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An interpretation of the energy conservation law for a point charge moving in a uniform electric field

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Contents

Abstract. **The standard interpretation of the energy conservation law accepted in the theory of electromagnetism maintains that a change in the field energy occurring in a volume is the sum of the energy increment in the volume due to the Poynting vector flow and the energy dissipated as heat in the interaction of charge carriers with the medium in which these carriers move. We show that the work done by an external electric field on a moving point charge is due to two sources: the energy arriving at the charge with the Poynting vector flow and a specific channel of energy supply to the charge originating from the interaction of the charge's field with the external field. Interestingly, at nonrelativistic charge velocities ($v \ll c$), 2/3 of the energy is supplied by the Poynting vector flow and 1/3 by the second channel. In the ultrarelativistic case ($v \lesssim c$), all of the energy comes with the Poynting vector flow.**

1. Introductory remarks

Physics abounds with startling patterns reflecting the amazing harmony of the world around us. As a revealing example, we recall the interference quenching of secondary side waves, a process that accounts for the propagation of light rays. A simpler illustration is provided by the slow charging of a capacitor, in which, practically always, exactly one half of the total work done by the power supply is converted to the energy of the capacitor electric field, while the other half is released in the form of heat. An apparently unexpected point is here that this result is independent of the circuit resistance (if this resistance tends to zero, the oscillations and emission of radiation must be taken into account). The pattern to be discussed below is also unexpected, in our opinion.

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The energy conservation law for an electromagnetic field, which is a consequence of the Maxwell equations [1], can be written as

$$\frac{\partial w}{\partial t} = -\mathbf{jE} - \nabla\mathbf{S}\,, \tag{1}$$

where

$$w = \frac{\varepsilon\varepsilon_0 E^2}{2} + \frac{\mu\mu_0 H^2}{2}\,, \qquad \mathbf{S} = [\mathbf{E}, \mathbf{H}] \tag{2}$$

are the field energy density and the energy flow density vector (the Poynting vector), and $\mathbf{j}$ is the current density vector. In an integral form, this law can be written as

$$\frac{\partial W}{\partial t} = -Q - \oint_F \mathbf{S}\,\mathrm{d}\mathbf{F}\,, \tag{3}$$

where $W = \int_V w\,\mathrm{d}V$ is the field energy in a volume V, Q is the heat power released in this volume, and $\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F}$ is the flow of the Poynting vector $\mathbf{S}$ through the surface F confining this volume. There are situations where the interpretation of Eqn (1) is fairly obvious. For instance, in the case of the slow charging of a plane capacitor, where Q is zero in the capacitor gap, the Poynting vector flow through the side surface of the gap is exactly equal to the increment of the electric field energy in this gap. One more example is provided by the following well-known problem: prove that the energy dissipated per unit time in a straight cylindrical current-carrying conductor ($Q = IU$) is supplied through the side surface of the conductor by the Poynting vector flow. In the steady state, the left-hand side of Eqn (1) is zero. Assuming the field $\mathbf{E}$ in the conductor to be constant and equal to the external field $E_0 = U/l$, and the magnetic field strength at the conductor side surface $H = I/2\pi r$, where l and r are the conductor length and radius, we obtain the flow of the Poynting vector $\mathbf{S}$ through the side surface of the conductor as $\Phi = 2\pi r l S = IU$. At the same time, the power IU can be written as

$$IU = j\pi r^2 E_0 l = eun\pi r^2 E_0 l = NeuE_0 = NQ_1 = -\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F}\,, \tag{4}$$

where $N = n\pi r^2 l$ is the total number of mobile carriers (electrons with charge e), $Q_1 = A_1 = euE_0$ is the power dissipated by one charge, which is numerically equal to the work done by the field on an electron per unit time, and n and u are the concentration and the drift velocity of electrons, respectively.

Relation (4) raises the following legitimate question: if the quantity IU is additive with respect to the number of charges, is the sum of Poynting vector energy flows absorbed by individual charges equal to the integral in the right-hand side of Eqn (4), or is relation (4) satisfied due to some contributing 'interference' effects? It turns out that this second situation is actually realized, and that it occurs in a peculiar way, indeed.

