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Spontaneous and stimulated emission induced by an electron, electron bunch, and electron beam in a plasma

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Contents

Abstract. **Two fundamental mechanisms — the Cherenkov effect and anomalous Doppler effect — underlying the emission by an electron during its superluminal motion in medium are considered. Cherenkov emission induced by a single electron and a small electron bunch is spontaneous. In the course of spontaneous Cherenkov emission, the translational motion of an electron is slowed down and the radiation energy grows linearly with time. As the number of radiating electrons increases, Cherenkov emission becomes stimulated. Stimulated Cherenkov emission represents a resonance beam instability. This emission process is accompanied by longitudinal electron bunching in the beam or by the breaking of an electron bunch into smaller bunches, in which case the radiation energy grows exponentially with time. In terms of the longitudinal size L_e of the electron bunch there is a transition region $\lambda < L_e < \lambda\delta_0^{-1}$ between the spontaneous and stimulated Cherenkov effects, where λ is the average radiation wavelength, and δ_0 is the dimensionless (in units of the radiation frequency) growth rate of the Cherenkov beam instability. The range to the left of this region is dominated by spontaneous emission, whereas the range to the right of this region is dominated by stimulated emission. In contrast to the Vavilov – Cherenkov effect, the anomalous Doppler effect should always (even for a single electron) be considered as stimulated, because it can only be explained by accounting for the reverse action of the radiation field on the moving electron. During stimulated emission in conditions where anomalous Doppler effect shows itself, an electron is slowed down and spins up; in this case, the radiation energy grows exponentially with time.**

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1. Introduction

In classical electrodynamics, the effect of spontaneous emission by an electron executing a preset motion depends on the work done by the radiation field on the electron, i.e., on the quantity

$$e\left\langle \mathbf{u}_0(t)\,\mathbf{E}\big(t,\mathbf{r}_0(t)\big)\right\rangle, \tag{1.1}$$

where e is the electron charge, $\mathbf{E}(t,\mathbf{r})$ is the strength of the electric radiation field, $\mathbf{r}_0(t)$ is the electron's radius vector unperturbed by emission, $\mathbf{u}_0 = \dot{\mathbf{r}}_0$, and averaging in formula (1.1) is taken over a time interval greater than the characteristic period of emitted waves. Quantity (1.1) is equal to instantaneous radiation power. Because the field $\mathbf{E}$ is determined by unperturbed electron motion, quantity (1.1) is a constant. Therefore, the main energy relation for

spontaneous emission is written down as

$$\frac{\mathrm{d}W}{\mathrm{d}t} = A_{\mathrm{sp}} = \mathrm{const}\,. \tag{1.2}$$

Here, A_{sp} depends on e^2, the characteristics of unperturbed electron motion, and the medium properties. By way of example, for spontaneous Vavilov – Cherenkov emission one finds $A_{\mathrm{sp}} \sim e^2 u\omega^2/c^2$, where u is the electron velocity, and ω is a certain characteristic frequency determined by the medium (see below). According to equation (1.2), the energy of spontaneous emission grows linearly with time.

When calculating (1.1), it is possible, in principle, to take into account the reverse action of the radiation field on the electron — that is, to substitute true electron radius $\mathbf{r}_{\mathrm{e}}(t)$ for unperturbed radius vector $\mathbf{r}_0$. Then, the right-hand side of equation (1.2) will be a time function. Nevertheless, the radiation should be regarded as spontaneous, as before.

Stimulated radiation develops in the case of passage from a single electron to an electron beam or a bunch. Calculation of (1.1) values for each electron in the beam and summation over all electrons give zero, namely

$$e\sum_j \left\langle \mathbf{u}_{0j}(t)\,\mathbf{E}\big(t, \mathbf{r}_{0j}(t)\big)\right\rangle = 0\,, \tag{1.3}$$

where j is the electron number in the beam (in the calculation of formula (1.3), the beam is assumed to be unmodulated). Therefore, in order to describe stimulated emission induced by the beam, it is necessary to replace $\mathbf{r}_{0j}$ in formula (1.3) by $\mathbf{r}_{\mathrm{e}j} = \mathbf{r}_{0j} + \tilde{\mathbf{r}}_j$, where $\tilde{\mathbf{r}}_j$ is the perturbation of the electron trajectory. Perturbation $\tilde{\mathbf{r}}_j$ being proportional to $\mathbf{E}$, quantity (1.3) is proportional to $E^2 \sim W$. Hence, the basic energy relation for spontaneous emission induced by an electron beam has the form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = B_{\mathrm{st}}\,W\,, \tag{1.4}$$

where B_{st} is a constant. It follows from equation (1.4) that the spontaneous emission energy grows exponentially with time, giving evidence that stimulated emission represents an instability and B_{st} coincides with the double increment of this instability.

The effects of spontaneous and stimulated emission by relativistic electron beams propagating in dispersive media were considered in our reviews [1, 2] in the context of formulas (1.1) – (1.4). Spontaneous emission was dealt with very fragmentarily for the sole purpose of classifying elementary emission events. In contrast, various effects of stimulated emission (and stimulated scattering) by dense relativistic electron beams in plasma and other media were examined in those reviews in great detail, both in the linear and in the nonlinear regimes. Spontaneous and stimulated effects were interpreted therein as certain independent processes even though there is no doubt that they represent limiting cases of a united group of phenomena occurring during the interaction of electric charges with an electromagnetic field. The recent monograph [3] (see also [4]) emphasizes the topical problem of transition from spontaneous to stimulated emission with an increasing number of emitting electrons and of determining the conditions of such a transition. In order to address this issue, we turned back again to the problem of emission of electromagnetic waves by fast plasma electrons and examined (with reference to Cherenkov emission by uniformly moving electrons) the transition from spontaneous emission by one electron to stimulated emission by an electron bunch and an electron beam with increasing electron 'density' (or the number of electrons per radiation wavelength). Naturally, we had to reproduce some results reported in reviews [1, 2] but treating them in the new context expounded in monograph [5], in line with the problem of transition from spontaneous to stimulated emission as viewed in the present paper whose outline is as follows.

Section 2 considers the model problem of potential wave emission in a one-dimensional plasma based on the Hamiltonian approach that has been developing in recent years [5, 6] in application to the problems of electrodynamics of radiation. Nonlinear self-consistent field equations and equations of field source motion are written in the framework of this approach. Spontaneous Cherenkov emission of a longitudinal wave by a plasma electron is considered in the zero-order approximation in the field strength, i.e., at a given source motion, while stimulated Cherenkov emission of a longitudinal wave by a uniform electron beam is treated in the next field approximation. In addition, this section deals with emission by an electron bunch in a one-dimensional model; here, application of the energy method analogous to the one used in reviews [1, 2] proves to be more convenient for the purpose. Also considered is transformation of spontaneous to stimulated emission upon passage from emission by a single electron to that by two, three, or more electrons. Finally, selected aspects of the quantum-mechanical interpretation of spontaneous and stimulated Cherenkov emission of longitudinal waves by fast electrons in a plasma are discussed.

In Section 3, the Hamiltonian approach is applied to a real three-dimensional case of emission of longitudinal and transverse waves both in isotropic and anisotropic media. The applicability of this approach to emission problems taking into account spatial dispersion of the medium is illustrated by the example of emission of ion-sound waves by a stream of fast particles in an anisothermic plasma. In addition, the Hamiltonian method in the theory of stimulated emission of electromagnetic waves is compared with the dispersion equation method developed in plasma electrodynamics and plasma electronics. Also, this section studies Cherenkov emission of sound waves by a supersonic gas flow in a gas. Cherenkov emission by supersonic gas flows in gas dynamics is shown to be analogous to Cherenkov emission by electron beams in electrodynamics.

Finally, Section 4 is devoted to the anomalous Doppler effect and its analogy with the collective stimulated Cherenkov effect. In circumstances where the collective Cherenkov effect shows itself, the emitter's eigenfrequency is the frequency of beam Langmuir oscillations, whereas for the anomalous Doppler effect the frequency of electron Larmor gyration in an external magnetic field plays the same role. The anomalous Doppler effect is shown to be a stimulated process regardless of the number of emitting electrons involved.

2. Cherenkov emission in a one-dimensional plasma: the model problem

2.1 Description of a one-dimensional system and derivation of nonlinear equations

Let us consider a one-dimensional model of perturbation (emission) of longitudinal Langmuir waves by a 'flat' electron bunch (inhomogeneous flat electron layer) propagating in a boundless cold electron plasma. Let us proceed from the

following system of equations for electric field strength $E_z(t,z)$ and current density $j_{pz}(t,z)$ in the plasma [7, 8]:

$$\frac{\partial E_z}{\partial t} + 4\pi j_{pz} = -4\pi j_{ez}(t,z)\,, \qquad \frac{\partial j_{pz}}{\partial t} - \frac{\omega_p^2}{4\pi} E_z = 0\,, \tag{2.1.1}$$

where ω_p is the Langmuir frequency of plasma electrons (described in a linear approximation [7]), $j_{ez}(t,z)$ is the electron current density in a bunch, and z is the direction of electron motion. If $E_z = \partial A/\partial t$, Eqn (2.1.1.) yields the equation for auxiliary function $A(t,z)$:

$$\frac{\partial^2 A}{\partial t^2} + \omega_p^2 A = -4\pi j_{ez}\,. \tag{2.1.2}$$

Plasma oscillation energy density is described by the formulas [8]

$$w = \frac{E_z^2}{8\pi} + \frac{2\pi}{\omega_p^2} j_{pz}^2 = \frac{1}{8\pi}\left[\left(\frac{\partial A}{\partial t}\right)^2 + \omega_p^2 A^2\right]. \tag{2.1.3}$$

Current density in the bunch is calculated using the Vlasov kinetic equation with a self-consistent field [8, 9] and integration over the initial data [10, 11]. It leads to the following expression for the current density:

$$j_{ez}(t,z) = e\int n_e(z_0)\, v_{ez}(t,z_0)\, \delta\big(z - z_e(t,z_0)\big)\, dz_0\,, \tag{2.1.4}$$

where $n_e(z)$ is the unperturbed electron density in the bunch. Coordinate $z_e(t,z_0)$ and velocity $v_{ez}(t,z_0)$ of the bunch electron located at point z_0 at the initial instant of time are found (relativistic effects being as yet disregarded) from the equations of motion

$$\frac{dz_e}{dt} = v_{ez}\,, \qquad \frac{dv_{ez}}{dt} = \frac{e}{m} E_z(t,z_e)\,, \tag{2.1.5}$$

supplemented by the initial conditions

$$z_e(0,z_0) = z_0\,, \qquad v_{ez}(t,z_0) = u\,. \tag{2.1.6}$$

Integration over initial condition z_0 in formula (2.1.4) is performed along the entire length of the number axis — in fact, over the region where unperturbed density $n_e(z_0)$ differs from zero. The establishment of the second initial condition in Eqn (2.1.6) means that all electrons of the beam have the same velocity u at $t = 0$.

Assuming that

$$S_1 n_e(z_0) = \sum_{j=1}^{N_e} \delta(z_0 - z_{0j})\,, \tag{2.1.7}$$

where S_1 is the unit area, and N_e is the total number of electrons, it is possible to put formula (2.1.4), Eqns (2.1.5), and initial conditions (2.1.6) in a form

$$j_{ez}(t,z) = eS_1^{-1}\sum_{j=1}^{N_e} v_{ezj}(t)\,\delta\big(z - z_{ej}(t)\big)\,, \tag{2.1.8}$$

$$\frac{dz_{ej}}{dt} = v_{ezj}\,, \qquad \frac{dv_{ezj}}{dt} = \frac{e}{m} E_z(t,z_{ej})\,, \qquad z_{ej}(0) = z_{0j}\,, \qquad v_{ezj}(0) = u\,. \tag{2.1.9}$$

Here, $z_{ej}(t) = z_e(t,z_{0j})$, and $v_{ezj}(t) = v_{ez}(t,z_{0j})$. Formula (2.1.4) and Eqns (2.1.5) are effective when the electron bunch is described as a continuous medium. Formula (2.1.8) and Eqns (2.1.9) are convenient for considering emission induced by a group of N_e free electrons, where N_e may be as small as $N_e = 1$.

2.2 Hamiltonian method

Let us rearrange Eqn (2.1.2) using the Hamiltonian method [6, 12]. To this effect, we shall assume that the field is enclosed in region $(z, z+L)$, where L is its spatial period, and write out the following expansion:

$$A(t,z) = \frac{1}{2}\sum_{n>0}\big[A_n(t)\exp(\mathrm{i}nk_0 z) + A_n^*(t)\exp(-\mathrm{i}nk_0 z)\big]\,, \tag{2.2.1}$$

where $k_0 = 2\pi/L$. Substituting expansion (2.2.1) into equation (2.1.2), multiplying the latter by $\exp(\mathrm{i}nk_0 z)$, and integrating over periodicity region L yield the following equations describing excitation of plasma oscillations $A_n(t)$ $(n = 1, 2, \ldots)$:

$$\frac{d^2 A_n}{dt^2} + \omega_p^2 A_n = -4\pi e\,\frac{2}{L}\begin{cases}\displaystyle\int n_e(z_0)\, v_{ez}(t,z_0)\exp\big(-\mathrm{i}nk_0 z_e(t,z_0)\big)\, dz_0\,,\\ \displaystyle S_1^{-1}\sum_{j=1}^{N_e} v_{ezj}(t)\exp\big(-\mathrm{i}nk_0 z_{ej}(t)\big)\,.\end{cases} \tag{2.2.2}$$

The right-hand side of equation (2.2.2) is given in two forms corresponding to two possible expressions (2.1.4) and (2.1.8) for electron current density in the bunch. Equations of electron motion corresponding to system (2.1.9) have the form

$$\frac{dz_{ej}}{dt} = v_{ezj}\,, \qquad \frac{dv_{ezj}}{dt} = \frac{1}{2}\frac{e}{m}\sum_n\big(\dot{A}_n(t)\exp(\mathrm{i}nk_0 z_{ej}) + \dot{A}_n^*(t)\exp(-\mathrm{i}nk_0 z_{ej})\big)\,, \qquad z_{ej}(0) = z_{0j}\,, \qquad v_{ezj}(0) = u\,. \tag{2.2.3}$$

Similar equations (but without subscript j) ensue from the system of equations (2.1.5).

Substitution of expansion (2.2.1) into formula (2.1.3) and integration over periodicity region L yield an expression for the total plasma oscillation energy (in volume LS_1):

$$W = wLS_1\,, \qquad w = \frac{1}{16\pi}\sum_n(\dot{A}_n\dot{A}_n^* + \omega_p^2 A_n A_n^*)\,. \tag{2.2.4}$$

The law of conservation of bunch electron kinetic energy and plasma oscillation energy is given by

$$W + \sum_{j=1}^{N_e}\frac{m v_{ezj}^2}{2} = \text{const} \tag{2.2.5}$$

and follows from Eqns (2.2.2) and (2.2.3). Formula (2.2.5) is derived multiplying equation (2.2.2) by $\dot{A}_n^*$, combining it with the complex-conjugate equation, and summing over all n.

Similarly, equation (2.2.3) is multiplied by v_{ezj} and summed up over all j; thereafter, the right-hand sides of the resulting expressions are compared.

When assuming that plasma field (2.2.1) has spatial period L, we actually postulated an identical period in the spatial distribution of electrons inducing this field. In other words, Eqns (2.2.2) describe the field of a sequence of similar electron bunches in plasma, which are spaced a distance L apart, with N_e being the number of electrons in one bunch. One may speak of a solitary electron bunch only with reference to the passage to the limit $L \to \infty$ or $k_0 \to 0$. This limiting case is implied throughout the rest of this review, except in Section 2.8.

2.3 Linear approximation equations

Equations (2.2.2) and (2.2.3) are exact in the framework of the proposed one-dimensional model; they will be numerically analyzed in Section 2.7. In the meantime, we shall derive and consider equations of the zero- and first-order approximations in the perturbations of bunch electron trajectories. Let us write

$$v_{ezj} = u + \tilde{v}_j\,, \qquad z_{ej} = z_{0j} + ut + \tilde{z}_j \tag{2.3.1}$$

and linearize Eqns (2.2.2), (2.2.3) with respect to perturbations $\tilde{v}_j$ and $\tilde{z}_j$. As a consequence, we arrive at the following equations

$$\begin{aligned}
&\frac{\mathrm{d}^2 A_n}{\mathrm{d}t^2} + \omega_p^2 A_n = -\frac{8\pi e}{LS_1}\, u(Q_n + \tilde{V}_n - \mathrm{i}n\tilde{Z}_n)\exp(-\mathrm{i}nk_0 ut)\,,\\
&n = 1, 2, \ldots,\\
&\frac{\mathrm{d}\tilde{V}_n}{\mathrm{d}t} = \frac{1}{2}\frac{e}{mu}\sum_{n'>0}\big[\dot{A}_{n'} Q_{n-n'}\exp(\mathrm{i}n'k_0 ut)\\
&\qquad + \dot{A}^*_{n'} Q_{n+n'}\exp(-\mathrm{i}n'k_0 ut)\big]\,,\\
&\frac{\mathrm{d}\tilde{Z}_n}{\mathrm{d}t} = k_0 u\tilde{V}_n\,.
\end{aligned} \tag{2.3.2}$$

Here, the notation was used:

$$\begin{aligned}
&Q_n = \sum_{j=1}^{N_e}\exp(-\mathrm{i}nk_0 z_{0j})\,,\\
&\tilde{V}_n = \sum_{j=1}^{N_e}\frac{\tilde{v}_j}{u}\exp(-\mathrm{i}nk_0 z_{0j})\,,\\
&\tilde{Z}_n = \sum_{j=1}^{N_e} k_0\tilde{z}_j\exp(-\mathrm{i}nk_0 z_{0j})\,.
\end{aligned} \tag{2.3.3}$$

The right-hand side of the first equation in system (2.3.2) contains three terms. One, the zero approximation term proportional to uQ_n, is determined by unperturbed electron motion in the bunch. This term describes the spontaneous radiation effect. In other words, spontaneous radiation in classical electrodynamics represents the excitation of medium eigenwaves by a given external source, i.e., an unperturbed electron bunch (or a single electron executing a preset motion). Two other, first approximation terms proportional to $(u\tilde{V}_n - \mathrm{i}un\tilde{Z}_n)$, take into account reverse action of the field on bunch electrons. It is precisely these terms that describe the stimulated emission effect as a process of self-consistent interaction between the emitters and the field. Notice that corrections for the first- and higher-order approximations arise in spontaneous emission, too. It is essential that spontaneous emission manifest itself even in the zeroth approximation, i.e., regardless of the reverse action of radiation on the source, whereas stimulated emission develops only in the next approximation in the perturbation of emitter motion. Stimulated emission in classical electrodynamics is usually associated with phasing of a group of emitters by the radiation field, for which at least two electrons are necessary (even if not sufficient).

It is easy to show that the following equation appears instead of the first equation in Eqn (2.3.2) in the description of the electron bunch as a continuous medium [see formula (2.1.4)]:

$$\frac{\mathrm{d}^2 A_n}{\mathrm{d}t^2} + \omega_p^2 A_n = -\frac{8\pi e}{L}\, u(Q_n + \tilde{V}_n - \mathrm{i}n\tilde{Z}_n)\exp(-\mathrm{i}nk_0 ut)\,, \tag{2.3.4}$$

where

$$\begin{aligned}
&Q_n = \int n_e(z_0)\exp(-\mathrm{i}nk_0 z_0)\,\mathrm{d}z_0\,,\\
&\tilde{V}_n = \int n_e(z_0)\,\frac{\tilde{v}(t, z_0)}{u}\exp(-\mathrm{i}nk_0 z_0)\,\mathrm{d}z_0\,,\\
&\tilde{Z}_n = \int n_e(z_0) k_0\,\tilde{z}(t, z_0)\exp(-\mathrm{i}nk_0 z_0)\,\mathrm{d}z_0\,,
\end{aligned} \tag{2.3.5}$$

whereas the third and the second equations in the (2.3.2) system are preserved [the dimensions of quantities (2.3.3) and (2.3.5), and of quantities A_n in Eqns (2.3.2) and (2.3.4) differ by S_1].

The systems of equations (2.3.2) and (2.3.4) are easy to analyze in certain special cases of importance (see Sections 2.4 and 2.5 below).

2.4 The theory of spontaneous Cherenkov emission by a free electron in a plasma

We shall begin from a single-electron case, i.e., consider the classical problem of Cherenkov emission of longitudinal waves by a free plasma electron [5, 8, 13] (by classical problem is meant the problem of emission of transverse waves in an isotropic dielectric [14]; see Section 3.2). At $N_e = 1$, it may be assumed, without loss of generality, that $z_{01} = 0$. Then, $Q_n = 1$ for all n and the system (2.3.2) is put in a form

$$\begin{aligned}
&\frac{\mathrm{d}^2 A_n}{\mathrm{d}t^2} + \omega_p^2 A_n = -\frac{8\pi e u}{LS_1}\exp(-\mathrm{i}nk_0 ut)\\
&\qquad + \frac{4\pi e^2}{mLS_1}(\tilde{V} - \mathrm{i}n\tilde{Z})\exp(-\mathrm{i}nk_0 ut)\,,\\
&\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = \sum_n\big[\dot{A}_n\exp(\mathrm{i}nk_0 ut) + \dot{A}^*_n\exp(-\mathrm{i}nk_0 ut)\big]\,,\\
&\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}t} = k_0 u\tilde{V}\,.
\end{aligned} \tag{2.4.1}$$

In writing equations (2.4.1), substitution $(\tilde{V}_n, \tilde{Z}_n) \to (e/2mu)(\tilde{V}, \tilde{Z})$ was made and account taken of the fact that $\tilde{V}_n$ and $\tilde{Z}_n$ do not depend on n, as follows from expressions (2.3.3) at $z_{01} = 0$.

In order to describe the effect of Cherenkov emission of plasma waves, one may confine oneself to the zero-order approximation, i.e., neglect the second term on the right-hand

side of the first equation in Eqn (2.4.1). In this case, the solution of the first equation in Eqn (2.4.1), meeting zero initial conditions $A_n(0) = 0$ and $\dot{A}_n(0) = 0$, is written out as

$$A_n = -\frac{8\pi eu(LS_1)^{-1}}{\omega_p^2 - (k_n u)^2} \times \left[\exp(-\mathrm{i}k_n ut) - \frac{1}{2}\left(1 - \frac{k_n u}{\omega_p}\right)\exp(\mathrm{i}\omega_p t) - \frac{1}{2}\left(1 + \frac{k_n u}{\omega_p}\right)\exp(-\mathrm{i}\omega_p t)\right] \approx -\frac{1}{2}\frac{8\pi eu(LS_1)^{-1}}{\omega_p(\omega_p - k_n u)}\left[\exp(-\mathrm{i}k_n ut) - \exp(-\mathrm{i}\omega_p t)\right], \quad (2.4.2)$$

where $k_n = nk_0$ is the wave number of the nth spatial field harmonic. In order to simplify the solution (2.4.2), we used inequality $|\omega_p - k_n u| \ll \omega_p$ determining the numbers of the largest resonant spatial field harmonics.

Substituting formula (2.4.2) into the expression for the plasma oscillation energy (2.2.4) gives

$$W = 4\pi\frac{e^2u^2}{LS_1}\sum_n \frac{1 - \cos\left[(\omega_p - k_n u)t\right]}{(\omega_p - k_n u)^2}. \quad (2.4.3)$$

Using further the rule of passage from summation over n to integration over k [5] (corresponding to the passage to the limit $L \to \infty$ or $k_0 \to 0$, $n = k_n/k_0 \to \infty$, $\Delta n = \Delta k/k_0$), viz.

$$\sum_n \ldots \to \int_0^\infty \ldots \mathrm{d}n = \frac{L}{2\pi}\int_0^\infty \ldots \mathrm{d}k, \quad (2.4.4)$$

we reduce formula (2.4.3) to the form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 2\frac{e^2u^2}{S_1}\int_0^\infty \frac{\sin\left[(\omega_p - ku)t\right]}{\omega_p - ku}\,\mathrm{d}k \underset{t\to\infty}{\longrightarrow} 2\pi\frac{e^2u^2}{S_1}\int_0^\infty \delta(\omega_p - ku)\,\mathrm{d}k. \quad (2.4.5)$$

The final expression for the total power of one-dimensional longitudinal plasma waves emitted by an electron moving rectilinearly has the form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{W}{t} = 2\pi\frac{e^2u}{S_1}. \quad (2.4.6)$$

To recall, S_1 in formula (2.4.6) stands for a unit quantity with the dimensions of area that appears because in a one-dimensional case emission is not induced by a single electron but by an 'electron plane' containing one electron at each site of area S_1. Formula (2.4.3) is an analog of the well-known classical Tamm – Frank formula for the power of Cherenkov emission of transverse electromagnetic waves by an electron in an isotropic medium [6, 14, 15] (see Section 3.2).

Now, we take into consideration corrections of the first-order approximation by substituting solution (2.4.2) into the second equation of system (2.4.1). This gives, using formulas (2.4.4) and (2.4.6), the following relationships

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = -\frac{8\pi eu}{LS_1}\sum_n \frac{\sin\left[(\omega_p - k_n u)t\right]}{\omega_p - k_n u} \underset{t\to\infty}{\longrightarrow} -4\pi\frac{eu}{S_1}\int_0^\infty \delta(\omega_p - ku)\,\mathrm{d}k = -\frac{2}{eu}\frac{\mathrm{d}W}{\mathrm{d}t}. \quad (2.4.7)$$

Bearing in mind the above substitution [see the explanatory text after equations (2.4.1)], relation (2.4.7) also follows from the energy conservation law (2.2.5). According to formulas (2.4.7), the electron is slowed down by radiation and its velocity linearly decreases with time:

$$v_{ez}(t) = u - \frac{2\pi e^2}{mS_1}t. \quad (2.4.8)$$

Based on formula (2.4.8), one may speculate that local Cherenkov resonance $\omega_p - k_n v_{ez}(t) = 0$ shifts with time into the region of large n. As a result, the spectrum of excited oscillations broadens. This assumption, based on the linear model (2.4.1), is fully confirmed by the solution of the exact nonlinear problem (see Section 2.7).

