocal propagation of a wave (see Section 6.4). Indeed, suppose that the incident and reflected rays are given by the vectors $\mathbf{V}_0$, $\mathbf{k}_0$ and $\mathbf{V}_1$, $\mathbf{k}_1$, respectively. If one finds a point in the isofrequency dependence that describes a ray with the wavevector $\mathbf{k}_2$ and the group velocity vector $\mathbf{V}_2$ directed oppositely to the vector $\mathbf{V}_1$, then it will turn out that $\mathbf{k}_2 \neq \mathbf{k}_1$ (as shown in Fig. 4). Therefore, if we now consider the ray given by $\mathbf{k}_2$, $\mathbf{V}_2$ vectors as the incident one and, by applying the momentum conservation law to the vector $\mathbf{k}_2$, find the new reflected ray with the parameters $\mathbf{k}_3$, $\mathbf{V}_3$, then it will turn out that $\mathbf{k}_3 \neq \mathbf{k}_0$ and $\mathbf{V}_3 \neq \mathbf{V}_0$. These considerations can be easily checked by the example of a metallized ferrite film whose isofrequency dependence is not centrally symmetric (Fig. 2a). One should keep in mind that, for some isofrequency dependences which are not centrally symmetric [for instance, those of an FDM structure (see Fig. 6a)], only certain sections can be much different from centrally symmetric (for instance, in an FDM structure these are sections close to the symmetry axis $k_y$). Other sections (in an FDM structure, these are sections lying far from the symmetry axis $k_y$) can be considered as almost centrally symmetric, and from the geometric considerations given above it follows that the vectors $\mathbf{k}_3$ and $\mathbf{k}_0$, as well as vectors $\mathbf{V}_3$ and $\mathbf{V}_0$, are almost the same for these sections of isofrequency dependences.

From the laws formulated above, it is clear that in the study of wave reflection in anisotropic media, the basic characteristic should be the dependence of the angle of reflection on the angle of incidence, $\alpha'_{\text{refl}}(\alpha'_{\text{i}})$. In isotropic media, this dependence is given by a straight line, $\alpha'_{\text{refl}} = \alpha'_{\text{i}}$. In anisotropic media, this is a complicated dependence which may have extrema and multivalued sections corresponding to the occurrence of several reflected rays (see, for instance, Refs [23, 25]).

[34] The right-hand curve in Fig. 2a has inflection points.

[35] The appearance of three points where the projecting line crosses an isofrequency curve with inflection points is clear from Fig. 10 of Section 9.4.

## 9. Laws of wave refraction

### 9.1 General remarks

From relation (11) obtained in Section 7, it follows that if both media are isotropic and their isofrequency dependences are represented by circles, then the vectors $\mathbf{k}_{\text{refr}}$ and $\mathbf{k}_{\text{i}}$ are radii of different circles and always

$$\frac{k_{\text{refr}}}{k_{\text{i}}} = n = \text{const}\,. \tag{33}$$

Therefore, from relation (11) we obtain for isotropic media that

$$\frac{\sin\beta_{\text{i}}}{\sin\beta_{\text{refr}}} = n\,. \tag{34}$$

The quantity $n$ in formulas (33) and (34) is called the refractive index. Since for a circular isofrequency dependence any corresponding vectors $\mathbf{k}$ and $\mathbf{V}$ are collinear, then for forward waves propagating in usual isotropic media, the following relationships always hold true:

$$\beta_{\text{i}} = \alpha_{\text{i}} = \alpha'_{\text{i}} \quad \text{and} \quad \beta_{\text{refr}} = \alpha_{\text{refr}} = \alpha'_{\text{refr}}\,. \tag{35}$$

Therefore, law (34) derived for the wavevectors turns out to be also valid for group velocities:

$$\frac{\sin\alpha'_{\text{i}}}{\sin\alpha'_{\text{refr}}} = n\,. \tag{36}$$

If the medium by which the wave is refracted is a hypothetical isotropic medium with negative $\varepsilon$ and $\mu$, where a backward wave propagates and the orientations of the $\mathbf{k}$ and $\mathbf{V}$ vectors are related through the equation $\beta_{\text{refr}} = \alpha'_{\text{refr}} + 180°$ (or $\beta_{\text{refr}} = \alpha'_{\text{refr}} - 180°$), then, by substituting this expression into Eqn (34), we obtain the relation for group velocities:

$$\frac{\sin\alpha'_{\text{i}}}{\sin\alpha'_{\text{refr}}} = -n\,. \tag{37}$$

Apparently, if the medium with a wave source also has negative $\varepsilon$ and $\mu$, then the group velocities also obey Eqn (36).