2. Field-based interpretation of the work done by an electric field on a moving charge

We consider the following model problem. A point charge q moves with a velocity $\mathbf{v}$ along a line of an external uniform electric field $\mathbf{E}_0$. The field does the work $A_1 = qE_0 v$ on the charge per unit time. We write Eqn (3) in the case of interest here as

$$qE_0 v = -\frac{\partial W}{\partial t} - \oint_F \mathbf{S}\,\mathrm{d}\mathbf{F}\,. \tag{5}$$

We now calculate the terms in the right-hand side of Eqn (5) separately. The volume is taken to be a sphere of radius R centered at the point O at which the charge q is located at a given instant. We start with the second term in the right-hand side of Eqn (5), which actually represents the flow of the vector $\mathbf{S}$ through the spherical surface within which this sphere is confined. We introduce a spherical coordinate system r, ϑ, φ such that the charge is at the origin and ϑ is the angle between the z axis aligned with $\mathbf{v} \parallel \mathbf{E}_0$ and the radius vector $\mathbf{r}$. Then,

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_q\,, \qquad \mathrm{d}\mathbf{F} = \mathrm{d}F\,\frac{\mathbf{r}}{r}\,, \tag{6}$$

where

$$\mathbf{E}_q = \frac{q}{4\pi\varepsilon_0\gamma^2(1-\beta^2\sin^2\vartheta)^{3/2}}\,\frac{\mathbf{r}}{r^3} \tag{7}$$

is the electric field of the moving charge. Accordingly,

$$\mathbf{H} = \mathbf{H}_q = \varepsilon_0[\mathbf{v}, \mathbf{E}_q] \tag{8}$$

is the magnetic field strength of the moving charge, $\beta = v/c$, and $\gamma = (1-\beta^2)^{-1/2}$.

Substituting Eqns (6) and (8) in the relation under consideration, we obtain

$$\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F} = \oint_F [\mathbf{E}, \mathbf{H}]\,\mathrm{d}\mathbf{F}$$
$$= \varepsilon_0 \oint_F \big[\mathbf{E}_0[\mathbf{v}, \mathbf{E}_q]\big]\,\mathrm{d}\mathbf{F} + \varepsilon_0 \oint_F \big[\mathbf{E}_q[\mathbf{v}, \mathbf{E}_q]\big]\,\mathrm{d}\mathbf{F}\,.$$

The second term in the right-hand side of this equation is zero; expanding the double vector product in the first term with the use of Eqns (6) and (7), and rearranging, we obtain

$$\oint_F [\mathbf{E}, \mathbf{H}]\,\mathrm{d}\mathbf{F} = J_1 + J_2\,, \tag{9}$$

$$J_1 = \varepsilon_0 \oint_F (\mathbf{E}_0, \mathbf{E}_q)(\mathbf{v}, \mathrm{d}\mathbf{F}) = \frac{qvE_0}{2\gamma^2}\int_0^{\pi} \frac{\cos^2\vartheta\sin\vartheta\,\mathrm{d}\vartheta}{(1-\beta^2\sin^2\vartheta)^{3/2}}$$
$$= -\frac{qvE_0(1-\beta^2)}{\beta^2}\left[1 - \frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right], \tag{10}$$

$$J_2 = -\varepsilon_0 v E_0 \oint_F (\mathbf{E}_q, \mathrm{d}\mathbf{F})$$
$$= -\frac{qvE_0}{2\gamma^2}\int_0^{\pi}\frac{\sin\vartheta\,\mathrm{d}\vartheta}{(1-\beta^2\sin^2\vartheta)^{3/2}} = -qvE_0\,. \tag{11}$$

Thus, we finally arrive at

$$\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F} = -\frac{qvE_0}{\beta^2}\left[1 - \frac{1-\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \tag{12}$$

Substituting Eqns (9) and (11) in Eqn (5) yields

$$\frac{\partial W}{\partial t} = -J_1\,. \tag{13}$$

Straightforward calculation of the integral

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}\left[\int_V \left(\frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2}\right)\mathrm{d}V\right] \tag{14}$$

confirms the validity of relation (13). Indeed, substituting Eqns (6) and (8) in Eqn (14), we have

$$W = \frac{1}{2}\left[\varepsilon_0\int E_q^2\,\mathrm{d}V + \mu_0\int H_q^2\,\mathrm{d}V\right.$$
$$\left. + \varepsilon_0\int E_0^2\,\mathrm{d}V + 2\varepsilon_0\int(\mathbf{E}_q\mathbf{E}_0)\,\mathrm{d}V\right]. \tag{15}$$

