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Eigenmode generation by a given current in anisotropic and gyrotropic media

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Contents

Abstract. **The theory of eigenmode generation by a given current was developed for a uniform, transparent, anisotropic and gyrotropic medium with a temporal and spatial dispersion. Different approaches were employed to determine the eigenmode dispersion relations and polarization vectors. A close interrelation was traced between the principal values and eigenvectors of the Maxwellian tensor and the properties of the linear eigenmodes of a given medium. The spectral energy density radiated in a given direction in the medium was calculated for different medium modes having different phase velocities and polarizations. Anisotropic factors were derived, which change the eigenmode radiation intensity in comparison with that for an isotropic medium with the same refractive index. Several typical examples were considered.**

1. Introduction

The problem of electromagnetic energy radiation by a given current in an anisotropic and gyrotropic medium with dispersion is nontrivial even under simplifying assumptions (a uniform medium, neglect of dissipation). The complexity of the problem in comparison with the similar problem for a vacuum or an isotropic medium stems from the fact that, in an anisotropic medium with dispersion, there exist, in the general case, a wealth of eigenmodes for which the dispersion relations and the geometry of polarization vectors may differ greatly. An analysis of the eigenmodes is therefore a necessary constituent part of the general radiation problem. Important special cases have been considered in the scientific literature, beginning with pioneering works (see, for instance, Refs [1 – 5]). Several helpful results and general relations concerning plasmas and crystal media are found in monographs and review papers [6 – 19]. However, the general solution to the problem and a comprehensive elucidation of the eigenmode generation problem for anisotropic and gyrotropic media are, as far as we know, lacking in the literature for general use (including learning aids [20 – 24]), although their significance for many practical purposes is evident.

To solve the problem formulated in the present paper we invoke both the permittivity tensor $\varepsilon_{\alpha\beta}(\omega, \mathbf{k})$ and the more general Maxwellian tensor $T_{\alpha\beta}(\omega, \mathbf{k})$ [see formula (7) in Section 2] which arises when the Maxwell equations are written down in the Fourier representation. A link is established between the medium eigenmodes and the principal values and eigenvectors of the Maxwellian tensor. Although there are only three eigenvectors of the Maxwellian tensor, they describe the complete set of polarization vectors of the eigenmodes, whose number is generally unlimited. The eigenvectors $\mathbf{b}(\omega, \mathbf{k})$ are shown to transform into polarization vectors $\mathbf{e}(\omega^\sigma(\mathbf{k}), \mathbf{k})$ when their arguments ω and $\mathbf{k}$ are assigned values that satisfy the dispersion relation $\omega = \omega^\sigma(\mathbf{k})$ corresponding to an eigenmode. These important relations, to our knowledge, are lacking in the literature, including the references cited above.

We derived general expressions for the spectral energy density radiated in a given direction. The integration rules were formulated for the singular expressions to satisfy the radiation principle. The general expressions were defined

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concretely and applied to the special cases of a medium without spatial dispersion, a uniaxial anisotropic medium, a uniaxial gyrotropic medium in different frequency ranges, etc.

So, let us calculate the spectral energy density radiated by some given current $\mathbf{j}(\mathbf{r}, t)$ in a certain direction in a uniform anisotropic medium with a negligible absorption, but with the inclusion of temporal and spatial dispersion. It is noteworthy that a dispersive medium, in principle, always exhibits absorption proportional to the anti-Hermitian part of the complex dielectric tensor. This fact follows explicitly from the Kramers – Kronig dispersion relations. However, absorption may be quite weak in certain frequency ranges. It is precisely these 'transparency windows' that we have in mind below. The radiation field will be considered at distances from the oscillation source that are shorter than the wave absorption length in the medium, and yet longer than the radiation formation zone.

We proceed from the energy balance in the transparent dispersion medium:

$$\frac{\partial w}{\partial t} + \nabla \mathbf{S} = -\mathbf{j}\,\mathbf{E}\,, \tag{1}$$

where w and $\mathbf{S}$ are the energy density and energy flux density of the electromagnetic field, respectively. The expressions for these quantities, which we will not employ below, may be found in Ref. [24]. The right-hand side of Eqn (1) may be treated as a source of field energy due to extraneous current, if the field $\mathbf{E}$ is generated by the same current. Having integrated both sides of Eqn (1) over the entire space and with respect to time we arrive at the field energy increment W during the whole life of the extraneous current:

$$W = -\int_{-\infty}^{\infty} \mathrm{d}t \int_V \mathbf{j}(\mathbf{r}, t)\,\mathbf{E}(\mathbf{r}, t)\,\mathrm{d}^3 r$$

$$= -\frac{2}{(2\pi)^4}\,\mathrm{Re}\int_0^{\infty}\mathrm{d}\omega \int \mathbf{j}^*(\omega, \mathbf{k})\,\mathbf{E}(\omega, \mathbf{k})\,\mathrm{d}^3 k\,. \tag{2}$$

Here, we passed on to the Fourier representation and made use of the properties of the Fourier components of an arbitrary real function $f^*(\omega, \mathbf{k}) = f(-\omega, -\mathbf{k})$. Equality (2) may be rewritten in the form

$$W = \int W_{\boldsymbol{\kappa}\omega}\,\mathrm{d}\omega\,\mathrm{d}\Omega\,, \tag{3}$$

where $\mathrm{d}\Omega$ is the solid angle of vector $\mathbf{k}$, and $W_{\kappa\omega}$ is the energy radiated by the extraneous current in the direction $\boldsymbol{\kappa} = \mathbf{k}/k$ at a frequency ω:

$$W_{\kappa\omega} = -\frac{2}{(2\pi)^4}\,\mathrm{Re}\int_0^{\infty} \mathbf{j}^*(\omega, \mathbf{k})\,\mathbf{E}(\omega, \mathbf{k}) k^2\,\mathrm{d}k\,. \tag{4}$$

The factor 2 and the sign denoting extraction of the real part in expressions (2) and (4) appeared because integration in expression (2) is performed only over positive frequencies.

2. Electric field calculations

We now calculate the electric field induced by an extraneous current $\mathbf{j}$ in an anisotropic medium. Let us consider a uniform medium whose electromagnetic properties at real values of $\mathbf{k}$ and ω are characterized by a Hermitian permittivity tensor $\varepsilon_{\alpha\beta}(\omega, \mathbf{k}) = \varepsilon^*_{\beta\alpha}(\omega, \mathbf{k})$ and a permeability $\mu = 1$. Electric vectors obey the relation

$$D_\alpha(\omega, \mathbf{k}) = \varepsilon_{\alpha\beta}(\omega, \mathbf{k})\,E_\beta(\omega, \mathbf{k})\,. \tag{5}$$

From the Maxwell vector equations it follows that the vector $\mathbf{E}$ of the macroscopic electric field in the Fourier representation satisfies the equation

$$T_{\alpha\beta}(\omega, \mathbf{k}) E_\beta(\omega, \mathbf{k}) = -\mathrm{i}\,\frac{4\pi}{\omega}\,j_\alpha(\omega, \mathbf{k})\,, \tag{6}$$

where the tensor

$$T_{\alpha\beta}(\omega, \mathbf{k}) = \varepsilon_{\alpha\beta}(\omega, \mathbf{k}) - \frac{c^2 k^2}{\omega^2}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) \tag{7}$$

will be referred to as Maxwellian for brevity [25].

We write down the solution of system (6) in terms of the inverse tensor $\widehat{T}^{-1}$ which satisfies the condition $T_{\alpha\beta}(\widehat{T}^{-1})_{\beta\nu} = \delta_{\alpha\nu}$:

$$E_\beta = -\mathrm{i}\,\frac{4\pi}{\omega}\,(\widehat{T}^{-1})_{\beta\nu}\,j_\nu\,. \tag{8}$$

As is known from linear algebra [26 – 29], for $\varDelta \neq 0$ one has

$$(\widehat{T}^{-1})_{\beta\nu} = \frac{\varDelta_{\nu\beta}}{\varDelta}\,. \tag{9}$$

Here, $\varDelta = |T_{\alpha\beta}|$ is the determinant of tensor $T_{\alpha\beta}$, and $\varDelta_{\beta\nu}$ are its algebraic adjuncts. The explicit form of the inverse tensor may turn out to be rather complicated; its different special cases for cool and hot magnetoactive plasmas were given in Ref. [25]. With the help of relationships (8) and (9) we write down the electric field in the form

$$E_\alpha = -\mathrm{i}\,\frac{4\pi}{\omega\varDelta}\,j_\nu \varDelta_{\nu\alpha}\,. \tag{10}$$

In the general case, this vector is not transverse relative to $\mathbf{k}$ in an anisotropic and gyrotropic medium. The induction vectors $\mathbf{D}$ and $\mathbf{B}$ are transversely aligned.

In view of expression (10), the energy loss of a *given current* $\mathbf{j}$ *in the medium* may, according to relation (4), be represented by the integral

$$W_{\boldsymbol{\kappa}\omega} = \frac{1}{2\pi^3}\,\mathrm{Re}\,\mathrm{i}\int_0^{\infty} \frac{j_\nu \varDelta_{\nu\alpha}\,j^*_\alpha}{\omega\varDelta(\omega, \mathbf{k})}\,k^2\,\mathrm{d}k\,. \tag{11}$$

The necessary and sufficient existence condition for the inverse tensor (9) is that the determinant $\varDelta$ is nonzero, $\varDelta \neq 0$. Since in the integration over real values of $\mathbf{k}$ and ω there are points in the integration path at which $\varDelta = 0$, it is required to introduce the rules of bypassing these points in the complex plane in such a way that the perturbations generated by extraneous current would asymptotically (at long distances) be represented by diverging spherical waves. A similar bypass rule is also introduced for a vacuum. To obtain the desired result in a medium, it would suffice to take into account weak damping, i.e., the small anti-Hermitian part of the permittivity tensor. We will revert to this issue when performing integrations, but prior to that we will derive a more lucid expression for the radiated energy and determine its relation to the eigenmodes of the medium.

