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Generation of squeezed (sub-Poissonian) light by a multimode laser

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Contents

Abstract. **Theoretical and experimental results of investigations into the quantum noise of multimode laser radiation are considered. The feasibility of generating light with a photon-number-squeezed (sub-Poissonian) photon distribution by a multimode laser with a homogeneously broadened line is analyzed. The conditions of noisy and noiseless (regular) pumping are considered. Photon-number fluctuations of the net laser radiation summed over all generated modes are calculated in the approximation of equidistant equal modes, as are photon-number fluctuations in an individual mode inside and outside the resonator. Output-radiation noise spectra and photon-number fluctuations are calculated for solid-state (neodymium glass, Nd:YAG) and semiconductor lasers. Theoretical results are compared with a number of experimental data obtained for semiconductor lasers in recent years.**

1. Introduction

1.1 Photon noise and squeezed light

The quantum nature of light introduces fundamentally irremovable fluctuations and noise into the radiation of both natural and artificial electromagnetic-field sources. The level of photon-number fluctuations in the electromagnetic field induced by deterministic classical currents corresponds to the shot-noise level. The electromagnetic field produced under these conditions resides in a quantum-mechanical coherent state with a Poissonian photon-number distribution, in which the variance of the number of photons is equal to their average number: $\langle(\Delta n)^2\rangle = \langle n\rangle$. The quantum coherent state, which is the eigenstate of the field photon annihilation operator, is of the fundamental importance in quantum optics. The nonclassical field states that are characterized by a photon-number fluctuation level below the standard shot-noise level which is characteristic of the coherent state, and these fluctuations are such that

$$\langle(\Delta n)^2\rangle < \langle n\rangle\,,$$

are referred to as sub-Poissonian states.

The variances of field quadratures $p = (a^+ - a)\,\mathrm{i}/2$, $q = (a^+ + a)/2$, the momentum and the coordinate in the phase space of the harmonic oscillator of normal field modes, where a^+ and a are the photon production and annihilation operators, are equal in the coherent state: $\langle(\Delta p)^2\rangle = \langle(\Delta q)^2\rangle = 1/4$, and minimize the Heisenberg uncertainty relation

$$\langle(\Delta p)^2\rangle\langle(\Delta q)^2\rangle \geqslant \frac{1}{16}\,.$$

The fluctuations in quadrature field components, corresponding to the coherent state, constitute the fundamental limit of the electromagnetic field fluctuation level and, consequently, define the limiting accuracy of its measurement. A lowering of noise in one of the field quadratures to a level below this limit and the corresponding rise in noise in the canonically conjugated quadrature represent essentially nonclassical effects.

The electromagnetic field states described in the foregoing and referred to as squeezed states are characterized by a lowered level of field photon-number and/or phase noise, as well as of quadrature field components [1 – 4].

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The pursuance of precision optical experiments employing different light sources generated a need for profound and systematic investigations into the quantum fluctuations of electromagnetic fields, which in turn led to the discovery of several new physical effects caused by the quantum nature of light and matter (see the following reviews and thematic collections of papers [1 – 3, 5 – 15]). Among them is the effect of light quadrature squeezing and the existence of light states with a photon-number fluctuation level below the level of quantum shot noise (sub-Poissonian states). This effect arises from well-known nonlinear optical processes like lasing [5, 14, 16 – 62], parametric scattering [1 – 3, 6 – 8], optical bistability [1, 2, 63, 64], and four-wave scattering [1 – 3, 65], as well as some others [15, 66].

The phenomenon of squeezing is of practical significance. Basic investigations into the quantum phenomena emerging in the interaction of light with a medium underlie the rapid progress along new promising lines of theoretical and experimental research into optical information transfer [10, 67, 68], quantum cryptography [67, 68], and quantum computers [11]. The use of electromagnetic field observables pertaining to a quadrature with a lowered noise level (for instance, the field intensity $E \propto a - a^{+}$) opens up the possibility of measuring low-intensity quantum fields with an accuracy which exceeds the fundamental limitation arising from the irremovable quantum fluctuations of a vacuum. Among the applications of squeezed light, mention should also be made of low-noise optical communication, high-precision atomic spectral measurements, and precise interferometric experiments [12, 13, 69, 70], as well as its employment in the photodetection of weak optical signals [65]. Quantum noise related to the preparation of a signal and its measurement defines the upper bound for the information capacity of a channel. The eigenstate of the photon-number operator, or the Fock state of light, whereby the photon-number fluctuations are equal to zero, is optimal for optical communication, because the ideal limit for the information capacity of an optical channel is realized in precisely the Fock state of a signal [10]. Furthermore, the employment of a single-photon light state for optical information transfer eliminates the possibility of intercepting the transmitted data, even for an open communication channel.

The squeezed-state electromagnetic field finds use in the implementation of quantum nondestructive measurements of electromagnetic field characteristics [12, 13, 71], as well as in high-precision experiments in atomic interferometry and in the refinement of frequency standards [12, 13, 69]. In particular, the employment of sub-Poissonian light in atomic Ramsey spectroscopy was shown to improve the signal-to-noise ratio [69].

The quest for reliable light sources possessing nonclassical properties, i.e., obeying sub-Poissonian photon-number statistics or squeezed in amplitude or phase quadrature of the field, is now one of the most topical problems of quantum optics. The mathematical apparatus elaborated during the last 30 years for the quantum theory of open systems, i.e., systems possessing fluctuations and dissipation, permits conducting quantitative investigations of quantum-statistical radiation properties. The progress in experimental investigations achieved in recent years causes us to anticipate that practically required sources of strongly squeezed light will be developed in the not distant future.

1.2 Quantum noise in single-mode lasers

At present, the main sources of coherent radiation are lasers and quantum optical traveling-wave amplifiers. Detailed quantitative investigations into the quantum fluctuations and noise of the electromagnetic field produced by these devices are a topical problem of today. The development and improvement of ways of solving this problem, as well as the quest for laser generation schemes which permit controlling the level of light noise and fluctuations arising from the interaction of the light with the amplifying medium, are the central problem of quantum optics. Solving this problem would open the way to the development of practical applications reliant on novel technologies and broaden the potentialities for employing lasers as instruments of basic physical research.