For an isotropic medium, refractive index $n$ is a constant, one of the most important characteristics of the medium.

If we try to apply similar reasoning for the case where one medium or both media are anisotropic, it will turn out that, first, the refractive index $n = k_{\mathrm{refr}}/k_{\mathrm{i}}$ is not constant for different geometries of the wave incidence and, second, the angles $\alpha_{\mathrm{i}}$ and $\alpha_{\mathrm{refr}}$ giving orientations of the rays and the angles $\beta_{\mathrm{i}}$ and $\beta_{\mathrm{refr}}$ are not equal, as in formulas (35), but are connected through the dispersion relation; therefore, the simple relationship (36) does not hold true either. Thus, the refractive index $n$ for anisotropic media loses one of its meanings, given by relation (36) — the value of $n$ does not characterize the refraction degree of the rays. However, its other meaning, given by relations (33) and (34), is preserved, namely, $n$ describes the refraction degree of wavevectors and shows how much wavenumbers (wavelengths) differ in both media for a given geometry of the wave incidence.

Sometimes, refraction is described by introducing the ray, or energy, refractive index $n_V = c/V$ [4, p. 734]. For instance, the index $n_V$ may be helpful for the study of light refraction from a vacuum into an anisotropic optical crystal whose isofrequency curves corresponding to extraordinary waves are elliptic. Indeed, in this case, first, no negative refraction occurs and, second, orientations of the vectors **k** and **V** are not much different for an ellipse (see Fig. 3, curves *2*), i.e., $n_V$ is always positive in this case and varies within a narrow range $n_{V\min} < n_V < n_{V\max}$. In those cases where negative refraction occurs (for instance, for MSW refraction from a free film to an FDM structure) one can introduce, by analogy with formula (36), the refractive index $n'_V$ of the group velocity vector **V** [36] (in order to somehow describe the degree of refraction for **V** vectors at different geometries of the wave incidence). However, this probably makes no sense since the value of $n'_V$ calculated according to formula (36) will vary within the range $-\infty < n'_V < +\infty$ [24]. By and large, a complete description of wave refraction for any anisotropic geometry can be obtained from the relationship between the angle of refraction and the angle of incidence, $\alpha'_{\mathrm{refr}}(\alpha'_{\mathrm{i}})$; therefore, in what follows we will use neither the index $n_V$ nor the index $n'_V$.

This reasoning shows that there is no sense in discussing wave refraction on the whole; refraction should be studied separately for every particular anisotropic geometry with given orientations of the symmetry axes of isofrequency dependences with respect to the medium interface.

However, let us ask: can one find some general rules typical for wave refraction in anisotropic geometries?

The answer turns out to be positive. These rules are formulated below and followed by brief comments.