3. Dispersion relations for the eigenmodes of an anisotropic and gyrotropic medium

The dispersion relations and polarization vectors $e_\beta(\omega, \mathbf{k})$ of the eigenmodes should be calculated from the system of homogeneous equations

$$T_{\alpha\beta}(\omega, \mathbf{k})\, e_\beta(\omega, \mathbf{k}) = 0\,, \qquad \alpha = 1, 2, 3\,. \tag{12}$$

The existence condition for a nontrivial solution of this system of equations is that its determinant is equal to zero:

$$\Delta(\omega, \mathbf{k}) = \left|T_{\alpha\beta}(\omega, \mathbf{k})\right| = 0\,. \tag{13}$$

The Hermitian tensor $\varepsilon_{\alpha\beta} = \varepsilon'_{\alpha\beta} + \mathrm{i}\varepsilon''_{\alpha\beta}$ has a symmetric, $\varepsilon'_{\alpha\beta} = \varepsilon'_{\beta\alpha}$, real part and an antisymmetric, $\varepsilon''_{\alpha\beta} = -\varepsilon''_{\beta\alpha}$, imaginary part. The latter may be written in terms of the gyration pseudovector g_α:

$$\varepsilon_{\alpha\beta} = \varepsilon'_{\alpha\beta} + \mathrm{i} e_{\alpha\beta\gamma} g_\gamma\,. \tag{14}$$

We take into consideration that the tensor determinant is invariant under spatial rotations and select the coordinate axes along the mutually perpendicular principal axes of the symmetric tensor $\varepsilon'_{\alpha\beta}$; we denote the tensor principal values as ε_1, ε_2, and ε_3. With the axes selected, the tensor $T_{\alpha\beta}$ takes the form

$$\widehat{T} = \begin{pmatrix} \varepsilon_1 - n^2(1-\kappa_1^2) & \mathrm{i}g_3 + n^2\kappa_1\kappa_2 & -\mathrm{i}g_2 + n^2\kappa_1\kappa_3 \\ -\mathrm{i}g_3 + n^2\kappa_1\kappa_2 & \varepsilon_2 - n^2(1-\kappa_2^2) & \mathrm{i}g_1 + n^2\kappa_2\kappa_3 \\ \mathrm{i}g_2 + n^2\kappa_1\kappa_3 & -\mathrm{i}g_1 + n^2\kappa_2\kappa_3 & \varepsilon_3 - n^2(1-\kappa_3^2) \end{pmatrix}, \tag{15}$$

where $\boldsymbol{\kappa} = \mathbf{k}/k$ is a unit vector in the direction of wave propagation, and $n = ck/\omega$ is the refractive index.

By equating the determinant Δ to zero we find the refractive indices for the eigenmodes of the medium under consideration. We expand the determinant to ascertain that the terms proportional to n^6 cancel out and the equation in n^2 takes on the form

$$\begin{aligned} &an^4 - \left[\varepsilon_1(\varepsilon_2+\varepsilon_3)\kappa_1^2 + \varepsilon_2(\varepsilon_1+\varepsilon_3)\kappa_2^2 + \varepsilon_3(\varepsilon_1+\varepsilon_2)\kappa_3^2 \right. \\ &\quad \left. + (\boldsymbol{\kappa}\mathbf{g})^2 - \mathbf{g}^2\right] n^2 + \varepsilon_1\varepsilon_2\varepsilon_3 - \varepsilon_1 g_1^2 - \varepsilon_2 g_2^2 - \varepsilon_3 g_3^2 = 0\,, \\ &a(\omega, \mathbf{k}) = \varepsilon_1\kappa_1^2 + \varepsilon_2\kappa_2^2 + \varepsilon_3\kappa_3^2\,. \end{aligned} \tag{16}$$

Equation (16) is a generalization of the Fresnel equation, well known in crystal optics, to the case of a gyrotropic medium (see Ref. [30]). The quantity $a(\omega, \mathbf{k}) = \varepsilon_{\alpha\beta}\kappa_\alpha\kappa_\beta$ is the permittivity longitudinal relative to the vector $\mathbf{k}$: $a = \varepsilon^{\mathrm{l}}(\omega, \mathbf{k})$.

If Eqn (16) is treated as a quadratic equation in the explicitly appearing quantity n^2, it is easy to find two roots n_1^2 and n_2^2 of this equation. This enables writing the Maxwellian tensor determinant (13) as a product of three factors:

$$\Delta = a(n^2 - n_1^2)(n^2 - n_2^2)\,; \tag{17}$$

setting any of the factors equal to zero leads to correct dispersion relations for possible eigenmodes.

The convenience of writing the determinant in the form of expression (17) manifests itself when the spatial dispersion is nonexistent, i.e., when the quantities ε_1, ε_2, ε_3, and g depend only on ω and not on the absolute value $|\mathbf{k}|$. Then, the relationships $n^2 = n_{1,2}^2$ turn out to be the *solutions* of the dispersion equation, so that in each direction (for a given $\boldsymbol{\kappa}$) two waves can propagate through the medium under consideration with two, generally different, phase velocities $v_{1,2} = c/n_{1,2}$. Here, $n_{1,2}(\omega, \boldsymbol{\kappa})$ are the positive solutions of the biquadratic equation (16), which depend only on the frequency and the propagation direction of the corresponding wave. The roots n_i^2 may be negative in some frequency ranges. For a Hermitian permittivity tensor, this signifies damping without dissipation, i.e., the absence of the corresponding mode. In this case, relation $a = 0$ corresponds to the oscillating modes of the medium, whose properties are independent of the magnitude of the wave vector, the electric vector in these modes being directed along the vector $\boldsymbol{\kappa}$, i.e., the oscillations are longitudinal.

In the presence of spatial dispersion, relations $a = 0$ and $n^2 = n_{1,2}^2$ are the *equations* for determining the refractive indices rather than the solutions of the dispersion equation, because a and $n_{1,2}$ themselves are functions of n. That is why, in principle, there is nothing to limit the number of eigenmodes, which are solutions of these equations. To determine the refractive indices in this case requires specifying the explicit dependence of the dielectric tensor on ω and $\mathbf{k}$.

4. Current energy losses by radiation

In this section we consider the case where the quantities $n_{1,2}(\omega, \boldsymbol{\kappa})$ are known and represent the refractive indices. We take advantage of formula (11) for the spectral energy density radiated in a given direction. In view of expression (17), the denominator of the integrand in integral (11) is written down as

$$\frac{1}{\omega\Delta} = \frac{\omega}{ac^2(n_1^2 - n_2^2)}\left(\frac{1}{k^2 - \omega^2 n_1^2/c^2} - \frac{1}{k^2 - \omega^2 n_2^2/c^2}\right). \tag{18}$$

Eventually, the expression for radiative losses (11) takes the form

$$\begin{aligned} W_{\kappa\omega} = \frac{1}{2\pi^3}\,\mathrm{Re}\,\mathrm{i}\int_0^\infty \frac{\omega j_\alpha \Delta_{\alpha\beta} j_\beta^*}{ac^2(n_1^2 - n_2^2)} \\ \times \left(\frac{1}{k^2 - \omega^2 n_1^2/c^2} - \frac{1}{k^2 - \omega^2 n_2^2/c^2}\right) k^2\,\mathrm{d}k\,. \end{aligned} \tag{19}$$

When performing integration we take into account the weak eigenmode damping resulting in the occurrence of small positive imaginary parts of refractive indices: $\mathrm{Im}\, n_\sigma^2 > 0$ for $\omega > 0$. These conditions give the rules of integration in the vicinity of singular points, because

$$\mathrm{Re}\,\frac{\mathrm{i}}{k^2 - \omega^2 n_\sigma^2/c^2}\bigg|_{\mathrm{Im}\, n_\sigma^2 \to +0} \to -\frac{\pi}{2k}\,\delta\left(k - \frac{\omega n_\sigma}{c}\right), \quad \sigma = 1, 2\,. \tag{20}$$

It must also be remembered that the tensor $\Delta_{\mu\nu}$ is Hermitian: $\Delta_{\mu\nu} = \Delta_{\nu\mu}^*$. That is why the convolution $j_\mu \Delta_{\mu\nu} j_\nu^*$ is real for real ω and $\mathbf{k}$. We integrate, in view of rules (20), expression (19) to obtain

$$W_{\kappa\omega} = -\frac{\omega^2}{4\pi^2 ac^3(n_1^2 - n_2^2)}\left(n_1 (j_\alpha \Delta_{\alpha\beta} j_\beta^*)_1 - n_2 (j_\alpha \Delta_{\alpha\beta} j_\beta^*)_2\right). \tag{21}$$

We now represent the algebraic adjuncts which appear in expression (21) in terms of the polarization vectors of the eigenmodes of the medium. We ascertain that the polarization vector may be written in the form

$$e_{\beta}^{(i)} = A^i \Delta_{\mu\beta}^{(i)} , \tag{22}$$

where i is the root number, A^i is the normalization constant, and the subscript μ takes an arbitrary value. On substituting the value of $n = n_i$ and solution (22) into system (12), it transforms into a system of identities. In this case, two equalities (with $\mu \neq \alpha$) are fulfilled irrespective of the value of n, and the third one (with $\mu = \alpha$) for $n = n_i$ is fulfilled because the determinant vanishes. Similarly, by taking advantage of the Hermitian character of the tensor $T_{\alpha\beta}$ we ascertain that the first subscript of the algebraic adjunct relates the components of the complex conjugate polarization vector e_{μ}^{*}. This permits expressing the algebraic adjuncts at $n = n_i$ in terms of the normalized polarization vectors:

$$\Delta_{\mu\nu}^{(i)} = C^{(i)} e_{\mu}^{*} e_{\nu} , \tag{23}$$

where $e_{\mu}^{*} e_{\mu} = 1$, and $C^{(i)} = \Delta_{\mu\mu}^{(i)}$ is a real normalization constant which may have different signs. We emphasize that the algebraic adjuncts reduce to polarization vectors *only* when the corresponding dispersion relation is fulfilled; in an arbitrary case, relationship (23) does not hold.