The fluctuations in laser radiation intensity are determined by two factors: intensity fluctuations due to technical factors, and those arising from the quantum nature of light – matter interactions, so-called quantum fluctuations. Considered as quantum fluctuations are the spontaneous emission of atoms of the active laser medium (polarization fluctuations in the active medium), fluctuations of the electromagnetic vacuum, and also the quantum fluctuations in laser pumping (fluctuations in level populations of the active medium). The fluctuations arising for technical reasons and the possible chaotic dynamics of lasing may be removed almost completely under experimental conditions, while quantum fluctuations are inherently fundamental and are inevitably present in the quantum-statistical laser dynamics. The Poissonian photon statistics (the shot noise), which are typical for laser radiation when the lasing threshold is exceeded, may be interpreted in the framework of quantum mechanics as a consequence of the quantum fluctuation processes mentioned above.

The feasibility of lowering the level of laser intensity fluctuations to a level lower than the fundamental Poissonian level has been carefully considered in several investigations during the last 20 years. These investigations suggest that the number of photons in the laser radiation when the lasing threshold is exceeded may, under special conditions, obey sub-Poissonian statistics. The level of quantum noise of single-mode laser radiation was shown to be in direct quantitative dependence on the parameters of the active medium, the excess over the lasing threshold, and the kind of pumping [16 – 29].

An efficient method for generating sub-Poissonian light is to employ noiseless (regular) pumping [16]. As shown in Refs [16, 28], in the absence of pump fluctuations for a large excess of the lasing threshold, the photon-number fluctuations in the resonator of a three-level laser may, neglecting the population depletion of the lower atomic level, attain the minimal level defined by only the electromagnetic vacuum fluctuations. This is an indication that the spontaneous emission fluctuations are completely suppressed under these conditions. In this case, the squeezing in the fluctuation spectrum of the output-radiation intensity turns out to be complete (100% squeezing). The influence of atomic coherence on the magnitude of photon-number and field-phase fluctuations was also considered in Refs [25, 26] for a pump with different statistical properties. As shown by the authors of Ref. [26], the atomic coherence created by preparing the active atoms in a coherent superposition of the states participating in lasing in the regular-pump case results in the complete suppression of spontaneous noise. In this case, the photon statistics are sub-Poissonian, i.e., the photon-number variance is below the shot-noise level.

It was found in Refs [18–24, 29] that the quantum photon noise is also settled below the standard Poissonian level under noisy (Gaussian) pumping in three- and four-level laser schemes upon exceeding the lasing threshold. Approximate numerical and analytical calculations have shown that the noise produced by spontaneous emission and the pumping is effectively suppressed for specific ratios between the spontaneous decay constants of atomic energy levels. The fluctuation minimum is equal to 3/4 of the average number of photons in the laser resonator in the three-level scheme, and to 2/3 in the four-level scheme. In this case, the noise level of a laser output radiation is two times lower than the shot-noise level for a three-level laser, and five times for a four-level laser. This phenomenon is referred to as the dynamic suppression of quantum laser noise [19, 21].

A single-atom laser, which has been actively investigated in recent years, may serve as a source of sub-Poissonian light [14, 30–40]. The feasibility of generating strongly squeezed light by a single-atom single-mode laser was theoretically demonstrated for two-, three-, and four-level schemes in Refs [30, 36, 37]. Experimental research which is being pursued at the present time as well has confirmed the feasibility of squeezed-state generation both by a single-atom laser [35] and a few-atom laser ($N_a \sim 10$) [36].

A quantum-dot laser can also be placed among single-atom lasers capable of generating sub-Poissonian light. It was predicted in Refs [31–33] that the level of the photon noise for this laser may, under certain lasing conditions, be substantially lower than the shot-noise level. The theoretical results were experimentally borne out in Ref. [34]. The theory of a quantum-dot laser demonstrates that this laser may be considered as a potential source of single-photon Fock states of light, which find use in optical information transfer and quantum cryptography [14].

Successful experiments on the production of sub-Poissonian light by means of semiconductor lasers with regular pumping were carried out in Refs [41–45]. Measurements of the photocurrent fluctuation spectrum and the photon statistics of the radiation from the semiconductor lasers revealed the existence of substantial squeezing. In particular, the attainment of 85% squeezing (−8.3 dB) with a complete suppression of pump noise was reported in Ref. [44].

The idea of producing sub-Poissonian light by means of a feedback laser was developed in Refs [15, 41, 48, 72]. The feasibility of lowering the photon noise of semiconductor laser radiation with the help of a feedback circuit was investigated in Refs [41–44]. When use is made of negative electronic feedback, the variable (fluctuation) component of the current in the photodetector which measures the output laser field intensity is, upon inverting its phase and amplification, mixed with the pump current which, in turn, modulates the intensity of output laser radiation received by the photodetector. The photon-number fluctuations measured in Ref. [41] in this closed feedback circuit turn out to be well below the Poisson level and may run to 0.26 of the average number of photons. However, extracting the squeezed light from the closed circuit by means of a beam splitter turned out to be impossible, because the vacuum field arriving at the free input of the beam splitter disturbed the quantum correlations between the intensities of the two fields emanating from the beam splitter. As a consequence, the field extracted with the help of the beam splitter exhibits super-Poissonian photon statistics for a fluctuation level substantially higher than the level of shot noise. The authors of Ref. [41] proposed a way of extracting squeezed light from the feedback loop with the aid of a nondestructive photon-number measurement using a nondestructive detector reliant on the optical Kerr effect. In this scheme, the photon flux of laser radiation is transmitted through the transparent Kerr medium to modulate the refractive index in the medium according to the temporal dependence of the photon flux $N(t)$. An additional probe field passes through the Kerr medium with the modulated refractive index and acquires a phase delay reflecting the temporal dependence of the photon flux $N(t)$. The probe field next arrives at an optical phase detector [41], which fulfills the function of reading the temporal dependence of the photon flux without destroying it.

