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Magnetic order and spin fluctuations in low-dimensional insulating systems

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Contents

Abstract. The current theoretical and experimental situations are reviewed for low-dimensional insulating systems with a low magnetic transition temperature T_M and pronounced short-range magnetic order above this temperature. Both the standard and self-consistent spin-wave theories are shown to be insufficient to quantitatively describe the experimental data on these systems. Field-theoretical approaches taking into account the contribution of the spin-fluctuation excitations (neglected in the spin-wave theories) to the thermodynamic properties of ferro- and antiferromagnets are discussed.

1. Introduction

The investigation of low-dimensional magnetism is an important problem of modern solid-state physics. The experimental interest to this problem is related to the unusual magnetic properties of layered perovskites such as Rb_2MnF_4, K_2NiF_4 [1, 2], K_2MnF_4 [2, 3] (easy-axis anisotropy), K_2CuF_4, $NiCl_2$, $BaNi_2(PO_4)_2$ [4] (easy-plane anisotropy), organic compounds [5, 6], ferromagnetic films, multilayers, and surfaces [7, 8]. In recent years, interest to low-dimensional compounds increased because of investigations of the magnetic properties of copper – oxygen planes in high-temperature superconductors such as La_2CuO_4 [9].

Another interesting class of low-dimensional magnetic systems with local moments comprises quasi-one-dimensional compounds containing chains of magnetic atoms with a weak interchain magnetic exchange. In particular, such experimentally well-studied compounds as $KCuF_3$ [10] and a number of recently studied systems on the basis of strontium, e.g., Sr_2CuO_3 ($S = 1/2$) [11, 12], and cesium, e.g., $CsNiCl_3$ ($S = 1$) [13] and $CsVCl_3$ ($S = 3/2$) [14] can be referred to this class. A related class of compounds is represented by recently synthesized systems with 'spin ladders' consisting of a limited number of chains of magnetic atoms coupled via exchange interaction [15].

Contrary to the three-dimensional systems, the possibility of magnetic ordering in low-dimensional systems is significantly restricted because of the strong fluctuations in the magnetic-order parameter. As is well known, the magnetic order in purely one-dimensional and two-dimensional isotropic systems is absent at finite temperatures. According to the Mermin – Wagner theorem, two-dimensional isotropic ferro- and antiferromagnets possess long-range order in the ground state only, and the exact results for one-dimensional isotropic antiferromagnets indicate the absence of long-range magnetic order even at $T = 0$. Real compounds exhibit a finite magnetic-transition temperature $T_M \ll |J|$ (where J is the magnitude of the exchange interaction in the chains or in the plane) which is caused by a weak interchain (interplane)

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exchange and (or) anisotropy. The smallness of the transition temperature gives rise to a number of specific features of these systems. In particular, the short-range magnetic order at temperatures above the magnetic-transition point is not completely destroyed (in the two-dimensional case it is retained at temperatures up to $T \sim |J|$), so that there exists a wide temperature range $T > T_{\mathrm{M}}$ with a strong short-range order [3, 9].

Substantial progress in the understanding of the properties of the ground state and the thermodynamics of one-dimensional and two-dimensional structures has been achieved owing to the use of numerical methods (for instance, the quantum Monte Carlo method). At the same time, these methods do not supersede analytical approaches which permit one to describe thermodynamic properties of layered systems in a wide temperature range and are useful for both the theoretical understanding of the physical properties of these systems (which are by no means always obvious from numerical calculations) and for the practical purposes of describing real compounds.

The simplest method for analytically investigating low-dimensional magnets is based on the standard spin-wave theory [16 – 18], which is applicable only at low temperatures $T \ll T_{\mathrm{M}}$. This theory neglects the interaction between spin waves, which leads, in particular, to a strong overestimation of the phase-transition temperatures for low-dimensional compounds. The problem of interaction of spin waves in ferromagnets was investigated for the first time by Dyson [17], who constructed a consistent theory of the thermodynamic properties of these systems at low temperatures. Later on, these results were reproduced by Maleev [18] who proposed a nonlinear boson representation of spin operators. The Dyson – Maleev formalism consistently takes into account the dynamic interaction of spin waves. This approach was applied to the description of three-dimensional [19] and two-dimensional antiferromagnets; special attention in papers [19, 20] was paid to the calculation of spin-wave damping which proved to be small within a wide region of momentum space at sufficiently low temperatures. In papers [21], nonanalytical corrections to the spectrum of spin waves and heat capacity of low-dimensional systems, arising as a result of dynamic interactions of magnons, were investigated.

At temperatures that are not small compared to the magnetic transition temperature, the kinematic interaction of spin waves, which arises due to constraints imposed on the number of bosons at a site, begins to play a substantial role. A representation that permits one to explicitly take into account the kinematic interaction of spin waves was suggested by Bar'yakhtar et al. [22, 23]. The introduction of auxiliary fermions in this representation removes the necessity of considering an additional constraint on the number of bosons at a site.

However, the spin-wave picture of excitations at not-too-low temperatures becomes completely inadequate and non-spin-wave excitations should be taken into account to correctly describe the thermodynamics of the system. To some extent, this situation is similar to that in the theory of itinerant-electron magnetism, where the Stoner (mean-field) theory proved to be inconvenient to adequately describe thermodynamic properties, which stimulated the development of spin-fluctuation theories [24]. Whereas the effects of non-spin-wave excitations on the thermodynamic properties of localized magnets were discussed many years ago in the context of a phenomenological theory [25, 26], the corresponding microscopic approach was developed only recently on the basis of the so-called $1/N$-expansion [27], where N is the number of spin components ($N = 3$ for the Heisenberg model). This approach proved to be very successful in the description of the thermodynamic properties of two-dimensional [27] and quasi-two-dimensional [28] ferro- and antiferromagnets.

Unlike two-dimensional magnets, the spectrum of excitations in one-dimensional antiferromagnets strongly depends on the spin S. Starting from the works of H Bethe, who constructed an exact wave function (the Bethe ansatz) for the one-dimensional antiferromagnetic chain, it is known that these systems do not possess long-range magnetic order even in the ground state.

Modern theoretical approaches to one-dimensional systems are based on Haldane's idea [29, 30] which reduces the problem of a chain to a nonlinear sigma model. According to Haldane, the cases of an integer and half-integer spin differ qualitatively. In the case of a half-integer spin, the so-called topological term appears in the effective action, which leads to an unusual magnetic behavior in such chains.

For a single chain with $S = 1/2$ (the same situation takes place at any half-integer spin), the ground state exhibits a 'quasi-long-range order' at which the spin correlations at long distances fall off according to power rather than exponential law. The spectrum of excitations in this case is gapless, although the magnetization is equal to zero (which resembles the classical two-dimensional xy model) at temperatures below the Berezinskii – Kosterlitz – Thouless point T_{BKT}. At the same time, the spectrum of excitations for integer values of spin S contains the so-called Haldane gap on the order of $\exp(-\pi S)$, and the structure of the excitation spectrum is close to that predicted by the spin-wave theory (SWT).

Because of their 'exotic' behavior, chains with a half-integer spin cannot be investigated in the context of the SWT; to consider them, fundamentally new physical approaches are required. For the limiting quantum case of $S = 1/2$ (which is also the most important from the practical viewpoint), there was developed a bosonization method which uses the Jordan – Wigner representation of spin operators through fermion operators with a subsequent transformation to boson operators that describe collective (non-spin-wave) magnetic excitations. This approach also proved to be successful in the investigation of spin ladders [15, 31, 32].

To investigate quasi-one-dimensional systems, techniques combining bosonization (and/or the Bethe ansatz) with the renormalization-group method [33 – 35], as well as the interchain mean-field approach [36], have been developed. These methods predict a finite value of the magnetic-transition temperature $T_{\mathrm{N}} \sim |J'|$ at any infinitely small magnitude of the interchain interaction J'. Whereas the renormalization-group method does not permit one to obtain quantitative estimates of T_{N}, the interchain mean-field approximation neglects spin correlations at different chains, which leads to a strong overestimation of the Néel temperatures as compared to their experimental values. Thus, the theory of interchain mean field leads to the same difficulties in describing quasi-one-dimensional magnets as the SWT does in describing quasi-two-dimensional magnets.

We see that substantially new approaches are required for the description of quasi-two-dimensional and quasi-one-dimensional magnets, and their consideration constitutes the chief aim of this review.

2. Quasi-two-dimensional magnets with an easy-axis anisotropy

To consider quasi-one-dimensional and quasi-two-dimensional magnets with localized magnetic moments, we avail ourselves of the Heisenberg model

$$H = -\frac{J}{2} \sum_{i,\delta=(\delta_\perp,\delta_{||})} \mathbf{S}_i \mathbf{S}_{i+\delta} + H_{3\mathrm{D}} + H_{\mathrm{anis}}\,, \tag{2.1}$$

$$H_{3\mathrm{D}} = -\frac{J'}{2} \sum_{i,\delta_\perp} \mathbf{S}_i \mathbf{S}_{i+\delta_\perp}\,,$$

$$H_{\mathrm{anis}} = -\frac{J\eta}{2} \sum_{i,\delta_{||}} S_i^z S_{i+\delta_{||}}^z - |J|\zeta \sum_i (S_i^z)^2\,, \tag{2.2}$$

where J is the in-plane exchange integral ($J > 0$ for ferromagnets, and $J < 0$ for antiferromagnets); $H_{3\mathrm{D}}$ corresponds to the Hamiltonian of the interchain (interplane) interaction; $J' = 2\alpha J$ is the exchange parameter between the chains (planes) (for definiteness, we consider the case of $\alpha > 0$); $\delta_{||}$ and $\delta_\perp$ correspond to the nearest neighbors within a single chain (plane) and for different chains (planes), respectively; H_{anis} is the anisotropic part of the interaction, which arises as a result of the effect of the crystalline field of surrounding ions, and η and $\zeta > 0$ are the exchange and single-ion anisotropy parameters, respectively.

2.1 Nonlinear boson representations in the theory of quasi-two-dimensional ferromagnets and antiferromagnets

At very low temperatures $T \ll T_{\mathrm{M}}$, the elementary excitations are spin waves. To describe these excitations, it is suitable to pass from spin to boson operators. Various representations of such type are used, in particular, a Dyson – Maleev representation [17, 18, 23] which is convenient for the description of a magnetically ordered phase:

$$S_i^+ = \sqrt{2S} b_i\,, \quad S_i^z = S - b_i^\dagger b_i\,, \tag{2.3}$$

$$S_i^- = \sqrt{2S}\left(b_i^\dagger - \frac{1}{2S} b_i^\dagger b_i^\dagger b_i\right),$$

where $b_i^\dagger$ and b_i are the magnon Bose operators. The boson operators in representation (2.3) should satisfy the constraint on the occupation number at a site: $b_i^\dagger b_i < 2S$, which leads to the so-called kinematic interaction of spin waves. In order to simplify the consideration of this interaction, Bar'yakhtar et al. [22, 23] introduced the following representation:

$$S_i^+ = \sqrt{2S} b_i\,, \quad S_i^z = S - b_i^\dagger b_i - (2S+1)\, c_i^\dagger c_i\,, \tag{2.4}$$

$$S_i^- = \sqrt{2S}\left(b_i^\dagger - \frac{1}{2S} b_i^\dagger b_i^\dagger b_i\right) - \frac{2\,(2S+1)}{\sqrt{2S}}\, b_i^\dagger c_i^\dagger c_i\,,$$

which contains, apart from the boson operators, auxiliary pseudofermion operators $c_i^\dagger$ and c_i, which take into account the kinematic interaction of spin waves. In the case of an antiferromagnet with two sublattices, representation (2.4) and a conjugate representation are used for each sublattice. The excitation energy of pseudofermions is on the order of $|J|$, so that at low temperatures their contribution to the thermodynamic quantities is exponentially small and can be neglected. At the same time, the kinematic interaction of spin waves becomes substantial at temperatures $T \sim |J|$.

Another useful representation of spin operators is a representation in terms of Schwinger bosons [37 – 39]:

$$\mathbf{S}_i = \frac{1}{2} \sum_{\sigma\sigma'} s_{i\sigma}^\dagger \boldsymbol{\sigma}_{\sigma\sigma'} s_{i\sigma'}\,, \tag{2.5}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, and $\sigma, \sigma' = \uparrow, \downarrow$, so that

$$S_i^z = \frac{1}{2}\left(s_{i\uparrow}^\dagger s_{i\uparrow} - s_{i\downarrow}^\dagger s_{i\downarrow}\right), \; S_i^+ = s_{i\uparrow}^\dagger s_{i\downarrow}\,, \; S_i^- = s_{i\downarrow}^\dagger s_{i\uparrow}\,. \tag{2.6}$$

The condition

$$s_{i\uparrow}^\dagger s_{i\uparrow} + s_{i\downarrow}^\dagger s_{i\downarrow} = 2S \tag{2.7}$$

restricts the number of spin states and should be fulfilled at each lattice site. Since the simultaneous change in the phases of the $s_{i\uparrow}$ and $s_{i\downarrow}$ bosons, $s_{i\sigma} \to s_{i\sigma} \exp(\mathrm{i}\phi_i)$, does not affect the physical results, the Schwinger-boson representation possesses gauge symmetry. This fact can be used for establishing the relation between the Schwinger-boson representation and the well-known Holstein – Primakoff representation [40]. To do this, we fix the gauge by assuming a Hermitian character of one of the operators $s_{i\sigma}$, for example, $s_{i\uparrow}$; then, we have from formula (2.7) that

$$s_{i\uparrow} = \sqrt{2S - s_{i\downarrow}^\dagger s_{i\downarrow}}\,. \tag{2.8}$$

Substituting Eqn (2.8) into Eqn (2.6), we obtain the Holstein – Primakoff representation. Thus, the Schwinger-boson representation and the Holstein – Primakoff representation are equivalent. This equivalence can, however, be violated in approximate approaches. Unlike the Holstein – Primakoff (or Dyson – Maleev) representation, the Schwinger-boson representation can easily be extended to an arbitrary number of boson species ($N \geqslant 2$), which leads to a model with $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ symmetry and makes it possible to construct a $1/N$-expansion [38]. At the same time, there exists no natural way to introduce Fermi operators into this representation to take into account the kinematic interaction. As in the Dyson – Maleev representation, in the case of an antiferromagnet we should pass to a local coordinate system using the substitutions [39]

$$s_{i\uparrow} \to -s_{i\downarrow}\,, \quad s_{i\downarrow} \to s_{i\uparrow}$$

in one of the two sublattices.

2.2 Self-consistent spin-wave theory of quasi-two-dimensional magnets

The interaction of magnons in the lowest (Born) approximation is considered in the so-called self-consistent spin-wave theory (SSWT). Many years ago this theory was first applied to the three-dimensional Heisenberg model [41]; the same results were later obtained in the context of the variational approach for the isotropic [42] and anisotropic [43] Heisenberg models. Similar ideas have recently been used for the description of two-dimensional magnets in the 'mean-field' theory for boson operators [38, 39, 44], based on the representation of spin operators through Schwinger bosons, and in the 'modified spin-wave theory' [45] based on the Dyson – Maleev representation. The results of these two theories agree well with those of the renormalization-group calculations [46, 47] and with available experimental data for the excitation spectrum of low-dimensional systems [4]. The SSWT was also applied to quasi-two-dimensional [48 – 51],

frustrated two-dimensional [52–56], and three-dimensional [53] antiferromagnets.

To derive the SSWT equations, we use the Dyson–Maleev representation (2.3). After substituting the representation of spin operators through boson operators into the Hamiltonian, it contains the terms of second and fourth power of boson operators. Whereas the contributions from the quadratic terms describe the propagation of free spin waves, the fourth-power terms are responsible for their interaction. Taking into account the interaction of spin waves in the lowest approximation, i.e., decoupling the fourth-power forms of boson operators using the Wick theorem, we arrive at a quadratic SSWT Hamiltonian

$$H = \sum_{i,\delta} J_\delta \gamma_\delta \left(b_i^\dagger b_i - b_{i+\delta}^\dagger b_i\right) - \mu \sum_i b_i^\dagger b_i \,, \tag{2.9}$$

where

$$\gamma_{\delta_\perp} = \gamma = \bar{S} + \langle b_i^\dagger b_{i+\delta_\perp}\rangle \,, \quad \gamma_{\delta_\parallel} = \gamma' = \bar{S} + \langle b_i^\dagger b_{i+\delta_\parallel}\rangle \tag{2.10}$$

are the short-range order parameters satisfying the equations

$$\gamma = \bar{S} + \sum_{\mathbf{k}} N_{\mathbf{k}} \cos k_x \,, \quad \gamma' = \bar{S} + \sum_{\mathbf{k}} N_{\mathbf{k}} \cos k_z \,. \tag{2.11}$$

To extend the theory to the case of a disordered phase, we introduced the chemical potential μ of bosons into Eqn (2.9); this makes it possible to satisfy the condition for a total number of bosons for $T > T_{\mathrm{C}}$ [39, 45] (for $T < T_{\mathrm{C}}$, we have $\mu = 0$, so that the number of bosons is not limited). The calculation of spin correlation functions shows [45] that the chemical potential directly determines the correlation length ξ_δ in the direction δ according to the relationship

$$\xi_\delta^{-1} = \sqrt{-\frac{\mu}{|J_\delta \gamma_\delta|}} \,. \tag{2.12}$$

The quantities γ and γ' define spin correlation functions at neighboring sites:

$$\left|\langle \mathbf{S}_i \mathbf{S}_{i+\delta}\rangle\right| = \gamma_\delta^2 \,. \tag{2.13}$$

The magnetization of a ferromagnet is determined by the total number of bosons:

$$\bar{S} = S - \sum_{\mathbf{k}} N_{\mathbf{k}} \,, \tag{2.14}$$

where $N_{\mathbf{k}} = N(E_{\mathbf{k}})$ is the Bose function, with the spin-wave spectrum having the form

$$E_{\mathbf{k}}^{\mathrm{SSWT}} = \Gamma_{\mathbf{k}=0} - \Gamma_{\mathbf{k}} + \Delta - \mu \,, \tag{2.15}$$

$$\Gamma_{\mathbf{k}} = 2S\left[\gamma\,|J|(\cos k_x + \cos k_y) + \gamma'|J'|\cos k_z\right] ,$$

$$\Delta = |J|\left[(2S-1)\,\zeta + \frac{4\eta S^2}{\gamma}\right]\left(\frac{\bar{S}}{S}\right)^2 .$$

For an antiferromagnet, the SSWT equations take the form [50, 51]

$$\gamma = \bar{S} + \sum_{\mathbf{k}} \frac{\Gamma_{\mathbf{k}}}{2E_{\mathbf{k}}} \cos k_x \coth\frac{E_{\mathbf{k}}}{2T} \,, \tag{2.16}$$

$$\gamma' = \bar{S} + \sum_{\mathbf{k}} \frac{\Gamma_{\mathbf{k}}}{2E_{\mathbf{k}}} \cos k_z \coth\frac{E_{\mathbf{k}}}{2T} \,,$$

$$\bar{S} = S + \frac{1}{2} - \sum_{\mathbf{k}} \frac{\Gamma_{\mathbf{k}=0} + \Delta - \mu}{2E_{\mathbf{k}}} \coth\frac{E_{\mathbf{k}}}{2T} \,,$$

where

$$\gamma = \bar{S} + \langle a_i b_{i+\delta_\perp}\rangle, \quad \gamma' = \bar{S} + \langle a_i b_{i+\delta_\parallel}\rangle, \tag{2.17}$$

a_i and b_i are the operators of the Dyson–Maleev representation in each of the sublattices, and the dispersion of spin waves is written as follows:

$$E_{\mathbf{k}}^{\mathrm{SSWT}} = \left[(\Gamma_{\mathbf{k}=0} + \Delta - \mu)^2 - \Gamma_{\mathbf{k}}^2\right]^{1/2} . \tag{2.18}$$

As in the case of ferromagnets, the chemical potential μ of bosons, which is nonzero at temperatures above the magnetic-transition point, determines the correlation length according to Eqn (2.12).