2.5 Stimulated Cherenkov emission by a uniform electron beam in a plasma (single-particle effect)

The second special case to be considered based on equations (2.3.2) covers Cherenkov emission by a uniform electron beam. In such a beam, electrons are uniformly distributed in space, which accounts for the uniform localization of their initial coordinates z_{0j} on a number axis. Then, it follows from the first formula in Eqn (2.3.3) that $Q_{n-n'} = \delta_{n,n'}N_e$, $Q_{n+n'} = 0$, and $Q_{n>0} = 0$ (where $\delta_{n,n'}$ is the Kronecker symbol), and the system of equations (2.3.2) is significantly simplified:

$$\frac{\mathrm{d}^2A_n}{\mathrm{d}t^2} + \omega_p^2 A_n = -\omega_e^2(\tilde{V}_n - \mathrm{i}n\tilde{Z}_n)\exp(-\mathrm{i}nk_0ut), \quad \frac{\mathrm{d}\tilde{V}_n}{\mathrm{d}t} = \frac{\mathrm{d}A_n}{\mathrm{d}t}\exp(\mathrm{i}nk_0ut), \quad \frac{\mathrm{d}\tilde{Z}_n}{\mathrm{d}t} = k_0u\tilde{V}_n. \quad (2.5.1)$$

Here, $\omega_e^2 = (4\pi e^2/m)(N_e/LS_1)$ is the square of the Langmuir frequency of beam electrons. Equations (2.5.1) are written out using the substitution $(\tilde{V}_n, \tilde{Z}_n) \to (eN_e/2mu)(\tilde{V}, \tilde{Z})$.

The right-hand side of the first equation in Eqn (2.5.1) lacks the free term responsible for spontaneous emission. This means that equations (2.5.1) describe only the stimulated radiation effect. Since equations (2.5.1) with different n are independent, one may speak about independent stimulated emission of different spatial harmonics of plasma waves by the beam. This inference holds only in the linear approximation.

Representing the solution of the system of equations (2.5.1) in the form

$$A_n(t) = A\exp(-\mathrm{i}\omega t)\exp(-\mathrm{i}k_nut), \quad \tilde{V}_n(t) = V\exp(-\mathrm{i}\omega t), \quad \tilde{Z}_n(t) = Z\exp(-\mathrm{i}\omega t), \quad (2.5.2)$$

where A, V, and Z are the constants, results in the following dispersion equation for determining complex frequency ω of a plasma-beam system [where each spatial harmonic has an eigenfrequency $\omega = \omega(n)$, which is taken into account in the forthcoming formulas (2.5.4), (2.5.6) and in Section 2.6]:

$$\omega^2 - \omega_p^2 = \omega_e^2\frac{\omega^2}{(\omega - k_nu)^2}. \quad (2.5.3)$$

Equation (2.5.3) coincides with the known dispersion equation of beam instability in plasma [16, 17]. When $\omega_e^2 \ll \omega_p^2$ and Cherenkov resonance condition $\omega_p \approx k_nu$ is satisfied, solu-

tions of the dispersion equation take the form

$$\omega \equiv \omega_{ns} = k_n u + \exp\left[\mathrm{i}\,\frac{2\pi}{3}(s-1)\right]\left(\frac{\omega_{\mathrm{e}}^2}{2\omega_{\mathrm{p}}^2}\right)^{1/3}\omega_{\mathrm{p}}\,, \qquad (2.5.4)$$
$$s = 1, 2, 3\,, \qquad \omega_{n4} \approx -\omega_{\mathrm{p}}\,.$$

The positive imaginary part of the complex frequency ω_{n2} defines the growth rate of instability referred to as the single-particle stimulated Cherenkov effect in the plasma theory [1, 2, 7]. This means that, in the one-dimensional case, stimulated Cherenkov emission induced by an electron beam (or flat layers of electrons, to be precise) in a plasma is a form of resonance beam instability.

The width of the spatial spectrum of stimulated emission depends on solutions (2.5.4) of the dispersion equation (2.5.3) existing only in the following range of wave numbers:

$$\frac{|\Delta k|u}{\omega_{\mathrm{p}}} = \frac{|\Delta n|}{n} < \left(\frac{\omega_{\mathrm{e}}^2}{2\omega_{\mathrm{p}}^2}\right)^{1/3} \equiv \delta_0\,. \qquad (2.5.5)$$

Plasma oscillations are not excited outside the range (2.5.5) because the corresponding frequencies ω_n have their imaginary part close to zero.

The general solution of system (2.5.1) is written out in the form

$$A_n(t) = \sum_{s=1}^{4} C_{ns}\exp(-\mathrm{i}\omega_{ns}t)\,, \qquad A(t) = \sum_n A_n(t)\,, \qquad (2.5.6)$$

where constants C_{ns} are calculated from the initial conditions, and ω_{ns} $(s = 1, 2, 3, 4)$ is defined in formula (2.5.4). Let us assume that an electron beam is unexcited at $t = 0$, and plasma has a certain noise background of Langmuir waves. Such a case is described by the following initial conditions for equations (2.5.1):

$$A_n(0) = A_{0n}\,, \quad \dot{A}_n(0) = 0\,, \quad \tilde{V}_n(0) = 0\,, \quad \tilde{Z}_n(0) = 0\,, \qquad (2.5.7)$$

where A_{0n} are the complex constants. Using conditions (2.5.7), it is easy to show that formulas $C_{n\,1,2,3} = A_{0n}/3$, $C_{n4} \approx 0$ are valid in the range (2.5.5) [7]. Thus, stimulated emission is induced by initial perturbations.

2.6 Energy approach in the theory of Cherenkov emission by an electron bunch in a plasma

At arbitrary N_{e}, i.e., in the most interesting case of an electron bunch (rather than a single electron and boundless beam), a strict analysis of the system of equations (2.3.2) encounters difficulty. It is more convenient to utilize qualitative energy relations. Let us derive the equation for the energy of one-dimensional plasma oscillations exited by an electron bunch. To this end, we shall differentiate relation (2.2.5) with respect to time and substitute into it the second expression from Eqn (2.2.3). Then, using representation (2.3.1), we obtain the following relation to within second-order quantities in A_n, inclusive:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{2}\,eu\sum_n\left[\dot{A}_n U_n^*\exp(\mathrm{i}k_n ut) + \dot{A}_n^* U_n\exp(-\mathrm{i}k_n ut)\right], \qquad (2.6.1)$$

where the quantity $U_n = Q_n + \tilde{V}_n - \mathrm{i}n\tilde{Z}_n$ is found from the second and third equations of system (2.3.2).

The first equation in system (2.3.2) can be formally considered as an inhomogeneous linear second-order differential equation (term $\sim Q_n$ is defined by the unperturbed state of the electron bunch and does not depend on A_n; term $\sim (\tilde{V}_n - \mathrm{i}n\tilde{Z}_n)$ is self-consistent and is linearly dependent on A_n and $A_{n'}$). The general solution of an inhomogeneous equation is the sum of its particular solution and the general solution of the corresponding homogeneous equation. Therefore, the following representation is applied for function $A_n(t)$:

$$A_n(t) = A_n^{(0)}(t) + \tilde{A}_n(t)\,,$$
$$A_n^{(0)}(t) = -\frac{4\pi eu(LS_1)^{-1}Q_n}{\omega_{\mathrm{p}}(\omega_{\mathrm{p}} - k_n u)}\left[\exp(-\mathrm{i}k_n ut) - \exp(-\mathrm{i}\omega_{\mathrm{p}}t)\right],$$
$$\tilde{A}_n(t) = A_{0n}\exp(-\mathrm{i}\omega_n t)\,. \qquad (2.6.2)$$

Here, $A_n^{(0)}(t)$ is the solution of the inhomogeneous equation [which describes the spontaneous field and differs from solution (2.4.2) only by factor Q_n], $\tilde{A}_n(t)$ is the solution of the homogeneous equation (stimulated field), A_{0n} are the constants [see conditions (2.5.7)], and ω_n are the complex frequencies found from Eqn (2.5.3). Quantities $\tilde{A}_n(t)$ describe plasma waves being excited; therefore, we take $\omega_n' \equiv \mathrm{Re}\,\omega_n \approx \omega_{\mathrm{p}}$, and $\omega_n'' \equiv \mathrm{Im}\,\omega_n > 0$.

Substituting formulas (2.6.2) into the second and the third equations of system (2.3.2) gives $\tilde{V}_n$ and $\tilde{Z}_n$. Substituting further $A_n = A_n^{(0)} + \tilde{A}_n$, $\tilde{V}_n$ and $\tilde{Z}_n$ into relation (2.6.1) brings it to the form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{4\pi e^2u^2}{LS_1}\sum_n |Q_n|^2\frac{\sin\left[(\omega_{\mathrm{p}} - k_n u)t\right]}{\omega_{\mathrm{p}} - k_n u}$$
$$+\,\mathrm{i}\,\frac{e^2}{4m}\sum_{n,n'}\left\{\frac{\omega_n\omega_{n'}^*\left[\omega_{n'}^* - (k_{n'} - k_n)u\right]}{(\omega_{n'}^* - k_{n'}u)^2}\,Q_{n-n'}^* A_{0n}A_{0n'}^*\right.$$
$$\left.\times\exp\left(-\mathrm{i}\left[(\omega_n - \omega_{n'}^*) - (k_n - k_{n'})u\right]t\right) - \mathrm{c.c.}\right\}. \qquad (2.6.3)$$

Relation (2.6.3) was obtained with the same accuracy as formula (2.4.6), i.e., without taking account of the reverse action of the spontaneous field $A_n^{(0)}(t)$ on bunch electrons [without considering the effect of radiative deceleration defined by formula (2.4.8)].

Nondiagonal terms with $n \neq n'$ in the double sum entering into relationship (2.6.3) describe the interference of different spatial harmonics of the stimulated field. The harmonics have different frequencies, which accounts for the appearance of an oscillating multiplier $\exp\left\{-\mathrm{i}\left[(\omega_n - \omega_{n'}^*) - (k_n - k_{n'})u\right]t\right\}$ in the double sum. Moreover, the initial phases of the harmonics may be regarded as random, namely

$$\sum_n A_{0n} = 0\,, \qquad \sum_{n,n'} A_{0n}A_{0n'}^* = \sum_n |A_{0n}|^2\,. \qquad (2.6.4)$$

For this reason, interference terms in the double sum in formula (2.6.3) make on average a zero contribution. For the same reason, relation (2.6.3) was written down without cumbersome terms arising from interference between spontaneous and stimulated fields.

Thus, discarding interference terms in formula (2.6.3) leads to the following equation for the energy of one-

dimensional plasma oscillations excited by an electron bunch:

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \frac{4\pi^2 e^2 u^2}{LS_1} \sum_n |Q_n|^2 \delta(\omega_\mathrm{p} - k_n u) - \frac{e^2 N_\mathrm{e}}{2m} \sum_n |\omega_n|^2 \frac{\omega_n'' \left[(\omega_n' - k_n u)(\omega_n' + k_n u) + \omega_n''^2\right]}{\left[(\omega_n' - k_n u)^2 + \omega_n''^2\right]^2} \left|\tilde{A}_n(t)\right|^2 . \tag{2.6.5}$$

Derivation of Eqn (2.6.5), just like Eqn (2.4.7), required a passage to the delta-function. The first term on the right-hand side of equation (2.6.5) describes spontaneous emission, while the second one is responsible for stimulated emission. Let us compare the roles of spontaneous and stimulated effects.

Neglecting the term describing stimulated emission and moving from summation over n to integration over k [see rule (2.4.4) and formula (2.4.7)] bring equation (2.6.5) to the form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 2\pi \frac{e^2 u}{S_1} \left[\sum_{j=1}^{N_\mathrm{e}} \exp\left(-\mathrm{i}\,\frac{\omega_\mathrm{p}}{u}\, z_{0j}\right)\right] \left[\sum_{j=1}^{N_\mathrm{e}} \exp\left(\mathrm{i}\,\frac{\omega_\mathrm{p}}{u}\, z_{0j}\right)\right] . \tag{2.6.6}$$

At $N_\mathrm{e} = 1$, formula (2.6.6) turns into formula (2.4.6). It should be noted that the product of the sums entering into formula (2.6.6) varies from 0 to N_e^2. In other words, the spontaneous emission intensity depends on the mutual position of electrons in the bunch, evidently due to interference of coherent waves emitted by individual electrons. If the positions of electrons in the bunch are such that the sum in formula (2.6.6) goes to zero, no spontaneous emission occurs. In the case of uniform continuous distribution of electrons in the bunch, one has

$$\left|\sum_{j=1}^{N_\mathrm{e}} \exp\left(-\mathrm{i}\,\frac{\omega_\mathrm{p}}{u}\, z_{0j}\right)\right| \to N_\mathrm{e} \left|\frac{\sin(\pi L_\mathrm{e}/\lambda)}{\pi L_\mathrm{e}/\lambda}\right| , \tag{2.6.7}$$

where L_e is the bunch length, and $\lambda = 2\pi u/\omega_\mathrm{p}$ is the average emission wavelength. Thus, spontaneous emission falls with increasing bunch length. The condition under which spontaneous emission induced by a uniform electron bunch may be regarded as essential is given by the inequality $L_\mathrm{e} < \lambda$. Spontaneous emission is insignificant for $L_\mathrm{e} > \lambda$, and altogether absent as $L_\mathrm{e} \gg \lambda$. Hence, one may speak of a boundless electron beam rather than of an electron bunch.

Let us discard the term describing spontaneous emission in formula (2.6.6), take $\omega_n' \approx k_n u \approx \omega_\mathrm{p}$ and $\omega_n'' \ll \omega_\mathrm{p}$, and bring expression (2.6.6) to the form [1, 2]

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = -\frac{e^2 N_\mathrm{e}}{m}\, \omega_\mathrm{p}^3 \sum_n \frac{\omega_n''(\omega_n' - k_n u)}{\left[(\omega_n' - k_n u)^2 + \omega_n''^2\right]^2} \left|\tilde{A}_n(t)\right|^2 . \tag{2.6.8}$$

The right-hand side of formula (2.6.8) describes the ponderomotive force acting on an electron bunch from the side of radiation. Spectral density of the ponderomotive force is determined by the function

$$f(k_n) = \frac{\omega_n''(\omega_n' - k_n u)}{\left[(\omega_n' - k_n u)^2 + \omega_n''^2\right]^2} \tag{2.6.9}$$

and for $\omega_n'' \ll \omega_n' \approx \omega_\mathrm{p}$ exhibits a well-defined resonant character. If $\omega_n' - k_n u < 0$ and $\omega_n'' > 0$, then $\mathrm{d}W/\mathrm{d}t > 0$ (i.e., the electrons radiate and the ponderomotive force slows down the bunch. Thus, stimulated emission is associated with excitation of waves having phase velocities ω_n'/k_n smaller than unperturbed electron velocity u.

When computing the sum in equation (2.6.8), it is necessary to distinguish between cases of wide and narrow radiation spectra. In the former case, quantities $|\tilde{A}_n(t)|^2$ on the right-hand side of Eqn (2.6.8) can be taken outside the summation sign. However, since $\sum f(k_n) = 0$, so does $\mathrm{d}W/\mathrm{d}t = 0$, suggesting the lack of stimulated emission. In the latter case, factor $f(k_n)$ is taken outside the summation sign. Then, using formula (2.2.4), Eqn (2.6.8) can be brought to

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -2\omega_\mathrm{e}^2 \omega_\mathrm{p} \omega_n'' f(k_n) W . \tag{2.6.10}$$

Setting the condition for the minimum of function (2.6.9), viz.

$$\omega_n' - k_n u = -\frac{1}{\sqrt{3}}\, \omega_n'' , \tag{2.6.11}$$

it is possible to derive from Eqn (2.6.10) the following law of maximum time-dependent growth of the energy of a plasma wave being emitted:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \left(\frac{\sqrt{3}}{2}\right)^3 \frac{\omega_\mathrm{e}^2 \omega_\mathrm{p}}{\omega_n''^2}\, W . \tag{2.6.12}$$

On the other hand, it can be seen from the third expression in Eqn (2.6.2) that the equation $\mathrm{d}W/\mathrm{d}t = 2\omega_n'' W$ must be valid. Its comparison with formula (2.6.12) leads to the expression for the imaginary part of the frequency, coinciding with the expression for the imaginary part of frequency (2.5.4). Thus, the width of the radiation spectrum and the type of emission are interrelated: the spectrum is wide when emission is spontaneous, and narrow when it is stimulated.

The width of the radiation spectrum is given either by inequality (2.5.5) or by the estimate

$$\frac{|\Delta k| u}{\omega_\mathrm{p}} \approx \frac{2\pi u}{L_\mathrm{e} \omega_\mathrm{p}} = \frac{\lambda}{L_\mathrm{e}} \tag{2.6.13}$$

based on the relation $\Delta k L_\mathrm{e} \approx 2\pi$ and solution (2.4.2): the maximum quantity from Eqn (2.5.5) or Eqn (2.6.13) is taken. For $L_\mathrm{e} < \lambda$, the spectrum is wide and stimulated emission is absent, while spontaneous emission prevails in accordance with formulas (2.6.6) and (2.6.7). If $\lambda < L_\mathrm{e} < \lambda\delta_0^{-1}$, spontaneous emission is low and the effect of stimulated emission induced by the electron bunch predominates. However, spectrum width and resonance width (2.6.9) are comparable in this intermediate case, which makes analysis of Eqn (2.6.8) difficult. Finally, as $L_\mathrm{e} > \lambda\delta_0^{-1}$, the waves are excited from a narrow (even if single-mode) range of wave numbers (2.5.5) because modes with different n are independent. Then, spectral function (2.6.9) can be taken outside the sign of summation over n and Eqn (2.6.8) transformed into (2.6.10). Thus, there is stimulated emission by an electron beam for $L_\mathrm{e} > \lambda\delta_0^{-1}$, i.e., resonance beam instability in the plasma due to the single-particle Cherenkov effect.

It may be concluded from the above analysis and the structure of the general equation (2.6.5) that radiation energy satisfies an equation of the form [see formulas (1.2) and (1.4)]

$$\frac{\mathrm{d}W}{\mathrm{d}t} = P(N_\mathrm{e}) + 2\delta(L_\mathrm{e}) W , \tag{2.6.14}$$

where $P(N_\mathrm{e})$ is the spontaneous radiation power defined by formula (2.6.6), and $\delta(L_\mathrm{e})$ is the growth rate of beam

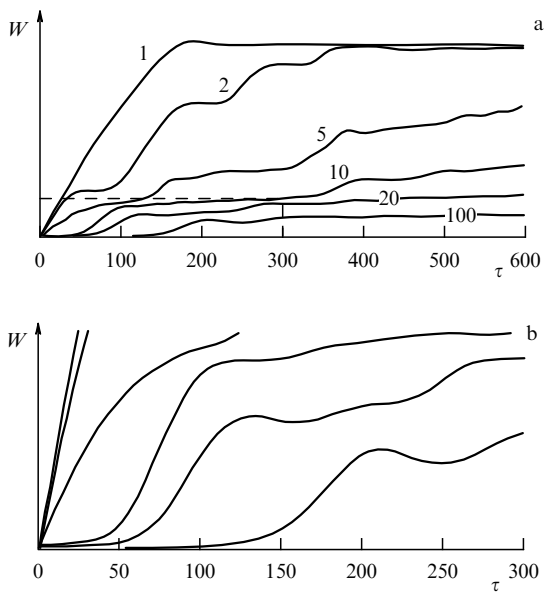

Figure 1. Cherenkov emission energy at different numbers of electrons in a bunch: (a) general view, and (b) enlarged portion.

instability. The increment is calculated in limiting cases, such as $L_e = 0$, in which it vanishes, and $L_e \to \infty$, in which it is given by formula (2.5.4). The general case lends itself to numerical analysis only.

2.7 Nonlinear dynamics of Cherenkov emission by an electron bunch in a plasma

Cherenkov emission induced by an electron bunch can be numerically investigated in full using general nonlinear equations (2.2.2) and (2.2.3). In dimensionless variables

$$\tau = \omega_p t\,, \qquad \xi = k_0 z\,, \qquad \xi_j = k_0 z_{ej}\,, \qquad \eta_j = \frac{v_{ezj}}{u}\,, \quad (2.7.1)$$

equations (2.2.2), (2.2.3) are written out in the form

$$\begin{aligned} &\frac{d^2 a_n}{d\tau^2} + a_n = -\nu\,\frac{2}{N_e}\sum_{j=1}^{N_e}\eta_j \exp\left(-in\xi_j\right), \\ &\frac{d\xi_j}{d\tau} = \alpha\eta_j\,, \\ &\frac{d\eta_j}{d\tau} = \frac{1}{2}\sum_{n=1}^{N_g}\left[\frac{da_n}{d\tau}\exp\left(in\xi_j\right) + \text{c.c.}\right], \end{aligned} \quad (2.7.2)$$

where a_n is the dimensionless amplitude proportional to dimensional amplitude A_n (in what follows, we shall not need its explicit form), and

$$\alpha = \frac{k_0 u}{\omega_p}\,, \qquad \nu = \frac{4\pi e^2 N_e}{\omega_p^2 S_1 L} = \frac{\omega_e^2}{\omega_p^2}\,. \quad (2.7.3)$$

Quantity N_g in system (2.72) specifies the maximum number of spatial harmonics of plasma oscillations. Let us assume that $N_g = 200$, $\alpha = 10^{-2}$, and $\nu = 10^{-5}$. Cherenkov resonance corresponds to a harmonic with $n = 100$. According to estimate (2.5.5), about four harmonics fall into the resonance band in the case of occurring the single-particle stimulated Cherenkov effect. It is sufficient to simulate even a continuous spectrum.

We shall vary the number N_e of emitters in a bunch. The right-hand side of the first equation in system (2.7.2) is built up so that a change in N_e at constant parameter ν leads to the spread of one and the same emitting charge over spatial regions of different sizes. Equations (2.7.2) are supplemented by the following initial conditions

$$\begin{aligned} &a_n(0) = 0\,, \qquad \dot{a}_n(0) = 0\,, \\ &\xi_j(0) = h(j-1)\,, \quad \eta_j(0) = 1\,, \quad j = 1, 2, \ldots, N_e\,. \end{aligned} \quad (2.7.4)$$

Here, h is the distance between electrons in the bunch (assumed equal to one-tenth of the resonant wavelength in the calculations). The electron bunch has the size $L_e = h(N_e - 1)$. Notice that the choice of parameters for the numerical solution of the model problem (2.7.2) – (2.7.4) was determined only by considerations of the maximum demonstrativeness of the results obtained.

Figure 1a presents, in relative units, energies $W(\tau)$ calculated from formula (2.2.4) for the number of electrons $N_e = 1, 2, 5, 10, 20$, and 100 (N_e values are indicated alongside the respective curves). Figure 1b shows an enlarged fragment framed in Fig. 1a. Dependences $W(\tau)$ vary qualitatively with an increasing number of emitting particles. For example, the radiation energy at $N_e = 1$ grows almost linearly up to saturation, which is reached rather quickly. In the initial stage, the case of $N_e = 1$ is clearly described by formula (2.4.6). In contrast, for $N_e \geqslant 10$, the radiation energy grows faster (exponentially) in the early stage. Thus, a rise in the number N_e of emitting particles causes transition of spontaneous emission by an electron (and thereafter by the bunch) to stimulated emission by the bunch, i.e., to Cherenkov beam instability. This observation is confirmed by the fact that $L_e = \lambda$ at $N_e = 10$, while the bunch length L_e at $N_e = 100$ compares with $\lambda\delta_0^{-1}$. This means that at the parameter values chosen for computation, $N_e = 10$ determines the borderline between spontaneous and stimulated effects, in full agreement with qualitative considerations used in writing equation (2.6.14).

Figure 2 depicts the amplitudes $\left|a_n(\tau)\right|$ of harmonics; the harmonic number n is indicated alongside the respective curve. For $N_e = 1$ (Fig. 2a), there is successive excitation of progressively higher spatial harmonics and simultaneous saturation of the amplitudes of harmonics with lower n. A similar picture is observed at $N_e = 2$ (Fig. 2b). The case of $N_e = 10$ is quite different because only resonant harmonic with $n = 100$ grows, while the rest remain at the background level (harmonics closest to the hundredth harmonic and residing within the range of $\pm 2 - 3$ also increase).

Figure 3 presents phase trajectories of electrons in a time interval τ from 0 to 600. Dimensionless electron velocities η_j are plotted on the vertical axis, and coordinates ξ_j on the horizontal one. The phase trajectory for $N_e = 1$ is shown in Fig. 3a under the dashed straight line. In the initial stage, the electron velocity decreases almost linearly, in excellent agreement with formula (2.4.8). Thereafter, the velocity becomes practically constant, with radiation energy saturation occurring simultaneously (Fig. 1a, curve 1). Saturation is associated with radiative deceleration of electrons. The electron velocity diminishes to the minimally possible phase velocity of the wave. The instant of saturation is found from the equation

$$v_{ez}(t) = u - \frac{2\pi e^2}{mS_1}\,t = \left(\frac{\omega(k_n)}{k_n}\right)_{\min} \equiv V_{\min}\,, \quad (2.7.5)$$

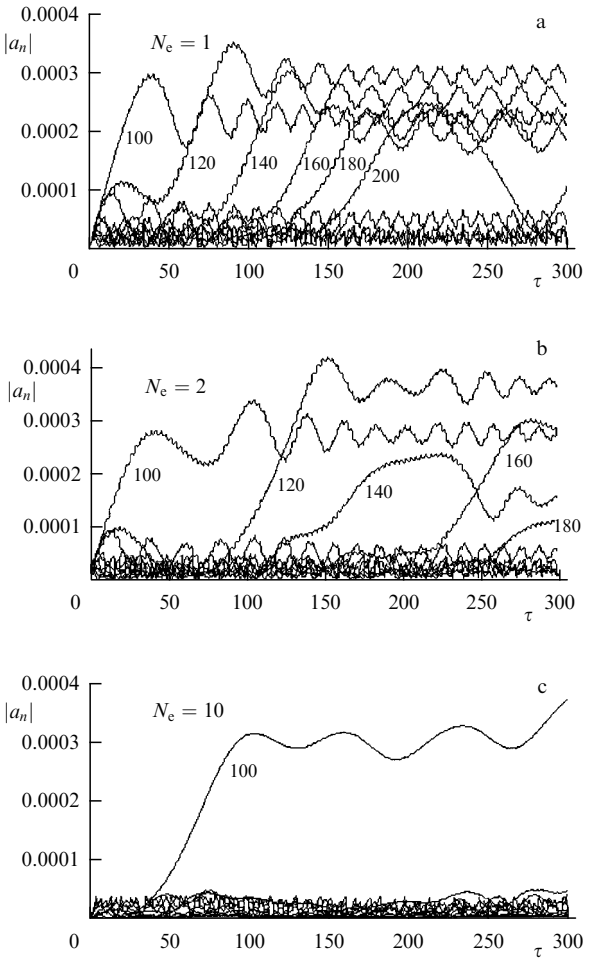

Figure 2. Amplitudes of Cherenkov emission harmonics at different numbers of electrons in the bunch.

where $\omega(k_n)$ is the dispersion law for emitted waves. The energy passing into emission equals

$$W_{\max} = \frac{m}{2}(u - V_{\min})^2 . \qquad (2.7.6)$$

Formulas (2.7.5) and (2.7.6) are in excellent agreement with the results presented in Fig. 3.