The current fed from the optical phase detector is then mixed with the pump current of the semiconductor laser in the electronic feedback circuit. As shown in Ref. [41], in circumstances where the uncertainty in the measurement data obtained by the phase detector is insignificant and the feedback circuit possesses a high gain, the fluctuations in laser output photon flux can be suppressed to an arbitrarily low level.

The method of nondestructive photodetection for lowering the laser radiation fluctuation level, proposed in Ref. [41], still remains experimentally unrealized. It is pertinent to note that the fundamental complexity of performing the nondestructive measurement is due to dissipation processes proceeding in the Kerr medium [71], as well as to the weakness of the cubic nonlinearity of the medium.

A different scheme of a feedback laser for generating sub-Poissonian light was conceived in Ref. [48]. The authors of this work considered a laser in which the output mirror transmission was controlled by the current of the photodetector measuring the intensity of the light transmitted through the mirror. Theoretical calculations predicted a photon-number fluctuation level 25% below the shot-noise level.

The level of quantum fluctuations in a single-mode ring neodymium laser (Nd:YAG) was theoretically and experimentally investigated in Refs [46, 47] for the injection of an external signal with different photon statistics. The results of calculations and experiments revealed that the use of a weak external signal permits lowering the level of laser radiation noise and results in sub-Poissonian light generation in this scheme.

1.3 Quantum noise in multimode lasers

The first works dedicated to the theoretical investigation of the quantum fluctuations in multimode laser radiation date back to the mid-1960s. In Ref. [73], the fluctuation spectra for the photon numbers inside a resonator for the total field, summed over all field modes, and for the field in an individual resonator mode were calculated on the basis of Langevin equations with a phenomenological inclusion of dissipation processes and the fluctuations in the field and atomic variables of the laser. The author of Ref. [73] arrived at the conclusion that the fluctuations in total laser radiation stabilize to attain the shot-noise level (the Poissonian photon distribution) on exceeding the lasing threshold. At the same time, the radiation in an individual mode resides in the thermal equilibrium state (the Bose–Einstein distribution) for an arbitrary excess of pump power over the lasing threshold and an arbitrary number of modes participating in the laser emission. To carry out the calculations of Ref. [73], use was made of rate (balance) equations for radiating modes with identical parameters.

A similar conclusion was reached in Ref. [74], where the calculations relied on the kinetic equations for the diagonal elements of the system's density matrix in the equivalent-mode approximation. The same approach was employed in the analysis of multimode laser fluctuations in a more recent work [75] for the case of different modes. The conclusions of the authors of Ref. [75] repeat the conclusions made in Ref. [74].

It has been known [76, pp. 523 – 525] that when the radiation contains Q equally filled independent modes of an electromagnetic field in an equilibrium state with the Bose – Einstein photon statistics, the field summed over all modes possesses a Poissonian photon distribution with the average $\langle n\rangle = Q\langle n_q\rangle$, where $\langle n_q\rangle$ is the average number of photons in the qth mode, for $Q \gg \langle n\rangle$. The calculations of Refs [73 – 75] for the radiation of a multimode laser led to the same results.

The theoretical calculations and experimental research carried out for a multimode dye laser in Ref. [77] allowed its authors to draw conclusions about the level of quantum fluctuations in multimode laser radiation, which are radically different from the theoretical predictions in Refs [73, 74]. The investigations showed that significant photon-number fluctuations in an individual laser mode are attributable to the complex chaotic dynamics of the multimode generation inherent in dye lasers, with the level of quantum fluctuations turning out to be insignificant in this case. Substantial fluctuations typical for thermal radiation and comparable to the average photon-number value emerge only in the immediate vicinity of the oscillation threshold for a given mode. Further theoretical and experimental investigations of the fluctuations in multimode lasers bore out the conclusions made in Ref. [77].

The development of the quantum theory of multimode laser generation opened the door to carrying out *ab initio* calculations of photon-number fluctuations. Invoking the general Heisenberg – Langevin equations which correctly describe the interaction of the atomic and field laser subsystems with the reservoirs responsible for dissipation processes and fluctuations resulted in the discovery of complex quantum-statistical properties of multimode laser radiation.

The authors of Ref. [50] predicted the feasibility of generating multimode laser radiation with nonclassical statistical properties. As shown in Refs [50, 51], the photon-number fluctuations may be both above the shot-noise level (the Poissonian distribution) and substantially below this level (the squeezed state of light), depending on the laser parameters. A similar conclusion was drawn by the authors of Ref. [52] who experimentally discovered the intensity squeezing of the total radiation from a diode laser with a regular pump which did not introduce additional quantum noise. The calculations carried out in Ref. [52] revealed that the fluctuations in both the overall intensity and the radiation intensity in an individual mode were substantially lower than the fluctuations in equilibrium thermal radiation.

Numerous experimental works [53 – 61], which followed Refs [50 – 52] and were carried out for different multimode semiconductor lasers, confirmed the conclusions made in these latter papers.

Sub-Poissonian radiation statistics in an individual mode of a multimode diode laser using an external optical signal are predicted, in particular, by the calculations performed in Ref. [62].

2. Quantum theory of a multimode laser in the approximation of a spectrum of equivalent modes

We treat an electromagnetic field in a resonator as its expansion in terms of normal resonator eigenmodes. For the atomic subsystem, advantage is taken of the Lax – Louisell four-level model whereby an atom excited by an incoherent pumping changes from the ground state $|0\rangle$ to the upper energy state $|3\rangle$. Then the electron experiences a nonradiative transition (with a high transition probability) from the latter state to the upper laser level $|2\rangle$ which is related to the lower laser level $|1\rangle$ by a radiative transition. Therefore, efficient pumping occurs directly to level $|2\rangle$, and state $|3\rangle$ may be excluded from the analysis of laser dynamics.