In the ground state of a ferromagnet, we have $\bar{S}_0 = S$ and $\gamma_0 = \gamma\,(T=0) = 1$; however, the sublattice magnetizations and the short-range order parameter of a two-dimensional antiferromagnet differ from these values because of the quantum zero-point spin fluctuations:

$$\bar{S}_0 = S - \frac{1}{2}\sum_{\mathbf{k}}\left[\frac{1}{\sqrt{1-\phi_{\mathbf{k}}^2}} - 1\right] \approx S - 0.1966 \,, \tag{2.19}$$

$$\gamma_0 = 1 + \frac{1}{2S}\sum_{\mathbf{k}}\left[1 - \sqrt{1-\phi_{\mathbf{k}}^2}\right] \approx 1 + \frac{0.0790}{S} \,, \tag{2.20}$$

where $\phi_{\mathbf{k}} = (\cos k_x + \cos k_y)/2$. The sublattice magnetization at $S = 1/2$ is equal to 40% of the result of the ferromagnetic case and coincides with its magnitude in the SWT [16], while the renormalization of the in-plane exchange parameter reaches 15%. As in the standard SWT, the spectrum of spin waves in the ordered phase in the absence of anisotropy ($\Delta = 0$) is gapless and at small wave vectors $\mathbf{k}$ has the form $E_{\mathbf{k}} = Dk^2$ in ferromagnets (FMs), and $E_{\mathbf{k}} = ck$ in antiferromagnets (AFMs). Here, D is the spin-wave stiffness, and c is the spin-wave velocity. In the SSWT, these parameters are expressed through the parameters γ and S according to the relationship

$$D = JS \,, \quad c = \sqrt{8}\,|J|\,\gamma S \,. \tag{2.21}$$

The spin stiffnesses of ferromagnets and antiferromagnets, determined from an analysis of transverse susceptibility, are expressed as

$$\rho_{\mathrm{s}} = \begin{cases} JS^2\,(\mathrm{FM}) \,, \\ |J|\,\gamma S\bar{S}_0\,(\mathrm{AFM}) \,. \end{cases} \tag{2.22}$$

The renormalized (observed) parameters of interplane exchange and anisotropy, found from the excitation spectrum, are written down as

$$f_{\mathrm{r}} = \frac{\Delta}{\gamma|J|S} = \frac{1}{\gamma S}\left[(2S-1)\,\zeta + \frac{4\eta S}{\gamma}\right]\left(\frac{\bar{S}}{S}\right)^2 , \tag{2.23}$$

$$\alpha_{\mathrm{r}} = \frac{2\gamma'}{\gamma} = \frac{\alpha\bar{S}}{S} \,. \tag{2.24}$$

Note that, unlike the in-plane exchange parameter, the renormalized parameters α, η, and ζ are proportional to the magnetization and, therefore, exhibit a strong temperature dependence.

At finite temperatures in the absence of interplane exchange and anisotropy ($J' = 0$, $\Delta = 0$), no long-range

order exists in accordance with the Mermin–Wagner theorem, so that $\bar{S} = 0$, $\mu < 0$ (solutions with $\bar{S} \neq 0$ and $\mu = 0$ do not exist because of the divergence of the integrals in Eqns (2.11) and (2.16) for $T > 0$ and $\mu = 0$). The magnitude of the chemical potential μ of bosons is determined by Eqn (2.13) for a ferromagnet, and by the last of the equations (2.16) with $\bar{S} = 0$ for an antiferromagnet. At low temperatures ($T \ll |J|S^2$), the absolute value of the chemical potential is exponentially small, so that the correlation length $\xi = \sqrt{-|J\gamma|/\mu}$ is exponentially large (the so-called renormalized classical regime):

$$\xi = \begin{cases} C_\xi^{\mathrm{F}} \sqrt{\dfrac{J}{T}} \exp\left(\dfrac{2\pi\rho_{\mathrm{s}}}{T}\right) & (\mathrm{FM}), \quad (2.25) \\ C_\xi^{\mathrm{AF}} \left(\dfrac{J}{T}\right) \exp\left(\dfrac{2\pi\rho_{\mathrm{s}}}{T}\right) & (\mathrm{AFM}), \quad (2.26) \end{cases}$$

where C_ξ^{F} and C_ξ^{AF} are the spin-dependent constants. Relations (2.25) and (2.26) agree with the results of the one-loop renormalization-group (RG) approach [46, 47]. A two-loop RG analysis changes only the preexponential factor which in the AFM case becomes a temperature-independent constant [46], whereas in the FM case it is proportional to $(T/J)^{1/2}$ (see Ref. [47]). In the presence of interplane exchange and at not-too-high temperatures ($T < T_{\mathrm{M}}$; the temperature T_{M} of magnetic ordering will be calculated below), a long-range magnetic order occurs; in this case, Eqns (2.11) and (2.16) have solutions with $\bar{S} > 0$. Although the SSWT can only be substantiated at temperatures $T \ll T_{\mathrm{M}}$ (at which a pronounced long-range order exists and the spin-wave interaction is small), we extrapolate the SSWT results to higher temperatures ($T \sim T_{\mathrm{M}}$), which will make it possible to compare the SSWT results with those of more intricate theories, which will be considered in Section 3.

At low temperatures ($T \ll |J'|S$) and at arbitrary J'/J, the corrections to the magnetization of the ground state of a ferromagnet are proportional to $T^{3/2}$, whereas the short-range order parameters exhibit a weaker $T^{5/2}$-dependence; for antiferromagnets, the corresponding dependences are proportional to T^2 and T^4, respectively [51]. For $T > T_{\mathrm{M}}$, we again have $\bar{S} = 0$ and $\mu < 0$, as in the two-dimensional case at finite T.

To study numerically the temperature dependence of magnetization and short-range order parameters at not-too-small magnitudes of interplane exchange, it is convenient to apply the approximation of an effective short-range order parameter, using the substitution [51]

$$\sum_\delta J_{i,\,i+\delta}\, \gamma_\delta\, (b_i^\dagger b_i - b_i^\dagger b_{i+\delta}) \to \gamma_{\mathrm{eff}} \sum_\delta J_{i,\,i+\delta}\, (b_i^\dagger b_i - b_i^\dagger b_{i+\delta}). \quad (2.27)$$

The temperature dependences of magnetization and the short-range order parameter of a ferromagnet for different J'/J are shown in Figs 1–3. At small positive $T - T_{\mathrm{M}}$, we have $-\mu \propto (T - T_{\mathrm{M}})^2$ (see Fig. 3 for the ferromagnetic case; the same situation takes place in antiferromagnets), so that according to Eqn (2.12) the critical exponent for the correlation length is $\nu = 1$. Since the magnetization varies linearly near T_{M}, the critical exponent for the magnetization is $\beta = 1$. The effect of corrections of higher orders in $1/S$ on the magnitude of critical exponents is discussed in Sections 2.4 and 2.5. In the classical limit ($S \to \infty$), the SSWT equations are simplified and for $T < T_{\mathrm{M}}$ ($\mu = 0$) the averaged (over

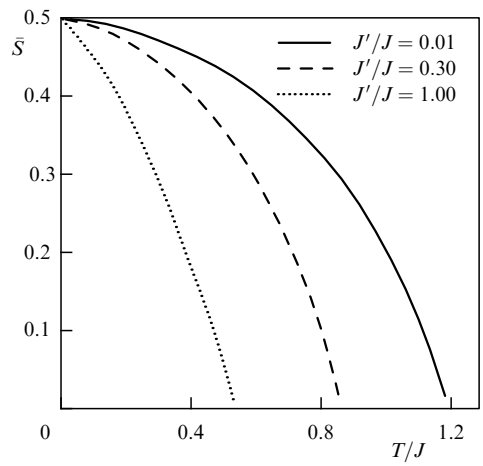

Figure 1. Temperature dependence of the magnetization of quasi-two-dimensional ferromagnets at different ratios between interplane and in-plane exchange integrals, J'/J ($S = 1/2$).

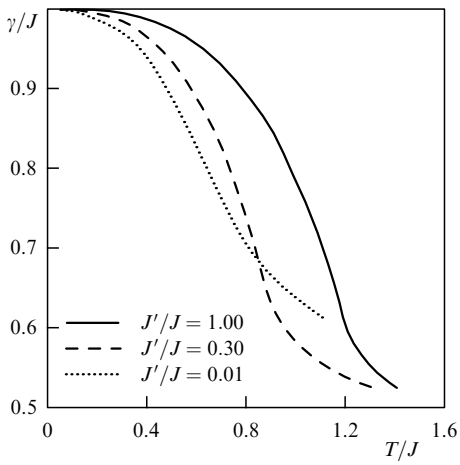

Figure 2. Temperature dependence of the short-range order parameter γ at the same values of parameters as in Fig. 1.

nearest neighbors) short-range order parameter

$$\gamma_{\mathrm{eff}}(T) = \frac{4J\gamma + 2J'\gamma'}{J_0} \quad (2.28)$$

(but not the magnetization!) satisfies the standard mean-field equation

$$\frac{\gamma_{\mathrm{eff}}}{S} = B_\infty\left(\frac{J_0\gamma_{\mathrm{eff}}S}{T}\right), \quad (2.29)$$

where $B_\infty(x) = \coth x - 1/x$ is the Langevin function (the Brillouin function in the classical limit). The temperature T^*, at which $\gamma_{\mathrm{eff}}(T^*) = 0$, proves to be higher than the magnetic-transition temperature T_{M}, so that $\gamma_{\mathrm{eff}}(T_{\mathrm{M}}) > 0$ and the behavior of γ_{eff} for $T > T_{\mathrm{M}}$ becomes more complex than that described by Eqn (2.29).

At small values of the interplane exchange ($J'/J \ll 1$) and of the anisotropy (η, $\zeta \ll 1$), analytical results for the temperature dependence of magnetization in a wide temperature range can be obtained [48, 51]. In this case, the SSWT leads to different results for magnetization in the so-called 'quantum' and 'classical' temperature regimes. It turns out that these regimes are not related directly to the cases of

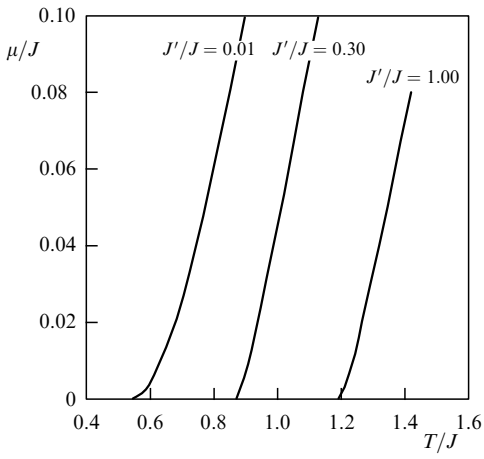

Figure 3. Temperature dependence of the gap in the spectrum of bosons at the same values of parameters as in Fig. 1.

quantum ($S \sim 1$) and classical ($S \gg 1$) spins (although the classical regime is realized only for $S \gg 1$), since the appropriate criteria also contain temperature (see below).

In the quantum regime, which takes place at not-too-low temperatures:

$$\begin{aligned} &J'S \ll T \ll JS \qquad \text{(FM)}\,, \\ &(JJ')^{1/2}S \ll T \ll |J|S \quad \text{(AFM)}, \end{aligned} \tag{2.30}$$

the magnetization (sublattice magnetization) is written out as follows:

$$\bar{S} = \begin{cases} S - \dfrac{T}{4\pi JS} \ln \dfrac{T}{J'\gamma' S} & \text{(FM)}, \\ \bar{S}_0 - \dfrac{T}{4\pi |J|\gamma S} \ln \dfrac{T^2}{8JJ'\gamma\gamma' S^2} & \text{(AFM)}\,. \end{cases} \tag{2.31}$$

The short-range order parameters are determined by the relations $\gamma \approx \gamma_0$ and

$$\gamma' = \begin{cases} S - \dfrac{T}{4\pi JS}\left(\ln \dfrac{T}{J'\gamma' S} - 1\right) & \text{(FM)}, \\ \bar{S}_0 - \dfrac{T}{4\pi |J|\gamma S}\left(\ln \dfrac{T^2}{8JJ'\gamma\gamma' S^2} - 1\right) & \text{(AFM)}\,, \end{cases} \tag{2.32}$$

so that $\gamma'_0 = \bar{S}_0$. Note that in the quantum regime (2.30) the integrals with respect to quasimomenta in the SSWT equations are determined by the contribution from quasimomenta q that are smaller than q_0, where

$$q_0 = \begin{cases} \left(\dfrac{T}{JS}\right)^{1/2} & \text{(FM)}, \\ \dfrac{T}{c} & \text{(AFM)}, \end{cases} \tag{2.33}$$

rather than by the quasimomenta in the entire Brillouin zone. For critical temperatures in the quantum regime (2.30), we obtain

$$T_{\mathrm{C}} = \frac{4\pi JS^2}{\ln\left(T/J'\gamma'_{\mathrm{c}} S\right)}\,, \tag{2.34}$$

$$T_{\mathrm{N}} = \frac{4\pi |J|\gamma_{\mathrm{c}}\bar{S}_0}{\ln\left(T^2/8JJ'\gamma_{\mathrm{c}}\gamma'_{\mathrm{c}} S^2\right)}\,,$$

where $\gamma_{\mathrm{c}} \approx \gamma_0$ and γ'_{c} are the renormalized exchange parameters for $T_{\mathrm{M}} = T_{\mathrm{C}}$ ($T_{\mathrm{M}} = T_{\mathrm{N}}$); the parameter γ'_{c} found from formulas (2.32) is written as

$$\gamma'_{\mathrm{c}} = \frac{T_{\mathrm{M}}}{4\pi} |J|\gamma_{\mathrm{c}} S^2 J'\,. \tag{2.35}$$

The renormalization of the interplane exchange interaction in formula (2.34) leads to a substantially lower value of the Curie (Néel) temperature as compared to its value in the SWT, since $\gamma_{\mathrm{c}}\gamma'_{\mathrm{c}}/JJ' = T_{\mathrm{M}}/4\pi JS^2 \ll 1$.

At large S (classical limit), we obtain for ferromagnets and antiferromagnets for $T \gg |J|S$ the following expressions

$$\bar{S} = S - \frac{T}{4\pi |J|S} \ln \frac{q_0^2 J}{J'\gamma'}\,, \tag{2.36}$$

$$\gamma' = S - \frac{T}{4\pi |J|S}\left(\ln \frac{q_0^2 J}{J'\gamma'} - 1\right). \tag{2.37}$$

In contrast to the results obtained in the quantum case, the expressions for the magnetization in this limit are nonuniversal, since they depend on the type of lattice through the cut-off parameter q_0^2 ($q_0^2 = 32$ for a square lattice). The corresponding expression for the critical temperature of a classical magnet with $1 \ll \ln\left(q_0^2 J/J'\right) \ll 2\pi S$ has the form

$$T_{\mathrm{M}} = \frac{4\pi |J| S^2}{\ln\left(q_0^2 J/J'\gamma'_{\mathrm{c}}\right)}\,, \tag{2.38}$$

where $\gamma'_{\mathrm{c}} = T_{\mathrm{M}}/4\pi |J|S$, and the critical temperature is the same for classical ferromagnets and antiferromagnets. In this case, the results obtained in the SWT, in which $\gamma'_{\mathrm{c}}/S \to 1$, are reproduced with a logarithmic accuracy.

As in the case of weak interplane exchange interaction, the separation of logarithmic contributions at small anisotropies leads to the following relations

$$\begin{aligned} &\bar{S} = S - \frac{T}{4\pi JS} \ln \frac{T}{\Delta} \qquad \text{(FM)}, \\ &\bar{S} = \bar{S}_0 - \frac{T}{4\pi |J|\gamma S} \ln \frac{T^2}{8\gamma JS\Delta} \qquad \text{(AFM)}. \end{aligned} \tag{2.39}$$

Unlike the quasi-two-dimensional case, a nonphysical result $\Delta(T_{\mathrm{M}}) = 0$ arises here, since the gap in the magnetic spectrum is proportional to $(\bar{S}/S)^2$ (in fact, the finite magnitude of the gap at $T = T_{\mathrm{M}}$ is related to topological excitations, namely, 'domain walls' which are not taken into account in the SWT). Thus, the SWT is unable to describe the temperature dependence of the gap $\Delta(T)$ near T_{M}. Introducing formally $\Delta_{\mathrm{c}} = \Delta(T_{\mathrm{M}})$, the critical temperature for $2\pi S \ll \ln(1/\Delta)$ becomes

$$\begin{aligned} &T_{\mathrm{C}} = \frac{4\pi JS^2}{\ln\left(T/\Delta_{\mathrm{c}}\right)}\,, \\ &T_{\mathrm{N}} = \frac{4\pi |J| S\bar{S}_0\gamma_{\mathrm{c}}}{\ln\left(T^2/8J\gamma_{\mathrm{c}} S\Delta_{\mathrm{c}}\right)}\,. \end{aligned} \tag{2.40}$$

In the limit of large S, we find for both ferromagnets and antiferromagnets:

$$\bar{S} = S - \frac{T}{4\pi |J| S} \ln \frac{|J| S q_0^2}{\Delta}\,. \tag{2.41}$$

The magnetization (2.41) vanishes at a temperature

$$T_{\mathrm{M}} = \frac{4\pi |J| S^2}{\ln\left(|J| S q_0^2/\Delta_{\mathrm{c}}\right)} \tag{2.42}$$

corresponding to the critical temperature for a classical magnet with an easy-axis anisotropy $\left[1 \ll \ln\left(|J|\, q_0^2/\Delta\right) \ll 2\pi S\right]$.