### 9.2 Refraction of a wave whose phase front is parallel to the boundary

*If a wave with wavevector* $\mathbf{k}_{\mathrm{i}}$ *and group velocity vector* $\mathbf{V}_{\mathrm{i}}$ *is incident on the interface between two media so that the* $\mathbf{k}_{\mathrm{i}}$ *vector is orthogonal to the interface, then the wavevectors of the refracted rays corresponding to forward waves will be directed the same way as the* $\mathbf{k}_{\mathrm{i}}$ *vector, while the wavevectors of the refracted rays corresponding to backward waves will be directed oppositely to* $\mathbf{k}_{\mathrm{i}}$. Moreover, if the incident wave is forward, then the angle of incidence $\alpha'_{\mathrm{i}}$ and the angle $\chi_i$ between the vectors $\mathbf{V}_{\mathrm{i}}$ and $\mathbf{k}_{\mathrm{i}}$ will be related as $\alpha'_{\mathrm{i}} = -\chi_{\mathrm{i}}$ (see Fig. 8), and if the incident wave is backward, then $\alpha'_{\mathrm{i}} = 180° - \chi_{\mathrm{i}}$. If the $j$th refracted ray is forward (as in Fig. 8), then the angle of refraction $\alpha'_{j\,\mathrm{refr}}$ and the angle $\chi_{j\,\mathrm{refr}}$ between the vectors $\mathbf{V}_{j\,\mathrm{refr}}$ and $\mathbf{k}_{j\,\mathrm{refr}}$ of this ray are related as $\alpha'_{j\,\mathrm{refr}} = -\chi_{j\,\mathrm{refr}}$, and if the $j$th refracted ray is backward, then $\alpha'_{j\,\mathrm{refr}} = 180° - \chi_{j\,\mathrm{refr}}$. Evidently, if the angles $\alpha'_{\mathrm{i}}$ and $\alpha'_{j\,\mathrm{refr}}$ have the same sign, then for the $j$th refracted ray, no matter if it relates to a forward or backward wave, positive refraction occurs (as in Fig. 8), and if the signs of $\alpha'_{\mathrm{i}}$ and $\alpha'_{j\,\mathrm{refr}}$ angles are opposite, then negative refraction occurs. The cases where $\alpha'_{\mathrm{i}} = 0$ or $\alpha'_{j\,\mathrm{refr}} = 0$ (which occurs when the wave propagates along the collinear axis in the first or the second medium or one of the media is isotropic), can be considered separately. In the first case, the incident wave will be orthogonal to the interface (which is not necessarily true for the refracted wave), while in the second case, the refracted wave (and not necessarily the incident one) will be orthogonal to the interface.

### 9.3 The emergence of two or more refracted rays

For the appearance of two refracted rays, it is necessary that the isofrequency dependence of the medium by which the wave is refracted cross the projecting line at two points where refraction condition (23) is satisfied. It is also evident that the same projecting line should, in this case, cross the isofrequency dependence of the initial medium at a point where incidence condition (21) is satisfied. One can formulate the following rule.

*For a given orientation of the interface with respect to the symmetry axis of the isofrequency dependence of the medium by which the wave is refracted, there are no fewer than two refracted rays if the isofrequency dependence of this medium possesses at least one of the following properties:* [37]

(1) *it consists of two curves, one of which is closed*;

(2) *it consists of three or more curves, and*

(3) *one of the curves of the isofrequency dependence has inflection points.*

Isofrequency dependences of optical crystals possess the first property from this list (see, for instance, Fig. 1b), which, for the appropriate polarization of the incident ray, causes birefringence [4, 5, 50].

Since the rule formulated above coincides with the rule describing the appearance of two reflected rays, comments on this rule can be found in the previous section. Furthermore, in what follows we consider an example where two refracted rays emerge due to the existence of the inflection point in the isofrequency dependence.

### 9.4 Positive and negative refraction

In order to formulate the conditions under which refraction of waves propagating from one anisotropic medium to another one is positive or negative (it is only important that refraction should indeed take place,[38] and not important which medium is initial and which is final), let us study the isofrequency dependences of both media from the viewpoint of extrema and singular points in the frame of reference $\Sigma'_k = \{0; k_b, k_n\}$. Let us first describe such a study for arbitrary isofrequency dependences and then consider it for the example of isofrequency dependences of a surface MSW refracted from a free ferrite film to an FDM structure.

[36] It may be convenient to introduce the index $n'_V$, for instance, for the study of wave refraction from a vacuum to an isotropic medium with negative $\varepsilon$ and $\mu$. In this case, one can assume $n'_V = -n$ and write down formula (37) in terms of $n'_V$.

[37] See footnote 27.

[38] For instance, one can theoretically consider refraction of an electromagnetic wave traveling from the air into a magnetic plate through its straight edge, by assuming that the refracted ray is an MSW. However, in reality such refraction is impossible.