The integrals in the right-hand side of Eqn (15) can be best analyzed by invoking some considerations that allow eliminating the divergences related to the infinite energy of the field generated by a point charge. The time derivative of the first three terms in the right-hand side of Eqn (15) is zero. For the third term, this is obvious. As regards the first two terms, we recall that as a charge approaches the center of the sphere under consideration, the self-energy of the charge inside this sphere increases, but as the charge moves away from the center, this energy decreases. This implies that at the instant of the charge passing through the center of the sphere, this energy reaches a maximum. It remains to consider the expression

$$\frac{\partial W}{\partial t} = \varepsilon_0\,\frac{\partial I}{\partial t}\,, \qquad I = \int(\mathbf{E}_q\mathbf{E}_0)\,\mathrm{d}V. \tag{16}$$

We are interested in the value of the derivative in the above relation at $t = 0$, when the charge passes through the center of the sphere (point O in Fig. 1). The quantity $\partial I/\partial t$ can then be represented as

$$\frac{\partial I}{\partial t} = \frac{I'' - I'}{\mathrm{d}t}\,, \tag{17}$$

where I' and I'' are the values of the integral I at the instants $t' = -\mathrm{d}t/2$ and $t'' = +\mathrm{d}t/2$.

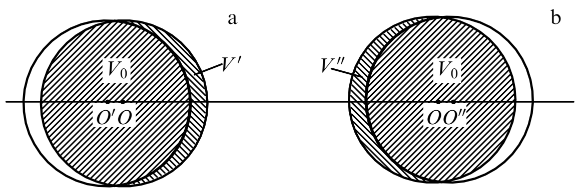

Figure 1.

Figure 1 displays spheres of radius R centered at a point O, as well as spheres with the same radius but centered at points O' (Fig. 1a) and O'' (Fig. 1b) at which the charge was located at the instants t' and t''. Evidently, the field $\mathbf{E}_q$ defined by Eqn (7) and entering the integrals I' and I'' describes the electric field of a moving charge at these instants only if the coordinates r, ϑ are referenced to the spherical coordinate frames with the origins at the respective points O' and O''. On the other hand, the integration is performed in I' and I'' over the volume of the sphere centered at O. As can be seen from Figs 1a, b, this volume consists of volumes V_0 and V' in Fig. 1a and of volumes V_0 and V'' in Fig. 1b. Because integration over V_0 is performed in both integrals, I' and I'', the integrals over this region cancel in subtraction in Eqn (17). Therefore,

$$I'' - I' = I(V'') - I(V')\,. \tag{18}$$

Because the charge passes the $O'O''$ distance in an infinitely short time dt, the V' and V'' regions are infinitely thin. The volume elements in the integrals I' and I'' can be written as

$$\mathrm{d}V = 2\pi R^2 \sin\vartheta \, \mathrm{d}\vartheta \, v \, \frac{\mathrm{d}t}{2} \cos\vartheta$$

and

$$\mathrm{d}V = -2\pi R^2 \sin\vartheta \, \mathrm{d}\vartheta \, v \, \frac{\mathrm{d}t}{2} \cos\vartheta$$

(the second expression involves the minus sign because $\cos\vartheta < 0$ in the V'' volume). Integration over the angle ϑ in the I' and I'' integrals is performed from 0 to $\pi/2$ and from $\pi/2$ to π, respectively. Equation (7) can be used to rewrite the scalar product $(\mathbf{E}_q\mathbf{E}_0)$ as

$$(\mathbf{E}_q\mathbf{E}_0) = \frac{qE_0 \cos\vartheta}{4\pi\varepsilon_0\gamma^2 R^2 (1 - \beta^2 \sin^2\vartheta)^{3/2}}\,. \tag{19}$$

Substitution of the expressions for dV and $(\mathbf{E}_q\mathbf{E}_0)$ in Eqn (18), followed by substitution of the relation thus obtained for the $I'' - I'$ difference in Eqn (17) and, subsequently, in Eqn (16), does indeed prove that $\partial W/\partial t = -J_1$.

Our analysis suggests that the energy flow derived from the Poynting vector flow is less than the total work done by the field on the charge. The moving charge acquires part of the energy as a result of a change in the energy of the interaction of the charge's field with the external field. As can be seen from Fig. 1, the 'interference' term $(\mathbf{E}_q\mathbf{E}_0)$ describing the energy of this interaction has opposite signs in the V' and V'' volumes, and this precisely accounts for the result $\partial W/\partial t = -J_1$ we have obtained.