Equations (2.34) and (2.38) can be compared with the results of the Tyablikov approximation for the magnetic-transition temperature of layered compounds [28, 57]:

$$T_{\mathrm{M}} \approx \frac{4\pi\,|J|\,S^2}{\ln\left(|J|\,q_0^2/J'\right)} \tag{2.43}$$

at $q_0^2 = 32$. The result (2.43) is smaller than that obtained in the SSWT [see Eqn (2.34)] and, therefore, better describes the experimental data (see Section 2.6). However, Eqn (2.43) coincides with the result obtained using the spherical model (which is adequate only in the classical limit $S \to \infty$ [64]) and with the result obtained in the spin-wave approximation in the classical regime. This implies that the Tyablikov approximation near the critical temperature does not take into account quantum fluctuations which are important at small values of S. Thus, the Tyablikov approximation can be to some extent satisfactory from the practical but not from the theoretical viewpoint.

Although all the above-mentioned approaches lead (within logarithmic accuracy) to the same value of the Néel temperature, this accuracy is insufficient for a quantitative description of experimental data. In addition, spin-wave theories incorrectly describe critical behavior. Formally, the SSWT corresponds to the limit $N \to \infty$ in the $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ generalization of the Heisenberg model [38]. To improve the description of the critical region and to more exactly calculate the Curie (Néel) temperature, we should consider fluctuation corrections to the SWT results with a higher accuracy than in the SSWT. The calculation of first-order corrections in $1/N$ in the $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ model makes it possible to describe the region of low and intermediate temperatures $T \lesssim T_{\mathrm{M}}$, but cannot ensure a correct description of the critical behavior [58]. The restrictions of this approach in the critical region are related to the fact that excitations of a non-spin-wave character in the above-mentioned generalization of the Heisenberg model are represented through bound states of spin waves [59] and their consideration is quite difficult in terms of the $1/N$ expansion. Therefore, approaches which make it possible to describe both the region of intermediate temperatures and the critical region should be developed. Such approaches will be considered in Sections 2.3 – 2.5.

2.3 Field-theoretical description of quasi-two-dimensional magnets with localized moments

To correctly describe thermodynamic properties of magnets in a wide temperature range, we should perform summation of the leading contributions to the thermodynamic quantities in all orders of the perturbation theory in magnon – magnon interaction. In this case, the kinematic interaction of spin waves becomes important in a wide temperature range only for systems in which T_{M} is not small as compared to $|J|\,S^2$ (e.g., in three-dimensional systems). For layered systems, the kinematic interaction is not very important because of the smallness of the magnetic-transition temperature: $T_M \ll |J|\,S^2$ (in fact, this interaction only plays a role in a small critical vicinity of T_{M}).

At the same time, the correct treatment of dynamic magnon interaction is critically important. In summing diagrams that describe the effect of this interaction beyond the lowest-order perturbation theory, the RG analysis can be used, which has successfully been employed for the description of classical and quantum isotropic magnets in spaces with a dimension of $d = 2$ [46, 60] and $d = 2 + \varepsilon$ [61, 62]. In these cases, the excitation spectrum only weakly differs from the spin-wave one. Thus, at $d = 2 + \varepsilon$, the corrections to the spectrum of spin waves are $\delta E_{\mathbf{k}} \sim |J|\varepsilon \ln k$. In this case, the relationship $T_{\mathrm{M}}/\left(|J|S^2\right) \sim \varepsilon$ is valid for the magnetic-transition temperature and the standard technique of the ε-expansion can be applied [61, 62]. The corresponding results of the RG analysis coincide with the results obtained in terms of the $1/N$ expansion in the $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ generalization of the Heisenberg model [58].

In the case of quasi-two-dimensional magnets with a weak interplane exchange interaction and/or weak easy-axis type anisotropy, the magnetic-transition temperature is also small compared to $|J|\,S^2$, but the excitation spectrum can substantially differ from the spin-wave spectrum. In particular, three temperature regimes can be distinguished [63].

At low temperatures ($T \ll T_{\mathrm{M}}$, first regime), the SWT alone is applicable. At intermediate temperatures ($T \sim T_{\mathrm{M}}$, outside the critical region, second regime), the interaction of spin waves becomes substantial, but the spin fluctuations are of a two-dimensional isotropic character (for which reason this regime is called below the 'two-dimensional Heisenberg regime'); the RG method can be applied when describing magnetic properties in this regime. Finally, in a narrow critical range near T_{M} there occurs a change from the above-mentioned two-dimensional Heisenberg regime to a three-dimensional Heisenberg *critical* regime in which fluctuations of a non-spin-wave character are essential (third regime). In the presence of anisotropy, the fluctuations in the critical regime are due to topological excitations such as domain walls, i.e., they are 'two-dimensional Ising' excitations. In both cases, the picture of spin waves becomes completely inadequate in a sufficiently narrow critical region. Thus, this region should be considered with account for the non-spin-wave excitations.

To consider excitations of this type, it is suitable to use, instead of the initial Heisenberg model, the so-called $\mathrm{O}(N)/\mathrm{O}(N-1)$ model with a large number N of spin components, which makes it possible to formally introduce a small parameter into the theory [27]. As $N \to \infty$, this model is equivalent to the spherical model [64]; at finite N, it properly takes into account corrections related to spin – spin interactions, since it is not based on the spin-wave picture of the excitation spectrum. This circumstance leads to important advantages at temperatures comparable to the phase transition temperature, but meets some difficulties at low and intermediate temperatures, at which the excitations are of a purely spin-wave nature. Thus, the RG approach and $1/N$-expansion in the $\mathrm{O}(N)/\mathrm{O}(N-1)$ model have advantages in different temperature regions and mutually supplement one another. Whereas the first well describes the two-dimensional regime which is realized at intermediate temperatures, the $1/N$-expansion technique satisfactorily describes the critical region.

The application of the RG approach and the $1/N$-expansion implies a reformulation of the initial problem of calculating the thermodynamic properties of the model with Hamiltonian (2.1) in terms of a path integral [65, 66]. In this formalism, the model is characterized by the generating functional $Z[h]$ which represents the partition function of the system in an external nonuniform magnetic field h, so that the magnetization and the correlation functions are deter-

mined by the logarithmic derivatives of $Z[h]$ with respect to the field. The generating functional is derived on the basis of coherent states [65, 66]

$$|\mathbf{n}_i\rangle = \exp(-\mathrm{i}\varphi_i S_i^z)\exp(-\mathrm{i}\theta_i S_i^y)|0\rangle\,, \tag{2.44}$$

which are parametrized by vectors $\mathbf{n}_i$ with a unit length and polar coordinates (θ_i, φ_i) that are determined for each lattice site i, and $|0\rangle$ is the eigenstate of the operator S_i^z with a maximum spin projection: $S_i^z|0\rangle = S|0\rangle$. The advantage of states (2.44) lies in the fact that the average value of spin operators over them has a simple form

$$\langle\mathbf{n}_i|S_i^{\mathrm{m}}|\mathbf{n}_i\rangle = Sn_i^{\mathrm{m}}\,, \tag{2.45}$$

i.e., the coherent states correspond to 'quasiclassical' spin states. It can be shown that for coherent states (2.44) the generating functional can be written down as

$$Z = \int D\mathbf{n}\exp\left\{\int_0^{1/T}\mathrm{d}\tau\left[\sum_i \mathbf{A}(\mathbf{n}_i)\,\frac{\partial\mathbf{n}_i}{\partial\tau} - \langle\mathbf{n}\,|H|\,\mathbf{n}\rangle\right]\right\}, \tag{2.46}$$

where the first term in the exponent takes into account the dynamics of spins related to their quantum character, and the second term describes the spin interaction; integration in formula (2.46) is performed over the angular variables of the vector $\mathbf{n}_i$ at each site and for each imaginary time τ, and $\mathbf{A}(\mathbf{n})$ is the vector potential of a unit magnetic monopole which satisfies the relation $\mathbf{n}\,\mathrm{rot}\,\mathbf{A}(\mathbf{n}) = 1$. The contribution to the action with a time derivative corresponds to the Berry phase [29].

The average over coherent states of Hamiltonian (2.1) can easily be calculated using relationship (2.45), which leads to the following expression for generating functional of the model (2.1):

$$\begin{aligned} Z[h] = \int D\mathbf{n}D\lambda\,\exp\Bigg\{\frac{JS^2}{2}\int_0^{1/T}\mathrm{d}\tau\sum_{i,\delta}\Bigg[&\frac{2\mathrm{i}}{JS}\,\mathbf{A}(\mathbf{n}_i)\,\frac{\partial\mathbf{n}_i}{\partial\tau} \\ &+ \mathbf{n}_i\,\mathbf{n}_{i+\delta_\parallel} + \frac{\alpha}{2}\,\mathbf{n}_i\,\mathbf{n}_{i+\delta_\perp} + \eta\,n_i^z n_{i+\delta_\parallel}^z \\ &+ \tilde{\zeta}(n_i^z)^2 + hn_i^z + \mathrm{i}\lambda_i(\mathbf{n}_i^2 - 1)\Bigg]\Bigg\}, \end{aligned} \tag{2.47}$$

where $\tilde{\zeta} = 2\zeta(1 - 1/2S)$. Functional (2.47) is a generalization of the well-known generating functional of the two-dimensional isotropic Heisenberg model [66] to the presence of interplane exchange (the term proportional to α) and anisotropy (the terms proportional to η and $\tilde{\zeta}$). The last term under the summation sign is due to the restriction $\mathbf{n}_i^2 = 1$, and λ_i are corresponding Lagrange multipliers. Functional (2.47) contains two scales of length:

$$\xi_{J'} = \frac{a}{\max(\alpha, \tilde{\zeta}, \eta)^{1/2}} \gg a\,, \tag{2.48}$$

and

$$L_\tau = \begin{cases} a\sqrt{\dfrac{JS}{T}} & (\mathrm{FM})\,, \\ \dfrac{c}{T} & (\mathrm{AFM})\,, \end{cases} \tag{2.49}$$

where a is the lattice parameter. These scales have different physical meaning. The $\xi_{J'}$ scale corresponds to a change in the type of fluctuations: the two-dimensional isotropic fluctuations change to three-dimensional isotropic or two-dimensional anisotropic fluctuations, depending on which of the parameters dominates in the denominator of Eqn (2.48), anisotropy or interplane exchange. At the same time, the quantity L_τ determines the role of quantum spin fluctuations: $L_\tau \ll a$ corresponds to the classical fluctuations at which the spin dynamics can be neglected, and $L_\tau > a$ corresponds to the quantum fluctuations.

Generating functional (2.47) can now be reduced to a form suitable for concrete calculations; the result is determined by the temperature regime for which the calculations are performed. In the classical regime ($T \gg JS$), we have $L_\tau \ll a$. Neglecting the dynamics of the field $\mathbf{n}$ (i.e., its dependence on imaginary time) yields a functional

$$\begin{aligned} Z_{\mathrm{cl}}[h] = \int D\mathbf{n}D\lambda\,\exp\Bigg\{\frac{\rho_{\mathrm{s}}^0}{2T}\sum_i\Bigg[&\mathbf{n}_i\,\mathbf{n}_{i+\delta_\parallel} + \frac{\alpha}{2}\,\mathbf{n}_i\,\mathbf{n}_{i+\delta_\perp} \\ &+ \eta n_i^z n_{i+\delta_\parallel}^z + \tilde{\zeta}\,(n_i^z)^2 + hn_i^z + \mathrm{i}\lambda_i(\mathbf{n}_i^2 - 1)\Bigg]\Bigg\} \end{aligned} \tag{2.50}$$

with a 'bare' spin stiffness $\rho_{\mathrm{s}}^0 = |J|S^2$. To derive expression (2.50) for the antiferromagnetic case, we should make substitutions $\mathbf{n}_i \to -\mathbf{n}_i$, and $\lambda_i \to -\lambda_i$ for one of the two sublattices. In the classical case, the results for Z are identical for ferromagnets and antiferromagnets. In the continuum limit, action (2.50) coincides with the action for the classical nonlinear sigma model [66]. However, the continuum limit cannot be used in a wide temperature range (not only in the critical region), since not only long-wavelength but also short-wavelength excitations contribute to the thermodynamic properties.

In the quantum case, we can pass to the continuum limit for each layer in view of the condition $\xi_{J'} \gg L_\tau$. For a ferromagnet, it is convenient to use the representation [65, 66]

$$\mathbf{A}(\mathbf{n}) = \frac{\mathbf{z}\times\mathbf{n}}{1 + (\mathbf{zn})}\,, \tag{2.51}$$

where $\mathbf{z}$ is the unit vector along the z-axis, and to introduce a two-component vector field $\pi = \mathbf{n} - (\mathbf{nz})\,\mathbf{z}$ describing fluctuations of the order parameter. For a quantum antiferromagnet, we should apply the Haldane procedure [29] (see also Ref. [66]) which permits integration over 'fast' components of the fields $\mathbf{n}_i$. To do this, the vector $\mathbf{n}_i$ is represented in the form $\mathbf{n}_i = \mathbf{L}_i + (-1)^i\,\boldsymbol{\sigma}_i$, where $\mathbf{L}_i$ and $\boldsymbol{\sigma}_i$ are uniform ('fast') and sublattice ('slow') components of the vector, with $\mathbf{L}_i\boldsymbol{\sigma}_i = 0$. To separate the 'fast' and 'slow' components at temperatures below the transition point, we use the parameter $\xi_{J'} \gg a$ instead of the standard correlation length equal to infinity. Using this procedure, we obtain the following generating functional of the quantum nonlinear sigma model:

$$\begin{aligned} Z_{\mathrm{AF}}[h] = \int D\sigma D\lambda\,\exp\Bigg\{-\frac{\rho_{\mathrm{s}}^0}{2}\int_0^{1/T}\mathrm{d}\tau \\ \times\int\mathrm{d}^2\mathbf{r}\sum_{i_z}\Bigg[\frac{1}{c_0^2}\,(\partial_\tau\sigma_{i_z})^2 + (\nabla\sigma_{i_z})^2 + \frac{\alpha}{2}\,(\sigma_{i_z+1} - \sigma_{i_z})^2 \\ - f(\sigma_{i_z}^z)^2 + h\sigma_{i_z}^z + \mathrm{i}\lambda(\sigma_{i_z}^2 - 1)\Bigg]\Bigg\}, \end{aligned} \tag{2.52}$$

where σ_{i_z} is the three-component field of unit length, i_z is the order number of the layer, and $c_0 = \sqrt{8}JS$ is the bare spin-wave velocity. Model (2.52) is a generalization of the quantum nonlinear sigma model for the case where inter-

plane exchange and anisotropy are present. Previously, this model was applied to describing two-dimensional isotropic antiferromagnets in the vicinity of quantum critical points [46]; its classical analog was used for the estimation of the Curie or Néel temperatures of isotropic classical magnets in a space of dimension $d = 2 + \varepsilon$ and for the determination of corresponding critical exponents [62]. In spite of the continuum character of model (2.52), in the case of quantum magnets, as shown in Sections 2.4 and 2.5, the Curie (Néel) temperature can be expressed through the parameters of the ground state, since in this case only excitations with a wave vector $k < (T/J)^{1/2} \ll 1$ play an essential role (see Section 2.2).

In the absence of anisotropy and an external magnetic field, model (2.52) corresponds to the O(3)/O(2) symmetry group corresponding to the possibility of rotation of one of the three base vectors in the three-dimensional space [O(2) is the group of rotations in the basal plane, which do not change the chosen base vector]. Unlike model (2.47), the model (2.52) can be extended to the case of an N-dimensional spin space $\mathrm{O}(N)/\mathrm{O}(N-1)$ with an arbitrary N by introducing a field $\sigma_i = \{\sigma_1, \ldots, \sigma_N\}$ and using a substitution $\sigma^z \to \sigma_N$.

2.4 Description of the intermediate temperature regime in the context of the renormalization group approach

In the intermediate two-dimensional Heisenberg regime, the interaction of spin waves is significant, but the spin-wave excitations are well defined. The existence of such a regime is a specific feature of quasi-two-dimensional systems with small interplane exchange and anisotropy. As was already seen from the results of spin-wave approaches (Section 2.2), logarithmic divergences, which depend on the parameters $\ln(\xi_{J'}/L_\tau)$ in the quantum case and $\ln(\xi_{J'}/a)$ in the classical case, exist in this regime in the magnetization (sublattice magnetization).

In order to go beyond the perturbation theory and determine the evolution of thermodynamic properties with increasing temperature, we should sum these divergences. The most efficient tool here is the RG approach [46, 60–63] which introduces a renormalized model containing no logarithmic divergences and considers the evolution of its parameters with changing scales of length. This model allows a description in terms of the perturbation theory (in the case under consideration, in terms of the SWT).

We first consider the case of a quantum antiferromagnet. As effective parameters in this case, we should consider the sublattice magnetization and the phase transition temperature; it is suitable to express these quantities through the observed parameters of the ground state: magnetization $\bar{S}_0$, spin stiffness ρ_s, spin-wave velocity c, interplane exchange α_r, and anisotropy $f_r = (\Delta/\rho_s)\,\bar{S}_0$, where Δ is the gap in the energy spectrum. At the first step of the RG transformation, parameters $Z_i = \{Z_{1,2,3}, Z_c, Z_\sigma\}$ of quantum renormalization are introduced according to the expressions

$$\bar{S}_0 = Z_\sigma S, \quad g_0 = gZ_1, \quad c_0 = cZ_c\,,$$
$$f = f_r Z_2, \quad \alpha = \alpha_r Z_3,$$

which relate the observed parameters g, c, α_r, and f_r of the ground state to the (bare) parameters g_0, c_0, α, and f of the initial model, where $g = c/\rho_s$ and $g_0 = c_0/\rho_s^0$ are the dimensionless renormalized and bare coupling constants of the model (2.52), respectively.