Two phase trajectories for $N_e = 2$ electrons are shown under the dashed straight line in Fig. 3b. Since distance h between electrons is small compared with the wavelength, virtually the same dynamics as at $N_e = 1$ are observed.

The picture is quite different at $N_e = 10$ (portion of Fig. 3a above the dashed straight line) and at $N_e = 20$ (portion of Fig. 3b above the dashed straight line). Specifically, the phase trajectories get intermixed, while all electrons are slowed down, suggesting phasing of bunch electrons by radiation. This phenomenon is characteristic of resonance beam instabilities stabilized by beam electron capture [7, 18, 19].

Figure 4 displays spatial spectra of plasma oscillation amplitudes $|a_n|$ at different instants of time τ indicated in the figure. In the case of $N_e = 1$ (Fig. 4a), the spectrum broadens with time due to electron deceleration and a shift of local Cherenkov resonance toward greater wave numbers. No spectrum broadening occurs at $N_e = 10$; moreover, the radiation spectrum is narrower than for $N_e = 1$.

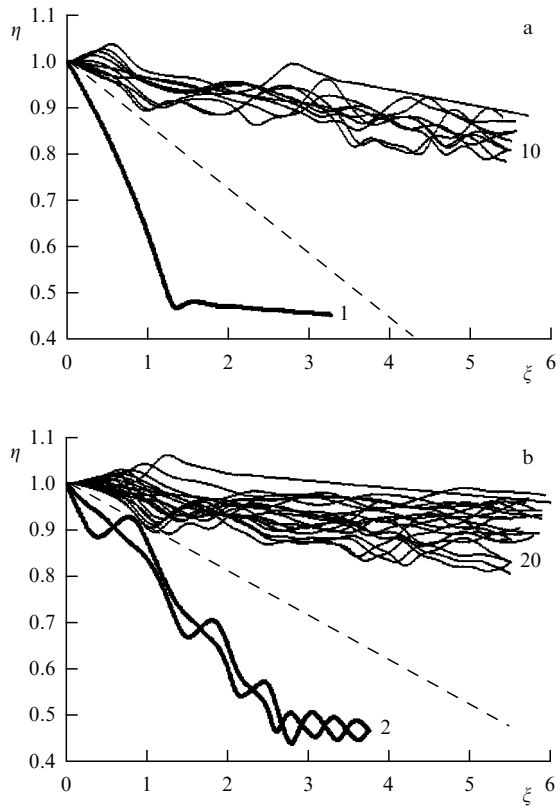

Figure 3. Phase trajectories of bunch electrons involved in Cherenkov emission.

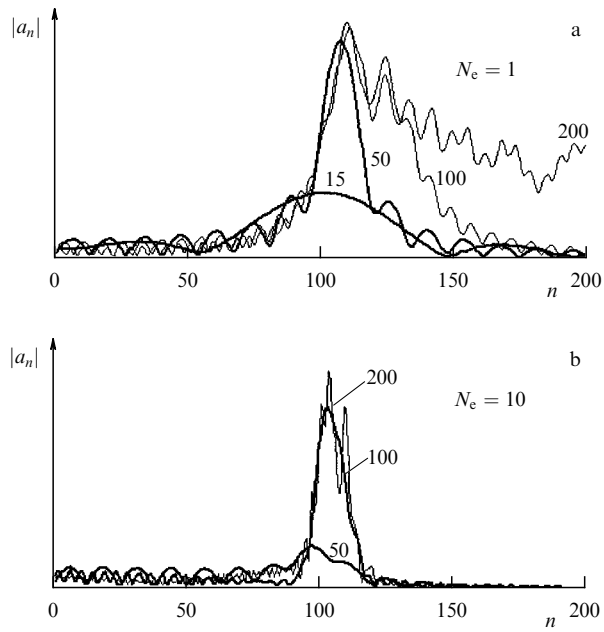

Figure 4. Spatial spectra of plasma oscillations accompanying Cherenkov emission.

Let us consider the spatio-temporal structure of plasma oscillations excited by a single electron. In a three-dimensional case, such a structure is traditionally characterized by a radiation pattern which is usually computed analytically [6, 15] (see Section 3.1). In a one-dimensional case, the spatial distribution of plasma oscillations can be described in full by electric field strength $E_z(t, z)$ or, in the dimensionless form, by

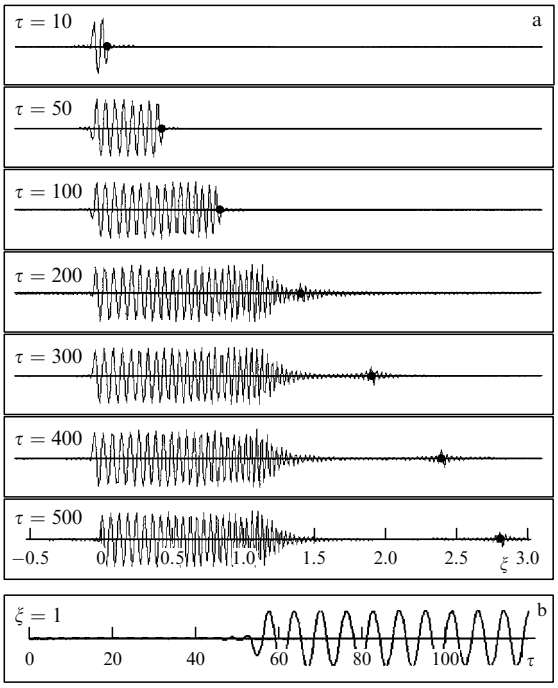

Figure 5. The structure of a one-dimensional plasma wave packet during Cherenkov emission by an electron.

the function

$$e_\xi(\tau,\xi) = \frac{1}{2}\sum_{n=1}^{N_g}\left(\dot{a}_n(\tau)\exp(\mathrm{i}n\xi) + \dot{a}_n^*(\tau)\exp(-\mathrm{i}n\xi)\right). \quad (2.7.7)$$

Figure 5a illustrates spatial distribution of wave packet (2.7.7) for successive instants of time τ. The position of an emitting electron at each instant is shown by a black circle. It can be seen that the electron occupies the forefront of the wave packet, whereas its rear front remains motionless. Such a situation can be attributed to the lack of dispersion in the waves being excited in plasma, which accounts for the vanishing of their group velocity [the left-hand side of the first equation in Eqn (2.3.2) contains constant ω_p^2 instead of function $\omega_p^2(n)$]. By about the time $\tau = 200$, the electron is slowed down so much that the Cherenkov resonance condition is no longer satisfied. The packet of plasma oscillations is detached from the electron which continues to move without radiating, accompanied by a weak entrained field.

Figure 5b depicts plasma oscillations at point $\xi = 1$, where no oscillations occurred until the electron reached it. After the electron passed this point, oscillations originated with a dimensionless period 2π, i.e., with a frequency equal to the plasma frequency.

2.8 Cherenkov emission in a rarefied spectrum; emission induced by a modulated electron beam in a plasma

For $L \gg \lambda = 2\pi u/\omega_p$, i.e., when the passage to the limit $L \to \infty$ is possible, bunch electron emission (2.2.1) belongs to the continuous spectrum. Indeed, the intermode distance $\Delta k = k_0 = 2\pi/L$ in spectrum is small compared with the resonant wave number $k_n = \omega_p/u$; therefore, it may be assumed that $\Delta k/k_n \to 0$. This limiting process is the basis for the transformation (2.4.4) of the sum over n to the integral over wave numbers k. We acted accordingly in Sections 2.4–2.6 when analyzing Eqns (2.3.2). In Section 2.7, the nonlinear system of equations (2.2.2), (2.2.3) was solved for the case of $\Delta k/k_n = 10^{-2}$ as well. The case opposing to that considered in the preceding paragraphs is emission in a line or rarefied spectrum. The case of a rarefied spectrum is realized at length L comparable with the wavelength $\lambda = 2\pi u/\omega_p$ (e.g., during emission in a short resonator); this case is of interest for microwave electronics [19]. Emission induced by a modulated beam also reduces to the case of small L.

Let $k_0 u = \omega_p$ (or $L = \lambda$), i.e., Cherenkov resonance occurs on mode $n = 1$. Then $\Delta k/k_n = 1$, and the spectrum is so rarefied that resonance on the second- and higher-order modes is impossible (for $\omega_e^2 \ll \omega_p^2$ and $\Delta k/k_n = 1$, inequality (2.5.5) cannot be fulfilled). This necessitates taking into consideration the wave A_1 alone (putting $A_{n>1} = 0$) in equations (2.3.2). Introducing the slow amplitude of a resonant plasma wave, $A_1(t) = A(t)\exp(-\mathrm{i}\omega_p t)$, and taking account of the resonance condition lead to a system of equations derived from Eqn (2.3.2):

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= -\mathrm{i}\,\frac{4\pi e u}{\omega_p L S_1}(Q_1 + \tilde{V} - \mathrm{i}\tilde{Z}),\\
\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} &= -\mathrm{i}\,\frac{e\omega_p}{2mu}(Q_0 A - Q_2 A^*),\\
\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}t} &= \omega_p \tilde{V}.
\end{aligned} \quad (2.8.1)$$

Here, $Q_{0,1,2}$ are defined by the first formula in Eqn (2.3.3), and, in particular, $Q_0 = N_e$. System of equations (2.8.1) differs from system (2.3.2) mainly by the absence of summation over the harmonics, which accounts for the possibility of new solutions to equations (2.8.1). It should be noted that taking into account only one wave in system (2.3.2) [e.g., as A_1 in system (2.8.1)] we pass from electron interaction with a wave packet to its interaction with a spatially monochromatic wave. This constitutes the main difference between radiation emission in continuous and rarefied (line) spectra.

Let us answer [based on the system of equations (2.8.1)] a methodically important question of how many electrons a uniform beam should contain per wavelength $\lambda = 2\pi u/\omega_p$ to enable stimulated emission to develop. Let us begin from the case with only one electron per wavelength, i.e., $N_e = 1$ (this number cannot be smaller since λ is the spatial period of the field). Then, $Q_0 = Q_1 = Q_2 = 1$ and equations (2.8.1) are written out as

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= -\mathrm{i}\,\frac{4\pi e u}{\omega_p L S_1}(1 + \tilde{V} - \mathrm{i}\tilde{Z}),\\
\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} &= -\mathrm{i}\,\frac{e\omega_p}{2mu}(A - A^*),\\
\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}t} &= \omega_p \tilde{V}.
\end{aligned} \quad (2.8.2)$$

The main feature of system (2.8.2) is its inhomogeneity, i.e., the presence of a free term on the right-hand side of the first equation. In the zero approximation in perturbations $\tilde{V}$ and $\tilde{Z}$, the first equation of system (2.8.2) yields

$$A(t) = -\mathrm{i}\,\frac{2e}{S_1}\,\omega_p t, \qquad W(t) = \frac{e^2 u}{S_1}\,\omega_p t^2. \quad (2.8.3)$$

In the next approximation, we find the following expression from the second equation of system (2.8.2) for the electron

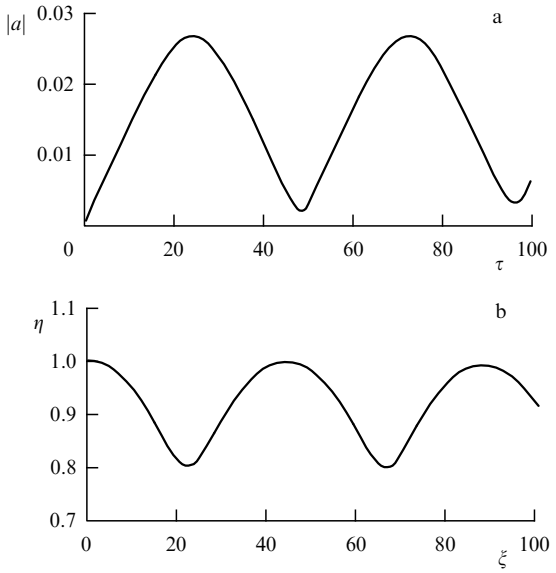

Figure 6. Nonlinear dynamics of spontaneous Cherenkov emission by an electron sequence comprising one electron per wavelength: (a) amplitude, and (b) phase trajectory.

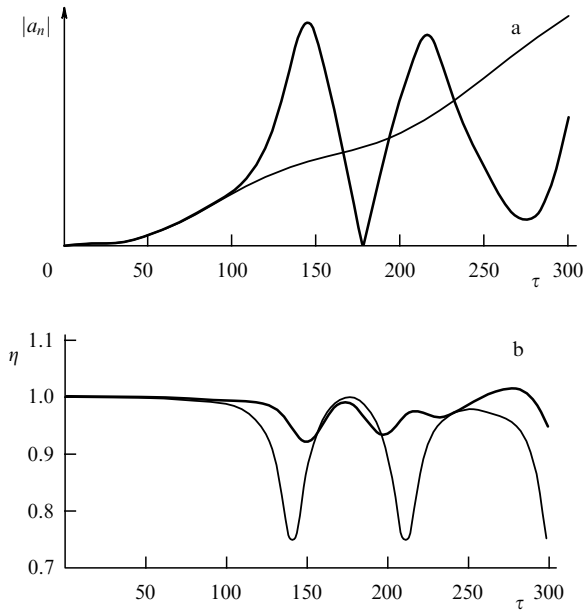

Figure 7. Dynamics of Cherenkov emission by a beam comprising two electrons per wavelength: (a) amplitude, and (b) phase trajectory.

velocity:

$$v_{ez}(t) = u - \frac{e^2}{mS_1}\,\omega_p t^2\,. \tag{2.8.4}$$

Formula (2.8.3) for W was obtained with the use of expressions (2.2.4). Certainly, formulas (2.8.3) and (2.8.4) agree with the law of conservation (2.2.5).

Formula (2.8.3) describes spontaneous emission induced by a uniform beam with a minimally possible number of electrons (one per wavelength), and formula (2.8.4) takes into consideration the reverse action of the field on each electron. It appears from the comparison of expressions (2.8.3) and (2.8.4) with formulas (2.4.6) and (2.4.8) that the dynamics of spontaneous Cherenkov emission by a sparse ($N_e = 1$) electron sequence in a rarefied spectrum differ from those of spontaneous Cherenkov emission by a single electron in a continuous spectrum. Nonlinear emission dynamics given by formulas (2.8.3) and (2.8.4) are described by the system of equations (2.7.2), the numerical solutions of which at $\alpha = 1$, $\nu = 10^{-3}$ are presented in Fig. 6, namely, the modulus of dimensionless plasma wave amplitude $|a| = |a_1(\tau)|$ in Fig. 6a, and the electron's phase trajectory in Fig. 6b. The initial segments ($\tau, \xi < 15-20$) of the curves presented are described by formulas (2.8.3) and (2.8.4).

Consider now a situation with two electrons per wavelength $\lambda = 2\pi u/\omega_p$, i.e., $N_e = 2$. In this case, $Q_0 = Q_2 = 2$, $Q_1 = 0$, and equations (2.8.1) take the form

$$\begin{aligned} \frac{dA}{dt} &= -i\,\frac{4\pi e u}{\omega_p L S_1}\,(\tilde{V} - i\tilde{Z})\,, \\ \frac{d\tilde{V}}{dt} &= -i\,\frac{e\omega_p}{mu}\,(A - A^*)\,, \\ \frac{d\tilde{Z}}{dt} &= \omega_p \tilde{V}\,. \end{aligned} \tag{2.8.5}$$

System of equations (2.8.5) being homogeneous, it cannot describe spontaneous effects. Consequently, spontaneous emission at $N_e = 2$ is absent.

In order to solve the system of equations (2.8.5), we shall set the following initial conditions: $\tilde{V}(0) = 0$, $\tilde{Z}(0) = 0$, and $A(0) = A_0$, where A_0 is the complex constant, meaning that the electron beam is not perturbed at $t = 0$ and that plasma initially possesses a certain background of Langmuir oscillations. System (2.8.5) with these initial conditions is readily integrated to give

$$\begin{aligned} &\operatorname{Im} A(t) = \operatorname{Im} A_0 \cos(\omega_{2e} t)\,, \\ &\operatorname{Re} A(t) = \operatorname{Re} A_0 + \omega_p t \operatorname{Im} A_0 \left[1 - \frac{\sin(\omega_{2e} t)}{\omega_{2e} t}\right], \end{aligned} \tag{2.8.6}$$

where $\omega_{2e} = \sqrt{4\pi e^2 2/(mLS_1)}$ is the 'Langmuir frequency' for two electrons. According to expressions (2.8.6), the amplitude of plasma oscillations grows linearly, and their energy grows as $\sim t^2$. This distinguishes radiation (2.8.6) from ordinary stimulated radiation that grows exponentially. Hence, the case of a uniform beam with two electrons per length of a spatially monochromatic resonant wave relates to a special one. Moreover, there is no emission whatever at $\operatorname{Im} A_0 = 0$. For this reason, the two-electron system is virtually stable.

Stability of the two-electron system with respect to resonant excitation of the spatially monochromatic wave is understandable. Indeed, the distance between any two neighboring electrons being $\lambda/2$, one of them resides in the decelerating and the other in the accelerating phase of the field; on the average, there is no energy exchange between electrons and the wave. However, such a situation holds true only in the linear approximation. As a result of nonlinear displacement of the electrons in the wave field, both eventually find themselves in the decelerating phase of the field, with a consequent emission of induced radiation. At $N_e = 2$, stimulated emission is described by the general nonlinear system of equations (2.7.2), whose numerical solutions at $\alpha = 1$, $\nu = 10^{-3}$ are presented in Fig. 7. The system was integrated under the same initial conditions as in obtaining solutions (2.8.6). In Fig. 7a showing a plasma wave

amplitude, the thick curve corresponds to the numerical solution, and the thin curve to the analytical one. The initial segments of both curves coincide, but the numerical solution then grows almost exponentially up to reaching saturation. Saturation and further lowering of the amplitude are due to the shift of both electrons to the accelerating phases of the wave, as appears from the phase trajectories displayed in Fig. 7b.

Consider finally the case of $N_e \geqslant 3$, where $Q_0 = N_e$, $Q_1 = Q_2 = 0$, and the system of equations (2.8.1) is transformed into the following system

$$\frac{dA}{dt} = -i\,\frac{4\pi eu}{\omega_p L S_1}(\tilde{V} - i\tilde{Z}),$$
$$\frac{d\tilde{V}}{dt} = -i\,\frac{e\omega_p N_e}{2mu}\,A, \qquad (2.8.7)$$
$$\frac{d\tilde{Z}}{dt} = \omega_p \tilde{V}.$$

The introduction of the frequency of a slow amplitude $A(t) \sim \exp(-i\tilde{\omega}t)$ into equations (2.8.7) leads to a dispersion equation

$$\tilde{\omega}^3 = \frac{1}{2}\,\omega_e^2(\omega_p + \tilde{\omega}) \approx \frac{1}{2}\,\omega_e^2\omega_p, \qquad (2.8.8)$$

which also follows from the dispersion equation (2.5.3), provided the inequality $|\tilde{\omega}| \ll \omega_p$ is fulfilled. Thus, stimulated emission effect, i.e., ordinary beam instability, occurs instead of a spontaneous one even with three beam electrons per wavelength. Such instability for $N_e \gg 1$ was comprehensively investigated in plasma physics and plasma microwave electronics [7, 18, 19]. By way of example, it is known that nonlinear saturation of single-mode Cherenkov instability of a uniform beam in plasma is due to the capture of beam electrons by a plasma wave. Cases of small $N_e = 3, 4$, etc. are not discussed here because they do not practically differ from that of $N_e \gg 1$.

Thus, the following statements are valid for Cherenkov emission in the rarefied spectrum of a uniform electron beam with N_e electrons per wavelength: emission is spontaneous at $N_e = 1$; stimulated emission arises at $N_e = 2$, being not associated with the development of instability, and, finally, stimulated emission for $N_e \geqslant 3$ appears due to the usual Cherenkov beam instability in plasma.

Let us consider now emission by a modulated electron beam. Such a beam comprises an infinite sequence of electron bunches, one per wavelength $\lambda = 2\pi u/\omega_p$. It radiates in the rarefied (line) spectrum, which distinguishes it from a solitary electron bunch (see Sections 2.4–2.7) having a continuous radiation spectrum. Emission by a modulated electron beam is described in the linear approximation by a system of equations (2.8.1) in which all three quantities, Q_0, Q_1, and Q_3, differ from zero, being determined by the character of modulation. Due to this, system (2.8.1) takes into consideration both single-electron spontaneous effects and the stimulated effects inherent in a uniform beam. The relationship between Q_0, Q_1, and Q_3 determines which effect prevails.

Let, for instance, each bunch have length L_e and electron density in the bunch be constant. Then, using the continuous medium model for the electron bunch gives us the following

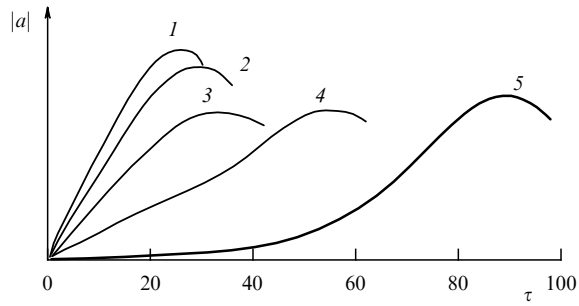

Figure 8. Plasma wave amplitudes in Cherenkov emission induced by electron bunches of different lengths.

relationships

$$Q_n = N_e \exp(-in\Delta_e)\sigma_n, \qquad \sigma_n = \frac{\sin(n\Delta_e)}{n\Delta_e},$$
$$\Delta_e = \frac{1}{2}\,k_0 L_e = \frac{\omega_p L_e}{2u} \leqslant \pi, \qquad n = 0, 1, 2. \qquad (2.8.9)$$

The equality $\Delta_e = \pi$ can be reached in the case of an unmodulated beam. Substituting expressions (2.8.9) into equations (2.8.1) leads to

$$\frac{dA}{dt} = -i\,\frac{1}{2}\,\omega_e^2(\sigma_1 + \tilde{V} - i\tilde{Z}),$$
$$\frac{d\tilde{V}}{dt} = -i(A - \sigma_2 A^*), \qquad (2.8.10)$$
$$\frac{d\tilde{Z}}{dt} = \omega_p \tilde{V}.$$

When moving from the system of equation (2.8.1) to equations (2.8.10), quantities A, $\tilde{V}$, $\tilde{Z}$ were redefined accordingly. This circumstance being unessential for the subsequent discussion, we omit the details. At $L_e = 0$, equality $\sigma_1 = \sigma_2 = 1$ is fulfilled and system (2.8.10) turns into system (2.8.2) except for notations, i.e., the small-sized electron bunch is equivalent to the single electron. Conversely, at $L_e = \lambda$, system (2.8.10) gives equations (2.8.7) describing a uniform electron beam.

Having no opportunity to analyze here the system of linear equations (2.8.10), we shall discuss certain characteristic cases. Not to be limited to consideration in a linear approximation, we shall make use of solutions to general equations (2.7.2), obtained at $\alpha = 1$ and $\nu = 10^{-3}$. Figure 8 depicts plasma wave amplitudes $|a| = |a_1(\tau)|$ for electron bunches of different sizes L_e: curve *1* — $L_e = 0.2\lambda$, *2* — $L_e = 0.4\lambda$, *3* — $L_e = 0.6\lambda$, *4* — $L_e = 0.8\lambda$, and *5* — $L_e = 0.98\lambda$. It can be seen that the growth character of amplitudes changes from linear to exponential as the bunch size increases. Specifically, curve *5* for $20 < \tau < 80$ is fairly well described by the theoretical dependence $\ln|a| = \delta\tau + \text{const}$, where δ is calculated by formula (2.5.4).

To conclude, we shall formulate selected necessary conditions and characteristics of stimulated Cherenkov emission in classical electrodynamics. First, growth of radiation energy elapses faster than proportional to t or t^2, and usually it is exponential. Second, there is phasing of electrons by the radiation field. Third, longitudinal size of the electron bunch is comparable with the mean radiation wavelength. Finally, the necessity of initial (priming) perturbation that may develop spontaneously as well.

2.9 Quantum theory of Cherenkov emission in a plasma

Quantum consideration of the Vavilov–Cherenkov effect was first reported by V L Ginzburg based on the laws of conservation of energy and momentum during electron interaction with a light quantum in a medium [20]. The electron was described classically, while the relationship between photon momentum and energy was established taking into account the effect of the medium. Because the optical wavelength is significantly greater than interatomic distances, the influence of the medium is described by its refraction index μ, and photon momentum is defined as $\mathbf{p} = (\mathbf{k}/k)\mu\hbar\omega/c$, where $\mathbf{k}$ is the wave vector. The result of quantum consideration is the following condition for Cherenkov emission of transverse electromagnetic waves in a medium [15]:

$$\omega = \mathbf{ku} - \frac{\hbar}{2m}\, k^2\, \frac{\varepsilon_\perp - 1}{\varepsilon_\perp} \sqrt{1 - \frac{u^2}{c^2}}\,, \tag{2.9.1}$$

where $\mathbf{u}$ is the electron velocity, and $\varepsilon_\perp = \mu^2$ is the transverse permittivity of the medium.