Let the isofrequency dependence of some wave in the initial medium be described by the dispersion relation $F(k_{y1}, k_{z1}) = 0$, and in the second medium, by which the wave is refracted, by the equation $G(k_{y2}, k_{z2}) = 0$.[39] In order to describe both isofrequency dependences in the frame of reference $\Sigma'_k = \{0; k_b, k_n\}$, let us change the variables, by analogy with formula (32). Then, the isofrequency dependences of the wave propagating in the two media will be described in each medium by equations $F(k_b, k_n) = 0$ and $G(k_b, k_n) = 0$ or, if $k_n$ can be explicitly written, by the dependences $k_n = P(k_b)$ and $k_n = R(k_b)$. Further, let the range[40] $k_{b\,\min} < k_b < k_{b\,\max}$ be the overlap of two ranges of $k_b$: one of which corresponds to the location of the dependence $P(k_b)$, and the other to the location of the dependence $R(k_b)$. Let us divide the whole range $k_{b\,\min} < k_b < k_{b\,\max}$ into the intervals separated by points $k_{b1}, k_{b2}, \ldots, k_{bm}$ whereat either at least one of the derivatives $\partial P/\partial k_b$, $\partial R/\partial k_b$ is equal to zero or tends to infinity or one of the dependences $P(k_b)$, $R(k_b)$ exhibits a discontinuity. Apparently, each one of the intervals $k_{b\,\min} < k_b < k_{b1}$, $k_{b1} < k_b < k_{b2}, \ldots, k_{bm} < k < k_{b\,\max}$ and each point $k_{b1}, k_{b2}, \ldots, k_{bm}$ corresponds to a certain type of wave refraction, since at each point $k_{b1}, k_{b2}, \ldots, k_{bm}$ either the group velocity vector $\mathbf{V}_{\rm i}$ of the incident wave or the group velocity vector $\mathbf{V}_{\rm refr}$ of the refracted wave will be collinear or orthogonal to the $\mathbf{k}_{n0}$ vector (and to the normal vector $\mathbf{n}_0$).

When analyzing the dependences $P(k_b)$ and $R(k_b)$, one should keep in mind that, as a rule, they are not functions, and each $k_b$ value lying within one of the above-listed intervals may correspond to several values of $k_n$. Therefore, it is convenient to assume that in every such interval the dependences $P(k_b)$ and $R(k_b)$ consist of separate sections $P_0(k_b), P_1(k_b), P_2(k_b), \ldots$ and $R_1(k_b), R_2(k_b), R_3(k_b), \ldots$.

Apparently, for refraction to be enabled for some of the above-given intervals or values of $k_b$, it is necessary that in the given interval of $k_b$ (or for the given value of $k_b$) incidence condition (21) be satisfied for at least one section, and refraction condition (23) for another section (if only for one of the other sections). The section of the $P(k_b)$ dependence satisfying condition (21) and corresponding to the incident ray will be denoted as $P_0(k_b)$, and the section satisfying condition (23) and corresponding to the $j$th refracted ray will be denoted by $R_j(k_b)$.[41] Now, one can formulate the rule determining the character of refraction.

*In those intervals of $k_b$ values where within the section $R_j(k_b)$ of the isofrequency dependence, corresponding to the jth refracted ray, the derivative $\partial R_j/\partial k_b$ has the same sign as the derivative $\partial P_0/\partial k_b$ within the section $P_0(k_b)$ corresponding to the incident ray, the jth refracted ray undergoes positive refraction. Otherwise, if the signs of the derivatives $\partial R_j/\partial k_b$ and $\partial P_0/\partial k_b$ are opposite, then the jth refracted ray undergoes negative refraction. For those values of $k_b$ from the set $\{k_{b1}, k_{b2}, \ldots, k_{bm}\}$, at which the derivative $\partial P_0/\partial k_b = 0$, the incident ray is oriented normally to the interface, and if $\partial R_j/\partial k_b = 0$, then the jth refracted ray is normal to the interface.*

[39] Symmetry axes (optical axes) of the first and the second media are not necessarily parallel to each other; therefore, in the general case the dispersion relations of these media can be written down in different frames of reference.