In the rotating-wave approximation, the Hamiltonian of the system comprising an electromagnetic field experiencing a dipole interaction with a two-level atom is represented in the form

$$H_{\mathrm{S}} = H_{\mathrm{S}0} + V_{\mathrm{int}} = \sum_q \hbar\omega_q a_q^+ a_q + \sum_{j=0,1,2} \hbar\omega_j \big(|j\rangle\langle j|\big)_n + \mathrm{i}\hbar \sum_q \mu_q \big[a_q^+ \big(|1\rangle\langle 2|\big)_n - \big(|2\rangle\langle 1|\big)_n a_q\big] ,$$

where $H_{\mathrm{S}0}$ is the sum of the energy operators of the atom and the field, V_{int} is their interaction operator, and a_q^+ is the production operator for the photons in the qth normal discrete cavity mode. In the derivation of the last term, which characterizes the multimode field – atom interaction, we employed the dipole approximation; in this case, the coupling constant is given by

$$\mu_q = \sqrt{\frac{2\pi\omega_q}{V\hbar}}\, d_{12} ,$$

where d_{12} is the transition matrix element of the atomic dipole moment operator, V is the resonator volume, $\omega_q = \pi c q/A$, A is the resonator length, c is the speed of light in vacuum, and q is an integer. The small difference in coupling constants for different modes will be disregarded in the subsequent calculations. Therefore, we will consider the equivalent mode approximation, which is valid for lasers with intermode frequency separations that are much smaller than the resonator eigenfrequencies, even for a large excess over the lasing threshold. We will also assume that the eigenfrequency of one of the resonator modes is close to the $|2\rangle \to |1\rangle$ atomic transition frequency.

We proceed from the assumption that the total Hamiltonian consists of the Hamiltonian H_{S} of the system and the Hamiltonians of the atomic (R_{A}) and field (R_{F}) subsystem's reservoirs, as well as the interaction of the corresponding reservoirs with the atomic and field subsystems:

$$H = H_{\mathrm{S}} + R_{\mathrm{A}} + R_{\mathrm{F}} + V_{\mathrm{A-R}} + V_{\mathrm{F-R}} .$$

We expand the atom – reservoir interaction operator in terms of the basis operators made up of the eigenvectors of the atomic states basis:

$$V_{\mathrm{A-R}} = \hbar \sum_{m,n} f_{mn} |m\rangle\langle n| .$$

Here, f_{mn} is the operator pertaining to the atomic reservoir. In a similar way, the field – reservoir interaction operator is taken as follows:

$$V_{\text{F-R}} = \mathrm{i}\hbar \sum_q (a_q^+ \Gamma_q - \Gamma_q^+ a_q)\,,$$

where Γ_q is the operator of the reservoir relating to mode q. The last expression was obtained in the rotating-wave approximation.

To exclude the reservoir variables from the differential equations of motion for the operators of the system, in the framework of a stochastic description of the system's dynamics we take advantage of the Markovian approximation for stochastic reservoir operators. The system's operator variations arising from their interaction with the reservoirs will be calculated on a finite time interval Δt longer than the reciprocals of the eigenfrequencies of these operators in the Heisenberg approximation, but shorter than the reservoir operator correlation times τ (the quantities corresponding to the system – reservoir 'collision times'), and then Δt will be turned to zero. The contribution from reservoirs to the system's operator dynamics is reflected in the presence of relaxation terms in the equations for average quantum-mechanical quantities. The interaction with the reservoirs gives rise to transitions between the atomic-subsystem states, as well as to field damping in the resonator modes. The stochastic equations of motion required for studying fluctuations may be derived according to Langevin's approach from the equations for the averages by supplementing them with random noise-source operators. We thereby obtain the system of quantum-mechanical stochastic equations for the operators of the field and atoms.

The Heisenberg – Langevin system of the equations of motion for a three-level multimode laser with a homogeneously broadened line, which was consistently derived in Ref. [50], is of the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\, a_q = -\frac{\gamma_q}{2}\, a_q + \mu_q \sigma \exp(\mathrm{i}\Delta_{q\mathrm{A}} t) + F_q(t)\,, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, (a_{q'}^+ a_{q''}) = -\frac{\gamma_{q'} + \gamma_{q''}}{2}\, a_{q'}^+ a_{q''} + B_{q'q''} + F_{q'}^+(t)\, a_{q''} + a_{q'}^+ F_{q''}(t)\,, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, N_1 = -\Gamma_1 N_1 + \sum_{q=1}^{Q} B_{qq} + F_{11}(t)\,, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, N_2 = -\Gamma_2 N_2 + N_{\mathrm{A}} w_{02} - \sum_{q=1}^{Q} B_{qq} + F_{22}(t)\,, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \sigma = -\Gamma\sigma + \sum_{q=1}^{Q} \mu_q a_q \exp(-\mathrm{i}\Delta_{q\mathrm{A}} t)\, N_{\mathrm{A}}(\sigma_{22} - \sigma_{11}) + F_{12}(t)\,, \tag{5}$$

if

$$B_{q'q''}(t) \equiv \mu_{q'} a_{q'}^+ \sigma \exp(\mathrm{i}\Delta_{q''\mathrm{A}} t) + \mu_{q''} \sigma^+ a_{q''} \exp(-\mathrm{i}\Delta_{q'\mathrm{A}} t), \tag{6}$$

$$\Delta_{q\mathrm{A}} \equiv \omega_q - \omega_{\mathrm{A}}\,,$$

where $a_q^+(a_q)$ are the photon production (annihilation) operators for the qth mode of the electromagnetic resonator field, $a_{q'}^+ a_{q''}$ are the cross operators for different field modes when $q'' \neq q'$ or the photon-number operators for an individual field mode when $q'' = q'$, N_1 and N_2 are the population operators of lower $|1\rangle$ and upper $|2\rangle$ atomic laser levels:

$$N_j = \sum_{n=1}^{N_{\mathrm{A}}} \big(|j\rangle\langle j|\big)_n\,, \quad j = 1, 2,$$

σ is the operator of atomic polarization (total induced dipole moment summed over all atoms) of the medium, $\sigma = \sum_{n=1}^{N_{\mathrm{A}}} (|1\rangle\langle 2|)_n \exp(\mathrm{i}\omega_{\mathrm{A}} t)$, N_{A} is the total number of active atoms in the medium, and w_{02} is the pumping rate of the upper laser state $|2\rangle$ produced by exciting the lower electronic state $|0\rangle$ of the three-level system. The dissipation coefficients $\gamma_q, \gamma_{q'}, \gamma_{q''}, \Gamma_1, \Gamma_2, \Gamma = \Gamma_{\mathrm{ph}} + (\Gamma_1 + \Gamma_2)/2$ (where Γ_{ph} is the phase relaxation constant for the atomic dipole moment, arising from elastic collisions) entering into the equations carry information about external reservoirs and are expressed in terms of the integrals of the correlators of reservoir-related operators. The operators $F_\alpha(t)$, $\alpha = q, q', q'', 11, 12, 22$ on the right-hand sides of Eqns (1) – (5) are the Langevin sources of quantum noise arising from the interaction of the system observed with external reservoirs. Due to the assumption of the Markov property made above, the Langevin sources possess δ-correlated second momenta which are time-dependent in the general case. The statistical properties of stochastic Langevin operators ensure the conservation in time of the commutation relations between Heisenberg operators for the system undergoing dynamical transformation with the inclusion of fluctuations and dissipation [78, 79].