In application to Cherenkov emission of longitudinal waves in plasma, formula (2.9.1) and the method of its derivation need refinement. Indeed, a longitudinal field cannot be quantized, and the relationship between energy w and plasmon momentum $\mathbf{p} = w\mathbf{k}/\omega$ has a classical nature. In order to take account of quantum effects, we confine ourselves to a one-dimensional model system described by classical equations for plasma electrons (Poisson equations and equations of cold hydrodynamics [8]) and by the Schrödinger equation for bunch electrons (neglecting relativistic effects for simplicity):

$$\begin{aligned} &\frac{\partial^2\varphi}{\partial z^2} = -4\pi\rho_\mathrm{p} - 4\pi e|\psi|^2\,, \\ &\frac{\partial^2\rho_\mathrm{p}}{\partial t^2} = \frac{\omega_\mathrm{p}^2}{4\pi}\frac{\partial^2\varphi}{\partial z^2}\,, \\ &\mathrm{i}\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} = e\varphi\psi\,. \end{aligned} \tag{2.9.2}$$

Here, $\varphi(t,z)$ is the scalar potential, $\rho_\mathrm{p}(t,z)$ is the perturbation of plasma electron charge density, and $\psi(t,z)$ is the electron bunch wave function. Representing ψ as the sum of unperturbed wave function ψ_0 and perturbation $\tilde{\psi}$ caused by interaction with plasma oscillations, namely

$$\psi = \psi_0 + \tilde{\psi}\,, \tag{2.9.3}$$

and discarding nonlinear terms $|\tilde{\psi}|^2$ and $\varphi\tilde{\psi}$ in Poisson and Schrödinger equations, we bring the first and the third equations of system (2.9.2) to a form

$$\begin{aligned} &\mathrm{i}\hbar\frac{\partial\tilde{\psi}}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\tilde{\psi}}{\partial z^2} = e\varphi\psi_0\,, \\ &\frac{\partial^2\varphi}{\partial z^2} + 4\pi\rho_\mathrm{p} = -4\pi e|\psi_0|^2 - 4\pi e(\psi_0^*\tilde{\psi} + \psi_0\tilde{\psi}^*)\,. \end{aligned} \tag{2.9.4}$$

The first term on the right-hand side of the second equation in Eqn (2.9.4), corresponding to the unperturbed state of the electron bunch, describes spontaneous emission, and the second term describes the effect of stimulated Cherenkov emission.

Let us define the unperturbed state of emitting electrons by the formula

$$\begin{aligned} &\psi_0 = H\exp(-\mathrm{i}\omega_\hbar t + \mathrm{i}k_\hbar z)\,, \\ &\omega_\hbar = \frac{mu^2}{2\hbar}\,, \qquad k_\hbar = \frac{mu}{\hbar}\,, \end{aligned} \tag{2.9.5}$$

to which obviously corresponds a uniform boundless monospeed electron beam rather than a bunch (here, H is the normalization constant). The constant term $\sim |\psi_0|^2$ in equations (2.9.4) is unrelated to emission and can be discarded. Then writing out the potential as

$$\varphi = \frac{1}{2}\left[\tilde{\varphi}\exp(-\mathrm{i}\omega t + \mathrm{i}kz) + \tilde{\varphi}^*\exp(\mathrm{i}\omega^* t - \mathrm{i}kz)\right]\,, \tag{2.9.6}$$

we will find from the first equation in Eqn (2.9.4) the expression for perturbation of the wave function:

$$\begin{aligned} \tilde{\psi} = \frac{1}{2}\, eA\Bigg\{ &\frac{\tilde{\varphi}\exp\left[-\mathrm{i}(\omega_\hbar + \omega)t + \mathrm{i}(k_\hbar + k)z\right]}{\hbar(\omega_\hbar + \omega) - \hbar^2(k_\hbar + k)^2/2m} \\ &+ \frac{\tilde{\varphi}^*\exp\left[-\mathrm{i}(\omega_\hbar - \omega^*)t + \mathrm{i}(k_\hbar - k)z\right]}{\hbar(\omega_\hbar - \omega^*) - \hbar^2(k_\hbar - k)^2/2m}\Bigg\}\,. \end{aligned} \tag{2.9.7}$$

Finally, we substitute expressions (2.9.6) and (2.9.7) into the second equations of systems (2.9.2) and (2.9.4) and obtain the following dispersion equation for determining the complex frequency ω in solution (2.9.6):

$$1 - \frac{\omega_\mathrm{p}^2}{\omega^2} - \frac{\omega_\mathrm{e}^2}{(\omega - ku)^2 - (\hbar^2/4m^2)k^4} = 0\,. \tag{2.9.8}$$

When writing out equation (2.9.8), we defined normalization in Eqn (2.9.5) by the formula $H = \sqrt{N_\mathrm{e}/LS_1}$. Evidently, equation (2.9.8) is different from the classical dispersion equation (2.5.3) in that it contains a quantum term. Thus, as $\omega_\mathrm{e} \to 0$ we have from equation (2.9.8) the following quantum condition for stimulated Cherenkov emission of longitudinal waves by a low-density beam in isotropic plasma:

$$\omega = \mathbf{ku} \pm \frac{\hbar}{2m}\, k^2\,, \tag{2.9.9}$$

and also other longitudinal waves in plasma-like media. Condition (2.9.9) exhibits an analog of the quantum condition (2.9.1) for the emission of transverse waves in isotropic media.

Describing an electron in terms of wave function (2.9.5), we automatically come to the case of stimulated radiation induced by an electron beam. For the purpose of quantum description of the spontaneous Vavilov–Cherenkov effect, it is necessary to consider a wave function packet, such that $|\psi_0|^2 = f(z - ut)$, where the function f is nonzero in a region equal in size to the bunch length L_e. In this case, the packet spreading time depending on electron de Broglie wave dispersion must be significantly greater than the plasma oscillation period in accordance with the inequality

$$\frac{mu^2}{\hbar\omega_\mathrm{p}} \gg \frac{\lambda^2}{L_\mathrm{e}^2}\,, \tag{2.9.10}$$

where $\lambda = 2\pi u/\omega_\mathrm{p}$ is the mean radiation wavelength. The results presented in Section 2.4 are valid given inequality (2.9.10) is fulfilled.

3. The theory of Vavilov – Cherenkov emission

3.1 Emission of longitudinal waves in an isotropic plasma

Now, we shall apply the methods and results presented in Section 2 to the theory of Cherenkov emission of three-dimensional waves by a bunch of relativistic electrons. Let us start from the emission of longitudinal waves in isotropic plasma. We proceed from the following equations for the auxiliary function $\mathbf{A}(t,\mathbf{r})$:

$$\frac{\partial^2\mathbf{A}}{\partial t^2}+\omega_{\mathrm{p}}^2\mathbf{A}=-4\pi e\sum_{j=1}^{N_{\mathrm{e}}}\mathbf{v}_{\mathrm{e}j}(t,z_{0j})\,\delta\big(\mathbf{r}-\mathbf{r}_{\mathrm{e}j}(t,z_{0j})\big)\,,$$
$$\frac{\mathrm{d}\mathbf{r}_{\mathrm{e}j}}{\mathrm{d}t}=\mathbf{v}_{\mathrm{e}j}\,,\qquad \frac{\mathrm{d}\mathbf{p}_{\mathrm{e}j}}{\mathrm{d}t}=e\mathbf{E}(t,\mathbf{r}_{\mathrm{e}j})\,,\qquad \mathbf{E}=\frac{\partial\mathbf{A}}{\partial t}\,, \tag{3.1.1}$$

where $\mathbf{p}_{\mathrm{e}j}=m\mathbf{v}_{\mathrm{e}j}(1-v_{\mathrm{e}j}^2/c^2)^{-1/2}$ is the relativistic momentum of a bunch electron. The plasma oscillation energy density is given by the formula

$$w_{\parallel}=\frac{1}{8\pi}\left[\left(\frac{\partial\mathbf{A}}{\partial t}\right)^2+\omega_{\mathrm{p}}^2\mathbf{A}^2\right]. \tag{3.1.2}$$

Equations of bunch electron motion are supplemented by the following initial conditions

$$\mathbf{r}_{\mathrm{e}j}(0)=\{0,0,z_{0j}\}\,,\qquad \mathbf{v}_{\mathrm{e}j}=\{0,0,u\}\,. \tag{3.1.3}$$

Then, the bunch residing in the unperturbed state is a linear ($x_{0j}=y_{0j}=0$) chain of electrons aligned along the z-axis parallel to the direction of unperturbed motion. A potential approximation is used to describe plasma waves, in accordance with equations (3.1.1). It is possible because waves in the isotropic plasma split into purely longitudinal and purely transverse ones [8, 13, 17], and Cherenkov excitation of transverse waves in such plasma does not occur [5, 6] (see Section 3.2).

Next we expand the function $\mathbf{A}(t,\mathbf{r})$ into plane waves using the Hamiltonian method:

$$\mathbf{A}(t,\mathbf{r})=\frac{1}{2}\sum_{\{n\}}\mathbf{e}_{\{n\}}\big[A_{\{n\}}(t)\exp(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r})$$
$$+A^*_{\{n\}}(t)\exp(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r})\big]\,. \tag{3.1.4}$$

Here, $\mathbf{k}_{\{n\}}=k_0\mathbf{n}$ is the wave vector, $\mathbf{n}=\{n_x,n_y,n_z\}\equiv\{n\}$ is the vector with integer-valued components, $k_0=2\pi/L$ is the elementary wave number, L^3 is the cavity volume in which the field is enclosed, and $\mathbf{e}_{\{n\}}=\mathbf{k}_{\{n\}}/|k_{\{n\}}|$ is the unit polarization vector for longitudinal plasma waves. Summation in formula (3.1.4) is taken over semispace $-\infty<n_x<\infty$, $-\infty<n_y<\infty$, $n_z>1$ [see Eqn (2.2.1)]. Substituting expansion (3.1.4) into equations (3.1.1) gives us

$$\frac{\mathrm{d}^2A_{\{n\}}}{\mathrm{d}t^2}+\omega_{\mathrm{p}}^2A_{\{n\}}$$
$$=-\frac{8\pi e}{L^3}\sum_{j=1}^{N_{\mathrm{e}}}\big(\mathbf{e}_{\{n\}}\mathbf{v}_{\mathrm{e}j}(t,z_{0j})\big)\exp\big(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j}(t,z_{0j})\big)\,, \tag{3.1.5}$$
$$\frac{\mathrm{d}\mathbf{r}_{\mathrm{e}j}}{\mathrm{d}t}=\mathbf{v}_{\mathrm{e}j}\,,\qquad \frac{\mathrm{d}\mathbf{p}_{\mathrm{e}j}}{\mathrm{d}t}=\frac{1}{2}\,e\sum_{\{n\}}\mathbf{e}_{\{n\}}\big[\dot{A}_{\{n\}}(t)\exp(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j})$$
$$+\dot{A}^*_{\{n\}}(t)\exp(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j})\big]\,.$$

Substitution of expansion (3.1.4) into formula (3.1.2) leads to the expression for plasma wave energy in volume L^3:

$$W_{\parallel}=w_{\parallel}L^3\,,\qquad w_{\parallel}=\frac{1}{16\pi}\sum_{\{n\}}\big(\dot{A}_{\{n\}}\dot{A}^*_{\{n\}}+\omega_{\mathrm{p}}^2A_{\{n\}}A^*_{\{n\}}\big)\,. \tag{3.1.6}$$

Relationships of importance for further discussion follow from the system symmetry with respect to the z-axis:

$$\sum_{\{n\}}n_xA_{\{n\}}=\sum_{\{n\}}n_yA_{\{n\}}=0\,, \tag{3.1.7}$$

meaning that components E_x and E_y of the electric field vanish on the symmetry axis.

Radius vectors and electron velocities can be represented in the form

$$\mathbf{r}_{\mathrm{e}j}(t)=\{\tilde{x}_j,\tilde{y}_j,z_{0j}+ut+\tilde{z}_j\}\,,$$
$$\mathbf{v}_{\mathrm{e}j}(t)=\{\tilde{v}_{xj},\tilde{v}_{yj},u+\tilde{v}_{zj}\}\,. \tag{3.1.8}$$

Substituting formulas (3.1.8) into equations (3.1.5), linearizing them over perturbations $\tilde{x}_j,\tilde{y}_j,\ldots$, and taking into account relations (3.1.7) lead to the following equations in linear approximation:

$$\frac{\mathrm{d}^2A_{\{n\}}}{\mathrm{d}t^2}+\omega_{\mathrm{p}}^2A_{\{n\}}$$
$$=-\frac{8\pi e}{L^3}\,u\,\frac{n_z}{|n|}(Q_{n_z}+\tilde{V}_{n_z}-\mathrm{i}n_z\tilde{Z}_{n_z})\exp(-\mathrm{i}n_zk_0ut)\,,$$
$$\frac{\mathrm{d}\tilde{V}_{n_z}}{\mathrm{d}t}=\frac{e\gamma^{-3}}{2mu}\sum_{\{n'\}}\frac{n'_z}{|n'|}\big[\dot{A}_{\{n'\}}Q_{n_z-n'_z}\exp(\mathrm{i}n'_zk_0ut)$$
$$+\dot{A}^*_{\{n'\}}Q_{n_z+n'_z}\exp(-\mathrm{i}n'_zk_0ut)\big]\,, \tag{3.1.9}$$
$$\frac{\mathrm{d}\tilde{Z}_{n_z}}{\mathrm{d}t}=k_0u\tilde{V}_{n_z}\,.$$

Here, quantities Q_{n_z}, $\tilde{V}_{n_z}$, and $\tilde{Z}_{n_z}$ are expressed through z_{0j} and perturbations $\tilde{v}_{zj}$ and $\tilde{z}_j$ by formulas of the form (2.3.3). It is essential for further calculations that Q_{n_z}, $\tilde{V}_{n_z}$, and $\tilde{Z}_{n_z}$ are explicitly dependent on index n_z alone. Equations (3.1.5) and (3.1.9) are three-dimensional analogs of nonlinear equations (2.2.2), (2.2.3) and linear equations (2.3.2), respectively. Here, we confine ourselves to considering only linear equations (3.1.9).

For spontaneous emission by a single electron, when $N_{\mathrm{e}}=1$, it may be assumed that $Q_{n_z}=1$ in equations (3.1.9) for all n_z. Then, neglecting perturbations V_{n_z} and Z_{n_z} in the first equation of system (3.1.9) and substituting its solution into formula (3.1.6) give the expression for the total plasma wave energy:

$$\frac{\mathrm{d}W_{\parallel}}{\mathrm{d}t}=\frac{4\pi e^2u^2}{L^3}\sum_{\{n\}}\frac{n_z^2}{n^2}\,\frac{\sin\big[(\omega_{\mathrm{p}}-n_zk_0u)t\big]}{\omega_{\mathrm{p}}-n_zk_0u}\,. \tag{3.1.10}$$

In conformity with the rule [5, 6] [see formula (2.4.4)]

$$\sum_{\{n\}}\ldots\to\iiint\ldots\,\mathrm{d}n_x\,\mathrm{d}n_y\,\mathrm{d}n_z=\frac{L^3}{(2\pi)^3}\iiint\ldots k^2\,\mathrm{d}k\,\mathrm{d}o\,, \tag{3.1.11}$$

expression (3.1.10) is brought to a form

$$\frac{\mathrm{d}W_{\parallel}}{\mathrm{d}t} = e^2u^2 \iint k^2\cos^2\theta\,\delta(\omega_{\mathrm{p}} - ku\cos\theta)\,\mathrm{d}k\sin\theta\,\mathrm{d}\theta\,. \tag{3.1.12}$$

Expressions (3.1.11) and (3.1.12) were written out in the spherical system of coordinates (k, θ, φ), where θ is the angle between wave vector and the z-axis, φ is the azimuthal angle, and $\mathrm{d}o = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the element of the solid angle; integration in formula (3.1.12) is taken over the azimuthal angle. In addition, passage to the delta-function was performed in Eqn (3.1.12), as in Eqn (2.4.5).

Integration over θ in equation (3.1.12) gives the final expression for the power of spontaneous Cherenkov emission of longitudinal waves by an electron in isotropic plasma:

$$\frac{\mathrm{d}W_{\parallel}}{\mathrm{d}t} = \frac{e^2\omega_{\mathrm{p}}^2}{u}\int_{k_{\min}}^{k_{\max}}\frac{\mathrm{d}k}{k}\,. \tag{3.1.13}$$

The lower limit of integration in formula (3.1.13) is determined from the condition of vanishing the delta-function argument in formula (3.1.12) — $k_{\min} = \omega_{\mathrm{p}}/u$. The upper limit is set based on the kinetic properties of Langmuir waves propagating in plasma [8, 13], such as strong collisionless damping (Landau damping) in the short-wave region for $k > r_{\mathrm{De}}^{-1} = \omega_{\mathrm{p}}/v_{\mathrm{Te}}$, where r_{De} is the electron Debye radius, and v_{Te} is the thermal velocity of plasma electrons. It is usually assumed that $k_{\max} = r_{\mathrm{De}}^{-1}$. Then, the following well-known expression ensues from formula (3.1.13) [8, 13]:

$$\frac{\mathrm{d}W_{\parallel}}{\mathrm{d}t} = \frac{e^2\omega_{\mathrm{p}}^2}{u}\ln\frac{u}{v_{\mathrm{Te}}}\,. \tag{3.1.14}$$

Formula (3.1.14) is analogous to formula (2.4.6) for spontaneous emission of one-dimensional waves in plasma.

An emitting electron loses energy. Electron energy losses in plasma, described by formula (3.1.14), are referred to as polarization or Bohr losses [13, 15, 21]. Formulas (3.1.14) of polarization losses are easy to generalize for the case of an electron bunch. Indeed, the first term on the right-hand side of the first equation in system (3.1.9) contains factor Q_{n_z} substituted by unity in derivation of relationship (3.1.12). This factor may be taken into account by putting

$$Q_{n_z} = \sum_{j=1}^{N_{\mathrm{e}}}\exp(-\mathrm{i}n_zk_0z_{0j}) = \sum_{j=1}^{N_{\mathrm{e}}}\exp(-\mathrm{i}k\cos\theta\,z_{0j}) \to \iiint n_{\mathrm{e}}(\mathbf{r})\exp(-\mathrm{i}k\cos\theta\,z)\,\mathrm{d}\mathbf{r}\,, \tag{3.1.15}$$

where $n_{\mathrm{e}}(\mathbf{r}) = n_{\mathrm{e}}(x, y, z)$ is the electron concentration in the bunch [in formulas (3.1.15), we moved to the general case of a three-dimensional electron bunch]. Substituting the quantity $Q_{n_z}Q_{n_z}^*$ into the integrand of equation (3.1.12) [energy (3.1.6) contains products $A_{\{n\}}A_{\{n\}}^*$, but $A_{\{n\}} \sim Q_{n_z}$] and literally repeating the derivation of formula (3.1.14), we obtain the expression for polarization losses by an electron bunch in plasma:

$$\frac{\mathrm{d}W_{\parallel}}{\mathrm{d}t} = \frac{e^2\omega_{\mathrm{p}}^2}{u}\left|\iiint n_{\mathrm{e}}(\mathbf{r})\exp\left(-\frac{\mathrm{i}z\omega_{\mathrm{p}}}{u}\right)\mathrm{d}\mathbf{r}\right|^2\ln\frac{u}{v_{\mathrm{Te}}}\,. \tag{3.1.16}$$

Let us consider further stimulated emission by a homogeneous beam in which all N_{e} electrons are uniformly distributed over length L, making relationships $Q_{n_z>0} = 0$, $Q_{n_z-n_z'} = \delta_{n_z,n_z'}N_{\mathrm{e}}$, $Q_{n_z+n_z'} = 0$ valid. Then, the system of equations (3.1.9) yields the following equations

$$\begin{aligned}&\frac{\mathrm{d}^2A_{\{n\}}}{\mathrm{d}t^2} + \omega_{\mathrm{p}}^2A_{\{n\}} = -\frac{8\pi e}{L^3}\,u\,\frac{n_z}{|n|}(\tilde{V}_{n_z} - \mathrm{i}n_z\tilde{Z}_{n_z})\exp(-\mathrm{i}n_zk_0ut)\,,\\ &\frac{\mathrm{d}\tilde{V}_{n_z}}{\mathrm{d}t} = \frac{e\gamma^{-3}N_{\mathrm{e}}}{2mu}\sum_{\{n_x,n_y\}}\frac{n_z}{|n|}\,\dot{A}_{\{n\}}\exp(\mathrm{i}n_zk_0ut)\,,\quad \frac{\mathrm{d}\tilde{Z}_{n_z}}{\mathrm{d}t} = k_0u\tilde{V}_{n_z}\,.\end{aligned} \tag{3.1.17}$$

Because there is no summation over n_z in equations (3.1.17), their solutions can be sought in the form

$$\begin{aligned}&A_{\{n\}} = \tilde{A}_{\{n_\perp\}}\exp(-\mathrm{i}\omega t)\exp(-\mathrm{i}n_zk_0ut)\,,\\ &\tilde{V}_{n_z} = V\exp(-\mathrm{i}\omega t)\,,\qquad \tilde{Z}_{n_z} = Z\exp(-\mathrm{i}\omega t)\,,\end{aligned} \tag{3.1.18}$$

where $\{n_\perp\} = \{n_x, n_y\}$. Substituting expressions (3.1.18) into system (3.1.17) and dropping the quantities V, Z give

$$(\omega^2 - \omega_{\mathrm{p}}^2)\tilde{A}_{\{n_\perp\}} = \frac{4\pi e^2N_{\mathrm{e}}}{mL^3}\,\frac{\omega^2\gamma^{-3}}{(\omega - n_zk_0u)^2} \times \sum_{\{n_\perp'\}}\frac{n_z^2\tilde{A}_{\{n_\perp'\}}}{\sqrt{(n_\perp^2 + n_z^2)(n_\perp'^2 + n_z^2)}}\,, \tag{3.1.19}$$

from which a dispersion equation for determining frequency ω explicitly follows:

$$\omega^2 - \omega_{\mathrm{p}}^2 = \frac{4\pi e^2N_{\mathrm{e}}}{mL^3}\,\frac{\omega^2\gamma^{-3}}{(\omega - n_zk_0u)^2}\sum_{\{n_\perp\}}\frac{n_z^2}{n_\perp^2 + n_z^2}\,. \tag{3.1.20}$$

Let us move in formulas (3.1.19) and (3.1.20) from summation over transverse indices n_x, n_y to integration over transverse wave numbers $\mathbf{k}_\perp = k_0\mathbf{n}_\perp$. Using the rule

$$\sum_{n_x,n_y}\ldots \to \iint\ldots\,\mathrm{d}n_x\,\mathrm{d}n_y = \frac{L^2}{(2\pi)^2}\iint\ldots\,\mathrm{d}k_x\,\mathrm{d}k_y = \frac{L^2}{(2\pi)^2}\iint\ldots k_\perp\,\mathrm{d}k_\perp\,\mathrm{d}\varphi\,, \tag{3.1.21}$$

where $k_\perp, \varphi$ are the cylindrical coordinates, it follows from Eqn (3.1.19) that

$$\tilde{A}_{\{n_\perp\}} \to \tilde{A}(k_\perp, k_z) = \mathrm{const}\,(k_\perp^2 + k_z^2)^{-1/2}\,, \tag{3.1.22}$$

and the dispersion equation (3.1.20) is brought to a form

$$\omega^2 - \omega_{\mathrm{p}}^2 = \frac{e^2N_{\mathrm{e}}}{mL}\,\frac{\omega^2\gamma^{-3}}{(\omega - k_zu)^2}\,k_z^2\,\ln\frac{k_{\perp\max}^2 + k_z^2}{k_z^2}\,, \tag{3.1.23}$$

where $k_{\perp\max}$ is the upper limit of integration over variable $k_\perp$.

If $k_{\perp\max} \to \infty$, equation (3.1.23) contains a divergence arising from the fact that the cylindrical wave field turns to infinity at the points on the z-axis on which the electron chain is located. In order to regularize the dispersion equation, it should be borne in mind that a real electron beam has a finite cross section of area S_{e}. Then, the minimal distance from the z-axis, at which plasma oscillations occur, is $r_{\perp\min} \sim \sqrt{S_{\mathrm{e}}}$. Because $k_{\perp\max} \sim 1/r_{\perp\min}$, one can put $k_{\perp\max}^2 = (gS_{\mathrm{e}})^{-1}$ (where g is a constant) in a logarithm of formula (3.1.23); hence, we

arrive at the dispersion equation

$$\omega^2 - \omega_{\rm p}^2 = \omega_{\rm e}^2 \gamma^{-3} \frac{\omega^2}{(\omega - k_z u)^2} G_{\rm e}(k_z)\,, \qquad G_{\rm e}(k_z) = \frac{k_z^2 S_{\rm e}}{4\pi} \ln \frac{1 + g k_z^2 S_{\rm e}}{g k_z^2 S_{\rm e}}\,, \tag{3.1.24}$$

having the structure of model dispersion equation (2.5.3). When writing out Eqn (3.1.24), beam electron density $n_{\rm e} = N_{\rm e}/LS_{\rm e}$ and Langmuir frequency $\omega_{\rm e} = \sqrt{4\pi e^2 n_{\rm e}/m}$ were introduced. The solution of equation (3.1.24) is given by formula (2.5.4) containing $\omega_{\rm e}^2\gamma^{-3}G_{\rm e}$ instead of $\omega_{\rm e}^2$. Evidently, factor $k_z^2 S_{\rm e}$ should be turned to infinity, $k_z^2 S_{\rm e} \to \infty$, to move to the one-dimensional case, and the dispersion equation (3.1.24) must go over into equation (2.5.3). This means that $g = (4\pi)^{-1}$. The structural similarity of equations (3.1.24) and (2.5.3) reflects the fact that stimulated Cherenkov emission of longitudinal waves in an isotropic plasma is a form of beam-plasma instability in terms of physical nature.

Notice that we estimated $k_{\perp\max}$ in equation (3.1.23) on the assumption that $r_{\perp\min} \sim S_{\rm e}^{1/2} \gg r_{\rm De}$. However, fulfillment of the reverse inequality implies $r_{\perp\min} \sim r_{\rm De}$ and it should be assumed that $k_{\perp\max} = r_{\rm De}^{-1}$. Then, geometric extent $G_{\rm e}$ in equation (3.1.24) is substituted by

$$G_{\rm e}(k_z) = \frac{k_z^2 S_{\rm e}}{4\pi} \ln \frac{1 + k_z^2 r_{\rm De}^2}{k_z^2 r_{\rm De}^2} \approx \frac{\omega_{\rm p}^2 S_{\rm e}}{4\pi u^2} \ln \frac{u}{v_{\rm Te}}\,, \tag{3.1.25}$$

where account was taken that, under Cherenkov emission in a plasma, $k_z \approx \omega_{\rm p}/u$ and $u \gg v_{\rm Te}$.

In order to elucidate the spatial structure of the plasma oscillation field, expression (3.1.22) needs to be introduced into expansion (3.1.4) and summation over $\{n_\perp\}$ performed; the latter is superseded by integration over $\mathrm{d}\mathbf{k}_\perp$ (integration over $\mathrm{d}k_z$ is infeasible in view of the independence of waves with different n_z). Simple computations yield the expression for the plasma oscillation field:

$$E_z(t, \mathbf{r}) = \text{const}\, \frac{\exp(-k_z r_\perp)}{r_\perp} \exp\left[-\mathrm{i}\omega(k_z)t + \mathrm{i}k_z z\right] + \text{c.c.}, \tag{3.1.26}$$

where $\omega(k_z)$ is the solution of dispersion equation (3.2.24), and $r_\perp^2 = x^2 + y^2$. At large times, the main contribution comes from perturbations with the maximum growth increment. Therefore, stimulated Cherenkov emission by an extended ($L_{\rm e} \gg 2\pi u/\omega_{\rm p}$) electron bunch in homogeneous isotropic plasma is defined by function (3.1.26) at $k_z = \omega_{\rm p}/u$.