[40] Hereinafter, see also remarks in the footnotes to Section 8.4.

[41] See footnote 27.

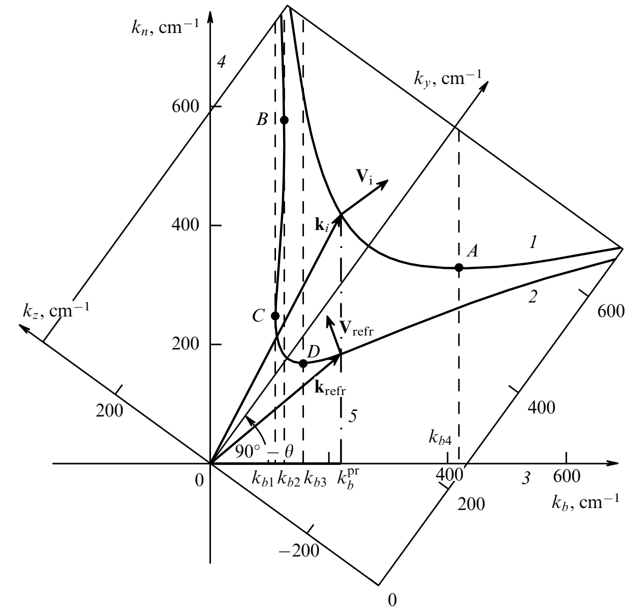


**Figure 10.** Example of determining the character of refraction: a surface MSW in the frame of reference connected with the interface and the normal: *1*, *2*, isofrequency curves; *3*, boundary of the film and the axis $k_b$ connected with it; *4*, normal to the boundary and the $k_n$-axis connected with it; *5*, projecting line; *A*, *B*, *C*, and *D*, extremum points of the isofrequency dependence in the reference system $k_b k_n$.

Let us briefly illustrate this rule using the specific example of a surface MSW refracted in traveling from a free ferrite film to an FDM structure (Fig. 10).[42] For the given orientation of the interface with respect to the common symmetry axis of the isofrequency dependences of both media, dependences $P(k_b)$ and $R(k_b)$ have six extremum points, four of which, labelled by *A*, *B*, *C*, and *D*, are shown in Fig. 10.[43] Depending on which vector $\mathbf{k}_{\rm i}$ (or its projection $k_b^{\rm pr}$ onto the interface) corresponds to the incident ray, the following cases of refraction arise:

(1) if the $k_b^{\rm pr}$ value lies within the interval $k_b^{\rm pr} \leqslant k_{b1}$, then the projecting line passing through the end of the $\mathbf{k}_{\rm i}$ vector crosses curve *2* of the $R(k_b)$ dependence at a single point which corresponds to positive refraction, since the derivatives $\partial P/\partial k_b$ and $\partial R/\partial k_b$ have the same signs [note that the $P(k_b)$ and $R(k_b)$ sections of isofrequency dependences corresponding to this case are located far from the origin and hence not seen in Fig. 10];

(2) if the $k_b^{\rm pr}$ value lies within the interval $k_{b1} < k_b^{\rm pr} < k_{b2}$, then the projecting line crosses curve *2* at three points falling on the sections $R_1(k_b)$ (below the *C* point), $R_2(k_b)$ (*BC* part of curve *2*), and $R_3(k_b)$ (above the *B* point) of the $R(k_b)$

[42] When describing this example of refraction, one can assume that all curves of the isofrequency dependences for a surface MSW both in the free film and in the FDM structure describe waves with the same polarization.