The equivalent-mode approximation which is employed in the subsequent calculations consists in the assumption that the field loss rates are equal, as are the atom – field coupling constants for all field modes, viz.

$$\gamma = \gamma_q \propto \omega_q^3\,, \quad \mu = \mu_q \propto \sqrt{\omega_q}\,, \quad q = 1, 2, \ldots, Q\,. \tag{7}$$

The cubic dependence of the field losses through the resonator mirror follows from the quantum-mechanical relationship $\gamma_q = 2\pi\rho(\omega_q)\big|\mu_q(\omega_q)\big|^2$, where $\rho(\omega_q)$ is the density of field states in empty space [76].

Under conditions where the decay rates of medium polarization (Γ) and atomic populations (Γ_1 and Γ_2) are much higher than the field damping rate γ, it is possible to perform the adiabatic elimination of atomic laser variables from Eqns (1) – (7) [50]. Then, the Heisenberg – Langevin equations for a multimode laser take on the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\, a_q = -\frac{\gamma}{2}\, a_q + \frac{\Pi D}{2} \sum_{q'} a_{q'} \exp(\mathrm{i}\Delta_{qq'} t) + G_q\,, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, (a_{q'}^+ a_{q''}) = -\gamma a_{q'}^+ a_{q''} + \frac{\Pi D}{2} \sum_q \big[a_q^+ a_{q'} \exp(\mathrm{i}\Delta_{qq'} t) + a_{q'}^+ a_q \exp(-\mathrm{i}\Delta_{qq''} t)\big] + \Pi N_2 + G_{q'q''}\,, \tag{9}$$

$$\frac{\mathrm{d}n_q}{\mathrm{d}t} = -\gamma n_q + \Pi D n_q + \frac{\Pi D}{2} \sum_{q' \neq q} \bar{B}_{q'q} + \Pi N_2 + G_{qq}\,, \tag{10}$$

$$\frac{dN}{dt} = -\gamma N + \Pi D \sum_q n_q + \frac{\Pi D}{2} \sum_q \sum_{q' \neq q} \bar{B}_{q'q} + Q\Pi N_2 + \sum_q G_{qq} \,, \tag{11}$$

$$\frac{dD}{dt} = -\Gamma_2 D + (\Gamma_1 - \Gamma_2) N_1 + R - 2\Pi D \sum_q n_q - \Pi D \sum_{q' \neq q''} \bar{B}_{q'q''} + G_2 - G_1 \,, \tag{12}$$

$$\frac{dN_1}{dt} = -\Gamma_1 N_1 + \Pi D \sum_q n_q + \frac{\Pi D}{2} \sum_{q' \neq q''} \bar{B}_{q'q''} + G_1 \,, \tag{13}$$

$$\frac{dN_2}{dt} = -\Gamma_2 N_2 + R - \Pi D \sum_q n_q - \frac{\Pi D}{2} \sum_{q' \neq q''} \bar{B}_{q'q''} + G_2 \,, \tag{14}$$

where the terms bilinear in field operators and responsible for the coherent field – atoms interaction are written as

$$\bar{B}_{q'q''} \equiv a_{q'}^+ a_{q''} \exp(i\Delta_{q'q''} t) + \text{H.c.} \tag{15}$$

Here, we introduced the notation

$$D \equiv N_2 - N_1 \,, \quad R \equiv w_{02} N_A \,, \quad \Pi \equiv \frac{2\mu^2}{\Gamma} \,.$$

New random sources in Eqns (8) – (14) are expressed in the following form

$$G_1 = N_A F_{11} + \sum_q N_A F_{B_q} \,, \quad G_2 = N_A F_{22} - \sum_q N_A F_{B_q} \,,$$

$$F_{B_q} = \frac{\mu}{\Gamma} \Big\{ a_{qc}^+ F_{12} \exp\left[-i(\omega_A - \omega_q)\, t\right] + F_{12}^+ a_{qc} \exp\left[i(\omega_A - \omega_q)\, t\right] \Big\} \,,$$

$$a_{qc} \equiv a_q(t - \varepsilon) \,, \quad \varepsilon \to 0 \,,$$

$$G_{a_q} = \frac{N_A \mu}{\Gamma} F_{12} \exp\left[i(\omega_q - \omega_A)\, t\right] + F_q \,,$$

$$G_{q'q''} = a_{q'c}^+ F_{q'}^+ a_{q''c} + \frac{\mu}{\Gamma} \Big\{ a_{q'c}^+ F_{12} \exp\left[-i(\omega_A - \omega_{q''})\, t\right] + F_{12}^+ a_{q''c} \exp\left[i(\omega_A - \omega_{q'})\, t\right] \Big\} = a_{q'c}^+ G_{a_{q'}} + G_{a_{q'}}^+ a_{q''c} \,.$$

For a complete description of the quantum laser dynamics by Eqns (8) – (14) it is also required to define the quantum-statistical properties of the operators of the random Langevin forces appearing in Eqns (8) – (14). To this end, we take advantage of the following approach (see also Refs [50, 51]).