In the model under consideration, it is easy to take into account the dispersion of plasma waves (e.g., due to thermal motion). To this end, it is sufficient to substitute a function $\omega_{\rm p}^2(n) \equiv \omega_{{\rm p}n}^2$ for the constant $\omega_{\rm p}^2$ in equations (3.1.17) and relationship (3.1.19). The result is the following dispersion equation instead of Eqn (3.1.20):

$$1 = \frac{\omega_{\rm e}^2 \gamma^{-3}}{(\omega - n_z k_0 u)^2} \frac{S_{\rm e}}{L^2} \sum_{\{n_\perp\}} \frac{\omega^2 n_z^2}{(n_\perp^2 + n_z^2)(\omega^2 - \omega_{{\rm p}n}^2)}\,. \tag{3.1.27}$$

In what follows, equation (3.1.27) will be generalized to the case of stimulated emission of longitudinal waves in an arbitrary isotropic medium.

3.2 Emission of transverse electromagnetic waves in an isotropic dielectric

Let us consider now the emission of transverse electromagnetic waves in an isotropic dielectric. We shall describe the electromagnetic field using vector $\mathbf{A}(t, \mathbf{r})$ and scalar $\psi(t, \mathbf{r})$ potentials that satisfy, in the Coulomb gauge $\nabla\mathbf{A} = 0$, the equations [21, 22]

$$\Delta\mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\, e \sum_{j=1}^{N_{\rm e}} \mathbf{v}_{{\rm e}j}(t, z_{0j})\, \delta\big(\mathbf{r} - \mathbf{r}_{{\rm e}j}(t, z_{0j})\big) + \frac{\varepsilon}{c}\, \nabla \frac{\partial\psi}{\partial t}\,, \qquad \Delta\psi = -\frac{4\pi}{\varepsilon}\, e \sum_{j=1}^{N_{\rm e}} \delta\big(\mathbf{r} - \mathbf{r}_{{\rm e}j}(t, z_{0j})\big)\,, \tag{3.2.1}$$

and relationships

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\psi\,, \qquad \mathbf{B} = \operatorname{rot}\mathbf{A}\,, \qquad \mathbf{E}_\parallel = -\nabla\psi\,, \qquad w_\perp = \frac{\varepsilon E_\perp^2 + B^2}{8\pi}\,, \tag{3.2.2}$$

where ε is the permittivity (assuming the form of an operator $\varepsilon(\hat\omega)$, $\hat\omega = \mathrm{i}\,\partial/\partial t$, when time-dependent dispersion is taken into account), and $w_\perp$ is the energy density of the transverse electromagnetic field. Equations of motion of bunch electrons can be written out as

$$\frac{\mathrm{d}\mathbf{r}_{{\rm e}j}}{\mathrm{d}t} = \mathbf{v}_{{\rm e}j}\,, \qquad \frac{\mathrm{d}\mathbf{p}_{{\rm e}j}}{\mathrm{d}t} = e\mathbf{E}(t, \mathbf{r}_{{\rm e}j}) + \frac{e}{c}\left[v_{{\rm e}j}\mathbf{B}(t, \mathbf{r}_{{\rm e}j})\right]. \tag{3.2.3}$$

Initial values of radius vectors and electron velocities are given by formulas (3.1.3).

Applying the Hamiltonian method, we make use of the expansions

$$\mathbf{A}(t, \mathbf{r}) = \frac{1}{2}\sum_{\{n\}}\left[\mathbf{A}_{\{n\}}(t)\exp(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}) + \mathbf{A}_{\{n\}}^*(t)\exp(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r})\right], \qquad \psi(t, \mathbf{r}) = \frac{1}{2}\sum_{\{n\}}\left[\psi_{\{n\}}(t)\exp(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}) + \psi_{\{n\}}^*(t)\exp(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r})\right]. \tag{3.2.4}$$

Suppose that radiation constitutes a superposition of only those waves that have a nonzero component of the electric field in the direction of unperturbed electron motion, and waves with $E_z = 0$ are not emitted. Moreover, suppose a symmetric spatial distribution of the electromagnetic field with respect to the z-axis. As is known, cylindrically symmetric waves with $E_z \neq 0$ (the so-called E type waves [23, 24]) have zero-field components E_x, E_y, and B_x, B_y on the symmetry axis, which gives the following conditions for expansion coefficients in formulas (3.2.4):

$$\sum_{\{n\}} A_{x\{n\}} = \sum_{\{n\}} A_{y\{n\}} = 0\,, \qquad \sum_{\{n\}} n_x A_{z\{n\}} = \sum_{\{n\}} n_y A_{z\{n\}} = 0\,, \qquad \sum_{\{n\}} n_x \psi_{\{n\}} = \sum_{\{n\}} n_y \psi_{\{n\}} = 0\,, \tag{3.2.5}$$

where $A_{x\{n\}}$, $A_{y\{n\}}$, $A_{z\{n\}}$ are the Cartesian components of the vector $\mathbf{A}_{\{n\}}$.

Substituting expansions (3.2.4) into equations (3.2.1) and eliminating the scalar potential yield the following Hamiltonian equations for excitation of electromagnetic field harmonic oscillators:

$$\frac{\mathrm{d}^2\mathbf{A}_{\{n\}}}{\mathrm{d}t^2}+\omega_n^2\mathbf{A}_{\{n\}}=\frac{8\pi ec}{L^3\varepsilon(n)}\sum_{j=1}^{N_\mathrm{e}}\left(\mathbf{v}_{\mathrm{e}j}-\frac{\mathbf{k}_{\{n\}}(\mathbf{k}_{\{n\}}\mathbf{v}_{\mathrm{e}j})}{k_{\{n\}}^2}\right)\times\exp\left(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j}(t,z_{0j})\right). \tag{3.2.6}$$

Here, ω_n is the frequency of the electromagnetic field oscillator:

$$\omega_n^2=\frac{c^2k_{\{n\}}^2}{\varepsilon(n)}=k_0^2c^2\,\frac{n^2}{\varepsilon(n)}\,, \tag{3.2.7}$$

and the dependence of permittivity ε on $n=(n_x^2+n_y^2+n_z^2)^{1/2}$ phenomenologically takes into consideration the frequency dispersion of the isotropic medium. The structure of the right-hand side of equation (3.2.6) is such that $\mathbf{k}_{\{n\}}\mathbf{A}_{\{n\}}=0$, which is equivalent to the Coulomb gauge. It follows from formulas (3.2.4) and (3.2.2) that the expression for the total energy of transverse electromagnetic waves in volume L^3 is given by

$$W_\perp=w_\perp L^3\,,\quad w_\perp=\frac{1}{16\pi c^2}\sum_{\{n\}}\varepsilon(n)\left(\dot{\mathbf{A}}_{\{n\}}\dot{\mathbf{A}}^*_{\{n\}}+\omega_n^2\mathbf{A}_{\{n\}}\mathbf{A}^*_{\{n\}}\right). \tag{3.2.8}$$

Equations (3.2.3) of electron motion are written out in the form

$$\frac{\mathrm{d}\mathbf{r}_{\mathrm{e}j}}{\mathrm{d}t}=\mathbf{v}_{\mathrm{e}j}\,,\qquad \frac{\mathrm{d}\mathbf{p}_{\mathrm{e}j}}{\mathrm{d}t}=\mathbf{F}_{1j}+\mathbf{F}_{2j}\,,$$
$$\mathbf{F}_{1j}=-\frac{e}{2c}\sum_{\{n\}}\left\{\left(\dot{\mathbf{A}}_{\{n\}}(t)-\mathrm{i}\left[\mathbf{v}_{\mathrm{e}j}\left[\mathbf{k}_{\{n\}}\mathbf{A}_{\{n\}}(t)\right]\right]\right)\times\exp\left(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j}\right)+\mathrm{c.c.}\right\}, \tag{3.2.9}$$
$$\mathbf{F}_{2j}=-\mathrm{i}\,\frac{4\pi e^2}{L^3}\sum_{\{n\}}\left(\frac{\mathbf{k}_{\{n\}}}{k_{\{n\}}^2\varepsilon(n)}\left(\sum_{j=1}^{N_\mathrm{e}}\exp\left(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j}\right)\right)\times\exp\left(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}_{\mathrm{e}j}\right)-\mathrm{c.c.}\right).$$

Equations (3.2.6) and (3.2.9) describe in the most general form the nonlinear dynamics of Cherenkov emission of transverse electromagnetic waves in an isotropic medium. Part of the force $\mathbf{F}_{2j}$ in Eqn (3.2.9) is due to the Coulomb (potential) interaction between bunch electrons; it also includes self-action. As concerns the force $\mathbf{F}_{1j}$, it is associated with the action of radiation on the bunch, i.e., radiative deceleration of type (2.4.8), and phasing of electrons by radiation that leads to developing instability.

Substituting representations (3.1.8) into equations (3.2.6) and (3.2.9), linearizing them with respect to perturbations, and taking into account relationships (3.2.5) give the following equations in the linear approximation:

$$\frac{\mathrm{d}^2A_{x,y\{n\}}}{\mathrm{d}t^2}+\omega_n^2A_{x,y\{n\}}=-\frac{8\pi ecu}{L^3}\,\frac{n_{x,y}n_z}{\varepsilon(n)n^2}\left(Q_{n_z}+\tilde{V}_{n_z}-\mathrm{i}n_z\tilde{Z}_{n_z}\right)\exp\left(-\mathrm{i}n_zk_0ut\right),$$

$$\frac{\mathrm{d}^2A_{z\{n\}}}{\mathrm{d}t^2}+\omega_n^2A_{z\{n\}}=\frac{8\pi ecu}{L^3}\,\frac{n_x^2+n_y^2}{\varepsilon(n)n^2}\left(Q_{n_z}+\tilde{V}_{n_z}-\mathrm{i}n_z\tilde{Z}_{n_z}\right)\exp\left(-\mathrm{i}n_zk_0ut\right),$$

$$\frac{\mathrm{d}\tilde{V}_{n_z}}{\mathrm{d}t}=-\frac{e\gamma^{-3}}{2mcu}\sum_{\{n'\}}\left(\dot{A}_{z\{n'\}}Q_{n_z-n_z'}\exp\left(\mathrm{i}n_z'k_0ut\right)+\dot{A}^*_{z\{n'\}}Q_{n_z+n_z'}\exp\left(-\mathrm{i}n_z'k_0ut\right)\right)-\Phi(t)\,, \tag{3.2.10}$$

$$\frac{\mathrm{d}Z_{n_z}}{\mathrm{d}t}=k_0uV_{n_z}\,,$$

$$\Phi(t)=\mathrm{i}\,\frac{4\pi e^2\gamma^{-3}}{muL^3k_0}\sum_{\{n'\}}\frac{n_z'}{\varepsilon(n')n'^2}\left[\left(Q_{n_z'}Q_{n_z-n_z'}-Q^*_{n_z'}Q_{n_z+n_z'}\right)-\mathrm{i}n_z'\left(\tilde{Z}_{n_z'}Q_{n_z-n_z'}+\tilde{Z}^*_{n_z'}Q_{n_z+n_z'}\right)+\mathrm{i}n_z'\left(Q_{n_z'}\tilde{Z}_{n_z-n_z'}+Q^*_{n_z'}\tilde{Z}_{n_z+n_z'}\right)\right].$$

If term $\Phi(t)$ having a Coulomb nature and appearing in the right-hand side of the equation for $\tilde{V}_{n_z}$ is neglected, unessential differences between equations (3.1.10) and (3.2.9) are due to a different polarization of radiation alone: systems of equations (3.1.9) and (3.2.10) describe emission of longitudinal and transverse waves, respectively. The term $\sim(Q_{n_z'}Q_{n_z-n_z'}-Q^*_{n_z'}Q_{n_z+n_z'})$ in $\Phi(t)$ depends on the static field of the electron bunch. In the single-electron case, this term, being the self-field of a point charge, becomes infinite. Evidently, the self-field (also called the entrained field) has nothing to do with the problem of emission of electromagnetic waves. Two other terms in $\Phi(t)$ describe the high-frequency self-field of the bunch (in microwave electronics it is referred to as the field of a high-frequency spatial charge [19, 25]), the field arising from modulation of the electron bunch with radiation. It will be shown in Section 4.2 that under certain conditions the high-frequency field has a marked effect on the mechanism of stimulated emission.

Let us now turn to the analysis of equations (3.2.10). In the case of spontaneous emission by a single electron, neglecting perturbations $\tilde{V}_{n_z}$, $\tilde{Z}_{n_z}$ in the first three equations of system (3.2.10) and substituting their solutions into formula (3.2.8) result in a following expression for the total energy of transverse electromagnetic waves:

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t}=\frac{4\pi e^2u^2}{L^3}\sum_{\{n\}}\frac{n_x^2+n_y^2}{n^2\varepsilon(n)}\,\frac{\sin\left[(\omega_n-n_zk_0u)t\right]}{\omega_n-n_zk_0u}\to\frac{e^2u^2}{2\pi}\iiint\frac{k_x^2+k_y^2}{k^2\varepsilon(k)}\,\delta\left(\omega(k)-k_zu\right)\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z\,. \tag{3.2.11}$$

In writing formula (3.2.11), we replaced summation over n_x, n_y, n_z by integration over k_x, k_y, k_z, introduced the frequency $\omega(k)=kc/\sqrt{\varepsilon}$, and passed to the delta function [see Eqn (2.4.5)]. Introducing further spherical coordinates k,θ,φ and integrating over azimuthal angle φ in formula (3.2.11) give

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t}=e^2u^2\iint\varepsilon^{-1}(k)\sin^3\theta\,\delta\left[\omega(k)-ku\cos\theta\right]k^2\,\mathrm{d}k\,\mathrm{d}\theta\,. \tag{3.2.12}$$

When writing out the initial equations (3.2.1), the medium was assumed to possess only frequency dispersion, i.e., $\varepsilon=\varepsilon(\omega)$. With this in mind, by $\varepsilon(k)$ in formula (3.2.12) one

should understand $\varepsilon[\omega(k)]$, where $\omega(k)$ is the solution of dispersion equation $\omega^2\varepsilon(\omega) = k^2c^2$ for transverse electromagnetic waves in an isotropic medium.

Because $\cos\theta \leqslant 1$, only those regions of k contribute to the integral in formula (3.2.12) in which inequality $\omega(k) \leqslant ku$ or $c/\sqrt{\varepsilon(k)} \leqslant u$ is fulfilled, the latter by taking into account the dispersion equation. Then, integration over angle θ in formula (3.2.12) gives the final formula

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t} = e^2u\int_{c/\sqrt{\varepsilon}\leqslant u}\left(1-\frac{c^2}{u^2\varepsilon(k)}\right)\frac{k\,\mathrm{d}k}{\varepsilon(k)} = \frac{e^2u}{c^2}\int_{c/\sqrt{\varepsilon}\leqslant u}\omega\left(1-\frac{c^2}{u^2\varepsilon(\omega)}\right)\mathrm{d}\omega\,. \qquad (3.2.13)$$

Formula (3.2.13), known as the Tamm – Frank formula [14], defines the power of spontaneous Cherenkov emission of electromagnetic waves in an isotropic medium. The losses of electron energy through emission, described by formula (3.2.13), are termed radiation losses [15, 21]. It is worth noting that in moving from integration over k to integration over ω in formulas (3.2.13) we used the relationship $\mathrm{d}k = (\sqrt{\varepsilon}/c)\,\mathrm{d}\omega$ valid when inequality $\left|(\varepsilon/\omega)\,\mathrm{d}\varepsilon/\mathrm{d}\omega\right| \ll 1$ is fulfilled. This means that the Tamm – Frank formula is applicable only to media with a weak frequency dispersion. Actually, this was expected from the very beginning. In particular, when substituting expansions (3.2.4) into equations (3.2.1), operator $\varepsilon(\hat{\omega})$ was replaced by the quantity $\varepsilon(n)$ which was later regarded as a constant.

In order to generalize the Tamm – Frank formula to the case of an electron bunch, account should be taken of quantity $Q_{n_z}Q^*_{n_z}$ in the integrand of formula (3.2.12), where Q_{n_z} is defined by Eqn (3.1.15). As a result, integration over the angle results in the formula

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t} = \frac{e^2u}{c^2}\int_{c/\sqrt{\varepsilon}\leqslant u}\omega\left(1-\frac{c^2}{u^2\varepsilon(\omega)}\right) \times\left|\iiint n_\mathrm{e}(\mathbf{r})\exp\left(-\frac{\mathrm{i}z\omega}{u}\right)\mathrm{d}\mathbf{r}\right|^2\mathrm{d}\omega\,. \qquad (3.2.14)$$

Let us consider now a radiation pattern for spontaneous Cherenkov emission of transverse waves in an isotropic medium. For this purpose, we assume the dispersion to be weak and move to the integration over frequency ω in formula (3.2.12). Then, one obtains

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t} = \frac{e^2u^2}{c^3}\iint\mu(\omega)\omega\delta\big(1-\beta\,\mu(\omega)\cos\theta\big)\sin^3\theta\,\mathrm{d}\omega\,\mathrm{d}\theta\,, \qquad (3.2.15)$$

where $\beta = u/c$, and $\mu(\omega) = \sqrt{\varepsilon(\omega)}$ is the index of refraction. We denote by ω^* the root of the equation

$$1-\beta\,\mu(\omega)\cos\theta = 0 \qquad (3.2.16)$$

and perform integration over the frequency in formula (3.2.15):

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}t} = \frac{e^2u^2}{c^3}\int\mu(\omega^*)\omega^*\left|\beta\cos\theta\,\frac{\mathrm{d}\mu(\omega^*)}{\mathrm{d}\omega^*}\right|^{-1}\sin^3\theta\,\mathrm{d}\theta\,, \qquad (3.2.17)$$

where ω^* as the solution of Eqn (3.2.16) is a function of angle θ. The radiation pattern is given by the integrand in formula (3.2.17), namely

$$D(\theta) = \mu(\omega^*)\omega^*\left|\beta\cos\theta\,\frac{\mathrm{d}\mu(\omega^*)}{\mathrm{d}\omega^*}\right|^{-1}\sin^3\theta\,. \qquad (3.2.18)$$

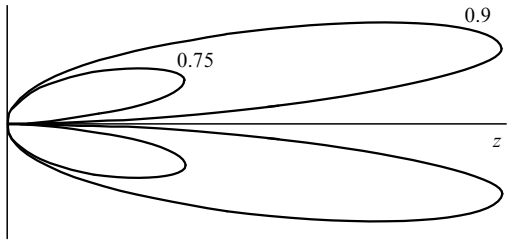

Figure 9. Vavilov – Cherenkov radiation patterns in an isotropic dielectric: $\beta = 0.75$, and $\beta = 0.9$.

Quantity $D(\theta)\,\mathrm{d}\theta$ defines the radiation energy flux traveling at angle θ with respect to the direction of electron motion within the solid angle $2\pi\,\mathrm{d}\theta$ (sometimes the radiation pattern does not include factor $\sin\theta$ pertaining to an element of the solid angle).

For further discussion, it is necessary to render concrete the dependence $\mu(\omega)$. Let us assume that

$$\mu^2(\omega) = 1+\frac{\omega_\mathrm{p}^2}{\omega_0^2-\omega^2}\,, \qquad (3.2.19)$$

where ω_p^2 and ω_0^2 are certain constants. The dependence (3.2.19) is characteristic of transparent media with normal dispersion in the optical frequency range. The solution of Eqn (3.2.16) has the form

$$\omega^*(\theta) = \omega_0\sqrt{1-\frac{\omega_\mathrm{p}^2}{\omega_0^2}\,\frac{\beta^2\cos^2\theta}{1-\beta^2\cos^2\theta}}\,. \qquad (3.2.20)$$

It follows from formula (3.2.20) that $\omega^* \leqslant \omega_0$. In this frequency region, $\mu(\omega) > 1$. In a frequency region of $\omega > \omega_0$, Cherenkov emission is impossible because either $\mu^2 < 0$ (nontransparency zone) or $\mu < 1$ (region of phase velocities higher than c). The index of refraction in the emission region being positive, so does $\cos\theta > 0$. Then, formula (3.2.20) gives the condition for the angles in which Cherenkov radiation is localized:

$$\arccos\left[\min\left(1,\frac{c}{u}\sqrt{\frac{\omega_0^2}{\omega_0^2+\omega_\mathrm{p}^2}}\right)\right] \leqslant \theta \leqslant \frac{\pi}{2}\,. \qquad (3.2.21)$$

At $\omega_0 = 0$ (as in the case of a cold isotropic plasma), the angular range covered by inequality (3.2.21) goes to zero, which implies the impossibility of Cherenkov emission of transverse waves in an isotropic plasma. This and the results of Section 3.1 indicate that only Cherenkov emission of longitudinal waves may occur in an isotropic plasma.

Substitution of formulas (3.2.19) and (3.2.20) into formula (3.2.18) and simple transformations give an expression for the radiation pattern:

$$D(\theta) = \omega_\mathrm{p}^2\,\frac{\beta\cos\theta\sin^3\theta}{(1-\beta^2\cos^2\theta)^2}\,. \qquad (3.2.22)$$

Formula (3.2.22) holds true only in the angular range (3.2.20); outside this range, $D(\theta) \equiv 0$. Radiation patterns (3.2.22) for two values of β are presented in Fig. 9. In a nonrelativistic

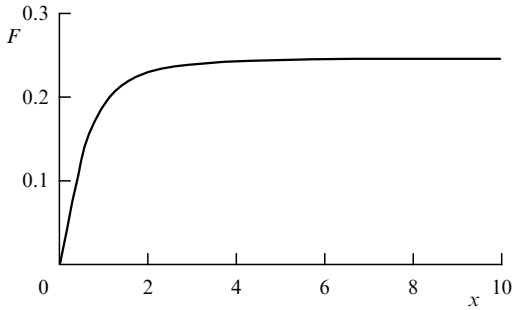

Figure 10. Graph of the function $F(x)$ determining the power of Vavilov–Cherenkov radiation in accordance with formula (3.2.24).

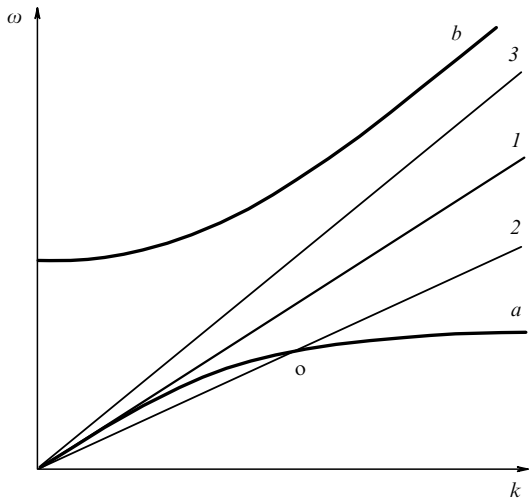

Figure 11. Frequencies and angles of Vavilov–Cherenkov radiation in a medium with refraction index (3.2.19).

case, for $\beta^2 \ll 1$, the maximum of the radiation pattern lies at $\theta = \arctan(\sqrt{2}) \approx 0.96$, while in the ultrarelativistic limit it falls on the angle $\theta \approx (1-\beta^2)^{1/2} \ll 1$.

Let us calculate the power of Vavilov–Cherenkov radiation by the Tamm–Frank formula (3.2.13) in a medium with refraction index (3.2.19). For simplicity, we shall confine ourselves to a special case of fulfilling the equality

$$\frac{c}{u}\sqrt{\frac{\omega_0^2}{\omega_0^2+\omega_{\rm p}^2}} = 1\,, \tag{3.2.23}$$

in which, in accordance with formulas (3.2.19) and (3.2.20), all frequencies (from zero to ω_0) are emitted into all angles from zero to $\pi/2$. Elementary integration in formula (3.2.13) yields

$$\frac{{\rm d}W_\perp}{{\rm d}t} = \frac{e^2 u}{c^2}\,\omega_0^2\,F\!\left(\frac{\omega_{\rm p}^2}{\omega_0^2}\right). \tag{3.2.24}$$

Here, $F(x) = (x/2)\big((1+x)\ln(1+x^{-1})-1\big)$ is the function plotted in Fig. 10.

Relationships between frequencies and angles of waves originated in Cherenkov emission by an electron in a medium with refraction index (3.2.19) are illustrated by Fig. 11 depicting dispersion curves of waves in the medium of interest, i.e., solutions $\omega(k)$ of the dispersion equation $\omega^2 = k^2c^2/\mu^2(\omega)$. There are two types of waves: one comprises optical waves (curve *b*), and the other acoustic waves (curve *a*). Vavilov–Cherenkov emission occurs only with acoustic waves. It is easy to see that the left-hand side of formula (3.2.23) is the ratio $V_{\rm ph}(0)/u$, where $V_{\rm ph}(0) = c/\mu(0)$ is the phase velocity of an acoustic type wave as $\omega \to 0$. Figure 11 also presents straight lines $\omega = ku = \beta kc$. When the equality (3.2.23) is fulfilled, the straight line $\omega = ku$ is fitted by line *1*. Straight lines $\omega = ku\cos\theta$ run below (e.g., line *2*), which accounts for the emission of all frequencies under condition (3.2.23), from zero to ω_0, and at all angles lying between zero and $\pi/2$. If the left-hand side of equality (3.2.23) is larger than unity, i.e., $V_{\rm ph}(0) > u$, the straight line $\omega = ku$ is positioned as line *2* and waves with frequencies lower than the frequency of point o shown in Fig. 11 are not emitted. This radiation travels through all the angles from zero to $\pi/2$. Finally, if the left-hand side of formula (3.2.23) is smaller than unity, i.e., $V_{\rm ph}(0) < u$, the dependence $\omega = ku$ has the form of straight line *3*. In this situation, all waves with frequencies from zero to ω_0 are emitted, but there is no emission in the region of small angles.