3. Limit cases

It is interesting to consider the following limit cases: (a) nonrelativistic velocity of charge motion $(v \ll c)$ and (b) the ultrarelativistic case $(v \lesssim c)$. In case (a), the above relations yield

$$\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F} \simeq -\frac{2}{3}\,qvE_0\,, \qquad \frac{\partial W}{\partial t} \simeq -\frac{1}{3}\,qvE_0\,. \tag{20}$$

In case (b), we have

$$\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F} \simeq -qvE_0\,, \qquad \frac{\partial W}{\partial t} \simeq 0\,. \tag{21}$$

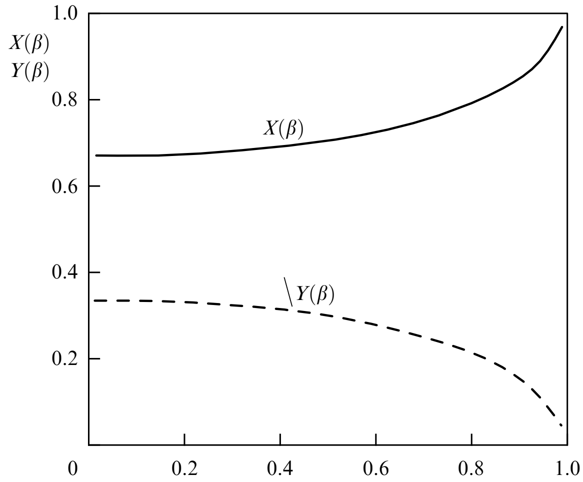

Figure 2.

Figure 2 plots the ratios

$$X(\beta) = \frac{1}{qvE_0}\left|\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F}\right|, \qquad Y(\beta) = \frac{1}{qvE_0}\left|\frac{\partial W}{\partial t}\right|$$

vs. the dimensionless velocity of charge motion $\beta = v/c$, which were calculated using Eqns (10), (12), and (13) derived above.

It can be seen from Fig. 2 that as β increases, passing from the nonrelativistic to the ultrarelativistic case, the progressively increasing fraction of the energy required for the field to do work on the charge is supplied by the Poynting vector flow. In the ultrarelativistic case, this fraction becomes equal to unity. By contrast, the fraction of the energy acquired by the moving charge in the interaction of its proper field with the external field decreases to zero with increasing β in the ultrarelativistic case. This result can be explained by a deformation of the electric field line pattern of the moving charge in passing from the nonrelativistic to the ultrarelativistic case. Indeed, as $v \to c$, it follows from Eqn (7) that $E_q \to 0$ everywhere except for the plane passing through the charge and perpendicular to the velocity vector $\mathbf{v}$ of its motion. Therefore, the scalar product $(\mathbf{E}_q\mathbf{E}_0) \to 0$ in Eqn (16), and hence the quantity $\partial W/\partial t$ also tends to zero.

4. Concluding comments

Returning to the question bearing on the supply of energy to a current-carrying conductor, the following comments may be in order. The proper field of the electrons generating the current in a conductor is canceled by the fields produced by positive ions. As a result, only the external field applied to the conductor remains uncompensated. Therefore, the mechanism by which the carrier proper fields interact with the external field is suppressed in the case of a conductor, while superposition of the magnetic fields created by a macroscopic ensemble of moving charges creates the characteristic pattern of field lines of the resultant magnetic field that ensure the equality

$$Q = -\oint_F \mathbf{S}\,\mathrm{d}\mathbf{F} \qquad (22)$$

for the current in a conductor.

Our previous consideration (Section 2) ignored the effects related to the acceleration acquired by the charge on which the external field does work. It can be shown that in most cases of practical significance, the radiation emitted by a charge moving with acceleration produces very small effects. Besides, in the particular case of a charge moving with acceleration, a term $\sim 1/r$ must be added to the electric field defined by Eqn (7); this term dominates over the term $\sim 1/r^2$ in the wave zone, i.e., for large r. As is evident from our previous analysis, the results in Eqns (9)–(12) are independent of the radius of the spherical surface that contained a moving charge at the center. Therefore, by choosing this radius small enough, we can entirely neglect the acceleration-induced term $\sim 1/r$ in the expressions for $\mathbf{E}$ and $\mathbf{H}$.

References

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