In the general case, the stochastic Heisenberg – Langevin equations for an arbitrary operator M_α of the system may be written as

$$\frac{d}{dt} M_\alpha(t) = A_\alpha(t) + G_\alpha(t) \,,$$

where A_α is the displacement operator for the given equation, and G_α is the Langevin operator of the equation for operator M_α. Then, the diffusion coefficients of the stochastic equations (8) – (14) of motion, having the form $\langle 2D_{\alpha\beta} \rangle = \big\langle \Delta M_\alpha(t)\, \Delta M_\beta(t) \big\rangle / \Delta t$, may be calculated from the generalized Einstein relation

$$\frac{d}{dt} \big\langle M_\alpha(t)\, M_\beta(t) \big\rangle = 2 \big\langle D_{\alpha\beta}(t) \big\rangle + \big\langle A_\alpha(t)\, M_\beta(t) \big\rangle + \big\langle M_\alpha(t)\, A_\beta(t) \big\rangle \,,$$

which follows from the identity

$$\Delta M_\alpha(t)\, \Delta M_\beta(t) = \Delta\big(M_\alpha(t)\, M_\beta(t)\big) + A_\alpha(t)\, \Delta M_\beta(t) + \Delta M_\alpha(t)\, A_\beta(t) \,,$$

where

$$\Delta M_x(t) \equiv M_x(t + \Delta t) - M_x(t) \,, \quad x = \alpha, \beta,$$

$$\Delta\big(M_\alpha(t)\, M_\beta(t)\big) \equiv M_\alpha(t + \Delta t)\, M_\beta(t + \Delta t) - M_\alpha(t)\, M_\beta(t) \,.$$

Assuming that the random Langevin sources G are δ-correlated (the Markovian approximation), we find for the corresponding diffusion coefficients [23]:

$$\big\langle G_i(t)\, G_j(u) \big\rangle = \langle 2D_{ij} \rangle\, \delta(t - u) \,, \tag{16}$$

$$\langle 2D_{11} \rangle = \Gamma_1 \langle N_1 \rangle + S\Pi \sum_q \langle n_q \rangle + \Pi \langle N_2 \rangle Q + \Sigma_1 \,, \tag{17}$$

$$\langle 2D_{22} \rangle = R + \Gamma_2 \langle N_2 \rangle + S\Pi \sum_q \langle n_q \rangle + \Pi \langle N_2 \rangle Q + \Sigma_1 \,, \tag{18}$$

$$\langle 2D_{12} \rangle = -S\Pi \sum_q \langle n_q \rangle - \Pi \langle N_2 \rangle Q - \Sigma_1 \,, \tag{19}$$

$$\langle 2D_{n_q n_q} \rangle = \gamma \langle n_q \rangle + S\Pi \langle n_q \rangle + \Pi \langle N_2 \rangle, \;\; \langle 2D_{n_q n_{q'}} \rangle = \Sigma_{2q}, \tag{20}$$

$$\langle 2D_{1n_q} \rangle = S\Pi \langle n_q \rangle + \Pi \langle N_2 \rangle + \Sigma_{2q} \,, \tag{21}$$

$$\langle 2D_{2n_q} \rangle = -S\Pi \langle n_q \rangle - \Pi \langle N_2 \rangle - \Sigma_{2q} \,, \tag{22}$$

where we introduced the notation

$$\Sigma_1 \equiv \Pi \sum_{q' \neq q''} \big\{ \langle a_{q'}^+ a_{q''} \rangle \exp(i\Delta_{q'q''} t) \langle N_1 \rangle + \langle a_{q'} a_{q''}^+ \rangle \exp(-i\Delta_{q'q''} t) \langle N_2 \rangle \big\}, \tag{23}$$

$$\Sigma_{2q} \equiv \Pi \sum_{q' \neq q} \big\{ \langle a_{q'}^+ a_q \rangle \exp(i\Delta_{q'q} t) \langle N_1 \rangle + \langle a_{q'} a_q^+ \rangle \exp(-i\Delta_{q'q} t) \langle N_2 \rangle \big\} \,, \tag{24}$$

$$S \equiv \langle N_1 \rangle + \langle N_2 \rangle \,.$$

3. Stationary averages of laser variables (the working point of a laser)

With the goal of subsequently calculating the photon-number fluctuations in laser radiation, we shall find the stationary solutions of exact equations (1) – (5) for the quantum-mechanical averages of the operators of the system. To do this, we set the time derivatives $d/dt \langle \ldots \rangle$ equal to zero in Eqns (1) – (7) averaged over the system's and reservoir variables. Considering that the average of Langevin source operators, taken over the variables of the corresponding reservoirs, is zero, we arrive at the following system of algebraic equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle a_q\rangle = -\frac{\gamma}{2}\,\langle a_q\rangle + \mu\langle\sigma\rangle\exp(\mathrm{i}\Delta_{q\mathrm{A}}t) = 0\,, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle a^+_{q'}a_{q''}\rangle = -\gamma\langle a^+_{q'}a_{q''}\rangle + \langle B_{q'q''}\rangle = 0\,, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\sigma\rangle = -\Gamma\langle\sigma\rangle + \sum_{q=1}^{Q}\mu\big\langle a_q(N_2-N_1)\big\rangle\exp(-\mathrm{i}\Delta_{q\mathrm{A}}t) = 0\,, \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle N_1\rangle = -\Gamma_1\langle N_1\rangle + \sum_{q=1}^{Q}\langle B_{qq}\rangle = 0\,, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle N_2\rangle = -\Gamma_2\langle N_2\rangle + N_{\mathrm{A}}w_{02} - \sum_{q=1}^{Q}\langle B_{qq}\rangle = 0\,, \tag{29}$$

$$D \equiv N_2 - N_1 = N_{\mathrm{A}}(\sigma_{22}-\sigma_{11})\,,\quad \Delta_{q\mathrm{A}} \equiv \omega_q - \omega_{\mathrm{A}}\,, \tag{30}$$

$$R \equiv N_{\mathrm{A}}w_{02}\,. \tag{31}$$

We find the stationary average population inversion from Eqns (25) and (26) for the field and polarization:

$$-\frac{\gamma}{2}\langle a_q\rangle + \mu\langle\sigma\rangle\exp(\mathrm{i}\Delta_{q\mathrm{A}}t) = 0\,, \tag{32}$$

$$-\Gamma\langle\sigma\rangle + \sum_{q=1}^{Q}\mu\langle a_qD\rangle\exp(-\mathrm{i}\Delta_{q\mathrm{A}}t) = 0\,. \tag{33}$$

Then we assume that the atomic and field variables exhibit no mutual correlation on exceeding the lasing threshold:

$$\langle a_qD\rangle \approx \langle a_q\rangle\langle D\rangle\,, \tag{34}$$

which is valid for a large number of active atoms and a large number of photons in every field mode [79].