Let us turn now to stimulated emission induced by a uniform electron beam. Bearing in mind that in the case of a uniform beam one has $Q_{n_z-n_z'} = \delta_{n_z,n_z'}N_{\rm e}$, $Q_{n_z+n_z'} = 0$, and $Q_{n_z>0} = 0$, the following equations can be obtained from system (3.2.10):

$$\begin{aligned}
&\frac{{\rm d}^2 A_{z\{n\}}}{{\rm d}t^2} + \omega_n^2 A_{z\{n\}} \\
&\quad = \frac{8\pi ecu}{L^3}\,\frac{n_x^2+n_y^2}{\varepsilon(n)n^2}\,(\tilde V_{n_z} - {\rm i}n_z\tilde Z_{n_z})\exp(-{\rm i}n_z k_0 ut)\,, \\
&\frac{{\rm d}\tilde V_{n_z}}{{\rm d}t} = -\frac{e\gamma^{-3}N_{\rm e}}{2mcu}\sum_{\{n_x,n_y\}}\dot A_{z\{n\}}\exp({\rm i}n_z k_0 ut) \\
&\quad - \frac{4\pi e^2\gamma^{-3}N_{\rm e}}{muL^3k_0}\sum_{\{n_x,n_y\}}\frac{n_z^2}{\varepsilon(n)n^2}\,\tilde Z_{n_z}\,, \\
&\frac{{\rm d}Z_{n_z}}{{\rm d}t} = k_0 u V_{n_z}\,.
\end{aligned} \tag{3.2.25}$$

Substituting expressions (3.1.18) into equations (3.2.25) and discarding V and Z lead to the relationship

$$\begin{aligned}
&(\omega^2-\omega_n^2)\tilde A_{\{n_\perp\}} \\
&\quad = \frac{4\pi e^2 N_{\rm e}}{mL^3\varepsilon(\omega)}\,\frac{\omega^2\gamma^{-3}}{(\omega-n_z k_0 u)^2-\Omega_{\rm b}^2}\,\frac{n_\perp^2}{n_\perp^2+n_z^2}\sum_{\{n_\perp'\}}\tilde A_{\{n_\perp'\}}\,,
\end{aligned} \tag{3.2.26}$$

which is slightly different from formula (3.1.19). Here, $\omega_n^2 = (n_\perp^2+n_z^2)k_0^2c^2/\varepsilon(\omega)$, and

$$\Omega_{\rm b}^2 = \frac{4\pi e^2 N_{\rm e}\gamma^{-3}}{mL^3\varepsilon(\omega)}\sum_{\{n_\perp\}}\frac{n_z^2}{n_\perp^2+n_z^2} = \frac{\omega_{\rm e}^2\gamma^{-3}}{\varepsilon(\omega)}\,\frac{S_{\rm e}}{L^2}\sum_{\{n_\perp\}}\frac{n_z^2}{n_\perp^2+n_z^2}\,. \tag{3.2.27}$$

The structures of formulas (3.1.19) and (3.2.26) differ in that the latter expression contains quantity (3.2.27). It will be shown in Section 4.2 that here physical factors are involved that influence the mechanism of stimulated Cherenkov emission. Suppose thus far that the inequality

$$|\omega - n_z k_0 u| \gg \Omega_{\rm b} \tag{3.2.28}$$

is fulfilled, allowing quantity (3.2.27) in relationship (3.2.26) to be neglected and the following dispersion relation to be

obtained:

$$1 = \frac{\omega_{\rm e}^2 \gamma^{-3}/\varepsilon(\omega)}{(\omega - n_z k_0 u)^2} \frac{S_{\rm e}}{L^2} \sum_{\{n_\perp\}} \frac{\omega^2 n_\perp^2}{(n_\perp^2 + n_z^2)(\omega^2 - \omega_n^2)} \,. \qquad (3.2.29)$$

Thus, the dispersion relation (3.2.29) is virtually analogous to Eqn (3.1.27), which gives the expected result, viz. stimulated Cherenkov emission of transverse waves in an isotropic medium, as of longitudinal waves in an isotropic plasma, represents beam instability of the same type as the single-particle stimulated Cherenkov effect. Inequality (3.2.28), implying a higher growth rate of instability compared with the beam Langmuir frequency, is a condition for the existence of the single-particle effect. In Section 3.6, we shall turn back to dispersion relation (3.2.29) and write it out in the explicit form (containing no infinite sums) based on different methods of the theory of Cherenkov beam instabilities.

3.3 Emission of transverse-longitudinal waves in an anisotropic plasma

Now, let us consider Cherenkov emission by electrons in an anisotropic medium with frequency dispersion, for example, in a cold electron plasma placed in a strong external uniform magnetic field. Assuming the electrons to be completely magnetized (i.e., their motion across the magnetic field to be forbidden), we shall consider a set of equations describing excitation of plasma waves by a linear bunch of free electrons traveling strictly along the external magnetic field:

$$\begin{gathered}
\frac{\partial}{\partial t}\left(\Delta_\perp + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi + 4\pi j_{{\rm p}z} \\
= -4\pi e \delta(\mathbf{r}_\perp) \sum_{j=1}^{N_{\rm e}} v_{{\rm e}zj}(t, z_{0j})\, \delta\big(z - z_{{\rm e}j}(t, z_{0j})\big)\,, \\
\frac{\partial j_{{\rm p}z}}{\partial t} - \frac{\omega_{\rm p}^2}{4\pi} E_z = 0\,, \qquad E_z = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi\,, \qquad (3.3.1) \\
\frac{{\rm d}z_{{\rm e}j}}{{\rm d}t} = v_{{\rm e}zj}\,, \qquad \frac{{\rm d}p_{{\rm e}zj}}{{\rm d}t} = eE_z(t, 0, z_{{\rm e}j})\,.
\end{gathered}$$

Here, $\psi(t, \mathbf{r}_\perp, z)$ is the Hertz polarization potential [24, 26], $\mathbf{r}_\perp = \{x, y\}$, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and the external magnetic field is aligned along the z-axis. System of equations (3.3.1) contains elements of both (3.1.1) and (3.2.1) systems, which reflects the coupling of longitudinal and transverse waves in an anisotropic plasma. Generally speaking, it makes no sense to categorize electron energy losses in an anisotropic medium into polarization (Bohr) and radiation ones [6, 13].

Comprehensive analytical treatment of the system of differential equations (3.3.1), having the fourth order in time, encounters difficulty. To clarify the simplification strategy adopted below, we shall consider the dispersion equation for eigenfrequencies of anisotropic plasma [19], which is written down in the form

$$\omega^2 = \omega_{\rm p}^2 \frac{k_z^2 - \omega^2/c^2}{k_\perp^2 + k_z^2 - \omega^2/c^2}\,. \qquad (3.3.2)$$

Here, $k_\perp^2 = k_x^2 + k_y^2$, k_z is the component of the wave vector $\mathbf{k} = \{k_x, k_y, k_z\}$ in the direction of the external magnetic field. Taking into consideration the condition $\omega = k_z u$ for Cherenkov emission, we shall write out an approximate solution of Eqn (3.3.2) (i.e., only one of the solutions describing waves with phase velocities lower than the speed of light, c) in the form

$$\omega^2(k_z, k_\perp) = \omega_{\rm p}^2 \frac{k_z^2}{k_z^2 + k_\perp^2 \gamma^2} \equiv \omega_{\rm p}^2(k_z, k_\perp)\,. \qquad (3.3.3)$$

In application to equations (3.3.1) and formulas (3.3.2), passage from Eqn (3.3.2) to Eqn (3.3.3) is equivalent to the replacement of the operator:

$$\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \to \frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\,. \qquad (3.3.4)$$

Because anisotropy of the system in question leads to other peculiarities too, we shall slightly change the design of the study in the present section; specifically, we shall resort to the energy conservation law instead of direct computation of the radiation energy.

Following the Hamiltonian method and taking into account the second equation of system (3.3.1), we expand the current density in plasma and the longitudinal component of the electric field strength into plane waves:

$$\begin{gathered}
j_{{\rm p}z}(t, \mathbf{r}) = \frac{1}{2}\frac{\omega_{\rm p}^2}{4\pi} \sum_{\{n\}} \big[A_{\{n\}}(t) \exp(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}) \\
+ A_{\{n\}}^*(t) \exp(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r})\big]\,, \\
E_z(t, \mathbf{r}) = \frac{1}{2}\sum_{\{n\}} \big[\dot{A}_{\{n\}}(t) \exp(\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r}) + \dot{A}_{\{n\}}^*(t) \exp(-\mathrm{i}\mathbf{k}_{\{n\}}\mathbf{r})\big]\,.
\end{gathered} \qquad (3.3.5)$$

Here, $\mathbf{k}_{\{n\}} = \{\mathbf{n}_\perp, n_z\}k_0$, and $\mathbf{n}_\perp = \{n_x, n_y\}$. Substituting expansions (3.3.5) into equations (3.3.1) and taking account of replacement (3.3.4) give the following equation for excitation of transverse-longitudinal waves:

$$\begin{gathered}
\frac{{\rm d}^2 A_{\{n\}}}{{\rm d}t^2} + \omega_{{\rm p}n}^2 A_{\{n\}} = -\frac{8\pi e}{L^3} \frac{n_z^2 \gamma^{-2}}{n_\perp^2 + n_z^2 \gamma^{-2}} \\
\times \sum_{j=1}^{N_{\rm e}} v_{{\rm e}zj}(t, z_{0j}) \exp\big(-\mathrm{i}nk_0 z_{{\rm e}j}(t, z_{0j})\big)\,, \\
\frac{{\rm d}z_{{\rm e}j}}{{\rm d}t} = v_{{\rm e}zj}\,, \qquad \frac{{\rm d}p_{{\rm e}zj}}{{\rm d}t} = e\,\frac{1}{2}\sum_{\{n\}} n_z^2 \big[\dot{A}_{\{n\}}(t) \exp(\mathrm{i}n_z k_0 z_{{\rm e}j}) \\
+ \dot{A}_{\{n\}}^*(t) \exp(-\mathrm{i}n_z k_0 z_{{\rm e}j})\big]\,.
\end{gathered} \qquad (3.3.6)$$

Here, $\omega_{{\rm p}n}^2$ is given by formula (3.3.3) with $k_z^2 = k_0^2 n_z^2$ and $k_\perp^2 = k_0^2 n_\perp^2$.

In a linear approximation, equations (3.3.6) after the substitution of representations (2.3.1) and (2.3.3) take the form

$$\begin{gathered}
\frac{{\rm d}^2 A_{\{n\}}}{{\rm d}t^2} + \omega_{{\rm p}n}^2 A_{\{n\}} \\
= -\frac{8\pi e}{L^3}\, u\, \frac{n_z^2 \gamma^{-2}}{n_\perp^2 + n_z^2 \gamma^{-2}} (Q_{n_z} + \tilde{V}_{n_z} - \mathrm{i}n_z \tilde{Z}_{n_z}) \exp(-\mathrm{i}n_z k_0 u t)\,, \\
\frac{{\rm d}\tilde{V}_{n_z}}{{\rm d}t} = \frac{e}{mu}\,\gamma^{-3}\,\frac{1}{2}\sum_{\{n'\}} \big[\dot{A}_{\{n'\}} Q_{n_z - n_z'} \exp(\mathrm{i}n_z' k_0 u t) \\
+ \dot{A}_{\{n'\}}^* Q_{n_z + n_z'} \exp(-\mathrm{i}n_z' k_0 u t)\big]\,, \\
\frac{{\rm d}\tilde{Z}_{n_z}}{{\rm d}t} = k_0 u \tilde{V}_{n_z}\,.
\end{gathered} \qquad (3.3.7)$$

In the case of spontaneous emission by a single electron, we put $Q_{n_z} = 1$ in the first equation of system (3.3.7), while the term $\tilde{V}_{n_z} - \mathrm{i}n_z\tilde{Z}_{n_z}$ is neglected. Substituting further the solution of Eqn (3.3.7) into the second expansion (3.3.5), we will obtain the electric field strength at the unperturbed electron trajectory $\mathbf{r}_\mathrm{e} = \{0, 0, ut\} \equiv \mathbf{r}_{\mathrm{e}0}$:

$$E_z(t, \mathbf{r}_{\mathrm{e}0}) = -\frac{4\pi eu}{L^3} \sum_{\{n\}} \frac{n_z^2\gamma^{-2}}{n_\perp^2 + n_z^2\gamma^{-2}} \frac{\sin\left[(\omega_{\mathrm{p}n} - n_z k_0 u)t\right]}{\omega_{\mathrm{p}n} - n_z k_0 u} . \tag{3.3.8}$$

However, in accordance with the law of conservation of energy, the work of field (3.3.8) done on an electron (taken with the opposite sign) equals the total energy of waves emitted by the electron. Passage from summation over n to integration over the wave number eventually leads to the expression for the power of spontaneous Cherenkov emission by an electron in an anisotropic plasma:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -euE_z(t, \mathbf{r}_{\mathrm{e}0}) = e^2u^2 \int k^2\delta\left[\omega_\mathrm{p}(k_z, k_\perp) - ku\cos\theta\right] \times \frac{\cos^2\theta}{\cos^2\theta + \gamma^2\sin^2\theta} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}k , \tag{3.3.9}$$

where the frequency $\omega_\mathrm{p}(k_z, k_\perp)$ is defined in Eqn (3.3.3).

In a nonrelativistic case, when $\gamma = 1$, formula (3.3.9) looks like formula (3.1.12). However, they are qualitatively different. First, formula (3.1.12) holds true at any electron relativity. Second, formula (3.1.12) has $\omega(k) = \omega_\mathrm{p}$, while formula (3.3.9) describes, in accordance with formula (3.3.3), Cherenkov emission of waves with an essentially different dispersion law. Setting the argument of the delta function in formula (3.3.9) equal to zero and taking account of formula (3.3.3) give the limitation on the range of wave numbers of the emitted transverse-longitudinal plasma waves:

$$k_z^2 + \gamma^2 k_\perp^2 = \frac{\omega_\mathrm{p}^2}{u^2} \to \frac{\omega_\mathrm{p}^2}{u^2\gamma^2} \leqslant k^2 = k_z^2 + k_\perp^2 \leqslant \frac{\omega_\mathrm{p}^2}{u^2} . \tag{3.3.10}$$

Integration first over k and then over θ in formula (3.3.9), bearing in mind solution (3.3.3), yields [27]

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{e^2\omega_\mathrm{p}^2}{u} \int_0^{\pi/2} D(\theta, \gamma^2)\,\mathrm{d}\theta = \frac{e^2\omega_\mathrm{p}^2}{2u\gamma^2} , \qquad D(\theta, \gamma^2) = \frac{\gamma^{-4}\cos\theta\sin\theta}{(1 - \beta^2\cos^2\theta)^2} . \tag{3.3.11}$$

Expression $D(\theta, \gamma^2)$ defines the radiation pattern of spontaneous Cherenkov emission in fully magnetized anisotropic plasma. Figure 12 presents functions $\gamma D(\theta, \gamma^2)$ in polar coordinates for a few values of γ. As $\gamma \gg 1$, the maximum of the radiation pattern, $\sim \gamma^{-1}$, falls on the angle

$$\theta_{\max} \underset{\gamma \gg 1}{\longrightarrow} (\sqrt{3}\,\gamma)^{-1} .$$

The half-height width of the radiation pattern is $\sim \gamma^{-1}$, too. Due to this, the Cherenkov radiation power (3.3.11) diminishes as γ^{-2}. It follows from formula (3.3.3) for $\gamma \gg 1$ that an electron radiates in the direction toward the maximum at frequency $\omega = \sqrt{3}\,\omega_\mathrm{p}/2$. Integration first over the angle in formula (3.3.9) gives

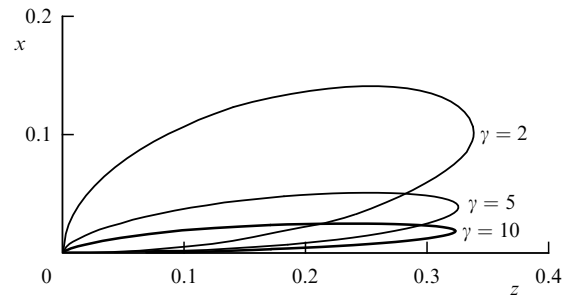

Figure 12. Vavilov–Cherenkov radiation patterns in an anisotropic plasma.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{e^2u}{\gamma^2 - 1} \int_{k_{\min}}^{k_{\max}} k\,\mathrm{d}k . \tag{3.3.12}$$

It makes sense to compare formula (3.3.12) with formula (3.1.13). If limits of integration in expression (3.3.12) are substituted from Eqn (3.3.10), one comes to the quantity presented in Eqn (3.3.11).

For stimulated emission induced by an electron beam, equations (3.3.7) are written out as

$$\frac{\mathrm{d}^2A_{\{n\}}}{\mathrm{d}t^2} + \omega_{\mathrm{p}n}^2 A_{\{n\}} = -\frac{8\pi e}{L^3}\, u\, \frac{n_z^2\gamma^{-2}}{n_\perp^2 + n_z^2\gamma^{-2}} (\tilde{V}_{n_z} - \mathrm{i}n_z\tilde{Z}_{n_z}) \exp(-\mathrm{i}n_z k_0 ut) ,$$
$$\frac{\mathrm{d}\tilde{V}_{n_z}}{\mathrm{d}t} = \frac{e\gamma^{-3}N_\mathrm{e}}{2mu} \sum_{\{n_x, n_y\}} \dot{A}_{\{n\}} \exp(\mathrm{i}n_z k_0 ut) , \qquad \frac{\mathrm{d}\tilde{Z}_{n_z}}{\mathrm{d}t} = k_0 u \tilde{V}_{n_z} . \tag{3.3.13}$$

The structure of equations (3.3.13) being even simpler than that of the previously considered equations (3.1.17) and (3.2.25), we immediately present a dispersion equation analogous to equations (3.1.27) and (3.2.29):

$$1 = \frac{\omega_\mathrm{e}^2\gamma^{-3}}{(\omega - n_z k_0 u)^2} \frac{S_\mathrm{e}}{L^2} \sum_{\{n_\perp\}} \frac{\omega^2 n_z^2\gamma^{-2}}{(n_\perp^2 + n_z^2\gamma^{-2})(\omega^2 - \omega_{\mathrm{p}n}^2)} . \tag{3.3.14}$$

The dispersion equation (3.3.14) describes the effect of stimulated Cherenkov emission of nonpotential transverse-longitudinal waves in an anisotropic plasma. Equations (3.3.14) and (3.1.27) differ in two ways: first, the change of $n_z^2 \to n_z^2\gamma^{-2}$ due to the nonpotentiality of plasma waves in anisotropic plasma, and, second, the difference in dispersion laws for $\omega_{\mathrm{p}n}$.

3.4 Emission of ion sound in an anisothermic plasma

It is commonly believed that radiation is nothing but excitation of transverse electromagnetic waves (i.e., light) by a charge (see Section 3.2). However, it follows from Sections 3.1 and 3.3 that resonant excitation of longitudinal and transverse-longitudinal waves in a plasma, i.e., waves of different nature when comparing them with light, should expediently be regarded as radiation as well, making no distinction between polarization and radiation losses in the radiator. Another example of radiation not generally accepted in electrodynamics is Cherenkov emission of ion-sound waves by a moving charge in an isotropic, anisothermic plasma.

As is known, ion-sound waves exist in a plasma in which electron temperature is much higher than ion temperature [8]. Such an anisothermic plasma is described in the framework of the Silin – Klimontovich one-fluid hydrodynamics model [8, 13]. The relevant hydrodynamic equations taking into account the potentiality of ion-sound waves are written out in the form

$$\frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{j}_{\mathrm{e}} + 4\pi \mathbf{j}_{\mathrm{i}} = -4\pi e_{\mathrm{r}} \sum_{j=1}^{N_{\mathrm{e}}} \mathbf{v}_{\mathrm{r}j}(t, z_{0j})\, \delta\big(\mathbf{r} - \mathbf{r}_{\mathrm{r}j}(t, z_{0j})\big)\,,$$
$$\frac{\partial \mathbf{j}_{\mathrm{i}}}{\partial t} = \frac{\omega_{\mathrm{i}}^2}{4\pi}\,\mathbf{E}\,, \qquad \Delta \mathbf{j}_{\mathrm{e}} = -\frac{1}{4\pi r_{\mathrm{De}}^2}\,\frac{\partial \mathbf{E}}{\partial t}\,. \tag{3.4.1}$$

Here, $\mathbf{j}_{\mathrm{e}}, \mathbf{j}_{\mathrm{i}}$ are the electron and ion current densities in plasma, ω_{i} is the ion Langmuir frequency, and $\mathbf{r}_{\mathrm{r}j}$ and $\mathbf{v}_{\mathrm{r}j}$ are the radius vector and velocity of the emitting charged particle. In view of the low speed of ion-sound waves, it would be interesting to consider their excitation by heavy nonrelativistic particles with mass m_{r} and charge e_{r}.

Introducing auxiliary function $\mathbf{A}(t, \mathbf{r})$ in accordance with formula $\mathbf{E} = \partial \mathbf{A}/\partial t$, making use of expansion (3.1.4), and taking into account the equation of motion for emitters, it is possible to derive from equations (3.4.1) Hamiltonian equations describing excitation of ion-sound oscillations $A_{\{n\}}$:

$$\frac{\mathrm{d}^2 A_{\{n\}}}{\mathrm{d}t^2} + \omega_{sn}^2 A_{\{n\}} = -\frac{8\pi e_{\mathrm{r}}}{L^3}\, g_{\{n\}} \sum_{j=1}^{N_{\mathrm{e}}} \big(\mathbf{e}_{\{n\}} \mathbf{v}_{\mathrm{r}j}(t, z_{0j})\big) \exp\big(-\mathrm{i}\mathbf{k}_{\{n\}} \mathbf{r}_{\mathrm{r}j}(t, z_{0j})\big)\,. \tag{3.4.2}$$

Given an evident change of notations $e \to e_{\mathrm{r}}$ and $m \to m_{\mathrm{r}}$, the equations of emitters' motion coincide with the respective equations of system (3.1.5). Equation (3.4.2) allows notations

$$g_{\{n\}} \equiv \frac{k_{\{n\}}^2 r_{\mathrm{De}}^2}{1 + k_{\{n\}}^2 r_{\mathrm{De}}^2}\,, \tag{3.4.3}$$

and $\omega_{sn}^2 \equiv \omega_{\mathrm{i}}^2 g_{\{n\}}$ is the frequency squared of ion-sound waves. In the linear approximation, equations (3.4.2) are written out in the form

$$\frac{\mathrm{d}^2 A_{\{n\}}}{\mathrm{d}t^2} + \omega_{sn}^2 A_{\{n\}} = -\frac{8\pi e_{\mathrm{r}}}{L^3}\, u g_{\{n\}}\, \frac{n_z}{|n|} \big(Q_{n_z} + \tilde{V}_{n_z} - \mathrm{i} n_z \tilde{Z}_{n_z}\big) \exp\big(-\mathrm{i} n_z k_0 u t\big)\,, \tag{3.4.4}$$

while equations for $\tilde{V}_{n_z}$ and $\tilde{Z}_{n_z}$ coincide with appropriate equations from system (3.1.9).

In view of the complete identity of equations (3.4.2) and (3.4.4) with equations describing Cherenkov emission of longitudinal waves in a cold plasma, we present here only the most important results of their analysis. The power of spontaneous Cherenkov emission of ion-sound waves is given by the formula

$$\frac{\mathrm{d}W_{\mathrm{s}}}{\mathrm{d}t} = \frac{4\pi e_{\mathrm{r}}^2 u^2}{L^3} \sum_{\{n\}} \frac{n_z^2}{n^2}\, g_{\{n\}}^2\, \frac{\sin\big[(\omega_{sn} - n_z k_0 u) t\big]}{\omega_{sn} - n_z k_0 u}\,, \tag{3.4.5}$$

or, by passing to integration in formula (3.4.5), by the formula

$$\frac{\mathrm{d}W_{\mathrm{s}}}{\mathrm{d}t} = e_{\mathrm{r}}^2 u^2 \iint k^2 g^2(k) \cos^2\theta\, \delta\big[\omega_{\mathrm{i}} \sqrt{g(k)} - k u \cos\theta\big]\, \mathrm{d}k \sin\theta\, \mathrm{d}\theta\,, \tag{3.4.6}$$

where $g(k) = k^2 r_{\mathrm{De}}^2 (1 + k^2 r_{\mathrm{De}}^2)^{-1}$. Integration first over the angle in formula (3.4.6) gives

$$\frac{\mathrm{d}W_{\mathrm{s}}}{\mathrm{d}t} = \frac{e_{\mathrm{r}}^2 \omega_{\mathrm{i}}^2}{u} \int_{x_{\min}}^{x_{\max}} \frac{x^5\, \mathrm{d}x}{(1 + x^2)^3} \approx \frac{e_{\mathrm{r}}^2 \omega_{\mathrm{i}}^2}{2u} \ln \frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\,, \tag{3.4.7}$$

where $x = k r_{\mathrm{De}}$. Setting the lower limit of integration in formula (3.4.7), we assumed that $u > V_{\mathrm{s}}$ ($V_{\mathrm{s}} = \omega_{\mathrm{i}} r_{\mathrm{De}}$ is the velocity of ion sound). In this case, the condition of Cherenkov resonance is fulfilled for all $k \geqslant 0$, i.e., $x_{\min} = 0$. The upper limit of integration is determined by the fact that for $k > r_{\mathrm{Di}}^{-1}$ ion-sound waves undergo strong damping (thermal ion Landau damping [8], r_{Di} is the ion Debye radius); it is therefore permissible to put $x_{\max}^2 = r_{\mathrm{De}}^2 / r_{\mathrm{Di}}^2 = T_{\mathrm{e}}/T_{\mathrm{i}} \gg 1$, where T_{i} is the ion temperature.

In the case of a beam of charged particles, Eqn (3.4.4) and the corresponding equations for $\tilde{V}_{n_z}$ and $\tilde{Z}_{n_z}$ give the following dispersion relation:

$$1 = \frac{\omega_{\mathrm{r}}^2}{(\omega - n_z k_0 u)^2}\, \frac{S_{\mathrm{r}}}{L^2} \sum_{\{n_\perp\}} \frac{\omega^2 n_z^2}{(n_\perp^2 + n_z^2)(\omega^2 - \omega_{sn}^2)}\, \frac{1}{1 + (n k_0 r_{\mathrm{De}})^{-2}}\,. \tag{3.4.8}$$

Evidently, Eqn (3.4.8) has the same structure as the dispersion equations obtained earlier. The introduction of longitudinal permittivity of anisothermic plasma, $\varepsilon^{\mathrm{l}}(\omega, k) = 1 + 1/(k r_{\mathrm{De}})^2 - \omega_{\mathrm{i}}^2/\omega^2$ [8], in the hydrodynamic model allows dispersion equation (3.4.8) to be rewritten in the generalized form

$$1 = \frac{\omega_{\mathrm{r}}^2}{(\omega - k_z u)^2}\, \frac{k_z^2 S_{\mathrm{r}}}{2\pi} \int \frac{k_\perp\, \mathrm{d}k_\perp}{(k_\perp^2 + k_z^2)\, \varepsilon^{\mathrm{l}}(\omega, k)}\,. \tag{3.4.9}$$

When writing Eqn (3.4.9), we used rule (3.1.21) to move from summation to integration with respect to transverse wave numbers. Evidently, dispersion equation (3.1.27) can also be rewritten in a similar generalized form. It is easy to show that stimulated Cherenkov emission of longitudinal waves in any isotropic medium is described by Eqn (3.4.9), taking account of spatial dispersion.