The renormalization constants Z_i are nonuniversal, i.e., they depend on the details of the crystal structure, and, therefore, can be determined only from the consideration of the initial lattice (noncontinuum) version of the generating functional (2.47). The above parameters can be calculated using SWT, which is in fact an expansion into a series in g ($g \sim 1/S$ for large S). For an antiferromagnet with a square lattice, the results of Section 2.2 yield the relations

$$Z_\sigma = \frac{1}{Z_1} = Z_2 = Z_3^{1/2} = 1 - \frac{0.197}{S}\,, \qquad (2.53)$$
$$Z_c = 1 + \frac{0.079}{S}$$

to first-order terms in $1/S$ [16, 38–45]. To take into account quantum renormalizations, it is convenient to have an equivalent of relations (2.53) determined in terms of the continuum model (2.52). In the first order in g, we find

$$Z_\sigma = 1 - (N-1)\,\frac{g\Lambda}{4\pi} + O(g^2),$$
$$Z_1 = 1 - (N-2)\,\frac{g\Lambda}{4\pi} + O(g^2), \quad Z_c = 1 + O(g^2), \qquad (2.54)$$
$$Z_2 = 1 + \frac{g\Lambda}{2\pi} + O(g^2), \quad Z_3 = 1 + \frac{3g\Lambda}{4\pi} + O(g^2),$$

where Λ is the ultraviolet cut-off parameter necessary for the regularization of divergences that arise upon calculation of the ground-state parameters. After following the procedure of quantum renormalization (2.54), the theory becomes completely universal, so that the thermodynamic properties in it are independent of the cut-off parameter Λ.

The resulting theory, however, contains logarithmic divergences $\ln(\xi_{J'}/L_\tau)$ which are the consequence of the two-dimensional character of spin fluctuations in the temperature regime under consideration. To sum these divergences in the framework of the RG approach, a formal infrared cut-off parameter Λ_R is introduced in the second step of the RG transformation, so that the divergences are replaced by $\ln[1/(\Lambda_R L_\tau)]$. Below, we will consider temperature-dependent renormalization parameters $\tilde{Z}_i$ introduced according to the field-theoretical RG formulation [61, 67],

$$g = g_R \tilde{Z}_1, \quad u_r = u_R \tilde{Z}_u,$$
$$\pi = \pi_R \tilde{Z}_\sigma, \quad h = h_R\,\frac{\tilde{Z}_1}{\sqrt{\tilde{Z}_\sigma}}\,,$$
$$f_r = f_R \tilde{Z}_2, \quad \alpha_r = \alpha_R \tilde{Z}_3\,, \qquad (2.55)$$

which are functions of Λ_R and are determined from the condition of the absence of logarithmic divergences in the renormalized theory; the subscript R corresponds to quantum- and temperature-renormalized quantities, and $u_r = c/T$. As in the case of the classical nonlinear sigma model [61], the introduction of five renormalization parameters for five independent parameters of the model proves to be sufficient to eliminate all divergences (see also Ref. [67]).

An infinitely small change in Λ_R generates a renormalization group transformation. For our purposes, it is required to take into account singular two-loop contributions in the diagrams for the renormalized quantities. The corresponding results for the effective temperature $t = g/(4\pi u) = T/(4\pi\rho_s)$ and for the sublattice magnetization in the renor-

malized theory are written down in the form

$$\Lambda_{\mathrm{R}} \frac{\mathrm{d} \ln Z_2}{\mathrm{d}\Lambda_{\mathrm{R}}} = -2t + O(t^2), \tag{2.56}$$

$$\Lambda_{\mathrm{R}} \frac{\mathrm{d} \ln Z_3}{\mathrm{d}\Lambda_{\mathrm{R}}} = -t + O(t^2). \tag{2.57}$$

In the renormalization of the parameters of interplane exchange and anisotropy, it is sufficient to consider singular contributions from the diagrams with a single loop, which yields

$$\Lambda_{\mathrm{R}} \frac{\mathrm{d} \ln Z_2}{\mathrm{d}\Lambda_{\mathrm{R}}} = -2t + O(t^2), \tag{2.58}$$

$$\Lambda_{\mathrm{R}} \frac{\mathrm{d} \ln Z_3}{\mathrm{d}\Lambda_{\mathrm{R}}} = -t + O(t^2). \tag{2.59}$$

Equations (2.56) – (2.59) determine the evolution of the model parameters upon RG transformation with the scaling parameter Λ_{R}; the final value of the parameter $\Lambda_{\mathrm{R}} \sim \max(\alpha, f^{1/2})$ is determined by the absence of logarithmic divergences in the effective model.

Solving Eqns (2.58) – (2.59) makes it possible to obtain temperature renormalization of the parameters of interplane exchange and anisotropy:

$$\frac{f_{\mathrm{t}}}{f_{\mathrm{r}}} = \bar{\sigma}_{\mathrm{r}}^{4/(N-1)} \left[1 + O\left(\frac{t_{\mathrm{r}}}{\bar{\sigma}_{\mathrm{r}}^{1/\beta_2}}\right)\right], \tag{2.60}$$

$$\frac{\alpha_{\mathrm{t}}}{\alpha_{\mathrm{r}}} = \bar{\sigma}_{\mathrm{r}}^{2/(N-1)} \left[1 + O\left(\frac{t_{\mathrm{r}}}{\bar{\sigma}_{\mathrm{r}}^{1/\beta_2}}\right)\right], \tag{2.61}$$

where $\bar{\sigma}_{\mathrm{r}}$ is the relative sublattice magnetization; it is determined by the solution to Eqns (2.56) – (2.57), which has the form [63]

$$\bar{\sigma}_{\mathrm{r}}^{1/\beta_2} = 1 - \frac{t_{\mathrm{r}}}{2} \left[(N-2) \ln \frac{2}{u_{\mathrm{r}}^2 \Delta(f_{\mathrm{t}}, \alpha_{\mathrm{t}})} + \frac{2}{\beta_2} \ln\left(\frac{1}{\bar{\sigma}_{\mathrm{r}}}\right) + 2\left(1 - \bar{\sigma}_{\mathrm{r}}^{1/\beta_2}\right) + O\left(\frac{t_{\mathrm{r}}}{\bar{\sigma}_{\mathrm{r}}^{1/\beta_2}}\right)\right], \tag{2.62}$$

where

$$\Delta(f, \alpha) = f + \alpha + \sqrt{f^2 + 2\alpha f} \tag{2.63}$$

is a universal function of anisotropy and interplane exchange. The quantity

$$\beta_2 = \frac{N-1}{2(N-2)} \tag{2.64}$$

is the limit of the critical exponent $\beta_{2+\varepsilon}$ of magnetization in the space of dimension $d = 2 + \varepsilon$ [61] as $\varepsilon \to 0$; in the physically important case of $N = 3$, we have $\beta_2 = 1$. The leading logarithmic term in brackets in expression (2.62) corresponds to the SSWT result (2.31), whereas the other two terms describe corrections to this theory. The expressions (2.60) – (2.62) at $N = 3$ coincide with the SSWT results (2.23) and (2.24).

As was already mentioned above, the Néel temperature cannot be calculated directly in the RG approach, since contributions from non-spin-wave excitations are important near T_{M}; this leads to the necessity of allowing for diagrams with an arbitrary number of loops. However, we can derive a general expression for the Néel temperature as follows. Let us first consider the crossover temperature t_{r}^* to the critical regime. In terms of the RG analysis, this temperature is determined by the condition of the crossover into the regime of strong coupling, i.e., by the condition $t \sim 1$ (which is equivalent to $t_{\mathrm{r}}^* \sim \bar{\sigma}_{\mathrm{r}}^{1/\beta_2}$). In the further RG transformation, the three-dimensional Heisenberg (or two-dimensional Ising) fluctuations can affect only the constant (nonsingular) term $O(t_{\mathrm{r}}/\bar{\sigma}_{\mathrm{r}}^{1/\beta_2})$, which is replaced by the universal function $\Phi_{\mathrm{AF}}(\alpha_{\mathrm{r}}/f_{\mathrm{r}}) \sim 1$. Thus, for the Néel temperature we find the expression

$$t_{\mathrm{N}} = 2\left[(N-2) \ln \frac{2}{u_{\mathrm{r}}^2 \Delta(f_{\mathrm{c}}, \alpha_{\mathrm{c}})} + 2 \ln \frac{2}{t_{\mathrm{N}}} + \Phi_{\mathrm{AF}}\left(\frac{\alpha_{\mathrm{r}}}{f_{\mathrm{r}}}\right)\right]^{-1}, \tag{2.65}$$

where α_{c} and f_{c} are the parameters of interplane exchange interaction and anisotropy at the magnetic-transition point, respectively, namely

$$f_{\mathrm{c}} = f_{\mathrm{r}} t_{\mathrm{N}}^{2/(N-2)}, \quad \alpha_{\mathrm{c}} = \alpha_{\mathrm{r}} t_{\mathrm{N}}^{1/(N-2)}, \tag{2.66}$$

with $t = T/(2\pi\rho_{\mathrm{s}})$, and $u_{\mathrm{r}} = c/T$. The second term in brackets in formula (2.65), which represents the correction to the SSWT, is on the order of $\ln\ln(2T_{\mathrm{N}}^2/\alpha)$ and substantially reduces the Néel temperature as compared to its value in the SSWT. The function Φ is determined by the non-spin-wave excitations and cannot be found in the context of the RG approach. In the quasi-two-dimensional case, the function Φ can be calculated using the $1/N$-expansion (see Section 3); a more general case requires numerical analysis, for example, applying the quantum Monte Carlo method.

The ferromagnetic case can be considered similarly to the antiferromagnetic one. In this case, the renormalizations caused by quantum fluctuations are absent, so that we can restrict ourselves to temperature renormalizations. In order to construct a perturbation theory for the generating functional (2.47), it is convenient to pass from real fields π_x and π_y to cyclic components

$$\pi^{\pm} = \pi_x \pm \mathrm{i}\pi_y. \tag{2.67}$$

The bare Green's functions of the fields π^+, and π^- are written down as

$$G^{(0)}(\mathbf{k}, \mathrm{i}\omega_n) = \frac{1}{g}\left[\mathrm{i}\omega_n + k^2 + \alpha(1 - \cos k_z) + f + h\right]^{-1}, \tag{2.68}$$

where $g = 1/S$. Since the quantum renormalization is absent ($Z_i \equiv 1$), the subscripts 'r' on the parameters of the ground state can be omitted. The multipliers $\tilde{Z}_i$ have the same form as in the antiferromagnetic case with $N = 3$. For the relative magnetization $\bar{\sigma} \equiv \bar{S}/S$ ($\bar{S} = \langle S^z \rangle$) and for the Curie temperature, we obtain Eqns (2.62) and (2.65) with substitutions $u_{\mathrm{r}}^2 \to u$ and $t_{\mathrm{r}} \to t$, respectively [63].

In the classical case, the bare Green's function of the field $\pi = \mathbf{n} - (\mathbf{n}\mathbf{z})\mathbf{z}$ has the form

$$G^{(0)}(\mathbf{k}) = \frac{1}{2\pi t}\left[2(2 - \cos k_x a - \cos k_y a) + \alpha(1 - \cos k_z a) + f + h\right]^{-1}, \tag{2.69}$$

where a is the lattice parameter. The renormalization constants in this case can be represented as

$$Z_i(t,a) = Z_{\mathrm{L}i}(t)\,\tilde{Z}_i(t_{\mathrm{L}},a)\,, \tag{2.70}$$

where $t_{\mathrm{L}} = tZ_{\mathrm{L}1}^{-1}$, and $Z_{\mathrm{L}i}$ contain all nonlogarithmic contributions which remain unchanged upon the RG transformation and are determined from the perturbation theory; the subscript 'L' labels nonuniversal renormalization parameters; $\tilde{Z}_i$ contain all other contributions determined by the structure of a concrete lattice. An equation for the magnetization in the two-loop RG approach for a classical magnet can be derived, as in the quantum case; to do this, substitutions $u_{\mathrm{r}}^2 \to 1/64$ and $t_{\mathrm{r}} \to t_{\mathrm{L}}$ should be made [63] in Eqns (2.62) and (2.65) for $\bar{\sigma}$ and T_{M}.

Thus, the RG approach is sufficient for calculating magnetization at temperatures that are not too close to the magnetic-transition temperature, at which spin-wave excitations play an important role; this approach also makes it possible to calculate the Curie (Néel) temperatures; the resulting expressions contain, however, an unknown constant which is universal in the quantum case.

2.5 Description of the critical regime and the calculation of the Néel temperature in quantum quasi-two-dimensional antiferromagnets

The treatment of non-spin-wave excitations, which are necessary for a correct description of the critical region and complete calculation of the Néel temperature, is possible in the framework of the $1/N$-expansion. This expansion, based on a simplification of the initial model in the limit $N \to \infty$, was first used for the calculation of critical exponents in the ϕ^4 model (see, e.g., Ref. [68]). Later on, the $1/N$-expansion was successfully employed for describing the properties of the quantum Heisenberg model near the quantum critical point [27].

For the Heisenberg model, the limit $N \to \infty$ coincides with the so-called spherical model [64] which neglects the connection between various spin components. In this case, the physical condition $\mathbf{S}_i^2 = S(S+1)$ is replaced by the condition for the average taken over sites:

$$\sum_i \mathbf{S}_i^2 = NS(S+1)\,. \tag{2.71}$$

Such an approximation leads to a significant simplification of the model and allows its exact solution. The further corrections calculated using expansion in the vicinity of the saddle point lead to successive improvement in approximation (2.71) with respect to the $1/N$ parameter.

Let us consider the application of the $1/N$-expansion to the calculation of the Curie (Néel) temperatures for quasi-two-dimensional magnets [28]. In the quantum case, an extension of the Heisenberg model to the model (2.52) with an $\mathrm{O}(N)/\mathrm{O}(N-1)$ symmetry is used. We first consider the isotropic case. In view of the presence of long-range order for $T < T_{\mathrm{M}}$, we perform a field shift $\sigma = \tilde{\sigma} + \bar{\sigma}$, where $\bar{\sigma}$ is the relative sublattice magnetization $\bar{S}/S$. The diagonal elements (the only nonzero elements) of Green's functions of the field $\tilde{\sigma}$ are written in the form

$$G^{mn}(\mathbf{k},k_z,\omega_n) = \rho_{\mathrm{s}}^0 \int \mathrm{d}\tau \left\langle T\left[\tilde{\sigma}^m_{\mathbf{k},k_z}(\tau)\,\tilde{\sigma}^n_{-\mathbf{k},-k_z}(0)\right]\right\rangle, \tag{2.72}$$

$$G^{mn}(\mathbf{k},k_z,\omega) = \frac{\rho_{\mathrm{s}}^0}{S^2}\,\chi^{mm}(\mathbf{k}+\mathbf{Q},k_z+\pi,\omega)\delta_{mn}\,, \tag{2.73}$$

where $\mathbf{Q} = (\pi,\pi)$ is the wave vector of the antiferromagnetic structure in the plane; for $N = 3$, they are proportional to the nonuniform dynamic spin susceptibility

$$\chi^{\alpha\beta}(\mathbf{k},k_z,\omega) = \sum_i \exp\left[\mathrm{i}\,(\mathbf{k}\mathbf{R}_i + k_z R_i^z)\right]\langle\langle\, S_0^\alpha | S_i^\beta \,\rangle\rangle_\omega\,, \tag{2.74}$$

where S_i^α are the spin operators; α, $\beta = x,y,z$, and $\mathbf{R}_i$ is the radius vector of the ith site. For certainty, we below assume that the sublattice magnetization is oriented along the Nth direction, i.e., $\bar{\sigma}^m = \bar{\sigma}\delta_{mN}$. Then, G^{NN} corresponds to longitudinal Green's function G_{l}, whereas the other diagonal components correspond to transverse Green's functions G_{t}. The condition $\mathbf{S}_i^2 = S(S+1)$ in these notations is written out as

$$1 - \bar{\sigma}^2 = \frac{T}{\rho_{\mathrm{s}}^0}\sum_n\sum_m \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\frac{\mathrm{d}k_z}{2\pi}\,G^{mm}(k,k_z,\omega_n). \tag{2.75}$$

After integration over $\tilde{\sigma}$, the generating functional takes the form

$$Z[h] = \int D\lambda \exp\left(NS_{\mathrm{eff}}[\lambda,h]\right), \tag{2.76}$$

$$S_{\mathrm{eff}}[\lambda,h] = \frac{1}{2}\ln\det\hat{G}_0 + \frac{1}{2g}\,(1-\bar{\sigma}^2)\,\mathrm{Sp}(\mathrm{i}\lambda) + \frac{1}{2g}\,\mathrm{Sp}\left[\left(\mathrm{i}\lambda\bar{\sigma} - \frac{h}{\rho_{\mathrm{s}}^0}\right)\hat{G}_0\left(\mathrm{i}\lambda\bar{\sigma} - \frac{h}{\rho_{\mathrm{s}}^0}\right)\right], \tag{2.77}$$

where

$$\hat{G}_0 = \left[\frac{\partial_\tau^2}{c_0^2} + \nabla^2 + \alpha\Delta_z\right]^{-1}, \tag{2.78}$$

$$\Delta_z\sigma_{i_z}(\mathbf{r},\tau) = \sigma_{i_z+1}(\mathbf{r},\tau) - \sigma_{i_z}(\mathbf{r},\tau).$$

Since N enters into action (2.77) only as a multiplier in the exponent, the limit $N \to \infty$ [spherical model (2.71)] corresponds to the saddle-point approximation of the functional $S_{\mathrm{eff}}[\lambda,h]$, which neglects the fluctuations in the field λ. For $T < T_{\mathrm{N}}$, the saddle point has the coordinates $\mathrm{i}\lambda = 0$ and $\bar{\sigma}^2 \neq 0$. In this case, for the transverse spin Green's function we have

$$G_0(k,k_z,\omega_n) = \left[\omega_n^2 + k^2 + \alpha\,(1-\cos k_z)\right]^{-1}. \tag{2.79}$$

The Néel temperature found from Eqn (2.75) is equal to

$$T_{\mathrm{N}}^0 = \frac{4\pi\rho_{\mathrm{s}}^{N=\infty}}{N\ln\left(2T_{\mathrm{N}}^2/\alpha c^2\right)}\,, \tag{2.80}$$

where $\rho_{\mathrm{s}}^{N=\infty} = N\,(1/g - 1/g_{\mathrm{c}})$ is the spin-wave stiffness in the zero-order in $1/N$, and $g_{\mathrm{c}} = 2\pi^2/\Lambda$ is a formal parameter of the theory. The value obtained from formula (2.80) is smaller by a factor of $N/(N-2)$ than those obtained in terms of the SSWT (2.34) and RG approach (2.62). This difference is due to the disadvantage of the spherical-model approximation which considers various spin components to be independent of one another.