[43] Figure 10 is obtained from Fig. 6 by rotating the reference system $k_y k_z$ by an angle of $90° - \theta$; in doing so, the $\theta$ value in Fig. 10 is chosen differently from the one in Fig. 6 to demonstrate the appearance of additional extremum points caused by the existence of inflection points in the isofrequency dependence $R(k_b)$. Four extremum points shown in Fig. 10 reside in the positive semiplane of $k_y$ values. Another two extremum points are in the isofrequency curves lying in the negative semiplane of $k_y$ (similarly to point *B* in Fig. 9). However, for this orientation of the interface, conditions of incidence (21) and refraction (23) are satisfied for neither of the points falling on these curves; therefore, for clarity, in Fig. 10 we only show the positive semiplane of $k_y$ values.

dependence; the second point will not correspond to refraction since condition (23) is not satisfied for this point, and the two other points will correspond to positive refraction (derivatives $\partial P/\partial k_b$ and $\partial R/\partial k_b$ have the same sign); thus, there are two rays with positive refraction;

(3) if the $k_b^{\rm pr}$ value lies within the interval $k_{b2} \leqslant k_b^{\rm pr} < k_{b3}$, then, similarly to the first case, the projecting line crosses curve *2* at a single point which corresponds to positive refraction (derivatives $\partial P/\partial k_b$ and $\partial R/\partial k_b$ have the same sign);

(4) if $k_b^{\rm pr} = k_{b3}$, i.e., the projecting line passes through the $D$ point, then the refracted ray will be normal to the interface, although the incident ray is directed at an angle to the interface;

(5) if the $k_b^{\rm pr}$ value lies within the interval $k_{b3} < k_b^{\rm pr} < k_{b4}$, then the projecting line crosses curve *2* at a single point which corresponds to negative refraction (derivatives $\partial P/\partial k_b$ and $\partial R/\partial k_b$ have opposite signs);

(6) if $k_b^{\rm pr} = k_{b4}$, i.e., the end of the $\mathbf{k}_{\rm i}$ vector passes through the $A$ point, then the incident ray will be normal to the interface, although the refracted ray is directed at an angle to the interface, and

(7) if the $k_b^{\rm pr}$ value lies within the interval $k_b^{\rm pr} > k_{b4}$, then, similarly to cases 1 and 3, the projecting line crosses curve *2* at a single point which corresponds to positive refraction (derivatives $\partial P/\partial k_b$ and $\partial R/\partial k_b$ have the same sign).

It should be noted that this rule is valid regardless of whether the wave in one medium or both media is forward or backward. This statement can be clearly demonstrated using the example of an MSW refracted in traveling from a free film to an FDM structure. It is known that if the parameters of the FDM structure are changed in a certain way, its isofrequency dependence becomes even more stretched towards the origin [47], and some sections of this dependence start to describe backward waves even before a separate oval is formed, as in Fig. 2c. Apparently, if such a dependence (with sections describing backward waves) were used in Fig. 10 instead of curve *2*, the analysis given above would be almost the same, with the only exception that in the isofrequency dependence there could appear points for which the vectors $\mathbf{k}_{\rm refr}$ and $\mathbf{V}_{\rm refr}$ would be orthogonal to each other and the wave would not propagate.

Negative refraction of an MSW falling from a free ferrite film on an FDM structure has been examined in experiment [24].

A similar analysis can be carried out for wave refraction in any other anisotropic two-dimensional geometries.

### 9.5 The absence of the refracted ray

The absence of a refracted ray (also called total internal reflection) is a well-known phenomenon in isotropic media. It arises, if we adopt the optics terminology, when a wave is refracted in traveling from any optically more dense medium (where $k$ is larger) to a medium that is optically less dense (its $k$ is smaller). This phenomenon also occurs in anisotropic media where the limiting angle is not a constant for the two given materials but depends on the orientations of the symmetry axes of their isofrequency dependences with respect to the medium interface. One can easily see from Fig. 6 that when an MSW is refracted upon falling from a free film on an FDM structure, a refracted ray emerges in all cases (at all orientations of the interface), while refraction of an MSW traveling from an FDM structure to a free film does not always create a refracted ray. Probably, one cannot formulate a general rule that predicts the absence of the refracted ray and is valid for any anisotropic medium. One can only state that either condition (23) should be violated for the absence of the refracted ray (i.e., there should be no intersection points of the projecting line with the isofrequency dependence of the second medium, by which the wave is refracted) or the incident ray should be polarized with respect to the characteristic directions of the second medium in such a way that the intersection points satisfying condition (23) correspond to waves of zero amplitude (see, for instance, Ref. [5, p. 383]).