3.5 The dispersion equation method

In the preceding sections, we have considered Cherenkov emission of waves of all possible types, namely, longitudinal waves in an isotropic medium (plasma), transverse waves in an isotropic medium (dielectric), transverse-longitudinal waves in an anisotropic medium (magnetized plasma), and ion-sound waves. Formulas for the power of spontaneous Cherenkov emission of all these waves differ solely due to their different physical nature and polarization. Stimulated Cherenkov emission represents in all cases a resonance beam instability described by a dispersion equation of the form

$$D(\omega, k_z) = G(\omega, k_z)\, \frac{\omega_{\mathrm{e}}^2 \gamma^{-3}}{(\omega - k_z u)^2}\,, \tag{3.5.1}$$

where $G(\omega, k_z)$ is the form factor depending on polarization of radiation, and $D(\omega, k_z)$ is the dispersion function, the zeros of which define eigenfrequencies of the waves being emitted. Equation (3.5.1) is a generalization of dispersion equations (3.1.27), (3.2.29), (3.3.14), and (3.4.8).

Dispersion equations of the (3.5.1) type were first derived and studied in plasma electrodynamics and plasma microwave electronics [8, 16, 19]. We shall try to elucidate the relationship between traditional, classical electrodynamics of radiative processes and the above fields of theoretical physics. Let us consider a circular metal waveguide of radius R along the z-axis of which runs a continuous cylindrical electron beam of radius $r_e \ll R$; in the region $r_e < r < R$, the waveguide is filled by a homogeneous medium with the permittivity tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_\perp(\omega) & 0 & 0 \\ 0 & \varepsilon_\perp(\omega) & 0 \\ 0 & 0 & \varepsilon_\parallel(\omega) \end{pmatrix}, \quad i, j = r, \varphi, z, \tag{3.5.2}$$

where r, φ, z are the cylindrical coordinates.

In the context of interaction with the beam, only waves with a nonzero longitudinal component of the electric field, i.e., $\mathbf{Eu} \neq 0$, are of interest. Bearing this in mind, let us assume that azimuthally symmetric waves of the E type are excited in the waveguide. As is known [23, 24], electromagnetic field components E_z, E_r, B_φ of such waves differ from zero, with

$$E_z(0) \neq 0, \qquad E_r(0) = 0, \qquad B_\varphi(0) = 0. \tag{3.5.3}$$

According to formulas (3.5.3), the longitudinal electric field around the waveguide axis is the basic one. It allows transverse field components in the beam region $0 < r < r_e$ for $r_e \ll R$ to be neglected and beam permittivity to be given in the form (3.5.2) with $\varepsilon_\perp = 1$ and

$$\varepsilon_\parallel(\omega) \equiv \varepsilon_e(\omega, k_z) = 1 - \frac{\omega_e^2 \gamma^{-3}}{(\omega - k_z u)^2}. \tag{3.5.4}$$

It follows from the Maxwell equations with permittivity tensor (3.5.2) that the electric field component E_z of the azimuthally symmetric wave of the E type satisfies the equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}E_z}{\mathrm{d}r} - \kappa_\perp^2 \frac{\varepsilon_\parallel}{\varepsilon_\perp} E_z = 0, \qquad \kappa_\perp^2 = k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp, \tag{3.5.5}$$

while other components of the electromagnetic field are computed from the formulas

$$E_r = -\mathrm{i}\frac{k_z}{\kappa_\perp^2}\frac{\mathrm{d}E_z}{\mathrm{d}r}, \qquad B_\varphi = -\mathrm{i}\frac{\omega}{c\kappa_\perp^2}\varepsilon_\perp \frac{\mathrm{d}E_z}{\mathrm{d}r}. \tag{3.5.6}$$

In accordance with Eqn (3.5.5), the electric field in the waveguide region $r_e < r < R$ is given by the expression

$$E_z = C_1 J_0(\sigma r) + C_2 N_0(\sigma r), \qquad \sigma = \sqrt{-\frac{\kappa_\perp^2 \varepsilon_\parallel}{\varepsilon_\perp}}, \tag{3.5.7}$$

where J_0, N_0 are the Bessel and Neumann functions, and C_1, C_2 are the constants. Taking into account the finiteness of the field on the waveguide axis, one has in the vicinity of the electron beam:

$$E_z = C_3 J_0\left(\sqrt{-\kappa_0^2 \varepsilon_e}\, r\right), \qquad \kappa_0^2 = k_z^2 - \frac{\omega^2}{c^2}. \tag{3.5.8}$$

The dispersion relation for the frequency spectra of symmetric E type waves is obtained by joining the solutions (3.5.7) and (3.5.8) at the boundary $r = r_e$ and eliminating constants C_1, C_2, and C_3 [7, 8]. Also used are the continuity of functions E_z and B_φ at the boundary and the vanishing of the field component E_z at the conducting wall of the waveguide, $r = R$. In the case of a thin beam (linear chain of electrons) localized on the waveguide axis, the following conditions are fulfilled:

$$|\kappa_0^2| r_e^2 \ll 1, \qquad \omega_e^2 S_e = \text{const}. \tag{3.5.9}$$

Taking them into account, we write out the dispersion equation in the form

$$\frac{\omega_e^2 \gamma^{-3}}{(\omega - k_z u)^2} S_e = 2\pi r_e \sqrt{-\frac{\varepsilon_\parallel \varepsilon_\perp}{\kappa_\perp^2}} \frac{J_1(\sigma r_e) N_0(\sigma R) - N_1(\sigma r_e) J_0(\sigma R)}{J_0(\sigma r_e) N_0(\sigma R) - N_0(\sigma r_e) J_0(\sigma R)}. \tag{3.5.10}$$

The passage to the limit $r_e \to 0$ ($k_z r_e \ll 1$) is needed in Eqn (3.5.10), the results of which vary between different cases.

In the case of a waveguide filled with a cold isotropic plasma, one has $\varepsilon_\perp = \varepsilon_\parallel = 1 - \omega_p^2/\omega^2$, $\sigma = \sqrt{-\kappa_\perp^2}$, $\kappa_\perp^2 > 0$, and simple Bessel functions go over to imaginary argument functions. Then, dispersion equation (3.5.10) reduces to equation (3.1.24).

For a waveguide filled with an isotropic dielectric, one has $\varepsilon_\perp = \varepsilon_\parallel = \varepsilon$, $\sigma = \sqrt{-\kappa_\perp^2}$, and $\kappa_\perp^2 < 0$. Bearing in mind that frequencies of an isotropic dielectric waveguide are found from the equation $J_0(\sigma R) = 0$, Eqn (3.5.10) can be brought into the form

$$J_0(\sigma R) = \frac{\omega_e^2 \gamma^{-3}}{\varepsilon(\omega)(\omega - k_z u)^2} S_e \kappa_\perp^2 \frac{1}{4} N_0(\sigma R), \qquad \kappa_\perp^2 = k_z^2 - \frac{\omega^2}{c^2}\varepsilon(\omega). \tag{3.5.11}$$

Finally, for a plasma waveguide in a strong external magnetic field, one has $\varepsilon_\perp = 1$, $\varepsilon_\parallel = 1 - \omega_p^2/\omega^2$, $\sigma = (-\kappa_0^2 \varepsilon_\parallel)^{1/2}$, and $\kappa_0^2 \varepsilon_\parallel < 0$. Because plasma waveguide frequencies are determined from the equation $J_0(\sigma R) = 0$, Eqn (3.5.10) is reduced to the following form

$$J_0(\sigma R) = \frac{\omega_e^2 \gamma^{-3}}{(\omega - k_z u)^2} S_e \kappa_0^2 \frac{1}{4} N_0(\sigma R). \tag{3.5.12}$$

Dispersion equations (3.5.11) and (3.5.12) possess the structure of a generalized dispersion equation (3.5.1) and correspond to equations (3.2.29) and (3.3.14) written out in an explicit form, i.e., after the computation of infinite sums. Using notations adopted in equation (3.5.1), solutions of equations (3.5.11) and (3.5.12) can be presented in the form

$$\omega = \omega_0 + \frac{-1 + \mathrm{i}\sqrt{3}}{2}\left[G_0\left(\frac{\partial D_0}{\partial \omega}\right)^{-1} \omega_e^2 \gamma^{-3}\right]^{1/3}. \tag{3.5.13}$$

Here, $D_0 = D(\omega_0, \omega_0/u)$, $G_0 = G(\omega_0, \omega_0/u)$, and ω_0 is the resonance frequency found from the system of equations

$$D(\omega, k_z) = 0, \qquad \omega = k_z u. \tag{3.5.14}$$

Formula (3.5.13) contains solution (2.5.4) and the solution of equation (3.1.24).

Solution (3.5.13) is only applicable in the case of a low-density beam. This observation refers first and foremost to dispersion equations (3.5.11) and (3.5.12). The left-hand sides of these equations define an infinite set of waves corresponding to different transverse numbers $\{n_\perp\}$; hence, $\omega_0 = \omega_{0\{n_\perp\}}$ in Eqn (3.5.13). If the difference between frequencies of neighboring waves is smaller than the increment in formula (3.5.13), waves with different $\{n_\perp\}$ merge and the dispersion equation requires complicated analysis [28]. The difference between frequencies of neighboring waves being $\sim c/R$, the passage to the limit $R \to \infty$ in equations (3.5.11) and (3.5.12) is as difficult as the passage to a free space $L \to \infty$ in equations (3.2.29) and (3.3.14).

For completeness, here is one more derivation of dispersion equation of the theory of stimulated Cherenkov emission based on the Hamiltonian method formulated for a waveguide. By way of example, let us consider an anisotropic plasma waveguide of arbitrary cross section. Suppose that an infinitely thin ('needle') electron beam having unperturbed density $n_e(\mathbf{r}) = n_{0e} S_e \delta(\mathbf{r}_\perp - \mathbf{r}_e)$ (where $\mathbf{r}_e = \{x_e, y_e\}$ is the beam coordinate in the waveguide cross section) passes through the waveguide parallel to the z-axis. We will proceed from equations (3.3.1) for the polarization potential $\psi(t, \mathbf{r}_\perp, z)$ and the plasma current density $j_{pz}(t, \mathbf{r}_\perp, z)$. The right-hand side of the equation for the potential ψ we may write out in the form $-4\pi e n_{0e} S_e \delta(\mathbf{r}_\perp - \mathbf{r}_e) j_{ez}(t, z)$. Function $j_{ez}(t, z)$ determining the current density in the beam is calculated based on the cold hydrodynamics model [8]; then, the following equation is obtained:

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial z}\right)^2 j_{ez} = \frac{e}{m}\,\gamma^{-3}\,\frac{\partial}{\partial t}\,E_z(t, \mathbf{r}_e, z)\,. \tag{3.5.15}$$

Potential ψ vanishes at the metal lateral surface of the waveguide.

The polarization potential may be presented as

$$\psi(t, \mathbf{r}_\perp, z) = \frac{1}{2}\left[\left(\sum_{\{n_\perp\}} A_{\{n_\perp\}}(t)\varphi_{\{n_\perp\}}(\mathbf{r}_\perp)\right)\exp(\mathrm{i}k_z z) + \text{c.c.}\right]. \tag{3.5.16}$$

Here, $\varphi_{\{n_\perp\}}(\mathbf{r}_\perp)$ are the eigenfunctions of the waveguide cross section that are at the same time solutions of the following eigenvalue problem:

$$\begin{aligned} &\Delta_\perp \varphi_{\{n_\perp\}} + k^2_{\{n_\perp\}}\varphi_{\{n_\perp\}} = 0\,, \\ &\varphi_{\{n_\perp\}}\big|_{\Sigma_W} = 0\,, \end{aligned} \tag{3.5.17}$$

where $k^2_{\{n_\perp\}}$ are the eigenvalues, and Σ_W is the lateral surface of the waveguide. For a waveguide with a rectangular cross section, $x, y \in [-L, L]$, one finds

$$\begin{aligned} &k_{\{n_\perp\}} = \{k_{n_x}, k_{n_y}\} = \{k_0 n_x, k_0 n_y\}\,, \\ &k_0 = \frac{\pi}{2L}\,, \qquad n_{x,y} = 1, 2, \ldots, \\ &\varphi_{\{n_\perp\}} = \frac{1}{4}\left[\exp(\mathrm{i}k_{n_x}x) - (-1)^{n_x}\exp(-\mathrm{i}k_{n_x}x)\right] \\ &\qquad\qquad \times \left[\exp(\mathrm{i}k_{n_y}y) - (-1)^{n_y}\exp(-\mathrm{i}k_{n_y}y)\right] \end{aligned} \tag{3.5.18}$$

and expansion (3.5.16) is a special case of the general representation (3.1.14) (summation over n_z in potential (3.5.16) is not required because the system is homogeneous in z, and different longitudinal modes are independent). Putting $A_{\{n_\perp\}}(t) = a_{\{n_\perp\}}\exp(-\mathrm{i}\omega t)$, and $j_{pz}, j_{ez} \sim \exp(-\mathrm{i}\omega t + \mathrm{i}k_z z)$, we will substitute expansion (3.5.16) into the first equation of system (3.3.1), express constants $a_{\{n_\perp\}}$, and substitute them into Eqn (3.5.15). The result is the dispersion equation

$$1 = \frac{\omega_e^2\gamma^{-3}}{(\omega - k_z u)^2}\sum_{\{n_\perp\}}\left[\frac{S_e\varphi^2_{\{n_\perp\}}(\mathbf{r}_e)}{\|\varphi_{\{n_\perp\}}\|^2}\,\frac{\kappa_0^2}{k^2_{\{n_\perp\}} + \kappa_0^2(1 - \omega_p^2/\omega^2)}\right], \tag{3.5.19}$$

where $\|\varphi_{\{n_\perp\}}\|$ is the norm of the eigenfunction. In the case of replacement (3.3.4), it should be assumed that $\kappa_0^2 = k_z^2\gamma^{-2}$. It is easy to see that dispersion equation (3.3.14) is a special case of Eqn (3.5.19). Indeed, for a waveguide with a rectangular cross section, it follows from formulas (3.5.18) that $\|\varphi_{\{n_\perp\}}\|^2 = L^2$, $k_z^2 = n_z^2 k_0^2$, $k^2_{\{n_\perp\}} = n_\perp^2 k_0^2$, and, at $\mathbf{r}_e = 0$, $\varphi^2_{\{n_\perp\}}(\mathbf{r}_e) = 1$. Then, Eqn (3.5.19) goes over to Eqn (3.5.14). This means that the Hamiltonian method in the theory of stimulated emission applied to an electron beam gives the same result as the method of dispersion equation in the theory of plasma and plasma microwave electronics.

Dispersion equation (3.5.19) has been thoroughly investigated in Refs [7, 19]. Poles on the right-hand side of the equation, i.e., zeroes of functions $k^2_{\{n_\perp\}} + \kappa_0^2(1 - \omega_p^2/\omega^2)$, determine plasma wave spectra in the absence of an electron beam; in other words, independent plasma waves correspond to different sets of numbers $\{n_\perp\}$. For the Cherenkov resonance instability of a low-density beam, the frequencies of emitted plasma waves are given by the formula

$$\begin{aligned} \omega = \omega_{0\{n_\perp\}} &+ \frac{-1 + \mathrm{i}\sqrt{3}}{2} \\ &\times\left(\frac{S_e\varphi^2_{\{n_\perp\}}(\mathbf{r}_e)}{\|\varphi_{\{n_\perp\}}\|^2}\,\frac{1}{2}\,\frac{\omega_e^2\gamma^{-3}}{\omega_p^2 + \beta^2 k^2_{\{n_\perp\}}u^2\gamma^4}\right)^{1/3}\omega_{0\{n_\perp\}}\,, \end{aligned} \tag{3.5.20}$$

where

$$\omega_{0\{n_\perp\}} = \sqrt{\omega_p^2 - k^2_{\{n_\perp\}}u^2\gamma^2}\,.$$

Cherenkov emission of waves for $\omega_p^2 < k^2_{\{n_\perp\}}u^2\gamma^2$ is impossible. Formula (3.5.20) is one of the main calculation formulas in relativistic plasma microwave electronics [19].

The following concrete geometry of a beam-plasma system is of special importance for applications: a waveguide of radius R that contains continuous magnetized plasma and a thin tubular electron beam of mean radius $r_e < R$ and thickness $\Delta_e \ll r_e$. The eigenfunctions and the corresponding eigenvalues in a circular waveguide are given by formulas $\varphi_{\{n_\perp\}} = J_l(k_{\{n_\perp\}}r)$, $\{n_\perp\} = \{l, s\}$, $k_{\{n_\perp\}} = \mu_{ls}/R$, where μ_{ls} is the root of the l-order Bessel function, and l, s are the azimuthal and radial wave numbers. We confine ourselves to the consideration of an azimuthally symmetric case of $l = 0$. Calculating the sum of a series with the help of known formulas of the theory of Bessel functions [29], we write out the dispersion equation (3.5.19) in the explicit form

$$1 = \frac{\omega_e^2\gamma^{-3}}{(\omega - k_z u)^2}\,S_e\kappa_0^2\,\frac{1}{4}\,J_0^2(\sigma r_e)\left(\frac{N_0(\sigma R)}{J_0(\sigma R)} - \frac{N_0(\sigma r_e)}{J_0(\sigma r_e)}\right), \tag{3.5.21}$$

where $\sigma = (-\kappa_0^2 \varepsilon_\parallel)^{1/2}$. Passing to the limit $r_e \to 0$ in equation (3.5.21) and assuming $\omega_e^2 S_e$ to be constant, Eqn (3.5.21) is transformed to dispersion equation (3.5.12), while the expression for frequency (3.5.20) is reduced to formula (3.5.13).

3.6 Stimulated Cherenkov emission in gas dynamics

In conclusion, we shall discuss a gasdynamic problem of sound wave emission by a supersonic gas flow, lying far apart from electrodynamics. We shall demonstrate that it is a stimulated Cherenkov emission and the methods for its description are identical with the methods of electrodynamics used in Sections 3.1–3.3. It should be emphasized that there is no spontaneous emission of a sound in gas dynamics similar to spontaneous charge emission in electrodynamics. Supersonic motion of a body in a gas produces a gas flow that represents in itself a source of sound. Therefore, we shall consider the problem of stimulated Cherenkov emission in gas dynamics, which is formulated very close to the problem of stimulated Cherenkov emission by a beam of charged particles in plasma electronics. The dispersion equations method appears most suitable for the purpose.

Let us consider a cylindrical channel of radius R with hard walls, filled with a gas. Let a cylindrical gas flow with velocity u and radius $r_0 \ll R$ be created along the z-axis of the channel. For simplicity, the gas flow and the 'bulk' gas are assumed to have equal density and temperature (only velocity along the z-axis undergoes a jump at $r = r_0$). Perturbations in the gas are given in the form of symmetric cylindrical waves running along the channel, namely

$$f(r) \exp(-\mathrm{i}\omega t + \mathrm{i}k_z z) . \tag{3.6.1}$$

The linearized equations of gas dynamics for perturbations of density ρ, velocity $\mathbf{v}$, and pressure p can be written out as [30]

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla(\rho_0 \mathbf{v} + \rho \mathbf{u}_0) = 0 , \\ &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u}_0 \nabla)\mathbf{v} = -\frac{1}{\rho_0} \nabla p , \qquad p = c_0^2 \rho , \end{aligned} \tag{3.6.2}$$

where c_0 is the speed of sound, ρ_0 is the equilibrium density, p_0 is the pressure, all assumed to be constant, and $\mathbf{u}_0 = \{0, 0, u_0(r)\}$ is the gas velocity along the z-axis, with

$$u_0(r) = \begin{cases} 0 , & r_0 < r < R , \\ u , & r \leqslant r_0 . \end{cases} \tag{3.6.3}$$

Taking into account the dependence (3.6.1), the system of equations (3.6.2) is reduced to the equation

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}p}{\mathrm{d}r} - \left[k_z^2 - \frac{\left(\omega - k_z u_0(r)\right)^2}{c_0^2} \right] p = 0 , \tag{3.6.4}$$

supplemented by the boundary conditions that also ensue from equations (3.6.2):

$$p\big|_{r=r_0} = 0 , \qquad \left[\frac{1}{\left(\omega - k_z u_0(r)\right)^2} \frac{\mathrm{d}p}{\mathrm{d}r} \right]_{r=r_0} = 0 , \qquad \frac{\mathrm{d}p}{\mathrm{d}r}\bigg|_{r=R} = 0 . \tag{3.6.5}$$

Furthermore, the problem is to find eigenfrequencies $\omega(k_z)$ of the gas channel and the instability condition at the fulfillment of which $\operatorname{Im} \omega > 0$.

Solutions of Eqn (3.6.4) in different parts of the gas channel have the form [see formulas (3.5.7) and (3.5.8)]

$$p(r) = \begin{cases} C_1 J_0(\sigma_1 r) + C_2 N_0(\sigma_1 r) , & r_0 < r < R , \\ C_3 J_0(\sigma_2 r) , & r \leqslant r_0 , \end{cases} \tag{3.6.6}$$

where

$$\sigma_1^2 = \frac{\omega^2}{c_0^2} - k_z^2 , \qquad \sigma_2^2 = \frac{(\omega - k_z u)^2}{c_0^2} - k_z^2 . \tag{3.6.7}$$

Substituting solutions (3.6.6) into boundary conditions (3.6.5) and eliminating constants C_1, C_2, and C_3 give the dispersion equation

$$J_1(\sigma_1 R) = \frac{\omega^2}{(\omega - k_z u)^2} S_0 \sigma_2^2 \frac{1}{4} N_1(\sigma_1 R) , \tag{3.6.8}$$

presented here, for simplicity, only for the case of $r_0 \ll R$; $S_0 = \pi r_0^2$ is the cross section area of the paraxial gas flow. The striking similarity of dispersion equations (3.6.8) and, say, (3.5.12) reflects the uniform wave nature of stimulated Vavilov–Cherenkov emission in the case of superluminal (supersonic) motion of the source in a medium regardless of the wave type, source structure, and mechanism of interaction between the source and the medium.

At $S_0 = 0$, it follows from Eqn (3.6.8) that $J_1(\sigma_1 R) = 0$ (dispersion equation for discrete sound wave frequencies of an acoustic waveguide). These frequencies are given by the formula

$$\omega \equiv \omega_{0s} = \sqrt{k_z^2 c_0^2 + \frac{\mu_{1s}^2}{R^2} c_0^2} , \qquad s = 1, 2, \ldots , \tag{3.6.9}$$

where μ_{1s} are the zeroes of the function $J_1(x)$, and s is the analog of a generalized index $\{n_\perp\}$. For $S_0 \neq 0$, dispersion equation (3.6.8) has complex roots solely for $u > c_0$, i.e., for a supersonic flow only. Putting $\omega = \omega_{0s} + \delta\omega = k_z u + \delta\omega$ and finding $\delta\omega$ from Eqn (3.6.8) lead to the following expression for the complex frequency [see formula (3.5.13)]:

$$\omega = \omega_{0s} + \frac{-1 + \mathrm{i}\sqrt{3}}{2} \left(\alpha \frac{S_0}{4R^2} \frac{k_z^2 c_0^2}{\omega_{0s}^2} \right)^{1/3} \omega_{0s} , \tag{3.6.10}$$

where $\alpha = \left| N_1(\mu_{1s}) / J_0(\mu_{1s}) \right| \approx 1$. The imaginary part of expression (3.6.10) is the growth increment of sound waves under Cherenkov instability of a supersonic gas flow in an acoustic waveguide.

4. The anomalous Doppler effect and collective stimulated Cherenkov effect

4.1 Emission in an isotropic medium in a magnetic field

Let us consider emission of electromagnetic waves in an isotropic medium (dielectric) in the presence of an external magnetic field $\mathbf{B}_0 = \{0, 0, B_0\}$ aligned with the unperturbed motion of emitting electrons. We shall confine our consideration to a one-dimensional model problem of excitation of circularly polarized transverse electromagnetic waves by a 'flat' electron bunch [inhomogeneous flat layer of electrons (see Section 2)]. The starting set of equations is

as follows [1, 2, 7]:

$$\frac{\partial^2 A}{\partial z^2} - \frac{\varepsilon}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c}\, e S_1^{-1} \sum_{j=1}^{N_e} v_{ej}(t)\, \delta\big(z - z_{ej}(t)\big)\,,$$

$$\frac{dv_{ej}}{dt} + i\Omega_e v_{ej} = -\frac{e}{mc}\left(\frac{\partial A}{\partial t} + v_{ezj}\frac{\partial A}{\partial z}\right), \qquad (4.1.1)$$

$$\frac{dv_{ezj}}{dt} = \frac{e}{2mc}\left(v_{ej}\frac{\partial A^*}{\partial t} + v_{ej}^*\frac{\partial A}{\partial z}\right), \qquad \frac{dz_{ej}}{dt} = v_{ezj}\,.$$

Here, $v_{ej}(t) = v_{exj}(t) + i v_{eyj}(t)$, $A(t,z) = A_x(t,z) + iA_y(t,z)$ is the vector potential of a circularly polarized electromagnetic field, and $\Omega_e = eB_0/mc$ is the electron cyclotron frequency (for certainty, we take $\Omega_e > 0$). The electron density in the bunch in Eqns (4.1.1) was assumed to be low, which permitted neglecting the longitudinal electric self-field of the bunch and disregarding relativistic effects.