The spectral radiation density is, in view of the last-derived relations, written down in terms of polarization vectors:

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2 a(\omega, \boldsymbol{\kappa}) c^3 (n_2^2 - n_1^2)} \times \left(n_1 C^{(1)} \left| (e_{\mu}^{(1)*} j_{\mu}) \right|^2 - n_2 C^{(2)} \left| (e_{\mu}^{(2)*} j_{\mu}) \right|^2 \right) . \tag{24}$$

Formula (24) describes the conversion of the energy of extraneous current to the energy of electromagnetic radiation and must, in passing to the vacuum case ($\varepsilon_{\alpha\beta} \to \delta_{\alpha\beta}$), give the radiation of transverse vacuum modes. In this passage to the limit, however, in expression (24) there emerges a 0/0 type indeterminate form; to evaluate this indeterminate form, it is expedient to revert to the previous stages of the calculations.

For $\varepsilon_{\alpha\beta} \to \delta_{\alpha\beta}$ we turn to expression (15) to obtain the corresponding limiting representations for the determinant and its algebraic adjuncts:

$$\begin{aligned} &\Delta \to \Delta^0 = (1 - n^2)^2 , \\ &\Delta_{\alpha\beta} \to \Delta_{\alpha\beta}^0 = (1 - n^2)(\delta_{\alpha\beta} - n^2 \kappa_{\alpha} \kappa_{\beta}) . \end{aligned} \tag{25}$$

On substitution of relations (25) into formula (10), the common multipliers in the numerator and denominator cancel and we obtain the Fourier transform of the electric field induced by extraneous current in a vacuum:

$$\begin{aligned} \mathbf{E} &= -\mathrm{i} \, \frac{4\pi}{\omega(1 - n^2)} \big(\mathbf{j} - n^2 (\boldsymbol{\kappa} \mathbf{j}) \boldsymbol{\kappa} \big) \\ &= -\mathrm{i} \, \frac{4\pi\omega}{\omega^2 - c^2 k^2} \left(\mathbf{j} - \frac{c^2 (\mathbf{k}\,\mathbf{j}) \mathbf{k}}{\omega^2} \right) . \end{aligned} \tag{26}$$

We substitute value (26) for a field in a vacuum into formula (4) to find

$$W_{\kappa\omega} = \frac{\omega^3}{2\pi^3 c^4} \, \mathrm{Re}\,(-\mathrm{i}) \int_0^{\infty} \frac{\left| \mathbf{j}_{\perp}(\omega, \mathbf{k}) \right|^2 \mathrm{d}k}{k^2 - \omega^2 / c^2} . \tag{27}$$

Selection of the retarded solution is made by adding an infinitesimal positive imaginary part to the frequency ω, which leads to the well-known expression

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2 c^3} \left| \mathbf{j}_{\perp}(\omega, \boldsymbol{\kappa}) \right|^2 , \tag{28}$$

which describes the emission of two transversely polarized vacuum modes.

The above-discussed waves in a medium may be termed quasitransverse, because on escaping from the medium they transform into purely transverse vacuum modes and have a nonzero magnetic vector, $\mathbf{B} \neq 0$. The latter condition was used in the derivation of Eqn (6). Modes with $\mathbf{B} = 0$ and a purely longitudinal (relative to the direction of propagation) electric field may also occur in media. In the Fourier representation these modes satisfy the equations

$$\mathbf{k} \times \mathbf{E} = 0 , \qquad k_{\alpha} \varepsilon_{\alpha\beta} E_{\beta} = -\mathrm{i} \, \frac{4\pi}{\omega} \, k_{\alpha} \, j_{\alpha}(\omega, \mathbf{k}) . \tag{29}$$

We will seek for the field in the form $\mathbf{E} = E_{\parallel} \boldsymbol{\kappa}$, and from the above equations we will find

$$E_{\parallel} = -\mathrm{i} \, \frac{4\pi \boldsymbol{\kappa} \mathbf{j}(\omega, \mathbf{k})}{\omega \varepsilon^{1}(\omega, \mathbf{k})} , \qquad \varepsilon^{1}(\omega, \mathbf{k}) \equiv a(\omega, \mathbf{k}) = \kappa_{\alpha} \kappa_{\beta} \varepsilon_{\alpha\beta}(\omega, \mathbf{k}) . \tag{30}$$

Calculating the spectral density of the generated longitudinal waves by formula (4) leads, with the inclusion of the infinitesimal imaginary part of the longitudinal permittivity ε^1, to the following result

$$W_{\kappa\omega}^{1} = \frac{1}{2\pi^2 \omega} \int_0^{\infty} \mathrm{d}k \left| \mathbf{k}\, \mathbf{j}(\omega, \mathbf{k}) \right|^2 \delta\big(\varepsilon^{1}(\omega, \mathbf{k}) \big) . \tag{31}$$

Here, unlike formula (24), account should be taken of the spatial dispersion. In its absence, the oscillations generated by the current cannot propagate in the form of waves and transfer energy in space. Neglecting spatial dispersion, such oscillations may occur only for specific discrete frequencies. Integration in expression (31) can be performed by employing the relationship

$$\delta\big(\varepsilon^{1}(\omega, \mathbf{k}) \big) = \sum_{a} \frac{\delta\big(k - k_a(\omega, \kappa) \big)}{|\partial \varepsilon^{1} / \partial k|} , \tag{32}$$

where summation is performed over all roots of the equation $\varepsilon^{1}(\omega, \mathbf{k}) = 0$, which defines the dispersion laws for longitudinal waves in the medium, i.e., over all longitudinal waves possible in this medium. The number of such modes may be quite large.

5. Principal values and eigenvectors of the Maxwellian tensor and their relation to the eigenmodes of the medium

It turns out that in many cases electromagnetic phenomena are conveniently considered in the system of orthogonal unit vectors, in which the Maxwellian tensor is diagonal in form.

In particular, this allows a consistent inclusion of spatial dispersion in the generation of quasitransverse modes and enables investigating the polarization of these modes.

We take into consideration that the Hermitian character of the permittivity tensor $\varepsilon_{\alpha\beta}(\omega, \mathbf{k})$ implies that the tensor $T_{\alpha\beta}(\omega, \mathbf{k})$ is also Hermitian. As is known from linear algebra, the principal values $\lambda^{(m)}(\omega, \mathbf{k})$ $(m = 1, 2, 3)$ of a Hermitian tensor are real and its eigenvectors $\mathbf{b}^{(m)}(\omega, \mathbf{k})$ are generally complex and mutually orthogonal:

$$\begin{aligned} &\mathbf{b}^{(m)*}(\omega, \mathbf{k})\, \mathbf{b}^{(m)}(\omega, \mathbf{k}) = 1\,, \\ &\mathbf{b}^{(m)*}(\omega, \mathbf{k})\, \mathbf{b}^{(n)}(\omega, \mathbf{k}) = 0 \quad \text{for} \quad m \neq n\,. \end{aligned} \tag{33}$$

Both sets should be calculated from the system of algebraic equations

$$T_{\alpha\beta} b_\beta = \lambda b_\alpha\,, \qquad \alpha = 1, 2, 3\,. \tag{34}$$

The principal values must obey a cubic algebraic equation, which is obtained by setting to zero the determinant of system (34):

$$|T_{\alpha\beta} - \lambda \delta_{\alpha\beta}| = 0\,. \tag{35}$$

Once Eqns (34) and (35) are solved, the tensor $T_{\alpha\beta}$ obtained from the Maxwell equations and its inverse tensor can be expressed in terms of the principal values and the complex eigenvectors:

$$T_{\alpha\beta}(\omega, \mathbf{k}) = \sum_{m=1}^{3} \lambda^{(m)}(\omega, \mathbf{k})\, b_\alpha^{(m)}(\omega, \mathbf{k})\, b_\beta^{(m)*}(\omega, \mathbf{k})\,, \tag{36}$$

$$(\widehat{T}^{-1})_{\alpha\beta} = \sum_{m=1}^{3} \frac{1}{\lambda^{(m)}(\omega, \mathbf{k})}\, b_\alpha^{(m)}(\omega, \mathbf{k})\, b_\beta^{(m)*}(\omega, \mathbf{k})\,. \tag{37}$$

On multiplying these tensors we obtain

$$\begin{aligned} &T_{\alpha\beta}(\omega, \mathbf{k})(\widehat{T}^{-1})_{\beta\nu}(\omega, \mathbf{k}) \\ &\quad = b_\alpha^{(1)} b_\nu^{(1)*} + b_\alpha^{(2)} b_\nu^{(2)*} + b_\alpha^{(3)} b_\nu^{(3)*} = \delta_{\alpha\nu}\,. \end{aligned}$$

The last equality follows from the completeness property of the Hermitian tensor eigenvectors.