In the first order in $1/N$, only the lowest corrections to condition (2.71) are taken into account, which are caused by the single exchange of the λ-field excitation, which allows for the coupling between various spin components at the site. The

general form of the equation for the magnetization for $T \gg \alpha^{1/2}$ and $\ln(2T_N^2/\alpha c^2) \gg 1$ in the first order in $1/N$ is given in Ref. [28]. For temperatures not too close to the magnetic-transition point, where

$$\frac{(N-2)\,T}{4\pi\rho_s} \ll \frac{\bar{\sigma}^2}{\bar{\sigma}_0^2}\,, \tag{2.81}$$

the equation for the magnetization reads as

$$\left(\frac{\bar{\sigma}}{\bar{\sigma}_0}\right)^{1/\beta_2}\left[1 - I_2(x_{\bar{\sigma}})\right] = 1 - \frac{NT}{4\pi\rho_s}\left[\left(1-\frac{2}{N}\right)\ln\frac{2T^2}{\alpha_r} + \frac{3}{N}\ln\frac{\bar{\sigma}_0^2}{\bar{\sigma}^2} - \frac{2}{N}\,\frac{\ln(2T^2/\alpha_r)}{\ln(2T^2/\alpha_r) + x_{\bar{\sigma}}} - I_1(x_{\bar{\sigma}})\right]. \tag{2.82}$$

Here, $I_{1,2}(x_{\bar{\sigma}})$ are some functions (see Ref. [28]) of the variable

$$x_{\bar{\sigma}} = \frac{4\pi\rho_s}{(N-2)\,T}\,\frac{\bar{\sigma}^2}{\bar{\sigma}_0^2}\,, \tag{2.83}$$

$\bar{\sigma}_0$ and ρ_s are the sublattice magnetization and spin stiffness of the ground state in the nonlinear sigma model of a quantum two-dimensional antiferromagnet [27]:

$$\frac{\bar{\sigma}_0^2}{\rho_s} = \frac{g}{N}\left(1 - \frac{8}{3\pi^2 N}\ln\frac{N\Lambda}{16\rho_s}\right), \tag{2.84}$$

$$\rho_s = \rho_s^{N=\infty}\left(1 + \frac{32}{3\pi^2 N}\ln\frac{N\Lambda}{16\rho_s}\right), \tag{2.85}$$

and α_r is the renormalized parameter of interplane exchange:

$$\alpha_r = \alpha\left(1 - \frac{8}{3\pi^2 N}\ln\frac{N\Lambda}{16\rho_s}\right). \tag{2.86}$$

As in the RG approach, the sublattice magnetization expressed in terms of quantum-renormalized quantities ρ_s, $\bar{\sigma}_0$, and α is independent of the cut-off parameter Λ, i.e., is a universal quantity. The result (2.82) obtained for magnetization differs from that of the RG approach (2.62) only in the coefficient of the subleading term $\ln(\bar{\sigma}_0/\bar{\sigma})$ ($6/N$ instead of $3/\beta_2$), which is beyond the accuracy of the first-order approximation in $1/N$. Equation (2.82) not only qualitatively correctly describes the two-dimensional regime, but also allows a description of the sublattice magnetization in the regime that is transient to critical.

The critical regime is governed by the condition which is inverse to inequality (2.81):

$$\frac{\bar{\sigma}^2}{\bar{\sigma}_0^2} \ll \frac{(N-2)\,T}{4\pi\rho_s}\,, \tag{2.87}$$

such that $x_{\bar{\sigma}} \ll 1$. In this regime, the result of a $1/N$-expansion for the sublattice magnetization is as follows:

$$\frac{\bar{\sigma}}{\bar{\sigma}_0} = \left[\frac{4\pi\rho_s}{(N-2)\,T_N}\right]^{(\beta_3/\beta_2-1)/2}\left[\frac{1}{1-A_0}\left(1-\frac{T}{T_N}\right)\right]^{\beta_3}, \tag{2.88}$$

where β_3 is the critical exponent of the magnetization:

$$\beta_3 = \frac{1}{2}\left(1 - \frac{8}{\pi^2 N}\right), \tag{2.89}$$

and $A_0 = 2.8906/N$. At $N = 3$, we have $\beta_3 \approx 0.36$, which coincides with the result of the $1/N$-expansion in the ϕ^4 model [68] at $d = 3$, in agreement with the hypothesis of universality. The equation for the Néel temperature T_N has the form

$$T_N = 4\pi\rho_s\left[(N-2)\ln\frac{2T_N^2}{\alpha_r} + 3\ln\frac{4\pi\rho_s}{(N-2)\,T_N} - 0.0660\right]^{-1}. \tag{2.90}$$

The results for the magnetization in the low-temperature [Eqn (2.82)] and critical [Eqn (2.88)] regions are smoothly joined with one another (see Section 2.6).

The excitation spectrum at the point of the magnetic phase transition is determined by the self-energy part $\Sigma(k, k_z, 0)$ at $T = T_M$. For $\alpha^{1/2} \ll k \ll T_N$, the corresponding Green's function ($G = G_t = G_l$) is written as

$$G(k, k_z, 0) = \frac{1}{k^2}\left[\frac{(N-2)\,T_N}{4\pi\rho_s}\ln\frac{2k^2}{\alpha}\right]^{1/(N-2)} \times \frac{N-1}{N}\left[1 - \eta\ln\frac{N\Lambda}{16\rho_s}\right]. \tag{2.91}$$

This equation differs from the previously considered results of the SWT by the nontrivial logarithmic dependence of $G(k, k_z, 0)$. The multiplier $N-1$ in expression (2.91) has a simple physical meaning: this is the number of the Goldstone modes in the N-component model. For $k \ll \alpha^{1/2}$ and $k_z \ll 1$, the momentum dependence of Green's function changes to become

$$G^{-1}(k, k_z, 0) = (1 + A_1)\,\alpha_c^{\eta/2}\left(k^2 + \frac{\alpha_c}{2}\,k_z^2\right)^{1-\eta/2}, \quad k \ll \alpha^{1/2}, \quad k_z \ll 1, \tag{2.92}$$

where

$$A_1 = \eta\ln\frac{N\Lambda}{16\rho_s} + \frac{1}{N}\ln\ln\frac{2T^2}{\alpha} + \frac{0.4564}{N}\,, \tag{2.93}$$

and η is the three-dimensional critical exponent in the first order in $1/N$ for the asymptotics of the correlation function at the transition point:

$$\eta = \frac{8}{3\pi^2 N}\,. \tag{2.94}$$

In this regime, as is seen from formula (2.92), the excitations have a three-dimensional nature and are characterized by the critical exponent η ($\eta \approx 0.09$ at $N = 3$). The other critical exponents can be found from scaling relationships which are retained in the framework of the regular $1/N$-expansion [68]. The quantity

$$\alpha_c = \frac{\alpha(1 + A_2)}{1 + A_1}\,, \tag{2.95}$$

where $A_2 = -0.6122/N$, can be interpreted as a renormalized parameter of interplane exchange at $T = T_N$.

Using formula (2.95), the relationship between the renormalized parameters of exchange at low temperatures and at $T = T_N$ is written as

$$\alpha_c = \alpha_r\left(1 + \frac{1.0686}{N}\right)\left[\frac{(N-2)\,T_N}{4\pi\rho_s}\right]^{1/(N-2)}. \tag{2.96}$$

As in the case of the SSWT (see Section 2.2), the renormalized value of the parameter of interplane exchange at $T = T_{\rm N}$ is lower than its low-temperature value, but the result obtained on the basis of the $1/N$-expansion at $N = 3$ differs from the SSWT result by a numerical factor approximately equal to 1.3.

In the presence of a weak easy-axis anisotropy, the spectrum of excitations in the zeroth order in $1/N$, which is determined by the poles of the unperturbed longitudinal and transverse Green's functions, contains a gap $\Delta = \Delta\,(\alpha_{\rm r}, f_{\rm r})$ for all the components of σ_m except for σ_N:

$$G_{\rm t}^0(k,\omega_n) = \left[\mathbf{k}^2 + \omega_n^2 + 2\alpha\,(1-\cos k_z) + \Delta\right]^{-1},$$
$$G_{\rm l}^0(k,\omega_n) = \left[\mathbf{k}^2 + \omega_n^2 + 2\alpha\,(1-\cos k_z)\right]^{-1}. \tag{2.97}$$

For sufficiently low temperatures

$$T \gg f_{\rm r}^{1/2}, \quad (N-2)\,T\,\frac{\ln\,(T^2/f_{\rm r})}{4\pi\rho_{\rm s}} \ll \frac{\bar\sigma^2}{\bar\sigma_0^2} \tag{2.98}$$

we obtain for the magnetization in the first order in $1/N$ the result (2.39) of the spin-wave theory. For intermediate temperatures

$$T\,(N-2)\,\frac{\ln\,(T^2/f_{\rm r})}{4\pi\rho_{\rm s}} \gg \frac{\bar\sigma^2}{\bar\sigma_0^2} \gg \frac{T\,(N-2)}{4\pi\rho_{\rm s}}\,, \tag{2.99}$$

we have

$$\left(\frac{\bar\sigma}{\bar\sigma_0}\right)^{1/\beta_2} = 1 - \frac{T}{4\pi\rho_{\rm s}}\left[(N-2)\,\ln\frac{T^2}{f_{\rm r}c^2} + B_2\ln\frac{\bar\sigma_0^2}{\bar\sigma^2} - 2 + 2\frac{\bar\sigma^2}{\bar\sigma_0^2} + O\left(\frac{NT}{4\pi\rho_{\rm s}}\,\frac{\bar\sigma_0^2}{\bar\sigma^2}\right)\right], \tag{2.100}$$

where

$$B_2 = 3 + \frac{f_{\rm r}}{\sqrt{f_{\rm r}^2 + 2\alpha_{\rm r} f_{\rm r}}}\,. \tag{2.101}$$

The temperature-dependent anisotropy parameter can be found from the relation

$$\left[G_{\rm t}(k,0)\right]^{-1} = \left[G_{\rm t}^0(k,0)\right]^{-1} + \Sigma_{\rm t}(k,0) - \Sigma_{\rm l}(0,0) \propto k^2 + f(T), \tag{2.102}$$

from which we obtain

$$\frac{f(T)}{f_{\rm r}} = \left(\frac{\bar\sigma}{\bar\sigma_0}\right)^{4/(N-2)}. \tag{2.103}$$

In the critical region determined by the inequality $\bar\sigma^2/\bar\sigma_0^2 \ll T\,(N-2)/4\pi\rho_{\rm s}$, the magnetization exhibits an Ising behavior and the $1/N$-expansion is inapplicable. This manifests itself, in particular, in the derivative $\partial\bar\sigma/\partial T$ diverging at a certain temperature $\widetilde{T}_{\rm M}$. However, the corresponding critical region is very small (as is the critical region for three-dimensional fluctuations in the isotropic quasi-two-dimensional case). Therefore, the Néel temperature can be estimated as $T_{\rm N} \approx \widetilde{T}_{\rm M}$.

Let us summarize the results of the RG approach and $1/N$-expansion for the practically important case of $N = 3$. In the spin-wave and two-dimensional regions, i.e., for

$$\bar\sigma_{\rm r} \gg \frac{T}{4\pi\rho_{\rm s}}\,, \quad \Gamma \gg \Delta\,, \tag{2.104}$$

the RG result for the relative (sublattice) magnetization is written as follows:

$$\bar\sigma_{\rm r} = 1 - \frac{T}{4\pi\rho_{\rm s}}\left[\ln\frac{2\Gamma(T)}{\Delta\,(f_{\rm t},\alpha_{\rm t})} + 2\ln\frac{1}{\bar\sigma_{\rm r}} + 2\,(1-\bar\sigma_{\rm r})\right]; \tag{2.105}$$

here, the function $\Delta(f,\alpha)$ is defined by formula (2.63), the temperature-renormalized values of the interplane exchange and anisotropy parameters are related as

$$\frac{f_{\rm t}}{f_{\rm r}} = \left(\frac{\alpha_{\rm t}}{\alpha_{\rm r}}\right)^2 = \bar\sigma_{\rm r}^2\,, \tag{2.106}$$

and the quantities $\Gamma(T)$, $\bar\sigma_{\rm r}$, $f_{\rm r}$, $\alpha_{\rm r}$, and $\rho_{\rm s}$ are listed in Table 1. The equation for $T_{\rm M}$ is written out as

$$T_{\rm M} = 4\pi\rho_{\rm s}\left[\ln\frac{2\Gamma(T_{\rm M})}{\Delta\,(f_{\rm c},\alpha_{\rm c})} + 2\ln\frac{4\pi\rho_{\rm s}}{T_{\rm M}} + \Phi\left(\frac{f}{\alpha}\right)\right]^{-1}, \tag{2.107}$$

where $\Phi(x)$ is a certain (universal in the quantum case) function which takes on values on the order of unity, and $f_{\rm c}$ and $\alpha_{\rm c}$ are the parameters of interplane exchange and anisotropy at $T = T_{\rm M}$, respectively, which are related as

$$\frac{f_{\rm c}}{f_{\rm r}} = \left(\frac{\alpha_{\rm c}}{\alpha_{\rm r}}\right)^2 = \left(\frac{T_{\rm M}}{4\pi\rho_{\rm s}}\right)^2. \tag{2.108}$$

Since $T_{\rm M}/4\pi\rho_{\rm s} \sim 1/\ln\,(1/\Delta) \ll 1$, temperature renormalizations are important for a correct description of experimental data. In particular, the interplane exchange and anisotropy parameters measured at different temperatures can differ quite significantly. The results (2.31)–(2.40) obtained in terms of the SSWT in the limit of zero anisotropy (zero interplane exchange) differ from Eqns (2.105) and (2.107) in the coefficients of $\ln\,(1/\bar\sigma_{\rm r})$ and $\ln\,(4\pi\rho_{\rm s}/T_{\rm M})$ in the second term in brackets, which are equal to 2 (1) in the SSWT, and 4 (3) in the $1/N$-expansion, respectively (with allowance for the temperature dependences of the parameters of interplane exchange and anisotropy). Thus, the role of corrections to the SSWT is more important in a two-dimensional anisotropic magnet than in an isotropic quasi-two-dimensional one.

The result obtained using the $1/N$-expansion in the O(N) model outside the critical region or, to be exact, for

$$\bar\sigma_{\rm r}^2 > \frac{T}{4\pi\rho_{\rm s}}\,, \quad \Gamma \gg \Delta \tag{2.109}$$

in the first-order expansion in $1/N$ is written down as

$$\bar\sigma_{\rm r} = 1 - \frac{T}{4\pi\rho_{\rm s}}\left[\ln\frac{2\Gamma\,(T)}{\Delta\,(f_{\rm t},\alpha_{\rm t})} + 2B_2\ln\frac{1}{\bar\sigma_{\rm r}} + 2\,(1-\bar\sigma_{\rm r}^2)\right]. \tag{2.110}$$

Table 1. Parameters of equations for the sublattice magnetization (2.105) and the magnetic-transition temperature (2.107) for various cases. $Z_{\rm L1} = Z_{\rm L2} = Z_{\rm L3} = 1 - T/8\pi\rho_{\rm s}^0$.

	$\Gamma\,(T)$	$\bar\sigma_{\rm r}$	$\rho_{\rm s}$	$f_{\rm r}$	$\alpha_{\rm r}$
Quantum AFM	T^2/c^2	$\bar S/\bar S_0$	$\gamma S\bar S_0$	$f\bar S_0^2/S^2$	$\alpha\bar S_0/S$
Quantum FM	T/JS	$\bar S/S$	$\rho_{\rm s}^0$	f	α
Classical FM, AFM	32	$\bar S/S$	$\rho_{\rm s}^0 Z_{\rm L1}$	$f Z_{\rm L2}^{-1}$	$\alpha Z_{\rm L3}^{-1}$

In the particular cases of $\alpha = 0$ and $f = 0$, the coefficient of the second term in brackets in expression (2.110) is twice as large as in the RG result (2.105), which ensures a more correct description of the temperatures close to the critical region and in the critical region itself. The equations for T_{M} have the form (2.107), identical in both approaches. In the isotropic case ($f = 0$), the result of the $1/N$-expansion for the sublattice magnetization in the critical region is given by

$$\bar{\sigma}_{\mathrm{r}} = \left(\frac{4\pi\rho_{\mathrm{s}}}{T_{\mathrm{N}}}\right)^{(\beta_3-1)/2}\left[\frac{1}{1-A_0}\left(1-\frac{T}{T_{\mathrm{N}}}\right)\right]^{\beta_3}, \qquad (2.111)$$

where $A_0 = 0.9635$, and $\beta_3 \approx 0.6$.

2.6 Theoretical description of experimental data on the magnetization and Néel temperatures of layered systems

Now we consider the application of field-theoretical methods for an analysis of experimental data. One of the well-studied layered compounds is La_2CuO_4 [9, 69]. The value of the renormalized exchange parameter for this compound, $\gamma\,|J| \approx 1850$ K, can be determined from the experimental data for the spin-wave spectrum at low temperatures [70], whereas the value of the interplane exchange parameter ($\alpha_{\mathrm{r}} = 10^{-3}$) can be found from a comparison of the magnetization in the SSWT with the experimental dependence $\bar{\sigma}_{\mathrm{r}}(T)$ at low temperatures [28, 63]. Figure 4 displays the experimental temperature dependence of the sublattice magnetization in La_2CuO_4 [69] and the results of spin-wave approximations (SWT, SSWT, and Tyablikov theory), as well as of the RG approach and $1/N$-expansion method for this compound. The result for the Néel temperature obtained in the first-order $1/N$-expansion ($T_{\mathrm{N}} = 345$ K) is significantly lower than those obtained in all spin-wave approximations, but agrees well with the experimental value $T_{\mathrm{N}}^{\mathrm{exp}} = 325$ K.

The RG approach correctly describes the $\bar{\sigma}_{\mathrm{r}}(T)$ dependence in the spin-wave region ($T < 300$ K) and in the region of two-dimensional fluctuations (which is quite narrow at the above-given small value of α_{r}), whereas at higher temperatures this approach yields overestimated values of $\bar{\sigma}_{\mathrm{r}}$. The curve obtained in terms of the $1/N$-expansion is closest to experimental data and correctly describes the critical behavior. The results of the numerical solution of Eqn (2.110) in the temperature region (2.109) and of the dependence (2.111) in the critical region coincide at the point $T = 330$ K, which is marked by a cross in Fig. 4. The difference between the theoretical and experimental curves in the range $320 < T < 340$ K can be due to the anisotropy effect. At a fixed Δ in formula (2.110) and B_2 determined from the best coincidence with experimental data at intermediate temperatures (see Fig. 4), we find the following values: $\alpha_{\mathrm{r}} = 10^{-4}$, and $f_{\mathrm{r}} = 5 \times 10^{-4}$. Thus, the approach under consideration permits us to estimate the relative role of the interplane exchange and magnetic anisotropy in layered compounds. Note that in Ref. [71] an alternative explanation of the difference between the theoretical and experimental results, based on the consideration of four-spin interaction, was suggested.