Following the Hamiltonian method, the vector potential of the field is expanded into one-dimensional plane waves:

$$A(t,z) = \sum_n A_n(t)\exp(ink_0 z)\,. \qquad (4.1.2)$$

Unlike expansion (2.2.1), function (4.1.2) is a complex one, while summation is performed over all integer n. Substituting expansion (4.1.2) into equations (4.1.1) gives, in the usual way, the Hamiltonian equations

$$\frac{d^2 A_n}{dt^2} + \omega_n^2 A_n = \frac{4\pi ec}{LS_1\varepsilon(n)}\sum_{j=1}^{N_e} v_{ej}\exp(-ink_0 z_{ej})\,,$$

$$\omega_n^2 = n^2 k_0^2\,\frac{c^2}{\varepsilon(n)}\,,$$

$$\frac{dv_{ej}}{dt} + i\Omega_e v_{ej} = -\frac{e}{mc}\sum_n (\dot{A}_n + ink_0 v_{ezj} A_n)\exp(ink_0 z_{ej})\,,$$

$$\frac{dz_{ej}}{dt} = v_{zej}\,, \qquad (4.1.3)$$

$$\frac{dv_{ezj}}{dt} = -\frac{e}{2mc}\, ik_0 \sum_n n\big[v_{ej}A_n^*\exp(-ink_0 z_{ej}) - v_{ej}^* A_n \exp(ink_0 z_{ej})\big]\,.$$

Let us analyze the system of equations (4.1.3) in a linear approximation. Let the transverse motion of bunch electrons be absent in the unperturbed state, i.e., let the complex transverse electron velocity $v_{ej} = v_{exj} + i v_{eyj} \equiv \tilde{v}_{ej}$ be a small perturbation. Using also formulas (2.3.1) and linearizing equations (4.1.3) over perturbations A_n, $\tilde{v}_{ej}$, $\tilde{v}_j$, and $\tilde{z}_j$ bring about a linear approximation system

$$\frac{d^2 A_n}{dt^2} + \omega_n^2 A_n = \frac{4\pi ec}{LS_1\varepsilon(n)}\,\tilde{V}_n\exp(-ink_0 ut)\,,$$

$$\frac{d\tilde{V}_n}{dt} + i\Omega_e \tilde{V}_n = -\frac{e}{mc}\sum_{n'}(\dot{A}_{n'} + ink_0 uA_{n'})Q_{n-n'}\exp(in'k_0 ut)\,, \qquad (4.1.4)$$

where the quantities $\tilde{V}_n$ and Q_n are defined in Eqn (2.3.3). Model system (4.1.4) describes in the one-dimensional approximation the interaction of an electron bunch in an isotropic medium with circularly polarized transverse electromagnetic waves in the external magnetic field.

In the case of a uniform electron beam, one has $Q_{n-n'} = N_e\delta_{nn'}$, and the system of equations (4.1.4) yields

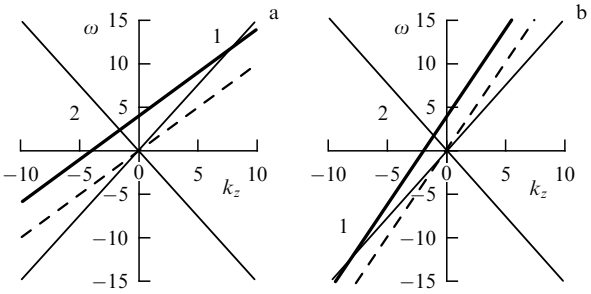

Figure 13. Resonance frequencies in anomalous and normal Doppler effects: (a) subluminal motion, and (b) superluminal motion.

the known dispersion equation [7, 31]:

$$\frac{\omega^2}{c^2}\,\varepsilon(\omega) - k_z^2 = \frac{\omega_e^2}{c^2}\,\frac{\omega - k_z u}{\omega - k_z u - \Omega_e}\,, \qquad (4.1.5)$$

into which the longitudinal wave number k_z is substituted instead of nk_0, and $\varepsilon(n)$ is replaced by the function $\varepsilon(\omega)$. Frequency $\omega = \omega_0$ and wave number $k_z = k_{z0}$ of the electromagnetic wave being in cyclotron resonance with electrons are found from the equations

$$\frac{\omega^2}{c^2}\,\varepsilon(\omega) - k_z^2 = 0\,, \qquad \omega - k_z u - \Omega_e = 0\,. \qquad (4.1.6)$$

Figure 13 illustrates resonance conditions (4.1.6) and the location of resonance points (points 1 and 2) on the plane (ω, k_z). The spectrum of electromagnetic waves is taken here in the simplest form:

$$\omega = \pm k_z c_0\,, \qquad c_0 = \frac{c}{\mu}\,, \qquad (4.1.7)$$

where $\mu = \sqrt{\varepsilon} = \text{const}$ is the index of refraction. Figure 13a illustrates subluminal electron motion with $u < c_0$. Spectra (4.1.7) are depicted by solid thin straight lines, the dashed line corresponds to the Cherenkov resonance line $\omega = k_z u$, and the thick straight line presents the cyclotron resonance line. Expressions for resonance frequencies follow from formulas (4.1.6) and (4.1.7):

$$\omega_{01} = \frac{\Omega_e}{1 - u/c_0}\,, \qquad \omega_{02} = \frac{\Omega_e}{1 + u/c_0}\,. \qquad (4.1.8)$$

For $u < c_0$, both resonance frequencies (4.1.8) are positive. Figure 13b presents a case of superluminal electron motion with $u > c_0$. The notations here are the same as in Fig. 13a. Resonance frequencies are given by formulas (4.1.8) as above, but the frequency ω_{01} for $u > c_0$ is negative. Notice that $A_n = A_{xn} + iA_{yn} \sim \exp(-i\omega t)$ in conformity with expansion (4.1.2); therefore, the sign of the frequency determines the direction of rotation of the field polarization plane, i.e., it has direct physical sense (the direction of electron rotation in a magnetic field is specified by the sign of Ω_e).

The solution of dispersion equation (4.1.5) in the vicinity of resonance points should be sought in the form

$$\omega = \omega_0 + \delta\omega = k_{z0}u + \Omega_e + \delta\omega\,, \qquad (4.1.9)$$

where ω_0 is one of the frequencies (4.1.8). Substituting formula (4.1.9) into equation (4.1.5) and assuming $\omega_e \ll \omega_0$

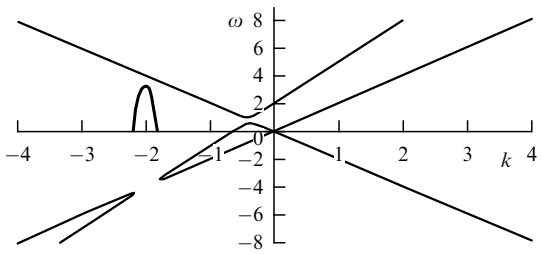

Figure 14. Dispersion curves for equation (4.1.5).

give the following relationship for the correction $\delta\omega$ to a resonance frequency ω_0:

$$(\delta\omega)^2 = \frac{\omega_e^2 \Omega_e}{2\varepsilon_0 \omega_0}\,, \tag{4.1.10}$$

where $\varepsilon_0 = \varepsilon(\omega_0)$. It follows from formula (4.1.10) that $(\delta\omega)^2 < 0$, i.e., instability occurs only for $\omega_0 < 0$. In turn, according to formulas (4.1.8), the frequency for $\Omega_e > 0$ is negative only if $u > c_0$; this is the necessary condition for the development of instability known as the anomalous Doppler effect [32]. There is no instability for $u < c_0$. The growth rate of instability in circumstances where the anomalous Doppler effect shows itself is given by the formula

$$\delta\omega = \mathrm{i}\,\frac{\omega_e}{\sqrt{2\varepsilon_0}}\sqrt{\frac{u^2}{c_0^2} - 1}\,. \tag{4.1.11}$$

To sum up, both the anomalous Doppler effect and the Vavilov–Cherenkov effect are possible only in the case of superluminal electron motion in a medium. In the case of the anomalous Doppler effect, the electrons initially moving rectilinearly acquire transverse motion, i.e., undergo rotation; simultaneously, the electromagnetic field becomes stronger. The source of energy is the energy of longitudinal electron motion [33]. For $u < c_0$, the energy is periodically pumped from the electromagnetic field into transverse electron motion and back. This stable process is referred to as the normal Doppler effect. The region of anomalous Doppler effect in Fig. 13 is located on the plane (ω, k_z) between straight lines $\omega = k_z u$ and $\omega = 0$. The rest of the plane is occupied by the normal Doppler effect. The dispersion curves for equation (4.1.5) are presented in Fig. 14, where the imaginary part of the frequency is depicted by the thick line.

Let us turn now to a single-electron case. Since all $Q_n = 1$, so does $\tilde{V}_n = \tilde{V}$ and equations (4.1.4) are transformed to

$$\begin{aligned} &\frac{\mathrm{d}^2 A_n}{\mathrm{d}t^2} + \omega_n^2 A_n = \frac{4\pi e c}{L S_1 \varepsilon(n)}\,\tilde{V}\exp\left(-\mathrm{i}nk_0 ut\right), \\ &\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} + \mathrm{i}\Omega_e \tilde{V} = -\frac{e}{mc}\sum_n (\dot{A}_n + \mathrm{i}nk_0 u A_n)\exp\left(\mathrm{i}nk_0 ut\right). \end{aligned} \tag{4.1.12}$$

Unlike equations (2.4.1), the system of equations (4.1.12) is homogeneous. Hence, the absence of spontaneous emission in the conditions where the anomalous Doppler effect shows itself; only one electron brought to rotation by the electromagnetic field produces stimulated emission. Clearly, no phasing is needed for that. Notice that spontaneous emission by an electron in a magnetic field may also take place, but for this the electron must have nonzero initial transverse velocity. Such radiation is called cyclotron radiation [22]. If electrons are located in Larmour orbits, they undergo angular phasing, and stimulated cyclotron radiation analogous to stimulated Cherenkov radiation appears (see Refs [1, 2] for details).

Here, it is appropriate to turn to the plasma theory and plasma microwave electronics in which an electron beam is described in terms of the permittivity tensor [13, 16, 17]. Specifically, the following expressions for the diagonal components of this tensor are known for an electron beam propagating in the finite external magnetic field [8, 34]:

$$\begin{aligned} &\varepsilon_{xx} = \varepsilon_{yy} = 1 + \varepsilon_\perp^{(+)} + \varepsilon_\perp^{(-)}\,, \\ &\varepsilon_\perp^{(\pm)} = -\frac{\omega_e^2 \gamma^{-1}}{2\omega}\left[\frac{\omega - k_z u}{\omega - k_z u \mp \Omega_e/\gamma} + \frac{1}{2}\,u_\perp^2\,\frac{k_z^2 - \omega^2/c^2}{(\omega - k_z u \mp \Omega_e/\gamma)^2}\right], \\ &\varepsilon_{zz} = 1 - \frac{\omega_e^2 \gamma^{-3}}{(\omega - k_z u)^2}\,, \end{aligned} \tag{4.1.13}$$

where $u_\perp$ is the electron velocity component transverse to the external magnetic field, and

$$\gamma = \left(1 - \frac{u^2}{c^2} - \frac{u_\perp^2}{c^2}\right)^{-1/2}$$

is the electron relativistic factor. Formulas (4.1.13) are given for the simplest case of an electromagnetic field independent of the coordinates x, y. Terms of the permittivity tensor having second-order poles describe radiative processes associated with the phasing of beam electrons by the field: the term with the pole $(\omega - k_z u)^{-2}$ is due to longitudinal phasing in stimulated Cherenkov emission, and the term with the pole $(\omega - k_z u \mp \Omega_e/\gamma)^{-2}$ describes transverse-longitudinal phasing in the magnetic field in stimulated cyclotron radiation. These two forms of stimulated emission in passing from a uniform beam to a bunch and a single electron are matched by the corresponding spontaneous processes, i.e., Cherenkov and cyclotron emission. It follows from Eqn (4.1.13) that cyclotron radiation is absent at $u_\perp = 0$. Only anomalous and normal Doppler effects remain, to which the terms with first-order poles $(\omega - k_z u \mp \Omega_e/\gamma)^{-1}$ correspond in permittivity.

Let us turn back to the system of equations (4.1.12). It differs from system (4.1.4) by the presence of summation over all integer n; this is understandable since the field of a single electron cannot be spatially monochromatic as a bunch field and is actually a wave packet. This circumstance accounts for an interesting feature. Let k_{z01} and k_{z02} be the solutions of the system of equations (4.1.6), i.e., resonance wave numbers corresponding to resonance frequencies (4.1.8). Field constituents with such wave numbers (or close to them) are indispensable in system (4.1.12). Let an electron in the regime of anomalous Doppler effect radiate at frequency ω_{01}. Emission at frequency ω_{02} under conditions of normal Doppler effect is possible only for an electron having a sufficient transverse velocity. However, the electron acquires such velocity due to the anomalous Doppler effect. In other words, radiation in the case of the anomalous Doppler effect at frequency ω_{01} stimulates normal Doppler radiation at frequency ω_{02}. In fact, induced scattering from an electron in a magnetic field occurs:

$$\omega_{01} - \omega_{02} = (k_{z01} - k_{z02})u\,. \tag{4.1.14}$$

This process takes place for an electron bunch too, but its efficiency for an infinite bunch (i.e., a uniform electron beam) approaches zero.

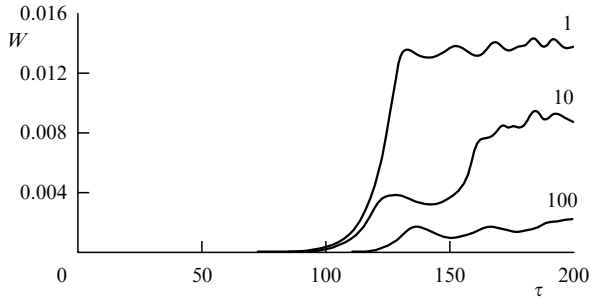

Figure 15. Radiation energy in the case of the anomalous Doppler effect at different numbers of electrons in the bunch.

In order to study radiation dynamics in the conditions of anomalous and normal Doppler effects, it is appropriate to use the general nonlinear system of equations (4.1.3) amenable to numerical solution in the same formulation as the nonlinear equations (2.2.2) and (2.2.3) in Section 2.7. Let us put $\varepsilon(n) = \varepsilon_0 = \text{const}$, fix the parameters

$$\nu = \frac{4\pi e^2 N_\mathrm{e}}{mLS_1\Omega_\mathrm{e}^2\varepsilon_0} = 0.01\,, \quad \sigma = \frac{k_0 u}{\Omega_\mathrm{e}} = 0.1\,, \quad \beta_0 = \frac{c}{u\varepsilon_0} = \frac{1}{2}\,,$$

and vary the number of electrons N_e in the bunch. Equations (4.1.6) give $n = -\sigma^{-1}(1 \mp \beta_0)^{-1}$ for resonant harmonic numbers. For the given values of parameters, harmonics with numbers $n = n_1 = -20$ and $n = n_2 = -7$ are in anomalous and normal Doppler resonances, respectively. When solving the system of equations (4.1.3), we shall take account of harmonics with numbers $n \in [-50, 50]$ alone. The 'distance' between electrons in the bunch is set to be one-tenth of the radiation wavelength at anomalous Doppler resonance.

Figure 15 presents dependences of the ratio of total radiation energy to kinetic energy of an electron bunch on dimensionless time $\tau = \Omega_\mathrm{e} t$ at different numbers N_e of bunch electrons (1, 10, 100) indicated alongside the respective curves. The curves are similar (exponential) at the linear stage, suggesting the development of instability, i.e., appearance of stimulated radiation regardless of the number of electrons in the bunch. Nonlinear saturation is associated with the escape from resonance upon a fall in longitudinal electron velocity.

Figure 16 shows characteristic spectral radiation densities n in relative units for different N_e at the linear stage of the process. Maximum spectral densities coincide with n values corresponding to anomalous and normal Doppler resonances. For $N_\mathrm{e} = 100$, however, no emission occurs under conditions of normal Doppler resonance. The point is that induced scattering (4.1.14) occurs on electron density inhomogeneity, and the size of a bunch inhomogeneity at $N_\mathrm{e} = 100$ is 10 times the radiation wavelength; in fact, a spatially homogeneous case takes place.

4.2 The collective stimulated Cherenkov effect

The anomalous Doppler effect and Vavilov – Cherenkov effect are the two main mechanisms of emission of electromagnetic waves by an electron executing rectilinear motion in a medium. In the case of the anomalous Doppler effect, the emitter is an electron-oscillator whose eigenfrequency is determined by a certain external impact, e.g., Ω_e, while moving over an external magnetic field. For the Vavilov – Cherenkov effect, the electron is not an oscillator or can it be

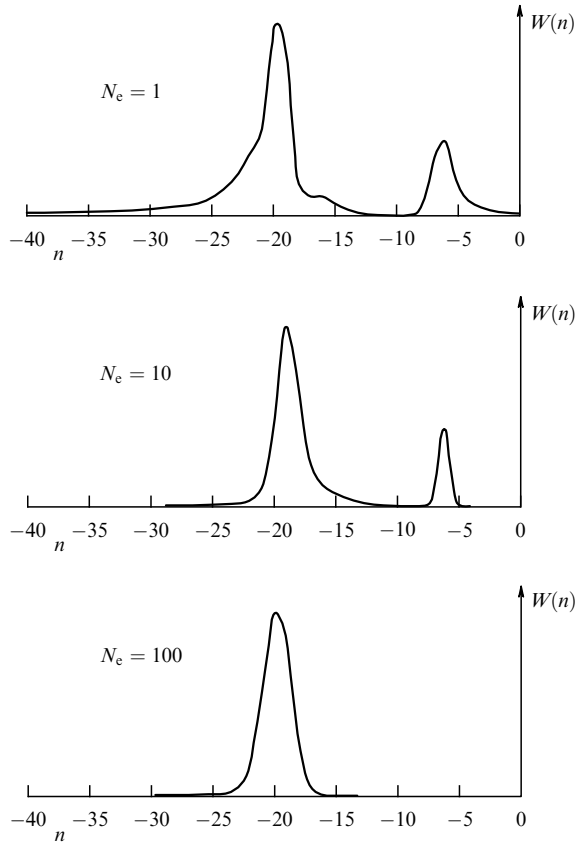

Figure 16. Spectral radiation densities in the case of the anomalous Doppler effect at different numbers of electrons in the bunch.

regarded as a zero-frequency oscillator. However, the Vavilov – Cherenkov effect under certain conditions resembles the anomalous Doppler effect, e.g., when a dense electron beam radiates and the eigenfrequency of the electron-oscillator is the frequency of plasma (Langmuir) oscillations of the beam as a whole. Such Cherenkov emission, called the collective stimulated Cherenkov effect [1, 2], is considered below as exemplified by Cherenkov emission of transverse waves in an isotropic dielectric (see Section 3.2).

An electron bunch (or beam) interacting with radiation undergoes density modulation, which leads to the appearance of an additional longitudinal field. If the radiation wavelength is small compared with the bunch size, the self-consistent changes in the longitudinal field and in density perturbations result in the formation of a plasma (Langmuir) charge density wave, the frequency of which can be found from a dispersion equation of the form [7, 32]

$$(\omega - k_z u)^2 - \Omega_\mathrm{b}^2 = 0\,, \tag{4.2.1}$$

where Ω_b^2 is defined by formulas (3.2.27). Quantity Ω_b represents the frequency of Langmuir beam oscillations in the moving coordinate system. Thus, neglecting the contribution from Ω_b^2 in dispersion equation (3.2.29), we thereby neglected natural oscillations of the beam, which is justified only when inequality (3.2.28) is satisfied. Evidently, this inequality is brought to the condition

$$|\delta\omega| \gg \Omega_\mathrm{b}\,, \tag{4.2.2}$$

where $\delta\omega$ is the growth rate of instability in stimulated Cherenkov emission.

Excitation of longitudinal waves in an electron beam (bunch) occurs in all forms of the Cherenkov emission examined earlier, not only in the emission of transverse waves in a dielectric. The fact is that the appearance of an additional longitudinal field of the beam against the background of a purely transverse radiation field is simply more noticeable in a dielectric than in another medium. In any case, if inequality $\omega_e \ll \omega$ is satisfied (ω is the characteristic radiation frequency), inequality (4.2.2) is also satisfied and then Langmuir beam oscillations may be neglected. Herein, beam instability is called the single-particle stimulated Cherenkov effect [1, 2].

The situation is different, for instance, in the case of emission by a beam propagating through a vacuum channel made in a dielectric. Because the transverse electromagnetic field of the dielectric undergoes exponential decay in a vacuum, the first term on the right-hand side of the second equation in system (3.2.25) acquires the 'screening' factor

$$\chi_1 \sim \exp\left[-\frac{\omega}{u\gamma}(r_c - r_e)\right], \tag{4.2.3}$$

where r_c is the radius of the vacuum channel. On the other hand, the beam current induced outside the dielectric less efficiently excites the transverse field inside it. As a result, the right-hand side of the first equation in system (3.2.25) admits the appearance of the factor $\chi_2 \sim \chi_1$. In the end, the right-hand side of dispersion equation (3.2.29) acquires the multiplier $\chi^2 = \chi_1\chi_2$ which can be significantly smaller than unity.

It is more convenient to use generalized dispersion equation (3.5.1) instead of equation (3.2.29) and, taking into account Langmuir beam waves as well as the weakening of beam – radiation interaction, write it down in the form [7]

$$D(\omega, k_z) = \tilde{G}(\omega, k_z)\,\frac{\omega_e^2\gamma^{-3}}{(\omega - k_z u)^2 - \Omega_b^2}\,, \tag{4.2.4}$$

where $\tilde{G}(\omega, k_z) = \chi^2 G(\omega, k_z)$ is a certain new form factor. It is easy to see that if inequality (4.2.2) is satisfied, the solution to equation (4.2.4) coincides with formula (3.5.13) (bearing in mind the alteration in the form factor notation).

Suppose now that the inequality opposite to Eqn (4.2.2) is satisfied and, besides, $\omega_e \ll \omega$. As $\omega_e \to 0$, equation (4.2.4) breaks down into two:

$$D(\omega, k_z) = 0\,, \qquad \omega - k_z u \mp \Omega_b = 0\,, \tag{4.2.5}$$

from which the resonance frequencies ω_0 are derived. Unlike system (4.1.6), equations (4.2.5) have two signs. We discuss here only positive solutions of equations (4.2.5), i.e., $\omega_0 > 0$. Representing the solution in the form [see expressions (4.1.9)]

$$\omega = \omega_0 + \delta\omega = k_{z0}u \pm \Omega_b + \delta\omega \tag{4.2.6}$$

and taking into account the inverse of inequality (4.2.2), we obtain from Eqn (4.2.4) the following relationship for the correction $\delta\omega$ to the resonance frequency:

$$(\delta\omega)^2 = \pm\frac{\omega_e^2\gamma^{-3}}{2\Omega_b}\,\tilde{G}_0\left(\frac{\partial D_0}{\partial\omega}\right)^{-1}. \tag{4.2.7}$$

As follows from independent physical considerations, $\partial D_0/\partial\omega > 0$ (e.g., in the case of transverse waves in a dielectric, one has $\partial D_0/\partial\omega = 2\varepsilon_0\omega_0$). According to formula (4.2.7), instability only develops when the lower minus sign is taken in Eqn (4.2.7), to which the lower signs in Eqns (4.2.5) and (4.2.6) correspond. Thus, the following relation is valid in the case of developing instability:

$$\omega_0 = k_{z0}u - \Omega_b \to \frac{\omega_0}{k_{z0}} < u\,. \tag{4.2.8}$$

In other words, the beam-excited wave belongs to the region of the anomalous Doppler effect [33]. The dispersion curves for equation (4.2.4) are presented in Fig. 17 which can be compared with Fig. 14. Substitution of formula (4.2.7) into the inequality opposite to Eqn (4.2.2) leads to the condition

$$\Omega_b^3 \gg \omega_e^2\gamma^{-3}\tilde{G}_0\left(\frac{\partial D_0}{\partial\omega}\right)^{-1}. \tag{4.2.9}$$

It can be seen from expressions (3.2.27) that $\Omega_b \sim \omega_e$; therefore, condition (4.2.9) means that the electron beam must have a high density.

Instability whose growth rate is found from relationship (4.2.7) and which is analogous to the anomalous Doppler effect is called the collective stimulated Cherenkov effect. Such an instability occurs only in high-density electron beams during their superluminal motion in a medium.

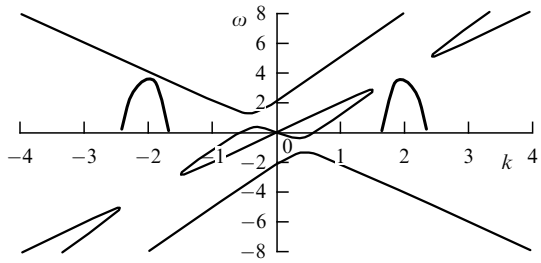

Figure 17. Dispersion curves for equation (4.2.4).

5. Conclusions

To summarize, we have considered two fundamental mechanisms of superluminal electron emission, the Vavilov – Cherenkov effect and the anomalous Doppler effect. Cherenkov emission by a single electron or by a small-sized electron bunch is spontaneous. During spontaneous Cherenkov emission, the translational motion of an electron is slowed down, and the radiation energy grows linearly with time. For a larger number of radiating electrons, the Cherenkov radiation becomes stimulated. Stimulated Cherenkov emission represents a resonance instability. Such an emission process is accompanied by longitudinal electron grouping in a bunch (or a beam), in which case the radiation energy grows exponentially with time.

In terms of a longitudinal size L_e of the bunch, there is a transition region $\lambda < L_e < \lambda\delta_0^{-1}$ between spontaneous and stimulated Cherenkov effects, where λ is the average radiation wavelength, and δ_0 is the growth rate of Cherenkov beam instability. The range to the left of this region is dominated by spontaneous emission, and the range to the right of it by stimulated emission. The first experimental study [35] was concerned with spontaneous Cherenkov emission. Numerous experimental data on stimulated Cherenkov emission available to date are in excellent agreement with the theory. Specifically, stimulated Cherenkov emission in a plasma

forms the basis for the existing relativistic plasma emitters developed in plasma microwave electronics [7, 10, 36, 37].

In contrast to Vavilov – Cherenkov effect, the anomalous Doppler effect should always (even for a single electron) be considered as stimulated because it can be explained only by accounting for the reverse action of radiation field on electron motion. During stimulated emission in the conditions where the anomalous Doppler effect occurs, the longitudinal motion of an electron is slowed down, it undergoes rotation, while the radiation energy grows exponentially with time. We are unaware of experimental studies on the anomalous Doppler effect proper. It certainly occurs when electrons are injected into a magnetoactive plasma but goes unnoticed against the background of more intense Cherenkov emission. Therefore, it appears appropriate to design experiments on plasma microwave electronics in order to obtain deeper insight into radiation spectra (Cherenkov and anomalous Doppler emissions occur at different frequencies), especially in conditions where Cherenkov emission is impossible. It is hoped that the present review will capture the interest of researchers, especially experimenters, in the fundamental problems of radiative physics.

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