From Eqn (32) it follows that

$$\langle a_q\rangle = \frac{2\mu}{\gamma}\,\langle\sigma\rangle\exp(\mathrm{i}\Delta_{q\mathrm{A}}t)\,,\ \forall q\,, \tag{35}$$

$$\langle\sigma\rangle = \frac{\gamma}{2\mu}\,\langle a_q\rangle\exp(-\mathrm{i}\Delta_{q\mathrm{A}}t)\,. \tag{36}$$

In view of approximation (34), from expression (33) we obtain

$$\langle\sigma\rangle = \frac{\mu}{\Gamma}\sum_q\langle a_q\rangle\langle D\rangle\exp(-\mathrm{i}\Delta_{q\mathrm{A}}t)\,. \tag{37}$$

From relationships (35) and (37) it follows that

$$\langle\sigma\rangle = \frac{\mu}{\Gamma}\sum_{q'}\langle a_{q'}\rangle\langle D\rangle\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t) = \frac{\gamma}{2\mu}\,\langle a_q\rangle\exp(-\mathrm{i}\Delta_{q\mathrm{A}}t),\quad \forall q\,. \tag{38}$$

Let us sum equality (38) over q:

$$\sum_q\langle\sigma\rangle = Q\langle\sigma\rangle = \frac{\mu}{\Gamma}\sum_q\left(\sum_{q'}\langle a_{q'}\rangle\langle D\rangle\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t)\right) = \frac{\gamma}{2\mu}\sum_q\langle a_q\rangle\exp(-\mathrm{i}\Delta_{q\mathrm{A}}t) = \frac{\mu Q}{\Gamma}\left(\sum_{q'}\langle a_{q'}\rangle\langle D\rangle\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t)\right).$$

Making the change $q \to q'$ in the last equality and introducing the notation for $S \equiv \sum_{q'}\langle a_{q'}\rangle\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t)$, we arrive at

$$\frac{Q\mu}{\Gamma}\,\langle D\rangle\,S = \frac{\gamma}{2\mu}\,S\,,$$

whence we obtain a relationship for the population inversion:

$$\langle D\rangle = \frac{\gamma\Gamma}{2\mu^2Q} = \frac{\gamma}{\Pi Q}\,. \tag{39}$$

From relationship (39) it follows that the population inversion in the case of multimode generation is substantially lower than for single-mode generation ($Q = 1$). Therefore, the effect of gain saturation in the multimode case is substantially stronger than in the single-mode case. It is pertinent to note that the suppression of photon-number fluctuations in the laser radiation upon exceeding the threshold of generation is due to the saturation effect.

Let us consider the terms $B_{q'q''}(t)$ which enter into the system of equations (1)–(5) and are responsible for stimulated emission; they are defined in expression (6) as

$$B_{q'q''}(t) \equiv \big[\mu_{q'}a^+_{q'}\sigma\exp(\mathrm{i}\Delta_{q''\mathrm{A}}t) + \mu_{q''}\sigma^+a_{q''}\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t)\big]\,. \tag{40}$$

In much the same way as was done in expression (34), it will be assumed that $\langle\sigma^+a_{q''}\rangle_0 \approx \langle\sigma^+\rangle\langle a_{q''}\rangle$, $\langle a^+_{q'}\sigma\rangle_0 \approx \langle a^+_{q'}\rangle\langle\sigma\rangle$. By also setting $\mu = \mu_q$, $\forall q$, we substitute formula (36) into expression (40). Then, for the quantum-mechanical average we obtain

$$\big\langle B_{q'q''}(t)\big\rangle \approx \mu\,\frac{\gamma}{2\mu}\,\big[\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t)\langle a^+_{q'}\rangle\langle a_{q'}\rangle\exp(\mathrm{i}\Delta_{q''\mathrm{A}}t) + \langle a^+_{q''}\rangle\langle a_{q''}\rangle\exp(\mathrm{i}\Delta_{q''\mathrm{A}}t)\exp(-\mathrm{i}\Delta_{q'\mathrm{A}}t)\big]\,.$$

Consequently, it can easily be shown that

$$\big\langle B_{q'q''}(t)\big\rangle \approx \frac{\gamma}{2}\big(\langle a^+_{q'}\rangle\langle a_{q'}\rangle + \langle a^+_{q''}\rangle\langle a_{q''}\rangle\big)\exp(\mathrm{i}\Delta_{q''q'}t) = \gamma\langle a^+_q\rangle\langle a_q\rangle\exp(\mathrm{i}\Delta_{q''q'}t)\,,\quad \forall q,\ q',\ q''\,, \tag{41}$$

because

$$\langle a_q\rangle = \langle a_{q'}\rangle\exp(\mathrm{i}\Delta_{qq'}t) = \langle a_{q''}\rangle\exp(\mathrm{i}\Delta_{qq''}t)\,, \tag{42}$$

$$\langle a^+_q\rangle = \langle a^+_{q'}\rangle\exp(-\mathrm{i}\Delta_{qq'}t) = \langle a^+_{q''}\rangle\exp(-\mathrm{i}\Delta_{qq''}t)\,.$$

Then we find the stationary averages for laser level populations and the number of photons from the equations for N_1, N_2 and $a^+_{q'}a_{q''}$.