It is significant that the eigenvectors $\mathbf{b}^{(m)}(\omega, \mathbf{k})$ in an anisotropic medium are essentially dependent on the frequency and the wave vector, with the result that this case is substantially different from the cases of vacuum and isotropic media. In an isotropic medium, polarization degeneracy occurs, making it possible to arbitrarily select the electromagnetic-wave polarization vectors in the plane perpendicular to the wave vector. An implication of this degeneracy is the possibility of selecting the same eigenvector basis for perturbations with any frequencies for a given direction $\mathbf{k}$. As a result, eigenvectors with different frequencies and wavenumbers, and not only with equal ones, turn out to be mutually orthogonal in an isotropic medium:

$$\mathbf{b}^{(m)}(\omega, k, \boldsymbol{\kappa})\, \mathbf{b}^{(n)*}(\omega', k', \boldsymbol{\kappa}) = 0 \quad \text{for} \quad m \neq n\,. \tag{38}$$

Polarization degeneracy arises from the symmetry of an isotropic medium about rotations by any angle in the plane perpendicular to the vector $\boldsymbol{\kappa}$. In an anisotropic medium, this symmetry is absent in the general case (although it may occur relative to the preferred directions), and the orthogonality of the eigenvectors of the Maxwellian tensor is ensured only for coinciding sets of ω and $\mathbf{k}$ [see expressions (33)].

Let us determine the link between the quantities introduced in the foregoing and the electromagnetic eigenmodes of an anisotropic medium. We will use the notation of the tensor in terms of its principal values (36) and express in these terms the determinant of the system:

$$\Delta(\omega, \mathbf{k}) = \lambda^{(1)}(\omega, \mathbf{k})\, \lambda^{(2)}(\omega, \mathbf{k})\, \lambda^{(3)}(\omega, \mathbf{k})\,. \tag{39}$$

Condition (13) that the determinant is equal to zero signifies that at least one of the principal values of the tensor should vanish:

$$\lambda^{(m)}(\omega, \mathbf{k}) = 0, \qquad m = 1, 2, 3\,. \tag{40}$$

Equations (40) give the dispersion relations $\omega^\sigma(\mathbf{k})$ for the eigenmodes of the medium. The number of these modes in the presence of spatial dispersion is unlimited in the general case.

Notice that both notation forms of the determinant $\Delta(\omega, \mathbf{k})$, expressions (17) and (39), represent it in the form of three factors. This leads to the temptation to identify the principal values $\lambda^{(m)}$ of the Maxwellian tensor with the factors a, $(n^2 - n_1^2)$, and $(n^2 - n_2^2)$ [31]. But this identification would be incorrect, because two other requisite relations would be violated in this case:

$$\begin{aligned} &\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = T_{11} + T_{22} + T_{33}\,, \\ &\lambda^{(1)}\lambda^{(2)} + \lambda^{(1)}\lambda^{(3)} + \lambda^{(2)}\lambda^{(3)} = \Delta_{11} + \Delta_{22} + \Delta_{33}\,, \end{aligned} \tag{41}$$

which become evident when determinant (35) is expanded. We return to the determination of the principal values $\lambda^{(m)}$ in the subsequent discussion.

Let us now express the eigenvector components in terms of the real basis vectors of the Cartesian coordinate system in which Maxwellian tensor (15) was written. Consider for definiteness the eigenvector which corresponds to the first principal number $\lambda = \lambda^{(1)}$. Then, by multiplying tensor (15) by $b_\beta^{(1)}$ we obtain, in view of expressions (33), the equation

$$T_{\alpha\beta} b_\beta^{(1)} = \lambda^{(1)} b_\alpha^{(1)}\,, \tag{42}$$

which is conveniently rewritten as

$$\tilde{T}_{\alpha\beta}\, b_\beta^{(1)} \equiv (T_{\alpha\beta} - \lambda^{(1)}\delta_{\alpha\beta}) b_\beta^{(1)} = 0\,. \tag{43}$$

Equation (43) always has a solution, because by definition of the principal value the determinant $\tilde{\Delta}$ of the tensor $\tilde{T}_{\alpha\beta}$ is always equal to zero. Writing equation (43) so as to solve it for the components we arrive at

$$\tilde{T}_{11} b_1^{(1)} + \tilde{T}_{12} b_2^{(1)} + \tilde{T}_{13} b_3^{(1)} = 0\,, \tag{44}$$

$$\tilde{T}_{21} b_1^{(1)} + \tilde{T}_{22} b_2^{(1)} + \tilde{T}_{23} b_3^{(1)} = 0\,. \tag{45}$$

We do not give the third equation, because it is a linear combination of these two owing to the zero value of the determinant $\tilde{\Delta}$. In these equations we express the y- and z-components of vector $\mathbf{b}^{(1)}$ in terms of the x-component $b_1^{(1)}$ to obtain

$$b_2^{(1)} = \frac{\tilde{T}_{21}\tilde{T}_{13} - \tilde{T}_{23}\tilde{T}_{11}}{\tilde{T}_{12}\tilde{T}_{23} - \tilde{T}_{13}\tilde{T}_{22}}\, b_1^{(1)}\,, \tag{46}$$

$$b_3^{(1)} = \frac{\tilde{T}_{12}\tilde{T}_{21} - \tilde{T}_{11}\tilde{T}_{22}}{\tilde{T}_{12}\tilde{T}_{23} - \tilde{T}_{13}\tilde{T}_{22}}\, b_1^{(1)}\,. \tag{47}$$

Considering that the combinations of the $\tilde{T}_{\alpha\beta}$ tensor components in the numerator and denominator of Eqns (46) and (47) are the algebraic adjuncts of the elements of the third line of tensor $\tilde{T}_{\alpha\beta}$ ($\tilde{\Delta}_{31} = \tilde{T}_{12}\tilde{T}_{23} - \tilde{T}_{13}\tilde{T}_{22}$, $\tilde{\Delta}_{32} = \tilde{T}_{21}\tilde{T}_{13} - \tilde{T}_{23}\tilde{T}_{11}$, and $\tilde{\Delta}_{33} = \tilde{T}_{11}\tilde{T}_{22} - \tilde{T}_{12}\tilde{T}_{21}$), we represent Eqns (46) and (47) in a compact form:

$$b_2^{(1)} = \frac{\tilde{\Delta}_{32}}{\tilde{\Delta}_{31}}\, b_1^{(1)}, \qquad b_3^{(1)} = \frac{-\tilde{\Delta}_{33}}{\tilde{\Delta}_{31}}\, b_1^{(1)}. \tag{48}$$

This allows writing the vector $\mathbf{b}^{(1)}$ in the form of a decomposition into the real unit vectors of the initial Cartesian coordinate system:

$$\mathbf{b}^{(1)} = C(\tilde{\Delta}_{31}\mathbf{e}_x + \tilde{\Delta}_{32}\mathbf{e}_y - \tilde{\Delta}_{33}\mathbf{e}_z), \tag{49}$$

where the constant C is determined from the condition that the vector $\mathbf{b}^{(1)}$ is normalized to unity, so that

$$\mathbf{b}^{(1)} = \frac{\tilde{\Delta}_{31}\mathbf{e}_x + \tilde{\Delta}_{32}\mathbf{e}_y - \tilde{\Delta}_{33}\mathbf{e}_z}{\sqrt{|\tilde{\Delta}_{31}|^2 + |\tilde{\Delta}_{32}|^2 + |\tilde{\Delta}_{33}|^2}}. \tag{50}$$

The other two eigenvectors, $\mathbf{b}^{(2)}$ and $\mathbf{b}^{(3)}$, are expressed in a similar manner, but the tensor $\tilde{T}_{\alpha\beta}$ should, in place of $\lambda^{(1)}$, contain the principal values $\lambda^{(2)}$ and $\lambda^{(3)}$, respectively.

Now let us determine the relation between the polarization vector $\mathbf{e}^{\sigma}(\omega, \mathbf{k})$ of a given eigenmode and the eigenvectors $\mathbf{b}^{(m)}(\omega, \mathbf{k})$ of the Maxwellian tensor. By substituting expression (36) for $T_{\alpha\beta}(\omega, \mathbf{k})$ in the homogeneous equation (12), we arrive at

$$\lambda^{(1)} b_{\alpha}^{(1)}\, \mathbf{b}^{(1)}\mathbf{e}^{\sigma} + \lambda^{(2)} b_{\alpha}^{(2)}\, \mathbf{b}^{(2)}\mathbf{e}^{\sigma} + \lambda^{(3)} b_{\alpha}^{(3)}\, \mathbf{b}^{(3)}\mathbf{e}^{\sigma} = 0, \tag{51}$$

where the arguments $(\omega, \mathbf{k})$ of variables $\lambda^{(m)}$, $\mathbf{b}^{(m)}$, and $\mathbf{e}^{\sigma}$ were omitted for brevity.

Consider for definiteness the mode which corresponds to the condition $\lambda^{(1)}(\omega, \mathbf{k}) = 0$ (there may be several such modes). In this case, $\omega = \omega^{\sigma}(\mathbf{k})$, $\lambda^{(2)}(\omega^{\sigma}, \mathbf{k}) \neq 0$, and $\lambda^{(3)}(\omega^{\sigma}, \mathbf{k}) \neq 0$. Then, equality (51) will be fulfilled only provided the polarization vector $\mathbf{e}^{\sigma}(\omega^{\sigma}, \mathbf{k})$ is orthogonal to the eigenvectors $\mathbf{b}^{(2)}(\omega^{\sigma}, \mathbf{k})$ and $\mathbf{b}^{(3)}(\omega^{\sigma}, \mathbf{k})$. This signifies that the polarization vector $\mathbf{e}^{\sigma}(\omega^{\sigma}, \mathbf{k})$ simply coincides (in view of normalization to unity) with the eigenvector $\mathbf{b}^{(1)}(\omega^{\sigma}, \mathbf{k})$ of the Maxwellian tensor $T_{\alpha\beta}(\omega, \mathbf{k})$. Attention is drawn to the fact that the other two eigenvectors, $\mathbf{b}^{(2)}(\omega^{\sigma}, \mathbf{k})$ and $\mathbf{b}^{(3)}(\omega^{\sigma}, \mathbf{k})$, do not represent the polarization vectors of some eigenmodes of the medium. They would turn into polarization vectors only under the conditions $\lambda^{(2)}(\omega, \mathbf{k}) = 0$ and $\lambda^{(3)}(\omega, \mathbf{k}) = 0$, respectively, i.e., when the dispersion law for the eigenmode with the same number is fulfilled, and not in an arbitrary case. In particular, the polarization vectors of the ordinary and extraordinary modes of equal frequency propagating through an anisotropic medium in the same direction are *nonorthogonal*, because they differ in wave vector magnitude (due to the difference in the refractive indices).