In layered perovskites such as K_2NiF_4, Rb_2NiF_4, and K_2MnF_4, magnetic anisotropy is known to be more important than the interplane exchange. The K_2NiF_4 compound has a spin $S = 1$; the neutron scattering data yield $|J| = 102$ K and $T_{\mathrm{N}}^{\mathrm{exp}} = 97.1$ K [4]. Figure 5 displays the experimental $\bar{\sigma}(T)$ dependence [1] and the results of spin-wave approaches, the RG approach, and a numerical solution to Eqn (2.110). The value $f_{\mathrm{r}} = 0.0088$ was obtained from a comparison of the results obtained in the context of the SSWT with the experimental data at low temperatures (this value agrees well with the experimental evidence: $f_r = 0.0084$ [4]. In the spin-wave and two-dimensional-fluctuation temperature ranges (2.104) ($T < 80$ K), the curves corresponding to the $1/N$-expansion and the RG approach are located somewhat higher than the experimental data points, since the $T^2/f_{\mathrm{r}}c^2$ values in this region are small. At the same time, the $1/N$-expansion curve quantitatively agrees well with experimental evidence. The procedure of extrapolation to the Ising critical behavior yields $T_{\mathrm{N}} = 91.4$ K, with the width

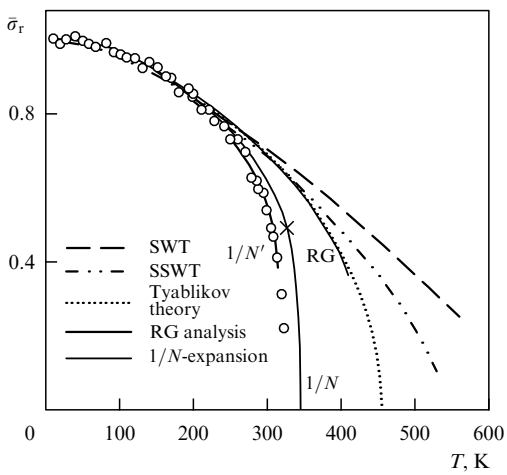

Figure 4. Theoretical temperature dependences of the relative sublattice magnetization $\bar{\sigma}_{\mathrm{r}}$ in various approximations (spin-wave theories, RG approach (2.105), and $1/N$-expansion in the O(N) model [Eqns (2.110) and (2.111)]) and experimental data points (circles) for La_2CuO_4 [3]. The RG curve is plotted up to the temperature at which the derivative $\partial\bar{\sigma}_{\mathrm{r}}/\partial T$ diverges. The curve designated as $1/N'$ lies closer to experimental data in the transition temperature region owing to the inclusion of anisotropy determined from the condition of the equality of T_{M} to its experimental value (see discussion in the main text).

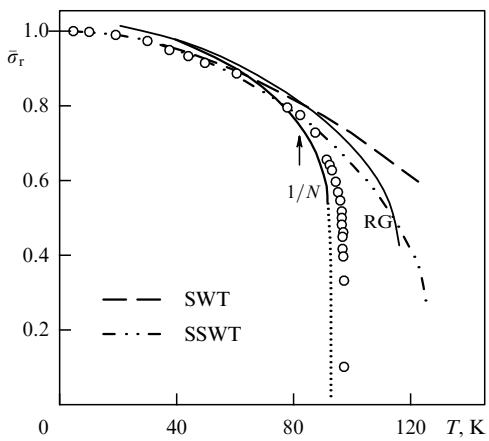

Figure 5. Relative sublattice magnetization $\bar{\sigma}_{\mathrm{r}}(T)$ for K_2NiF_4 (circles) as compared to the relative magnetization obtained in the context of the standard SWT (dashed curve), the SSWT (dot-and-dash curve), and the RG approach, and by solving Eqn (2.110) in the intermediate temperature region (solid curve). Dotted line shows the extrapolation of the results of the $1/N$-expansion to the Ising critical region. The boundary between the region with fluctuations of the two-dimensional type and the behavior that is transient to critical is indicated by an arrow.

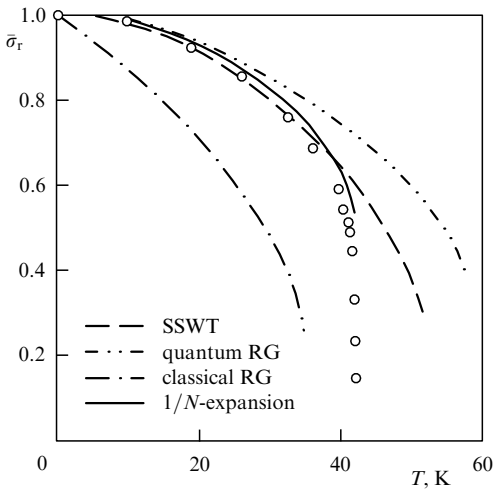

Figure 6. Experimental temperature dependence $\bar{\sigma}_r(T)$ for K_2MnF_4 (circles) as compared to the results obtained in terms of the SSWT (dashed line), the quantum RG analysis (double-dot-and-dash line), and the classical RG analysis (dot-and-dash line), and by solving Eqn (2.110) in the intermediate temperature region (solid curve).

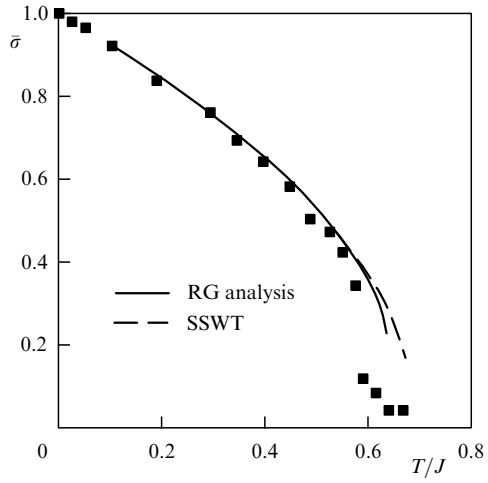

Figure 7. Results obtained in the context of the RG approach (solid curve) and SSWT (dashed line) for the relative magnetization $\bar{\sigma}$ of a classical anisotropic two-dimensional magnet ($\zeta = 0$, $\eta = 0.001$) as compared to those calculated by the Monte Carlo method [72]. The RG and SSWT curves are plotted to the temperature at which $\partial\bar{\sigma}/\partial T = \infty$.

of the critical Ising region being 1 K. Note that the allowance for the terms of order $1/x_{\bar{\sigma}}$ in expression (2.110) leads to a value of $T_N = 92.7$ K. In the region ($80 < T < 90$ K) that is transient to the critical behavior, the theoretical O(3) curve for K_2NiF_4, in contrast to the curve for La_2CuO_4, lies somewhat lower than the experimental curve. This fact can be ascribed to the interplane exchange effect. The determination of the corresponding parameters in the transition region yield $\alpha_r = 0.0017$ and $f_r = 0.0069$, which correspond to $T_N = 97$ K and the bare parameters $\alpha|J| = 0.1$ K, and $\zeta|J| = 0.76$ K. The appropriate experimental results for α are absent and, therefore, the comparison with experiment in this case is difficult.

The Rb_2NiF_4 compound exhibits strong magnetic anisotropy: according to Ref. [4], we have $|J| = 82$ K, $|J|f_r = 3.45$ K, and $T_N^{exp} = 94.5$ K. A comparison of the experimental $\bar{\sigma}_r(T)$ dependence with the SSWT results at low temperatures yields a magnitude of the anisotropy parameter $f_r = 0.046$, which agrees well with the above-given experimental value. Equation (2.107) yields $T_N = 95.5$ K, which is also close to the experimental data for the Néel temperature.

The K_2MnF_4 compound has a spin $S = 5/2$; therefore, it can be used to demonstrate a situation intermediate between the quantum and classical cases. The exchange and anisotropy parameters $|J| = 8.4$ K and $|J|f_r = 0.13$ K can be found from the neutron-scattering data [4]. Figure 6 compares the results of different approaches with experimental data for this compound. It can be seen that the $1/N$-expansion leads to the results that describe well the experimental situation in the entire temperature range. At the same time, the experimental points are located between the quantum curve and the classical RG curve, with the quantum approximation being more satisfactory. This confirms the quantum character of the corrections to the magnetization even at a relatively large magnitude of spin. In the case under consideration, the SSWT, which correctly allows for excitations on the scale of an order of the lattice parameter, yields better results as compared to those of the RG approach. Thus, an accurate consideration of systems with large spins in the framework of continual models requires a numerical calculation of integrals over momenta and summation over Matsubara frequencies.

Figure 7 displays a comparison of the SSWT and RG-approach results for the magnetization of a classical magnet with the results calculated by the Monte Carlo method [72]. It can be seen that, except for a small critical region, the RG curve is sufficiently exact, although topological excitations were neglected when constructing it. The range of applicability of the RG approach in the classical case is wider than in the quantum case, so that there is no need in the use of the $1/N$-expansion to describe the transient and critical regions.

The results of a comparison of theoretical and experimental data for layered perovskites (see Table 2) show that the RG approach and the $1/N$-expansion in the O(N) model lead to quantitatively correct results which agree well with the experimental evidence for the magnetic-transition temperatures and magnetization of these systems.

Table 2. Experimental parameters and magnetic-transition temperatures for layered magnets and corresponding theoretical values of T_M in the standard spin-wave theory (SWT), self-consistent spin-wave theory (SSWT), and $1/N$-expansion approach.

Compound	S	J, K	J', K	η	T_{SWT}, K	T_{SSWT}, K	$T_{1/N}$, K	T_{exp}, K
La_2CuO_4	1/2	1600	0.8	0	672	537	343	325
K_2NiF_4	1	102	0	0.0088	160	125	90.0	97.1
Rb_2NiF_4	1	82	0	0.046	180	118	88.4	94.5
K_2MnF_4	5/2	8.4	0	0.015	74.8	52.1	42.7	42.1
$CrBr_3$	3/2	12.4	1	0.024	79.2	51.2	39.0	40.0

3. Quasi-two-dimensional magnets with an easy-plane anisotropy

Another important class of low-dimensional systems comprises two-dimensional systems with an easy-plane anisotropy. The classical two-dimensional xy model corresponding to the limiting case of strong easy-plane anisotropy was studied in much detail in earlier works [73–75]. It was demonstrated in these works that the elementary excitations in the xy model are topological vortex-like structures and that there exists a transition (called the Berezinskii–Kosterlitz–Thouless transition) which is related to the dissociation of vortex pairs at the temperature

$$T_{\mathrm{BKT}} = \frac{\pi}{2}\,|J|\,S^2. \tag{3.1}$$

At the same temperature, the power-law dependence of the correlation function of spins on the distance is replaced by an exponential dependence (the situation is more complex in the quantum xy model, since not only transverse but also longitudinal spin components should be taken into account).

A physically more realistic situation is, however, described by the two-dimensional Heisenberg model (2.1) with a weak easy-plane anisotropy (Fig. 8), i.e., for η, $\zeta < 0$, and $|\eta|$, $|\zeta|$, $\alpha \ll 1$. In this case, the temperature dependence of the (sublattice) magnetization is mainly determined by spin-wave excitations at low temperatures. As in the case of an easy-axis anisotropy, at temperatures that are not too low compared to the magnetic-phase-transition temperature a correct allowance for the dynamic interaction of spin waves is required.

However, the Berezinskii–Kosterlitz–Thouless transition precedes the magnetic phase transition in the case of weak easy-plane anisotropy. In this case, owing to the existence of a quasi-long-range order for $T < T_{\mathrm{BKT}}$, the introduction of an arbitrarily weak interplane exchange leads to the occurrence of a magnetic transition for $T > T_{\mathrm{BKT}}$. The simple expression for the Berezinskii–Kosterlitz–Thouless temperature, obtained in the limit of small anisotropy, has the form [76]

$$T_{\mathrm{BKT}} = \frac{4\pi\,|J|\,S^2}{\ln(\pi^2/\eta)} \tag{3.2}$$

(for convenience, hereinafter in this section we make the substitutions $\eta \to -\eta$ and $\zeta \to -\zeta$). As in the case of isotropic and easy-axis magnets, formula (3.2) is insufficient to quantitatively describe the experimental data.

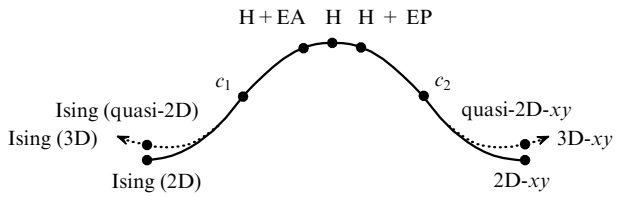

Figure 8. A schematic of RG trajectories in layered magnets. On the left-hand side: crossover from the two-dimensional Heisenberg model with an easy-axis anisotropy (H + EA) to the two-dimensional Ising model [Ising (2D)]. On the right-hand side: crossover from the two-dimensional Heisenberg model with an easy-plane anisotropy (H + EP) to the two-dimensional xy model. The inflection points c_1 and c_2 mark transition regions. Dotted lines correspond to quasi-two-dimensional models.

As is the case with easy-axis magnets, we can expect that the thermodynamic properties of these systems, except for a small vicinity of T_{BKT}, are determined by spin-wave excitations and to take into account the interaction effect of spin waves at temperatures that do not belong to the critical region we also can apply the RG method [77].

The RG analysis again is performed on the basis of functional (2.47). In the classical case (i.e., when neglecting the dynamic part of the action, which contains the time derivative) there are two types of excitations, namely, the field n_y which describes gapless in-plane excitations, and the field n_z which describes excitations with a spin rotation at right angles to the plane, with a gap in their energy spectrum. Expanding functional (2.47) in terms of $n_{y,z}$ [it is assumed that the axis of quantization of (sublattice) magnetization is oriented along the x-axis] results, in the leading order in $1/S$, in the action

$$L_{\mathrm{st}} = \frac{1}{2}\,S^2 \int_0^{1/T} \mathrm{d}\tau \sum_{\mathbf{k}} \left[(J_0 - J_{\mathbf{k}})\,\pi_{y,\mathbf{k}}\,\pi_{y,-\mathbf{k}} + (J_0 - J_{\mathbf{k}} - \eta J_{\mathbf{k}})\,\pi_{z,\mathbf{k}+\mathbf{Q}}\,\pi_{z,-\mathbf{k}-\mathbf{Q}} \right], \tag{3.3}$$

where $\mathbf{Q}$ is the wave vector of the magnetic structure, and the vector $\mathbf{n}$ is represented as $\mathbf{n}_{\mathbf{k}}(\tau) = \{\sigma_{\mathbf{k}}(\tau),\ \pi_{y,\mathbf{k}}(\tau),\ \pi_{z,\mathbf{k}}(\tau)\}$.

At not-too-small temperatures, the logarithmic contributions to the magnetization (sublattice magnetization) occur; their summation is a subject of the RG approach. The characteristic energy scales of π_y and π_z excitations, unlike those in the easy-axis case, are different. In this connection, two types of logarithmic contributions occur in the magnetization: logarithms of anisotropy, and logarithms of interplane exchange. The situation where there exist two types of excitations with different characteristic scales is typical of systems demonstrating a crossover between two regimes with different types of fluctuations [67]. In the model under consideration, this corresponds to the crossover from Heisenberg (almost isotropic) behavior to xy behavior.

To describe correctly this crossover, anisotropy should be introduced into all renormalization parameters [67]. Because of the anisotropic character of the model, there are two renormalization parameters of the field π: Z_{xy} and Z_z, which are determined by the relationships $\pi_{x\mathrm{R}}/\pi_x = \pi_{y\mathrm{R}}/\pi_y = Z_{xy}$ and $\pi_{z\mathrm{R}}/\pi_z = Z_z$. For these parameters, as well as for the effective temperature and anisotropy, we find the following set of RG equations which determine changes in the temperature, parameters of anisotropy and interplane exchange, and renormalization parameters of the field π with a change in scale [77]:

$$\begin{aligned}
\Lambda_{\mathrm{R}}\,\frac{\mathrm{d}(1/t_{\Lambda_{\mathrm{R}}})}{\mathrm{d}\Lambda_{\mathrm{R}}} &= (1 + t_{\Lambda_{\mathrm{R}}})\,f(\eta_{\Lambda_{\mathrm{R}}}, \Lambda_{\mathrm{R}}) + O(t_{\Lambda_{\mathrm{R}}}^2)\,,\\
\Lambda_{\mathrm{R}}\,\frac{\mathrm{d}\ln Z_{xy}}{\mathrm{d}\Lambda_{\mathrm{R}}} &= t_{\Lambda_{\mathrm{R}}}\left[1 + f(\eta_{\Lambda_{\mathrm{R}}}, \Lambda_{\mathrm{R}})\right] + O(t_{\Lambda_{\mathrm{R}}}^3)\,,\\
\Lambda_{\mathrm{R}}\,\frac{\mathrm{d}\ln \eta_{\Lambda_{\mathrm{R}}}}{\mathrm{d}\Lambda_{\mathrm{R}}} &= 2t_{\Lambda_{\mathrm{R}}}\,f(\eta_{\Lambda_{\mathrm{R}}}, \Lambda_{\mathrm{R}}) + O(t_{\Lambda_{\mathrm{R}}}^2)\,,\\
\Lambda_{\mathrm{R}}\,\frac{\mathrm{d}\ln \alpha_{\Lambda_{\mathrm{R}}}}{\mathrm{d}\Lambda_{\mathrm{R}}} &= -t_{\Lambda_{\mathrm{R}}} + O(t_{\Lambda_{\mathrm{R}}}^2)\,,
\end{aligned} \tag{3.4}$$

where Λ_{R} is the scaling parameter, $f(\eta_{\Lambda_{\mathrm{R}}}, \Lambda_{\mathrm{R}}) = \eta_{\Lambda_{\mathrm{R}}}\Lambda_{\mathrm{R}}^2/(\eta_{\Lambda_{\mathrm{R}}}\Lambda_{\mathrm{R}}^2 + \eta)$, $t_{\Lambda_{\mathrm{R}}}$ is the effective temperature, and $Z_z \equiv 1$.