### 9.6 Irreversibility of the refracted ray path

*If the isofrequency dependence of at least one medium is not centrally symmetric, then the refracted ray path becomes irreversible*. This rule follows from the law of nonreciprocal propagation of a wave (see Section 6.4). Let us assume, for definiteness, that the isofrequency dependence of the medium by which the wave is refracted is not centrally symmetric (as in Fig. 6) and also that the incident and refracted rays are described by vectors $\mathbf{V}_{\rm i}$, $\mathbf{k}_{\rm i}$ and $\mathbf{V}_{\rm refr}$, $\mathbf{k}_{\rm refr}$, respectively. If there is a point in the isofrequency dependence of the FDM structure that corresponds to a ray with the wavevector $\mathbf{k}_{\rm i}'$ and group velocity vector $\mathbf{V}_{\rm i}'$ directed oppositely to the $\mathbf{V}_{\rm i}$ vector, then it will turn out that $\mathbf{k}_{\rm i}' \neq \mathbf{k}_{\rm refr}$ (this point will lie in curve *4* in the top left quadrant of the $k_y k_z$ plane in Fig. 6.) Therefore, if we now consider a ray with the parameters $\mathbf{k}_{\rm i}'$, $\mathbf{V}_{\rm i}'$ as the incident one and, by applying the momentum conservation law to the $\mathbf{k}_{\rm i}'$ vector, find the new refracted ray with the parameters $\mathbf{k}_{\rm refr}'$, $\mathbf{V}_{\rm refr}'$, then it will turn out that $\mathbf{k}_{\rm refr}' \neq \mathbf{k}_{\rm i}$ and $\mathbf{V}_{\rm refr}' \neq \mathbf{V}_{\rm i}$.

## 10. Conclusions

By analyzing the properties of various isofrequency dependences (often called sections of wavevector surfaces), described in some experimental and theoretical works, we have presented the laws of geometrical optics for two-dimensional geometries in anisotropic media. We have considered the effect on wave propagation, reflection, and refraction of such geometric and mathematical properties of isofrequency dependences as the existence of their asymptotes, inflection points, and centers of symmetry, the number of symmetry axes, single- or multi-valuedness of the dependences, and the existence and number of extremum points in the frame of reference connected with the normal to the medium interface. It was shown that if the isofrequency dependence of an anisotropic medium possesses some of these properties, then propagation, reflection, and refraction of waves can be accompanied by such phenomena as the impossibility of wave propagation in some directions or within some angular sector, nonreciprocality of propagation (when counter-propagating rays have different parameters), unidirectional propagation (when for a given ray there exists no counter-propagating ray and, in some cases, there is only a single direction, in the vicinity of which the energy can be transferred), negative reflection and refraction (when the incident, reflected, and refracted rays are on the same side of the normal to the interface), the emergence of two (or more) reflected or refracted rays, the absence of reflection altogether, and the irreversibility of ray paths in reflection or refraction. In particular, if the isofrequency dependence is not centrally symmetric, then reflection and refraction are nonreciprocal and manifest irreversibility of ray paths. If the isofrequency dependence is a function in some frame of reference (i.e., every abscissa value corresponds to a single ordinate value), the propagation of waves is unidirectional and, if the interface is parallel to the abscissa axis of this frame of reference, reflection is absent. The existence of inflection points in at least one curve of the isofrequency dependence always leads, for a certain orientation of the interface, to the appearance of two reflected rays. For a certain orientation of the interface and a certain polarization of the incident ray, two reflected rays also appear when the isofrequency dependence contains no fewer than two curves, one of which is closed. It was shown that, for a given orientation of the boundary between two media with respect to the characteristic directions of the anisotropic medium, one can find out whether reflection and refraction are positive or negative at various angles of incidence by searching for the extremums and singular points of the isofrequency dependence in the frame of reference connected with the normal to the interface.

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