Eventually, we showed that the polarization vectors of the eigenmodes of a medium are constructed from three eigenvectors $\mathbf{b}^{(m)}(\omega^{\sigma}, \mathbf{k})$ $(m = 1, 2, 3)$ of the Maxwellian tensor by assigning their arguments ω and $\mathbf{k}$ the values corresponding to the dispersion relation $\omega = \omega^{\sigma}(\mathbf{k})$ for the corresponding eigenmode.

Lastly, let us obtain in explicit form the expressions for the polarization vectors of the electromagnetic eigenmodes of a given anisotropic medium. For some eigenmode, say, with number 1, we have $\lambda^{(1)} = 0$, and therefore the tensor $\tilde{T}_{\alpha\beta}$ reduces merely to the Maxwellian tensor $T_{\alpha\beta}$. Therefore, the polarization vector $\mathbf{e}^{(1)}$ is expressed similarly to $\mathbf{b}^{(1)}$ in terms of the algebraic adjuncts of the Maxwellian tensor (i.e., in terms of the quantities Δ_{ab} without a tilde):

$$\mathbf{e}^{(1)} = \frac{\Delta_{31}\mathbf{e}_x + \Delta_{32}\mathbf{e}_y - \Delta_{33}\mathbf{e}_z}{\sqrt{|\Delta_{31}|^2 + |\Delta_{32}|^2 + |\Delta_{33}|^2}}. \tag{52}$$

Of course, expression (52) is obtained from formula (50) on substituting into it $\lambda^{(1)} = 0$ and taking into account in the remaining terms the dispersion law corresponding to this condition. The polarization vector $\mathbf{e}^{(2)}$ of the second eigenmode is expressed similarly to expression (52) but under the condition $\lambda^{(2)} = 0$ and with the dispersion law corresponding to this condition. It may turn out that the polarization vectors are conveniently decomposed into other sets of algebraic adjuncts, for instance

$$\mathbf{e}^{(1)} = \frac{-\Delta_{11}\mathbf{e}_x + \Delta_{12}\mathbf{e}_y + \Delta_{13}\mathbf{e}_z}{\sqrt{|\Delta_{11}|^2 + |\Delta_{12}|^2 + |\Delta_{13}|^2}} = \frac{\Delta_{21}\mathbf{e}_x - \Delta_{22}\mathbf{e}_y + \Delta_{23}\mathbf{e}_z}{\sqrt{|\Delta_{21}|^2 + |\Delta_{22}|^2 + |\Delta_{23}|^2}}. \tag{53}$$

The polarization vectors of different eigenmodes of anisotropic media, including magnetoactive plasma, are given in many monographs and review papers (see, for instance, Refs [6, 7, 19, 25]). In particular, monograph [6, p. 59] gives the components of the polarization vectors for several specific eigenmodes of magnetoactive plasma. This monograph makes extensive use of quantum-mechanical language and introduces, in particular, the quantity $N_k^{\sigma}(\mathbf{r}, t)$ — the number density of the plasmons of a given mode σ with the wave vector $\mathbf{k}$. Use is also made of the notion of permittivity $\varepsilon_k^{\sigma} \equiv \varepsilon^{\sigma}(\mathbf{k})$ for the given mode σ, which already includes the dispersion relation for the wave under consideration. In our paper we restrict ourselves to classical electrodynamics and do not resort to the notion of plasmons. However, the permittivity $\varepsilon^{\sigma}(\mathbf{k})$ of individual modes can naturally be introduced in our treatment, too. We shall show how this can be done.

From relationships (36) and (33) we express, for the general case, the eigenvalues of the Maxwellian tensor in terms of its eigenvectors and the permittivity tensor:

$$\lambda^{(m)}(\omega, \mathbf{k}) = \varepsilon^{(m)}(\omega, \mathbf{k}) - \frac{c^2 k^2}{\omega^2}, \tag{54}$$

where

$$\varepsilon^{(m)}(\omega, \mathbf{k}) = b_{\alpha}^{(m)*}(\omega, \mathbf{k})\, \varepsilon_{\alpha\beta}(\omega, \mathbf{k})\, b_{\beta}^{(m)}(\omega, \mathbf{k}) + \frac{c^2}{\omega^2}\, k_{\alpha} b_{\alpha}^{(m)*}(\omega, \mathbf{k})\, k_{\beta} b_{\beta}^{(m)}(\omega, \mathbf{k}). \tag{55}$$

Equating eigenvalue (54) to zero corresponds to the dispersion law which can define several eigenmodes of a medium when spatial dispersion is taken into account. For every specific mode, as shown above, the eigenvector $\mathbf{b}$ turns into the polarization vector $\mathbf{e}$ of this mode and the dispersion relation (40) takes, on substituting expression (54) into it, the form

$$\omega^2 \varepsilon^{(m)}(\omega, \mathbf{k}) - c^2 k^2 = 0. \tag{56}$$

By the index σ we denote the eigenmode frequencies which are generated by the eigenvalue $\lambda^{(m)}(\omega, \mathbf{k})$ and are calculated from Eqn (56) to obtain the quantities

$$\varepsilon_\sigma^{(m)}(\mathbf{k}) \equiv \varepsilon^{(m)}\big(\omega_m^\sigma(\mathbf{k}), \mathbf{k}\big) = e_\alpha^{(m)*}\varepsilon_{\alpha\beta}e_\beta^{(m)} + \frac{c^2}{k^2}\left|(\mathbf{k}\,\mathbf{e}^{(m)})\right|^2 . \tag{57}$$

Here, on the right-hand side the frequency ω is everywhere replaced by the mode frequency $\omega_m^\sigma(\mathbf{k})$, i.e., practical use of these relations can be made only when the dispersion law is known for the mode under consideration. Quantities (57), as follows from Eqn (56), are the squares of the refractive indices corresponding to individual modes, and they may therefore be termed the permittivities by analogy with the case of an isotropic medium. Of course, the two indices, m and σ, may be combined into one, as was done in monograph [6], and the permittivity of the given mode may be denoted by $\varepsilon^\sigma(\mathbf{k})$.

6. Spectral energy density of the generated modes

We revert to calculations of the radiated energy. With the help of formulas (4), (8), and (37) we obtain

$$W_{\kappa\omega} = -\frac{1}{2\pi^3\omega}\,\mathrm{Im}\int_0^\infty k^2\,\mathrm{d}k \sum_{m=1}^{3} \frac{\left|\mathbf{b}^{(m)*}(\omega, \mathbf{k})\,\mathbf{j}(\omega, \mathbf{k})\right|^2}{\lambda^{(m)}(\omega, \mathbf{k})} . \tag{58}$$

Recall that the Maxwellian tensor is Hermitian in a nonabsorbing medium, and its principal values $\lambda^{(m)}(\omega, \mathbf{k})$ are real. This signifies that for any values $\lambda^{(m)}(\omega, \mathbf{k}) \neq 0$ the integral in expression (58) is also real, so that its imaginary part (and hence the radiated energy) vanishes. An imaginary contribution emerges only from the integration in the vicinity of the points at which the quantities $\lambda^{(m)}(\omega, \mathbf{k})$ vanish. However, it is from the conditions $\lambda^{(m)}(\omega, \mathbf{k}) = 0$ that the dispersion laws of the eigenmodes, $\omega = \omega_m^\sigma(\mathbf{k})$, result. Therefore, it is evident that the energy losses of extraneous current $\mathbf{j}(\omega, \mathbf{k})$ in a nonabsorbing medium are entirely due to the excitation of the eigenmodes of this medium (including the traveling high-frequency electromagnetic waves which escape to infinity, as well as all kinds of modes which vanish when passing to empty space).

To take into account the imaginary contributions to integral (58), we introduce weak mode damping in the medium and assign a small positive imaginary part to the refractive indices: $n \to n + \mathrm{i}\epsilon$, $\epsilon \to +0$. Then we will obtain

$$\lambda^{(m)}(\omega, \mathbf{k}) = \lambda^{(m)}\left(\frac{ck}{n + \mathrm{i}\epsilon}, \mathbf{k}\right) \approx \lambda^{(m)}(\omega - \mathrm{i}\gamma, \mathbf{k}) \approx \lambda^{(m)}(\omega, \mathbf{k}) - \mathrm{i}\gamma\,\frac{\partial\lambda^{(m)}}{\partial\omega} .$$

We next resort to the Sokhotskii formula

$$\mathrm{Im}\,\frac{1}{\lambda^{(m)}(\omega - \mathrm{i}\gamma, \mathbf{k})}\bigg|_{\gamma\to 0} \to +\pi\,\mathrm{sign}\left(\frac{\partial\lambda^{(m)}}{\partial\omega}\bigg|_{\omega=\omega_m^\sigma(\mathbf{k})}\right)\delta\big(\lambda^{(m)}(\omega, \mathbf{k})\big) . \tag{59}$$

Considering that the eigenvectors $\mathbf{b}^{(m)}(\omega, \mathbf{k})$ coincide with the polarization vectors of the corresponding mode when the dispersion law $\lambda^{(m)}(\omega, \mathbf{k}) = 0$ is fulfilled, we find the energy expended by the extraneous current for the generation of medium eigenmodes:

$$W_{\kappa\omega} = \sum_{m=1}^{3} W_{\kappa\omega}^{(m)} , \tag{60}$$

where $W_{\kappa\omega}^{(m)}$ is the energy emitted in the form of an eigenmode with index m:

$$W_{\kappa\omega}^{(m)} = -\frac{1}{2\pi^2\omega}\int_0^\infty k^2\,\mathrm{d}k\,\left|\mathbf{e}^{(m)*}(\omega, \mathbf{k})\,\mathbf{j}(\omega, \mathbf{k})\right|^2 \times \mathrm{sign}\left(\frac{\partial\lambda^{(m)}}{\partial\omega}\bigg|_{\omega=\omega_m^\sigma(\mathbf{k})}\right)\delta\big(\lambda^{(m)}(\omega, \mathbf{k})\big) . \tag{61}$$

Therefore, the resulting expressions represent the decomposition of the radiated energy into the medium eigenmodes. We note that every dispersion equation may have several roots, and therefore the number of terms in sum (60) may be greater than three.