Similar to Eqns (2.56)–(2.59), the equations (3.4) describe the evolution of parameters of the renormalized model that

occur with a change in scale. The expression for the effective temperature in this model has the following form:

$$\frac{1}{t_{\Lambda_R}} = \frac{1}{t} + \frac{1}{2}\ln\frac{\Lambda_R^2 t_{\Lambda_R}^2 + t^2\eta}{\Lambda_0^2 t_{\Lambda_R}^2 + t^2\eta} + \ln\frac{t}{t_{\Lambda_R}} + \Phi(\Lambda_R)\,, \tag{3.5}$$

where t is the dimensionless temperature:

$$t = \begin{cases} \dfrac{T}{2\pi JS^2} & \text{(FM)}, \\ \dfrac{T}{2\pi\rho_s} & \text{(AFM)}, \end{cases} \tag{3.6}$$

$\rho_s \approx S(S+0.079)|J|$ is the spin stiffness constant, the function $\Phi(\Lambda_R) = O(t_{\Lambda_R})$ corresponds to higher-order contributions, and $\Lambda_0 = q_0$ is the initial scale: $q_0 = \sqrt{T/JS}$ in the FM case, and $q_0 = T/c$ in the AFM case. In the two-dimensional Heisenberg regime ($\Lambda_R \gg \sqrt{\eta}$), the effective temperature t_{Λ_R} is small, so that

$$\frac{t}{t_{\Lambda_R}} = \frac{1}{t} + \ln\frac{\Lambda_R t}{\Lambda_0 t_{\Lambda_R}}\,. \tag{3.7}$$

At the same time, for $\Lambda_R \ll \sqrt{\eta}$ we find

$$\frac{1}{t_{\Lambda_R}} = \frac{1}{t} - \ln\frac{\Lambda_0}{\sqrt{\eta}} + 2\ln\frac{t}{t_{\Lambda_R}} + \Phi(\Lambda_R)\,. \tag{3.8}$$

In this regime, t_{Λ_R} depends on Λ_R only through the function $\Phi(\Lambda_R)$. The quantity $1/\eta^{1/2}$ is the characteristic scale of the transition (crossover) from the Heisenberg behavior to xy behavior, so that formula (3.8) describes the behavior of the effective temperature t_{Λ_R} in the xy regime. Similar to the temperature of the magnetic transition for magnets with an easy-axis anisotropy, the Berezinskii–Kosterlitz–Thouless temperature can be estimated from the condition for the transition to the regime of strong coupling, $t_{\Lambda_R} \sim 1$:

$$t_{BKT} = \left(\ln\frac{\Lambda_0}{\sqrt{\eta}} + 2\ln\frac{2}{t_{BKT}} + C\right)^{-1}, \tag{3.9}$$

where C is the universal constant.

Result (3.9) can also be obtained from a comparison of expression (3.8) with the solution to the RG equations for the effective classical xy model [74, 75]. Indeed, even if the initial model is quantum, the effective xy model is classical on the scales of order $\Lambda_R \ll \sqrt{\eta} \ll L_\tau^{-1}$, since L_τ defines a characteristic scale separating quantum fluctuations from classical ones. Therefore, the standard set of RG equations of the two-dimensional classical xy model [73–75]:

$$\Lambda_R\,\frac{d(1/t_{\Lambda_R})}{d\Lambda_R} = 32\pi^2 y_{\Lambda_R}^2\,,$$
$$\Lambda_R\,\frac{dy_{\Lambda_R}}{d\Lambda_R} = -y_{\Lambda_R}\left(2 - \frac{1}{2t_{\Lambda_R}}\right) \tag{3.10}$$

can be used for the description of the RG transformation in the xy regime. Note that the coupling constant for the system of vortices is $y = \exp(-E_0/T)$ (where E_0 is the vortex energy) rather than t (as in the case of spin waves). By choosing $\Lambda'_R \ll \sqrt{\eta}$ as the scale on which the passage from Eqns (3.6) to (3.10) occurs, and using the equation of the separatrix:

$$8\pi y_1 = \frac{1}{t_1} - 4\,, \quad t = t_{BKT}\,, \tag{3.11}$$

which separates the low-temperature and high-temperature phases ($t_1 \equiv t_{\Lambda'_R}$, $y_1 \equiv y_{\Lambda'_R}$), we can reproduce result (3.9) for the Berezinskii–Kosterlitz–Thouless temperature.

Equations (3.4) also make it possible to determine the temperature dependence of the correlation length at temperatures exceeding the Berezinskii–Kosterlitz–Thouless temperature. In the critical region, for $t > t_{BKT}$, i.e., for

$$\frac{1}{8\pi}\left(t_{BKT}^{-1} - t^{-1}\right) \ll 1\,, \tag{3.12}$$

the expression for the correlation length takes the form

$$\xi \approx \frac{1}{\sqrt{\eta}}\exp\left(\frac{A}{2\sqrt{t_{BKT}^{-1} - t^{-1}}}\right) \tag{3.13}$$

(where A is a constant), which is similar to the result for the classical xy model. Under the condition which is the reverse of inequality (3.12), there takes place standard Heisenberg behavior [46]:

$$\xi = \frac{C_\xi}{\Lambda_0}\,t\,\exp\frac{1}{t}\,. \tag{3.14}$$

In the presence of interplane exchange interaction and at sufficiently low temperatures, a magnetic order arises. Because of topological effects, the transition temperature in the case of small interplane exchange tends, however, to T_{BKT} rather than to zero. The description of the RG transformation at temperatures close to the critical region is difficult because of the complex geometry of vortex loops in a three-dimensional space. Instead of the direct calculation of RG trajectories, the same arguments can be used as in the easy-axis case. The transition temperature is determined from the requirement that the correlation length of the model without interplane exchange ($\alpha = 0$) be coincident with the characteristic scale of the transition from the two-dimensional xy model to three-dimensional model. Then, for the critical temperature $t_c = T_C/(2\pi JS^2)$ [or $T_N/(2\pi\rho_s)$] with the proviso that $\alpha \ll \eta$ we find [77]

$$t_c = \left[\ln\frac{\Lambda_0}{\sqrt{\eta}} + 2\ln\frac{2}{t_{BKT}} + C - \frac{A^2}{\ln^2(\eta/\alpha)}\right]^{-1}. \tag{3.15}$$

The last term in brackets in formula (3.15) determines the difference between t_c and t_{BKT}. Since this term is not small, no expansion of Eqn (3.15) in this term is performed.

The result (3.15) is qualitatively correct up to α values of order η (in this case, the last term in brackets in formula (3.15) leads only to a renormalization of the constant C). In the inverse limit, $\alpha \gg \eta$, the corrections to the RG result for quasi-two-dimensional magnets are determined, because of the presence of an easy-plane anisotropy, as

$$t_c = \left[\ln\frac{\Lambda_0}{\sqrt{\alpha}} + 2\ln\frac{2}{t_c} + C' + O\left(\frac{\eta^{1/\psi}}{\alpha^{1/\psi}}\right)\right]^{-1}, \tag{3.16}$$

where $\psi = \nu_3(2 - \gamma_\eta)$ is the critical exponent for the crossover region between isotropic and anisotropic behavior, ν_3 is the appropriate critical exponent of the three-dimensional Heisenberg model, and γ_η is the anomalous dimension of the anisotropy parameter in the three-dimensional Heisenberg model. The result of an ε-expansion in the anisotropic ϕ^4 model with a dimension of $4 - \varepsilon$ at $\varepsilon = 1$ is $\psi \approx 0.83$ [67]. For

an antiferromagnet, the constant $C \approx -0.066$ by relation (2.90). The last term on the right-hand side of formula (3.16), unlike the last term in formula (3.15), exhibits a power-law dependence on the anisotropy parameter. This is a consequence of the fact that the correlation length in the three-dimensional Heisenberg model has a power-law temperature dependence near the magnetic-phase-transition temperature. For this reason, the correction in formula (3.16) is small and can be neglected in the case of small anisotropy.

Now, we turn to the experimental situation. The most thoroughly experimentally studied system with an easy-plane anisotropy is the K_2CuF_4 compound which is a ferromagnet with a spin $S = 1/2$, $T_{BKT} = 5.5$ K, $T_C = 6.25$ K, and the parameters $J = 20$ K, $\eta = 0.04$, and $\alpha = 6 \times 10^{-4}$ [4]. By inserting these values into formulas (3.9) and (3.15), we obtain $C \approx -0.5$ and $A \approx 3.5$. These values of the constants can be verified using other systems.

Another example of a quasi-two-dimensional ferromagnet with an easy-plane anisotropy is $NiCl_2$ with $S = 1$. According to Ref. [4], its parameters are $J = 20$ K, $\eta = 8 \times 10^{-3}$, and $\alpha = 5 \times 10^{-5}$. Using the values of A and C found for K_2CuF_4, we arrive at $T_{BKT} = 17.4$ K and $T_C = 18.7$ K, which agree with the experimental data (both values of T_{BKT} and T_C lie in the range of 18 – 20 K). At the same time, the calculations with a leading logarithmic accuracy according to expression (3.2) yield $T_{BKT} = 35.3$ K, which is twice as large as the experimental value.

The $BaNi_2(PO_4)_2$ compound is, according to Ref. [4], an antiferromagnet with $S = 1$, $|J| = 22.0$ K, and anisotropy $\eta = 0.05$, $\alpha = 10^{-4}$. The calculations performed in Ref. [78] yield $T_{BKT} = 23.0$ K coinciding with the experimental value, and $T_N = 24.3$ K, which also agrees well with $T_N^{exp} = (24.5 \pm 1)$ K. Although $T_{BKT} \sim |J|\,S$ for this compound, this case should also be considered to be quantum according to the criterion $(T/JS)^2 \ll 32$ (see Section 2.2) for the quantum regime.

4. Quasi-one-dimensional isotropic antiferromagnets

4.1 The model and its bosonization

Although the physical situation for quasi-one-dimensional magnets differs substantially from that in the quasi-two-dimensional case, a theoretical description of quasi-one-dimensional magnets can also be made on the basis of the Heisenberg model (2.1). Below, we consider the simplest case of isotropic antiferromagnets ($\eta = \xi = 0$) with a spin $S = 1/2$ and small interchain exchange: $|J'| \ll J$. In this case, it is convenient to write down the Hamiltonian of the system as follows:

$$H = J \sum_{n,i} \mathbf{S}_{n,i}\,\mathbf{S}_{n+1,i} + \frac{1}{2} J' \sum_{n,\langle ij\rangle} \mathbf{S}_{n,i}\,\mathbf{S}_{n,j}\,, \tag{4.1}$$

where n is the order number of a site in the chain; i and j are the indices of the chains, and $J > 0$ and J' are the intrachain and interchain exchange integrals, respectively.

When studying elementary excitations of the model, the spin operators in each chain can be represented in terms of the Bose operators $\varphi_i(x)$ (the so-called 'bosonization'). In the bosonization method, the spin operators are first represented through the Fermi operators with the aid of the so-called Jordan – Wigner transformation (see, e.g., Ref. [79]). This transformation makes it possible to reduce the Heisenberg model to a system of interacting fermions. In this case, the transverse part of spin exchange leads to a Hamiltonian of free fermions, while the longitudinal part is responsible for their interaction. In turn, the Fermi operators are represented through new Bose operators φ_i via relationships which permit the reproduction of commutation relations between the initial Fermi operators. As a result, we obtain formulas for the initial spin operators expressed in terms of the Bose operators [79]:

$$\mathbf{S}_{n,i} = \mathbf{J}_i(x) + (-1)^n\,\mathbf{n}_i(x)\,, \tag{4.2}$$

where

$$J_i^z(x) = \frac{\beta}{2\pi}\,\partial_x\varphi_i(x)\,,$$
$$J_i^{\pm}(x) = \frac{\Lambda}{\pi}\exp\left[\pm \mathrm{i}\beta\theta_i(x)\right]\cos\beta\varphi_i(x) \tag{4.3}$$

are the so-called homogeneous components of spin operators;

$$n_i^z(x) = \frac{\Lambda}{\pi}\cos\beta\varphi_i(x)\,,$$
$$n_i^{\pm}(x) = \frac{\Lambda}{\pi}\exp\left[\pm \mathrm{i}\beta\theta_i(x)\right] \tag{4.4}$$

are appropriate sublattice components; Λ is a constant of an order of the inverse lattice parameter; $\beta = \sqrt{2\pi}$, and θ_i satisfies the relation $\partial_x\theta_i = -\Pi_i$, where Π_i is a momentum canonically conjugate to φ_i.

Hamiltonian (4.1) written in terms of Bose operators $\varphi_i(x)$ has the form

$$H = \frac{v}{2}\sum_i \int \mathrm{d}x\left[\Pi_i^2 + (\partial_x\varphi_i)^2\right] + g_u \sum_i \int \mathrm{d}x\,\cos 2\beta\varphi_i$$
$$- \frac{J'\Lambda^2}{2\pi^2}\sum_{i,\delta_\perp}\int \mathrm{d}x\left[\cos(\beta\varphi_i)\,\cos(\beta\varphi_{i+\delta_\perp})\right.$$
$$\left. + \cos\beta(\theta_{i+\delta_\perp} - \theta_i)\right], \tag{4.5}$$

where $v = \pi J/2$. The first two terms on the right-hand side of Eqn (4.5) correspond to a system of isolated chains and represent the Hamiltonian of the standard sine-Gordon model. The first term describes the free Bose system, and the second term describes the interaction of bosons along the chains. The last interaction arises as a result of umklapp scattering (which is accompanied by the process of electron flip) in the system of fermions introduced via the Jordan – Wigner transformation; this interaction is marginal from the RG viewpoint and is responsible for logarithmic corrections to thermodynamic quantities [36, 80 – 84]. Numerical estimations (see Refs [36, 80]) yield $g_u/(2\pi) \approx 0.25$. The last sum in Eqn (4.5) describes the interaction of spins of different chains.

4.2 Interchain mean-field approximation for a bosonized Hamiltonian

The simplest way to consider interchain exchange interaction is the use of the so-called interchain mean-field approximation [36]. By decoupling the interaction term:

$$\cos(\beta\varphi_i)\,\cos(\beta\varphi_{i+\delta_\perp}) \to 2\left\langle\cos(\beta\varphi_{i+\delta_\perp})\right\rangle\cos(\beta\varphi_i)\,, \tag{4.6}$$

we arrive at

$$H_{\mathrm{MF}} = \frac{v}{2}\sum_i \int \mathrm{d}x\left[\Pi_i^2 + (\partial_x\varphi_i)^2\right] + g_{\mathrm{u}}\sum_i \int \mathrm{d}x\,\cos(2\beta\varphi_i) - \frac{\Lambda}{\pi}\,h_{\mathrm{MF}}\sum_i \int \mathrm{d}x\,\cos(\beta\varphi_i), \tag{4.7}$$

where

$$h_{\mathrm{MF}} = \frac{z_\perp J'\Lambda\langle\cos(\beta\varphi_i)\rangle}{\pi}, \tag{4.8}$$

and $z_\perp$ is the number of nearest neighbors in the direction perpendicular to the chain ($z_\perp = 4$ for the tetragonal lattice). Approximation (4.6) makes it possible to reduce the problem of many chains to the problem of a single chain in an effective sublattice magnetic field. By introducing a function

$$B(h;T) = \frac{\Lambda}{\pi}\left\langle\cos(\beta\varphi_i)\right\rangle_h \tag{4.9}$$

calculated in the presence of a magnetic field [the last term in Eqn (4.7)], we obtain a self-consistent equation for the sublattice magnetization $\bar{S}$:

$$\bar{S}_{\mathrm{MF}} = B(z_\perp J'\bar{S}_{\mathrm{MF}}, T). \tag{4.10}$$

Although the Hamiltonian H_{MF} describes a single-chain interaction, the calculation of the function $B(h,T)$ (which is an analog of the Brillouin function in the conventional mean-field theory of Heisenberg magnets) at arbitrary temperatures is a sufficiently difficult problem. According to the dimensional estimate $B(h,T) = h^{1/3}f(h^{2/3}/T)$, with a certain function $f(x)$: $f(x) \sim x$ as $x \to 0$, and $f(\infty) = \mathrm{const}$. At $g_{\mathrm{u}} = 0$ (in which case we have the standard sine-Gordon model or the massive Thirring model, which is equivalent), $B(h,T)$ was determined using the Bethe ansatz [85]. As $h \to 0$, we have

$$B(h,T) = h\chi_0(T), \tag{4.11}$$

where $\chi_0(T)$ is the sublattice susceptibility of the system in the absence of the field h [36, 84]:

$$\chi_0(T) = \frac{\tilde{\chi}_0}{T}\,L\left(\frac{\Lambda J}{T}\right), \quad \tilde{\chi}_0 = \frac{\Gamma^2(1/4)}{4\Gamma^2(3/4)} \approx 2.1884, \tag{4.12}$$

$$L\left(\frac{\Lambda J}{T}\right) = C\left[\ln\frac{\Lambda J}{T} + \frac{1}{2}\ln\ln\frac{\Lambda J}{T} + O(1)\right]^{1/2}, \tag{4.13}$$

where Γ is the gamma-function. The constants C and Λ can be determined numerically [86]: $C \approx 0.137$, and $\Lambda \approx 5.8$.

The result (4.11) makes it possible to calculate T_{N} in the context of the mean-field theory, since $h_{\mathrm{MF}} \to 0$ as $T \to T_{\mathrm{N}}$. The equation for the Néel temperature takes the form [36]

$$T_{\mathrm{N}}^{\mathrm{MF}} = z_\perp J'\,\tilde{\chi}_0 L\left(\frac{\Lambda J}{T_{\mathrm{N}}^{\mathrm{MF}}}\right). \tag{4.14}$$

Thus, according to the interchain mean-field theory, we have $T_{\mathrm{N}} \propto |J'|$; the sublattice magnetization of the ground state, $\bar{S}_0 \propto \sqrt{|J'|/J}$, also depends on J' according to a power law, which means the development of long-range order at arbitrarily small $|J'|$. At the same time, the standard spin-wave theory makes no distinction between integer and half-integer values of spins and predicts a finite critical value $J'_{\mathrm{c}} \sim J\exp(-\pi S)$ [30, 53], so that for $|J'| < J'_{\mathrm{c}}$ the sublattice magnetization $\bar{S}_0$ vanishes; for $|J'| > J'_{\mathrm{c}}$, we have

$$\bar{S}_0 \propto \ln\left|\frac{J'}{J'_{\mathrm{c}}}\right|, \quad T_{\mathrm{N}} \propto \bar{S}_0\sqrt{|J'|}. \tag{4.15}$$

This contradiction was removed using the RG method [33–35], which showed that the standard spin-wave theory is indeed applicable on the scale of the inverse length $\Lambda \gg J'_{\mathrm{c}}/J$ and the renormalization factor for magnetization is $Z_\Lambda^{-1/2} \propto \ln\Lambda$. At the same time, for half-integer spins and $\Lambda \ll J'_{\mathrm{c}}/J$ we obtain the dependence $Z_\Lambda^{-1/2} \propto \Lambda^{1/2}$ [33, 34] which means the validity of the results of the interchain mean-field theory for $|J'| \ll J'_{\mathrm{c}}$.