In principle, expression (61) permits finding the generation energy of mode m, because all quantities that appear in it can be calculated by performing algebraic operations, and the calculation of integrals with delta functions presents no fundamental problems. However, these calculations for an arbitrary anisotropic medium are rather cumbersome. Especially complicated in the general case are Cardano formulas for the solution of cubic equation (35) which defines the principal values of the Maxwellian tensor. Simplifications emerge only when some additional symmetry properties are inherent in the medium under consideration.

Nevertheless, the problem of finding the principal values of the Maxwellian tensor may be simplified (reduced to the solution of a quadratic equation) when one of these numbers tends to zero. In the radiation problem this turns out to be sufficient, because the vanishing of the principal value corresponds, according to condition (40), to the excitation of medium modes.

Writing equation (35) so as to solve it for the Maxwellian tensor components and resorting to the representation (17) of its determinant, we derive the characteristic equation for the determination of the principal values λ:

$$\lambda^3 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 2n^2)\lambda^2 + \left[n^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + a)n^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3 - g^2\right]\lambda - a(n^2 - n_1^2)(n^2 - n_2^2) = 0 . \tag{62}$$

As noted in the foregoing, the principal values in the general case do not reduce to a, $(n^2 - n_1^2)$, or $(n^2 - n_2^2)$ and are found by solving characteristic equation (62). We consider the special case of interest when dispersion law (40) is fulfilled for one of the medium eigenmodes, for instance, $n^2 = n_1^2$. Then, the constant term in Eqn (62) vanishes and λ can be factorized, so that the cubic equation reduces to a quadratic of the form

$$\lambda\Big\{\lambda^2 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 2n_1^2)\lambda + \left[n_1^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + a)n_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3 - g^2\right]\Big\} = 0 , \tag{63}$$

whose solutions are easily found:

$$\lambda^{(1)} = 0\,, \tag{64}$$

$$\lambda^{(2,3)} = \frac{1}{2}\Big[\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 2n_1^2 \pm \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 4g^2 - 2(\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3) + 4an_1^2}\,\Big]\,. \tag{65}$$

According to expression (61), the contribution to radiation is made only by the values of $\lambda = 0$, and in this case we are therefore concerned with the root $\lambda^{(1)}$ whose behavior as $n^2 \to n_1^2$ must be defined more precisely. To this end we represent equation (62) as

$$\lambda\Big\{\lambda^2 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 2n^2)\lambda + \big[n^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + a)n^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3 - g^2\big]\Big\} = a(n^2 - n_1^2)(n^2 - n_2^2)\,. \tag{66}$$

Condition $\lambda \to 0$ permits putting $\lambda = 0$ in braces in Eqn (66). After that, the vanishing principal value $\lambda^{(1)}$ takes on the form

$$\lambda^{(1)} = \frac{a(n_1^2 - n_2^2)}{n_1^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + a)n_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3 - g^2} \times (n^2 - n_1^2) = \frac{a(n_1^2 - n_2^2)}{\lambda^{(2)}\lambda^{(3)}}(n^2 - n_1^2)\,, \tag{67}$$

i.e., the principal value $\lambda^{(1)}$ is proportional to the difference $n^2 - n_1^2$ with a factor which is generally not equal to unity; the latter equality in formula (67) was written with the use of the Vièta theorem applied to Eqn (63):

$$\lambda^{(2)}\lambda^{(3)} = n_1^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + a)n_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3 - g^2\,, \tag{68}$$

i.e., finding the eigenvalue $\lambda^{(1)}$ does not require finding the other two eigenvalues separately; all we need to know is their product (68). It should be noted that one can easily verify, considering the explicit form of formulas (65) and (67), that all the necessary relationships (41) for the principal values of the Maxwellian tensor are fulfilled.

The second principal value, which is determined in a similar way, has the form

$$\lambda^{(2)} = \frac{a(n_2^2 - n_1^2)}{\lambda^{(1)}\lambda^{(3)}}(n^2 - n_2^2)\,,\qquad \lambda^{(1)}\lambda^{(3)} = n_2^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + a)n_2^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3 - g^2\,. \tag{69}$$

Finally, the third principal value is obtained under the condition that $a = 0$ and corresponds to the emission of longitudinal medium eigenmodes considered at the end of Section 4. As already indicated, to correctly describe the generation of longitudinal modes requires the inclusion of spatial dispersion. Only two quasitransverse modes can be generated in the absence of spatial dispersion, as shown in Section 4. Then, with the help of formulas (67) and (69), taking into account the small imaginary parts of the refractive indices n_1 and n_2 we find

$$\mathrm{Im}\,\frac{1}{\lambda^{(1)}} = \frac{\pi\omega\lambda^{(2)}\lambda^{(3)}}{2acn_1(n_1^2 - n_2^2)}\,\delta\Big(k - \frac{\omega n_1}{c}\Big)\,,\qquad \mathrm{Im}\,\frac{1}{\lambda^{(2)}} = \frac{\pi\omega\lambda^{(1)}\lambda^{(3)}}{2acn_2(n_2^2 - n_1^2)}\,\delta\Big(k - \frac{\omega n_2}{c}\Big)\,. \tag{70}$$

We calculate the total radiation energy of the quasitransverse modes from formula (58):

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2 a(\omega, \boldsymbol{\kappa})c^3(n_2^2 - n_1^2)} \times \Big(n_2\lambda^{(1)}\lambda^{(3)}\big|_{n=n_2}\big|(e_\mu^{(2)*}j_\mu)\big|^2 - n_1\lambda^{(2)}\lambda^{(3)}\big|_{n=n_1}\big|(e_\mu^{(1)*}j_\mu)\big|^2\Big)\,. \tag{71}$$

Expression (71) is consistent with the previously derived formula (24) when the relation between the constants is taken into account: $C^{(1)} = \lambda^{(2)}\lambda^{(3)}|_{n=n_1}$, $C^{(2)} = \lambda^{(1)}\lambda^{(3)}|_{n=n_2}$. Attention is drawn to the following fact: when the refractive indices of the anisotropic medium are close to unity (the case considered in book [31]), the factors

$$F_1 = \frac{\lambda^{(2)}\lambda^{(3)}|_{n=n_1}}{a(n_2^2 - n_1^2)}\,,\qquad F_2 = \frac{\lambda^{(1)}\lambda^{(3)}|_{n=n_2}}{a(n_1^2 - n_2^2)} \tag{72}$$

tend to unity. Similarly, these factors take the unit value in empty space and in an isotropic medium. However, in the general case of an anisotropic and gyrotropic medium these factors may significantly depart from unity, with the effect that the radiation intensity and polarization differ strongly from those in an isotropic medium. In summary, we represent the radiation intensity (71) in a more compact form, directly in terms of the anisotropic factors (72) introduced above:

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2 c^3}\Big(n_1F_1\big|(e_\mu^{(1)*}j_\mu)\big|^2 + n_2F_2\big|(e_\mu^{(2)*}j_\mu)\big|^2\Big)\,. \tag{73}$$

It is pertinent to note that, although in an isotropic medium the anisotropic factors turn into unity and the radiation intensities of two orthogonal eigenmodes of this medium are accordingly defined by two terms in expression (73) with equal refractive indices, this by no means implies that the emitted radiation would turn out to be unpolarized. Quite the opposite, the polarization of radiation emitted in a given direction may be quite high (and even one hundred percent, as in the case of Vavilov – Cherenkov radiation), depending on the relative directions of the electric current vector, the radiation wave vector, and other vectors of significance in the problem involved.

7. Special cases

(1) Radiation in a uniaxial anisotropic (nongyrotropic) medium at an angle θ to the optical axis ($\varepsilon_1 = \varepsilon_2 = \varepsilon_\perp$, $\varepsilon_3 = \varepsilon_\parallel$, $\mathbf{g} = 0$, $a(\omega, \boldsymbol{\kappa}) = \varepsilon_\parallel\cos^2\theta + \varepsilon_\perp\sin^2\theta$). From Eqns (16), (67), and (69) we find

$$n_1^2 = \varepsilon_\perp > 0\,,\qquad n_2^2 = \frac{\varepsilon_\perp\varepsilon_\parallel}{a(\omega,\theta)} > 0\,,$$
$$a(\omega,\theta) = \varepsilon_\parallel + (\varepsilon_\perp - \varepsilon_\parallel)\sin^2\theta\,,$$
$$C^{(1)} = \varepsilon_\perp(\varepsilon_\parallel - \varepsilon_\perp)\sin^2\theta\,,$$
$$C^{(2)} = \varepsilon_\perp(\varepsilon_\parallel + \varepsilon_\perp) - \frac{\varepsilon_\perp\varepsilon_\parallel(2\varepsilon_\perp + \varepsilon_\parallel)}{a(\omega,\theta)} + \frac{(\varepsilon_\perp\varepsilon_\parallel)^2}{a^2(\omega,\theta)}\,. \tag{74}$$

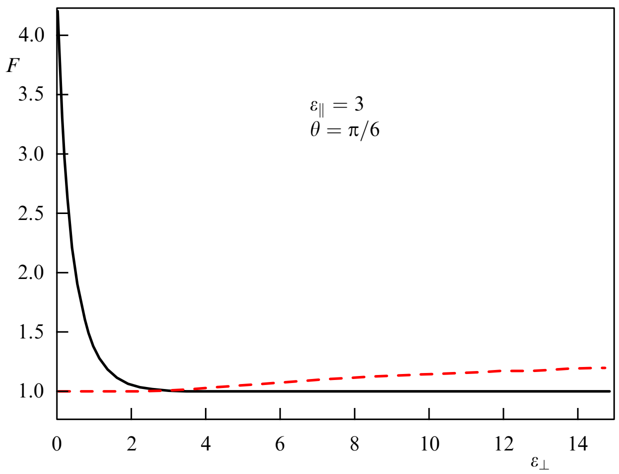

Figure 1. Anisotropic factors (72) for the ordinary (solid curve) and extraordinary (dashed curve) waves as functions of transverse permittivity. The values of longitudinal permittivity and emission angle are given in the figure.

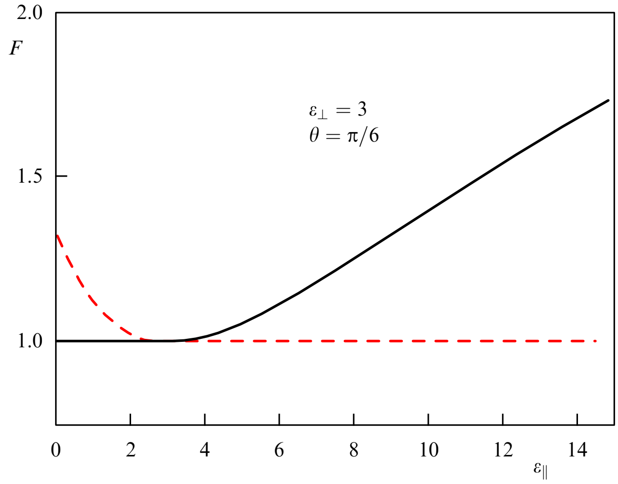

Figure 2. Anisotropic factors (72) for the ordinary (solid curve) and extraordinary (dashed curve) waves as functions of longitudinal permittivity. The values of transverse permittivity and emission angle are given in the figure.

In what follows we consider real positive refractive indices. When $n_i^2 < 0$, the corresponding mode cannot be radiated and makes no contribution to the spectral radiation density. The quantities $C^{(1)}$ and $C^{(2)}$ for $0 < \theta < \pi$ are nonzero, are opposite in sign, and vanish at $\theta = 0, \pi$. Therefore, the spectral density (71) is, as it must be, positive for $0 < \theta < \pi$, irrespective of the relation between $\varepsilon_\perp$ and $\varepsilon_\parallel$. Figures 1 and 2 depict the 'anisotropic factors' $F_{1,2}$ entering into formulas (72) for ordinary and extraordinary wave radiation at an angle to the optical axis of a uniaxial crystal as functions of $\varepsilon_\perp$ and $\varepsilon_\parallel$. One can see that the anisotropic factors differ markedly from unity and from one another in the general case, which may result in the generation of strongly polarized radiation in the anisotropic medium. Interestingly, for $\varepsilon_\perp > \varepsilon_\parallel$ the anisotropic factor F_1 of the ordinary wave is close to unity, while for $\varepsilon_\perp < \varepsilon_\parallel$ it is the anisotropic factor F_2 of the extraordinary wave which tends to unity. Of course, both factors are close to unity when $\varepsilon_\perp \sim \varepsilon_\parallel$, which corresponds to an isotropic medium.

In the radiation emission along the optical axis ($\theta = 0, \pi$) we have $n_1 = n_2 = n = \sqrt{\varepsilon_\perp}$, formula (71) yields a $0/0$ type indeterminant form. In this case, it is required to revert to formula (11). Then we will find $\varDelta_{11} = \varDelta_{22} = \varepsilon_\parallel(\varepsilon_\perp - n^2)$, $\varDelta_{33} = (\varepsilon_\perp - n^2)^2$, and $\varDelta_{\mu\nu} = 0$ for $\mu \neq \nu$. Upon cancellation of the common multiplier and integrating, we arrive at

$$W_{\kappa\omega} = \frac{\omega^2 \sqrt{\varepsilon_\perp}}{4\pi^2 c^3} \left|\mathbf{j}_\perp(\omega, \boldsymbol{\kappa})\right|^2 . \tag{75}$$

Again, this formula describes emission of radiations with two possible polarizations transverse relative to the optical axis. The radiation itself may be polarized — this depends on the current which excites the radiation; in this case, the radiation intensity of two mutually orthogonal modes is described by the two terms in expression (73) if it is taken into account that the anisotropic factors $F_{1,2}$ become unity.

In the case of propagation transverse to the optical axis ($\theta = \pi/2$) we obtain a spectral radiation density

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2 c^3} \left(\sqrt{\varepsilon_\perp}\, |e_\mu^{(1)*} j_\mu|^2 + \sqrt{\varepsilon_\parallel}\, |e_\mu^{(2)*} j_\mu|^2\right); \tag{76}$$

the polarization vector $\mathbf{e}^{(2)}$ is found from formula (52):

$$\mathbf{e}^{(2)} = \frac{\varepsilon_\parallel - \varepsilon_\perp}{|\varepsilon_\parallel - \varepsilon_\perp|}\, \mathbf{e}_z .$$

However, for the polarization vector $\mathbf{e}^{(1)}$ formula (52) gives a $0/0$ type indeterminant form. In this case, advantage should be taken of another equivalent representation (53) free from indeterminacy, which yields, given $\lambda^{(1)} = 0$, the correct result $\mathbf{e}^{(1)} = \pm(\mathbf{e}_x \sin\varphi - \mathbf{e}_y \cos\varphi)$, where $\cos\varphi$ and $\sin\varphi$ are the projections of vector $\boldsymbol{\kappa}$ onto the axes 1 and 2.

We consider what changes may be brought about by the existence of spatial dispersion of the permittivity. By way of illustration we analyze a simplified model wherein only the transverse component of the permittivity tensor depends on the wavenumber. Let

$$\varepsilon_\perp = \varepsilon_0(\omega) + \frac{\alpha(\omega)}{\sigma(\omega) + \gamma(\omega)k^2} \equiv \varepsilon_0(\omega) + \frac{\alpha(\omega)}{\sigma(\omega) + \beta(\omega)n^2}\,, \tag{77}$$

where $\varepsilon_0(\omega)$, $\alpha(\omega)$, $\beta(\omega)$, and $\sigma(\omega)$ are some functions of the frequency. Although we do not set for ourselves the task of modeling some specific situation, it should be noted that this structure of the permittivity emerges, in particular, when the contribution of excitons is taken into account [7, 32]. Furthermore, in terms of spatial dispersion it is equally possible to describe the electromagnetic properties of magnetics, in which there also exist specific eigenmodes (magnons, spin waves).

The equation for the refractive index of the first mode now takes on the form

$$n^2 = \varepsilon_\perp = \varepsilon_0(\omega) + \frac{\alpha(\omega)}{\delta(\omega) + \beta(\omega)n^2}\,.$$

This equation has become biquadratic:

$$\beta n^4 + (\delta - \beta\varepsilon_0)n^2 - (\delta\varepsilon_0 + \alpha) = 0\,, \tag{78}$$

and may generally possess two positive roots n_1^2, n_2^2, rather than one. This signifies that the occurrence of spatial dispersion gives rise to new *quasitransverse* medium eigenmodes, and the extraneous current energy may be expended for their emission.

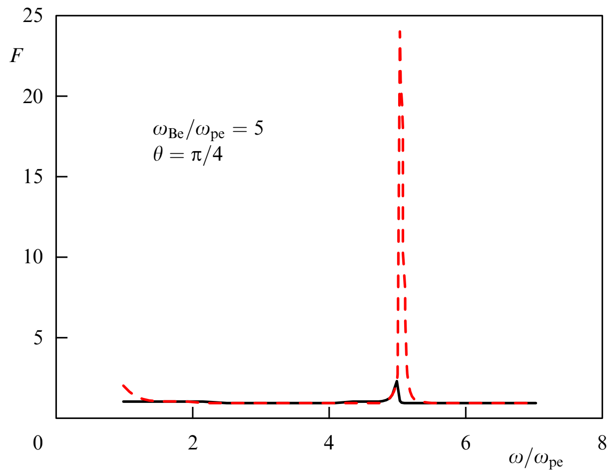

Figure 3. Anisotropic factors (72) for the oblique emission of ordinary (solid curve) and extraordinary (dashed curve) waves as functions of radiation frequency in magnetoactive plasma. The anisotropic factor for the extraordinary wave rises sharply in the vicinity of the gyrofrequency, which is five times the electron plasma frequency in this example.

(2) In the medium there is one preferred direction along which the gyration vector $\mathbf{g}$ is oriented. This is precisely the case realized in a uniform magnetoactive plasma. Expressions for an arbitrary emission angle θ are cumbersome, which is why the radiation along and perpendicular to the direction of *vector* $\mathbf{g}$ is considered in greater detail below. Figure 3 depicts the frequency dependence of anisotropic factors in the general case of emission at some angle to the direction of the external magnetic field. These factors are close to unity almost everywhere, with the exception of the neighborhoods of the plasma frequency and the gyrofrequency, in which these factors may quite substantially depart from unity, especially for extraordinary waves. This should be taken into consideration, in particular, in the calculation of the transition radiation in magnetoactive plasma [31, 33].