The values of the Néel temperature in the interchain mean-field theory turn out to be overestimated compared to experimental data, since this theory does not take into account the correlation effects between spins located on different chains. In particular, the value of the Néel temperature given by formula (4.14) is insensitive to the spatial dimensionality of the system; although in the two-dimensional case, the Néel temperature should be $T_{\mathrm{N}} = 0$, and in the three-dimensional case the values of T_{N} turn out to be overestimated compared to experimental data.

The correlations between spin positions on different chains manifest themselves in the existence of collective excitations which make contributions to the thermodynamic properties. In this case, the disadvantages of the interchain mean-field theory are similar to the disadvantages of the Stoner theory for itinerant-electron magnets that neglects the contribution of collective excitations which later have been taken into account in the Moriya theory [24]. As in the Moriya theory, the collective excitations in the Heisenberg theory can be considered in terms of the random-phase approximation (RPA) in which they are determined by the poles of spin susceptibilities [36, 81]:

$$\chi^{\pm}(k_z,\omega) = \frac{\chi_0^{\pm}(k_z,\omega)}{1 - J'(k_x,k_y)\,\chi_0^{\pm}(k_z,\omega)/2}, \tag{4.16}$$

$$\chi^{zz}(k_z,\omega) = \frac{\chi_0^{zz}(k_z,\omega)}{1 - J'(k_x,k_y)\,\chi_0^{zz}(k_z,\omega)}, \tag{4.17}$$

where for the tetragonal lattice we have

$$J'(k_x,k_y) = 2J'(\cos k_x + \cos k_y), \tag{4.18}$$

and $\chi_0(k,\omega)$ is the dynamic sublattice susceptibility in model (4.7). As $h \to 0$, the susceptibility $\chi_0(k,\omega)$ is also determined by simple analytical expressions [83, 84]:

$$\chi_0(k_z,\omega) = \frac{1}{T}\,L\left(\frac{\Lambda}{T}\right)\tilde{\chi}_0\left(\frac{k_z}{T},\frac{\omega}{T}\right),$$

$$\tilde{\chi}_0(q,r) = \frac{1}{4}\,\frac{\Gamma(1/4+\mathrm{i}k_+)\,\Gamma(1/4+\mathrm{i}k_-)}{\Gamma(3/4+\mathrm{i}k_+)\,\Gamma(3/4+\mathrm{i}k_-)}, \quad k_\pm = \frac{r \pm q}{4\pi}, \tag{4.19}$$

where $\chi_0^{zz}(k_z,\omega) = \chi_0^{\pm}(k_z,\omega)/2 = \chi_0(k_z,\omega)$, with $\chi_0(0,0) = \chi_0(T)$.

To determine corrections to the interchain mean-field theory, related to the contribution of collective excitations, the $1/z_\perp$-expansion can be applied ($z_\perp$ is the number of nearest neighbors in the directions perpendicular to the

chains) [78]. This approach, which is similar to the $1/z$-expansion (or the expansion in terms of the inverse radius of interaction) used many years ago to improve the standard mean-field theory of Heisenberg magnets [87, 88], makes it possible to determine the Néel temperature in quasi-one-dimensional systems to a greater accuracy than in the case of interchain mean-field approximation.

4.3 Corrections of first order in $1/z_\perp$ to the interchain mean-field approximation

Now, we consider the perturbation theory in $J'/\max(h_{\rm MF}, T) \sim 1/z_\perp$, which is an analog of the expansion in terms of $J/\max(h_{\rm MF}, T) \sim 1/z$ for three-dimensional Heisenberg magnets [88]. To expand the sublattice magnetization

$$\bar{S} = \frac{\Lambda \langle \cos(\beta\varphi_i)\rangle_h}{\pi} \tag{4.20}$$

into a series in terms of J', it is convenient to use the expression for $\bar{S}$ in the formalism of a continuum integral:

$$\bar{S} = \frac{\Lambda}{\pi}\,\frac{\int D\varphi \cos\big(\beta\varphi_i(0)\big)\exp\big(-L[\varphi]\big)}{\int D\varphi \exp\big(-L[\varphi]\big)}\,, \tag{4.21}$$

where $L[\varphi]$ is the Lagrangian function corresponding to Hamiltonian (4.5). In the zero order in J' (i.e., at $J' = 0$), we have

$$\bar{S}_0 = B(h, T)\,, \tag{4.22}$$

where the function B is defined by formula (4.9). By expanding expression (4.21) into a series in J', we find that each term can be represented by a certain diagram; the diagrammatic technique in this case coincides with that used for spin operators [87, 88].

Some elements of this technique are shown in Fig. 9. All the diagrams are classified according to their order in $J'/\max(h_{\rm MF}, T) \sim 1/z_\perp$. The diagrams displayed in Fig. 10 are of zero order in $1/z_\perp$. Summation of these diagrams leads to a shift of the external magnetic field by the magnitude of the mean field:

$$h \to \tilde{h} = h + h_{\rm MF}\,, \quad h_{\rm MF} = z_\perp J' \bar{S} \tag{4.23}$$

[the same result can be obtained by excluding the contribution of the mean field directly from expression (4.21)]. The diagrams that are of first order in $1/z_\perp$ (Fig. 11a) contain one line of RPA interaction, which is a sum of the irreducible diagrams shown in Fig. 11b. In the analytical form, this interaction is defined as follows:

$$V^{+-,zz}(\mathbf{k}, \omega) = \frac{J'(k_x, k_y)}{1 + \delta - J'(k_x, k_y)\,\chi_0^{\pm,zz}(k_z, \omega)}\,, \tag{4.24}$$

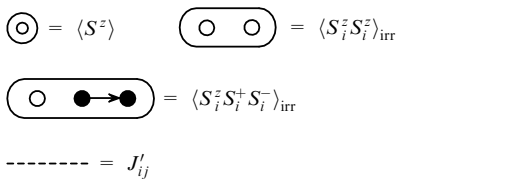

Figure 9. Some elements of the diagrammatic technique for spin operators (for details, see Ref. [88]).

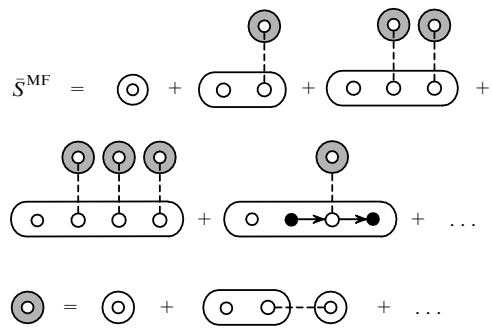

Figure 10. Diagrams for sublattice magnetization in the zero order in $1/z_\perp$ (mean-field approximation).

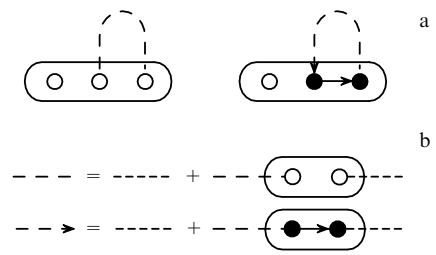

Figure 11. (a) Diagrams in the first order in $1/z_\perp$ for sublattice magnetization. (b) Equations for RPA lines of interaction.

where

$$\chi_0^{zz}(k_z, \omega) = \frac{\Lambda^2}{\pi^2}\int {\rm d}^2x\,\langle \cos\beta\varphi_i(0)\cos\beta\varphi_i(x)\rangle_{0,\,\rm irr} \times \exp(-{\rm i}k_z x + {\rm i}\omega_n\tau)\,,$$

$$\chi_0^{\pm}(k_z, \omega) = \frac{\Lambda^2}{\pi^2}\int {\rm d}^2x\,\langle \exp\{{\rm i}\beta[\theta_i(0) - \theta_i(x)]\}\rangle_0 \times \exp(-{\rm i}k_z x + {\rm i}\omega_n\tau)\,, \tag{4.25}$$

and $\langle AB\rangle_{\rm irr}$ is the irreducible average of two operators:

$$\langle AB\rangle_{\rm irr} = \langle AB\rangle - \langle A\rangle\langle B\rangle\,. \tag{4.26}$$

As in the Moriya theory [24], a correction $\delta = z_\perp J'\chi_0^{\pm}(0,0) - 1$ was introduced into the denominator of expression (4.24) in order to satisfy the Goldstone theorem which requires the presence of a pole of effective interaction at $k = 0$, $\omega = 0$, and for $T \leqslant T_{\rm N}$.

Taking into account formula (4.24), we arrive at the result for the sublattice magnetization

$$\bar{S} = \frac{1}{T}\,h_{\rm MF}\tilde{\chi}_0 L\left(\frac{\Lambda}{T}\right)\left\{1 + \frac{\pi^2}{2T\tilde{\chi}_0}\,L\left(\frac{\Lambda}{T}\right)\int {\rm d}^2r V(r) \times \left[\frac{1}{8}\,F(r) + \frac{1}{2}\,G(r)\right]\right\}, \tag{4.27}$$

where

$$V(r) = \int_{-\infty}^{\infty}\frac{{\rm d}k_z}{2\pi}\sum_n\sum_{k_x,k_y}\frac{\cos k_x + \cos k_y}{2\tilde{\chi}_0 - (\cos k_x + \cos k_y)\,\tilde{\chi}_0(k_z, 2\pi{\rm i}n)} \times \exp({\rm i}k_z r - 2\pi{\rm i}n\tau)\,, \tag{4.28}$$

$$\tilde{\chi}_0 = \frac{\pi}{2}\int {\rm d}^2z\,\frac{1}{|\sinh(\pi z)|} \approx 2.1184\,, \tag{4.29}$$

and the functions $F(r)$ and $G(r)$ were defined in Ref. [78]. Using the relation (4.23) between the mean field and sublattice magnetization, the result for the Néel temperature in the first order in $1/z_\perp$, after gathering all corrections into the denominator, takes the form

$$T_{\mathrm{N}} = rJ' z_\perp \tilde{\chi}_0 L\left(\frac{\Delta J}{T_{\mathrm{N}}}\right). \tag{4.30}$$

Equation (4.30) differs from the result of the mean-field theory (4.14) by a factor r which depends on the structure of the lattice in the direction perpendicular to the chains. The numerical calculation for a tetragonal lattice yields $r \approx 0.70$. Thus, a decrease in T_{N} due to fluctuation effects is 25% of its mean-field value, which agrees well with the results of the numerical analysis performed in Ref. [89]. In the two-dimensional case, we find $r = 0$, so that $T_{\mathrm{N}} = 0$.

The corrections to the sublattice magnetization of the ground state can be calculated in a similar way [78]. For the dynamic susceptibility of a single chain at $T = 0$, we have [36, 81]

$$\chi_0^{+-} = \frac{1}{4\,|J'|}\,\frac{\Delta^2}{\omega^2 + v^2k^2 + \Delta^2}\,, \tag{4.31}$$

$$\chi_0^{zz} = \frac{Z'/Z}{4\,|J'|}\,\frac{\Delta^2}{\omega^2 + v^2k^2 + 3\Delta^2}\,, \tag{4.32}$$

where $\Delta \approx 6.175\,|J'|$ is the gap in the spectrum of spin excitations; Z and Z' are the spectral weights of longitudinal and transverse one-magnon excitations, respectively ($Z'/Z \approx 0.49$), and $\bar{S}_0 \approx 1.017\,|J'|$ is the sublattice magnetization in the ground state.

Using again the RPA for the interaction potential (4.27) between excitations at different chains, we find

$$\bar{S} = \bar{S}_{\mathrm{MF}} - \frac{\Delta}{4\pi}\,\frac{\partial\Delta}{\partial h_{\mathrm{MF}}}\,I\,,$$
$$I = \sum_k \left[\left(1 - \frac{\Gamma_k}{2}\right)\ln\frac{1}{1-\Gamma_k} + \left(3 - \frac{Z'\Gamma_k}{2Z}\right)\ln\frac{1}{1 - Z'\Gamma_k/(3Z)}\right], \tag{4.33}$$

where $h_{\mathrm{MF}} = z_\perp J' \bar{S}_0$, $\Gamma_k = \cos k$ for the two-dimensional lattice, and $\Gamma_k = (\cos k_x + \cos k_y)/2$ for the tetragonal lattice. Numerical integration yields [78]

$$\bar{S}_0 = (0.677 - I)\,h_{\mathrm{MF}}^{1/3}\,. \tag{4.34}$$

The last term in the parentheses in formula (4.34) represents a $1/z_\perp$-correction to the magnetization of the ground state:

$$I = \begin{cases} 0.011 & (3\mathrm{D})\,, \\ 0.060 & (2\mathrm{D})\,. \end{cases} \tag{4.35}$$

It follows from relationship (4.34) that the ground-state magnetization decreases by almost 10% as compared to its value in the mean-field theory for the two-dimensional lattice, and only by 2% for the three-dimensional lattice. Thus, the fluctuation corrections for the sublattice magnetization of the ground state are much less important compared to the corrections for the Néel temperature and, therefore, can be neglected in the three-dimensional case.

4.4 Comparison with experimental data

Let us consider the application of the theoretical results to the description of experimental data for magnetic quasi-one-dimensional systems. The most thoroughly studied quasi-one-dimensional compound is $KCuF_3$ with a spin $S = 1/2$. From neutron-scattering experiments [10] for this compound, it follows that the parameter of magnetic exchange along the chains equals $J = 406$ K and the ground-state magnetization is $\bar{S}_0/S = 0.25$. As was discussed in Ref. [36], this magnitude of $\bar{S}_0/S$ corresponds to $J'/J = 0.047$, so that $J' = 19.1$ K. The interchain mean-field approximation (4.14) with these parameters yields $T_{\mathrm{N}} = 47$ K, which is somewhat higher than the experimental value $T_{\mathrm{N}} = 39$ K [10]. At the same time, the value obtained using the $1/z_\perp$-expansion (4.30), namely $T_{\mathrm{N}} = 37.7$ K, is much closer to the experimental evidence. Thus, the approach under consideration slightly overestimates fluctuation effects but significantly improves the interchain mean-field approximation. The contribution from the double-logarithmic term to formula (4.13), which is equal to approximately 5% of the Néel temperature, improves the agreement with experimental data.

Another compound with $S = 1/2$ that has been widely discussed in the literature, Sr_2CuO_3, possesses the following parameters [11, 12]: $J = 2600$ K, and $T_{\mathrm{N}} = 5$ K. The reliable experimental data for J' are absent, but using Eqn (4.30) and the experimental value of T_{N}, we obtain $J' = 1.85$ K. Then, it follows from formula (4.34) that $\bar{S}_0/S = 0.042$, which agrees with experimental data ($\bar{S}_0/S \lesssim 0.05$).

For the Ca_2CuO_3 compound, the experimental parameters are as follows [11, 12]: $S = 1/2$, $J = 2600$ K, and $T_{\mathrm{N}} = 11$ K, from which we obtain $J' = 4.3$ K and $\bar{S}_0/S = 0.062$, which also agrees well with experimental data [12]: $\bar{S}_0\,(Ca_2CuO_3)/\bar{S}_0\,(Sr_2CuO_3) = 1.5 \pm 0.1$. Thus, result (4.30) is sufficient for a quantitative description of realistic quasi-one-dimensional magnetic systems.

5. Conclusions

Quasi-one-dimensional and layered magnets represent an example of systems with strong fluctuations and an unusual behavior of thermodynamic and magnetic properties. The investigation of these systems is a nontrivial problem from the viewpoint of theoretical physics. The ordinary spin-wave theory (SWT), and even its improved self-consistent version (SSWT), although leading to a correct result for the magnetic-transition temperature T_{M} in the leading logarithmic approximation, prove to be quantitatively applicable only for temperatures $T \ll T_{\mathrm{M}}$. At higher temperatures, dynamic interaction of spin waves should be taken into account, which is beyond the framework of the lowest (Born) approximation and of substantially non-spin-wave excitations.

The problem of the description of the thermodynamic properties of quasi-one-dimensional and layered magnets has been substantially developed in terms of field – theoretical methods that were applied to the widely common model of magnetism of these systems — the Heisenberg model. The use of such approaches makes it possible to obtain simple analytical results for the temperature dependence of the magnetization and for the quantity T_{M}, which can be used in processing experimental data. In quasi-two-dimensional magnets in a wide temperature range below T_{M}, the spin-wave picture of the excitation spectrum is adequate; the interaction of spin waves leads to the appearance of correction terms in the expressions for the magnetization

and inverse transition temperature $1/T_M$, which significantly improves the agreement with experimental data. The small critical region near T_{M} can be described, allowing for non-spin-wave excitations, e.g., in terms of the $1/N$-expansion. In quasi-one-dimensional magnets, the consideration of the Bose (non-spin-wave) excitations makes it possible to construct a systematic expansion in inverse coordination number of the lattice in the directions perpendicular to chains.

Thus, from the theoretical viewpoint a good understanding of the physical picture of the spectrum and properties of low-dimensional magnets has been achieved in a wide temperature range, which gives a basis for a quantitative description of the properties of real systems, and one of our aims was to attract the attention of experimentalists to this fact. At the same time, for a detailed analysis of concrete compounds the consideration of the dipole interactions, relativistic interactions (such as Dzyaloshinski – Moriya interactions), and some others is required. Although the first attempts at describing the systems with such interactions in terms of the self-consistent spin-wave and field-theoretical approaches have already been made [90, 91], this problem still awaits further development. Note also that some compounds with a complex crystal structure, which have been intensively studied recently, as well as systems such as thin films and multilayers, which were considered earlier in the context of the SWT [92], require a more thorough study in the framework of the above-considered approaches.

Interesting problems arise upon the description of systems with frustrated magnetic structures on a two-dimensional square lattice with due regard for exchange interactions between next-nearest neighbors [93 – 97], as well as on a two-dimensional triangular lattice [98 – 103], Kagome lattices, pyrochlore systems [104, 105], etc. The presence of spin frustrations in such systems leads, as in low-dimensional compounds, to the suppression of long-range magnetic order (with some short-range order having been retained) and, consequently, to very unusual thermodynamic properties. Frustrated systems were also considered in the context of spin-wave theories [106 – 111].

One more problem, which is important, in particular, upon the investigation of high-temperature superconductivity and has not been touched in this review, is the interaction of charge carriers with magnetic moments. The specificity of low-dimensional systems (strong short-range magnetic order) leads to corresponding features in their electron spectrum [112, 113]. Strong electron – electron interaction under these conditions is an additional factor leading to the formation of incoherent electron states and to the possibility of metal – insulator transition. In this connection, intense theoretical and experimental investigations of conducting low-dimensional systems near such a transition point are being widely performed [114 – 116] now. These investigations require the development of substantially new approaches in which, however, the theoretical methods of description of a subsystem of localized moments, considered in this review, can successfully be used.

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