(2a) The longitudinal case: $\theta = 0$, $n_1^2 = \varepsilon_\perp + g > 0$, $n_2^2 = \varepsilon_\perp - g > 0$, $C^{(1)} = -C^{(2)} = -2g\varepsilon_\parallel$, and $a(\omega, \boldsymbol{\kappa}) = \varepsilon_\parallel$. With the help of formula (71) we find

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2 c^3}\left(\sqrt{\varepsilon_\perp + g}\,\left|\mathbf{e}_\mu^{(1)*}\,\mathbf{j}_\mu(\omega, \boldsymbol{\kappa})\right|^2 + \sqrt{\varepsilon_\perp - g}\,\left|\mathbf{e}_\mu^{(2)*}\,\mathbf{j}_\mu(\omega, \boldsymbol{\kappa})\right|^2\right). \tag{79}$$

The mode polarization vectors $\mathbf{e}_\mu^{(1,2)}$ are easily determined from formula (53):

$$\mathbf{e}^{(1,2)} = \mp \frac{\varepsilon_\parallel g}{\sqrt{2}\,|\varepsilon_\parallel g|}(\mathbf{e}_x \mp \mathrm{i}\mathbf{e}_y)\,, \tag{80}$$

which corresponds to circularly polarized waves. When one or both of n^2 are negative, such a mode is not excited and the corresponding term should be removed from formula (79).

(2b) The transverse case: the emitted waves propagate along axis 1, $\theta = \pi/2$. Then one has

$$n_1^2 = \varepsilon_\parallel > 0\,, \qquad n_2^2 = \varepsilon_\perp - \frac{g^2}{\varepsilon_\perp} > 0\,,$$

$$C^{(1)} = -\varepsilon_\perp(\varepsilon_\parallel - \varepsilon_\perp) - g^2\,,$$

$$C^{(2)} = \varepsilon_\perp(\varepsilon_\parallel - \varepsilon_\perp) + \frac{g^2\varepsilon_\parallel}{\varepsilon_\perp} + \frac{g^4}{\varepsilon_\perp^2}\,.$$

By calculating the algebraic adjuncts in formulas (52) and (53) we find that the first (ordinary) wave is transverse and polarized along axis 3, $\mathbf{e}^{(1)} = \pm\mathbf{e}_z$. The second (extraordinary) wave is nontransverse and has the polarization vector

$$\mathbf{e}^{(2)} = \pm\left(\frac{g}{\sqrt{\varepsilon_\perp^2 + g^2}}\,\mathbf{e}_x + \frac{\mathrm{i}\varepsilon_\perp}{\sqrt{\varepsilon_\perp^2 + g^2}}\,\mathbf{e}_y\right). \tag{81}$$

The radiation energy, which is given by formula (71), is positive for both modes when $n_i^2 > 0$. As in the previous case, the eigenmodes for which $n^2 < 0$ are missing.

(3) Low-frequency magnetohydrodynamic waves in a cold magnetoactive plasma ($\omega \ll \omega_{\rm i,e}$, $c_{\rm s} \ll v_{\rm A}$), where $\omega_{\rm i,e}$ are the ion and electron cyclotron frequencies, respectively, $c_{\rm s}$ and $v_{\rm A}$ are the sound and Alfvén velocities. Neglecting damping, the permittivity tensor has components (see, for instance, Ref. [19, p. 191])

$$\varepsilon_\perp \approx \frac{c^2}{v_{\rm A}^2}\,, \qquad \varepsilon_\parallel \approx -\frac{\omega_{\rm pe}^2}{\omega^2}\,, \qquad |\varepsilon_\parallel| \gg \varepsilon_\perp\,,$$
$$g \approx \frac{c^2\omega}{v_{\rm A}^2\omega_{\rm i}} \ll \varepsilon_\perp\,, \qquad \omega_{\rm pe}^2 = \frac{4\pi n e^2}{m_{\rm e}}\,. \tag{82}$$

We neglect the terms $g^2/\varepsilon_\perp^2 \ll 1$ in Eqn (16) to find

$$\Delta = (\varepsilon_\perp - n^2)\left[\varepsilon_\perp\varepsilon_\parallel - n^2(\varepsilon_\parallel\cos^2\theta - \varepsilon_\perp\sin^2\theta)\right] = 0\,,$$

from whence we obtain two values of the refractive index:

$$n_1^2 = \varepsilon_\perp\,, \qquad n_2^2 \approx \frac{\varepsilon_\perp}{\cos^2\theta} > 0\,. \tag{83}$$

The former corresponds to the fast magnetosonic mode, and the latter to the Alfvén one. Formula (83) remain valid only for angles $\pi/2 - \theta \gg (\varepsilon_\perp/|\varepsilon_\parallel|)^{1/2}$. For a transverse propagation ($\theta \to \pi/2$), the Alfvén wave is impossible.

The normalization constants have the values

$$C^{(1)} = \varepsilon_\parallel\varepsilon_\perp\sin^2\theta \leqslant 0\,, \qquad C^{(2)} = -\varepsilon_\parallel\varepsilon_\perp\tan^2\theta \geqslant 0\,.$$

The spectral density (71) is positive and takes the following form upon substitution of all the quantities obtained:

$$W_{\kappa\omega} = \frac{\omega^2}{4\pi^2c^2v_{\rm A}}\left(\left|e_\mu^{(1)*}\,j_\mu(\omega, \boldsymbol{\kappa})\right|^2 + |\cos\theta|^{-3}\left|e_\mu^{(2)*}\,j_\mu(\omega, \boldsymbol{\kappa})\right|^2\right),$$
$$\cos^2\theta \gg \frac{\varepsilon_\perp}{|\varepsilon_\parallel|}\,. \tag{84}$$

The polarization vectors for $\theta > 0$ are easily calculated from algebraic adjuncts: $\mathbf{e}^{(1)} = (0, 1, 0)$; $\mathbf{e}^{(2)} = (1, 0, 0)$. The electric fields of the two modes are mutually orthogonal and nontransverse relative to the propagation direction $\boldsymbol{\kappa}$. The magnetic vectors of both modes are transverse relative to $\boldsymbol{\kappa}$, but the magnetosonic wave field is nontransverse relative to the uniform background field. When the waves travel strictly along the field, the refractive indices in approximation (83) degenerate, and to determine the wave polarization requires including the terms with the gyration vector, as was done in example 2. In this case, the polarization is circular, which follows from formula (80).

8. Conclusions

In our paper we have applied the standard methods of linear algebra to the solution of the inhomogeneous system of

Maxwell equations in the problem of radiation emission and energy losses by a given current in an arbitrary infinite anisotropic and gyrotropic medium with the inclusion of temporal and spatial dispersion. We emphasize that the 'given' current is not necessarily 'extraneous' relative to the medium. On the contrary, these currents may be nonlinear plasma currents responsible, in particular, for the generation of transition radiation or polarization bremsstrahlung [31, 33–37]. An important point is that our approach enables a consistent inclusion of the spatial dispersion of a medium even when its effect is by no means weak. In particular, it permits calculating the radiation intensity of those modes which emerge only due to the spatial dispersion and vanish in its absence.

An important constituent of this treatment is bringing the Hermitian Maxwellian and its inverse tensors to the diagonal real form on the orthonormal basis of the complex eigenvectors of the Maxwellian tensor. The proposed analysis is a generalization of the well-known eigenmode method [6, 19], which enjoys wide use in different areas of physics, including crystal optics [7] and plasma physics [6, 19, 25]. In particular, the eigenmode method has proven to be highly fruitful in the consideration of nonlinear phenomena in plasmas [25], including the theory of turbulent plasma [6].

In the framework of our approach we make use of the decomposition of the electromagnetic field not into the propagating modes of a given medium (i.e., the solutions of the homogeneous system of Maxwell equations), but into the eigenvectors and principal values of the eigenvectors of the Maxwellian tensor, which are the solutions of inhomogeneous Maxwell equations (34). The eigenmodes of this medium are the special cases of these general solutions, provided that the corresponding principal value λ of the Maxwellian tensor vanishes.

Our approach is more general and consistent, for it allows for the existence of perturbations in the medium, whose frequency and wave vector are not related by the dispersion law of some of the eigenmodes of this medium. In particular, this method makes it possible to easily elucidate the origination of the apparent contradiction between the orthogonality of the eigenvectors of any Hermitian tensor for given ω and $\mathbf{k}$ and the nonorthogonality of the eigenmodes of an anisotropic medium, which travel in the same direction and have the same frequency ω (see Section 5).

Furthermore, the method developed in our work has enabled us to explicitly calculate the anisotropic factors $F_{1,2}$ entering into formulas (72), whose availability is of significance both from the methodical and practical viewpoints. Indeed, when these factors are different from unity, some mode with a given polarization and a refractive index $n(\omega)$ will be radiated with different intensity in anisotropic and isotropic media. It is noteworthy that we can always select an isotropic medium in such a way as to make its refractive index coincide with one of the refractive indices of the anisotropic medium. Of course, the second refractive index of the anisotropic medium will be different. This signifies that the radiation intensity of this eigenmode is generally defined not only by its own refractive index and polarization vector, but also by the properties of the anisotropic medium as a whole, because the anisotropic factors of each of the modes depend on the differences in the refractive indices squared of both modes. From the practical standpoint, this effect may be employed for the development of strongly polarized incoherent radiation sources.

Therefore, the approach outlined permits describing from a unified standpoint a broad range of phenomena in the anisotropic media, many of which have been well and widely known but some of which are described here for the first time. Seemingly, the material set forth in our work may prove to be helpful when lecturing on the subject of radiation emission to students in physics specialties, because it contains, in a beautifully elegant and compact form, the general solution to the problem of electromagnetic energy radiation in a medium. This enables one to easily obtain all results of practical interest in the form of special cases of our general expressions.

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