To do this, we take advantage of Eqns (28) and (29) and relationships (39) and (41). Straightforward calculations yield

$$-\Gamma_2N_{20} + R - \sum_q B_{qq} = -\Gamma_2N_{20} + R - \sum_q\gamma\langle a^+_q\rangle\langle a_q\rangle = 0, \tag{43}$$

$$-\Gamma_1N_{10} + \sum_q\gamma\langle a^+_q\rangle\langle a_q\rangle = 0\,. \tag{44}$$

We introduce the designation $\rho \equiv \langle a_q^+ \rangle \langle a_q \rangle$. Using expressions (42), from Eqn (43) we obtain

$$N_{20} = \frac{R - Q\gamma\rho}{\Gamma_2} . \tag{45}$$

Substituting formula (45) into Eqn (44) gives

$$\rho = \frac{\bar{\Gamma}}{\Pi Q^2} \left(\frac{\Pi R Q}{\gamma \Gamma_2} - 1 \right) . \tag{46}$$

So, we have obtained the stationary average values of the atomic variables D_0, N_{10}, N_{20}, and ρ. Now, from Eqns (26) for $\langle a_{q'}^+ a_{q''} \rangle$ we find $\langle n_{q0} \rangle$ and $N_{\mathrm{ph}\,0} = \sum_{q=1}^{Q} \langle n_{q0} \rangle$ — the average value of the number of photons in an individual field mode and the total number of photons in all field modes, respectively.

For $q' = q''$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle n_q \rangle = -\gamma \langle n_q \rangle + \langle B_{qq} \rangle = 0 ,$$

$$\langle B_{qq} \rangle = \gamma \langle a_q^+ \rangle \langle a_q \rangle = \gamma\rho ,$$

whence it follows that

$$n_0 \equiv \langle n_{q0} \rangle = \frac{\langle B_{qq} \rangle}{\gamma} = \frac{\gamma\rho}{\gamma} = \rho . \tag{47}$$

Therefore, we have shown that

$$\langle n_{q0} \rangle = \langle a_q^+ a_q \rangle = \langle a_q^+ \rangle \langle a_q \rangle = \frac{\bar{\Gamma}}{\Pi Q^2} \left(\frac{\Pi R Q}{\gamma \Gamma_2} - 1 \right) . \tag{48}$$

Similarly, one finds

$$\langle a_{q'}^+ a_{q''} \rangle = n_0 \exp\left(\mathrm{i} \varDelta_{q''q'} t\right) . \tag{49}$$

4. Quasilinearization procedure for balance equations and calculation of photon-number fluctuations in a resonator

To calculate photon fluctuations, let us resort to the balance equation approximation. To do this, we average over time the terms rapidly oscillating in time and being proportional to $\bar{B}_{q',q''}(t)$, which are defined by expression (15) and enter into the right-hand sides of Eqns (10)–(14). We apply the ergodic theorem and replace the time averaging with the quantum-mechanical averaging over the ensemble of realizations, namely

$$\begin{aligned} &\overline{\frac{\Pi D}{2} B_{q'q''}(t)} \\ &= \overline{\frac{\Pi D}{2} a_{q'}^+ a_{q''} \exp\left(\mathrm{i}\varDelta_{q'q''} t\right) + \frac{\Pi D}{2} a_{q''}^+ a_{q'} \exp\left(-\mathrm{i}\varDelta_{q'q''} t\right)} \\ &= \left\langle \frac{\Pi D}{2} B_{q'q''}(t) \right\rangle = \left\langle \frac{\Pi D}{2} a_{q'}^+ a_{q''} \right\rangle \exp\left(\mathrm{i}\varDelta_{q'q''} t\right) \\ &+ \left\langle \frac{\Pi D}{2} a_{q''}^+ a_{q'} \right\rangle \exp\left(-\mathrm{i}\varDelta_{q'q''} t\right) . \end{aligned} \tag{50}$$

We next employ the assumption concerning the factorization of atomic and field variable correlators of the form $\langle a_{q'}^+ a_{q''} D \rangle \approx \langle a_{q'}^+ a_{q''} \rangle \langle D \rangle$. Since the time-averaged quantities do not depend on this assumption, by taking into account relationship (49) and substituting the average stationary values of D_0 and n_0 into the averaged terms we arrive at the following system of stochastic balance equations for laser variables:

$$\frac{\mathrm{d}n_q(t)}{\mathrm{d}t} = -\gamma n_q(t) + \Pi D(t)\, n_q(t) + C_{n_q} + G_{n_q}(t) , \tag{51}$$

$$\frac{\mathrm{d}N_{\mathrm{ph}}(t)}{\mathrm{d}t} = -\gamma N_{\mathrm{ph}}(t) + \Pi D(t)\, N_{\mathrm{ph}}(t) + C_{N_{\mathrm{ph}}} + G_{N_{\mathrm{ph}}}(t) , \tag{52}$$

$$\begin{aligned} \frac{\mathrm{d}D(t)}{\mathrm{d}t} &= -\Gamma_2 D(t) + (\Gamma_1 - \Gamma_2)\, N_1(t) - 2\Pi D(t)\, N_{\mathrm{ph}}(t) + R \\ &\quad + C_D + G_2(t) - G_1(t) , \end{aligned} \tag{53}$$

$$\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = -\Gamma_1 N_1(t) + \Pi D(t)\, N_{\mathrm{ph}}(t) + C_1 + G_1(t) . \tag{54}$$

The values of constants $\{C_i\}$ (average values) depend on the selection of the working point of laser oscillation. In particular, for the working point (39), (45), (48) we obtain $C_{n_q} = \gamma n_0 (1 - 1/Q) = C_{N_{\mathrm{ph}}}/Q$, $C_{N_1} = -C_{N_2} = C_{N_{\mathrm{ph}}} = -C_D/2$.

It is easy to verify that the stationary solutions of quantum-mechanical averaged approximate kinetic equations (51)–(54), namely

$$\left\langle \frac{\mathrm{d}n_q}{\mathrm{d}t} \right\rangle, \left\langle \frac{\mathrm{d}N_{\mathrm{ph}}}{\mathrm{d}t} \right\rangle, \left\langle \frac{\mathrm{d}D}{\mathrm{d}t} \right\rangle, \left\langle \frac{\mathrm{d}N_1}{\mathrm{d}t} \right\rangle = 0 ,$$

coincide with the working point (39), (45), (48), which was found above by way of the solution of exact equations of motion.

We write down the equations in the following variables

$$\begin{aligned} &\Delta n_q(t) = n_q(t) - n_0 , \quad \Delta N_{\mathrm{ph}}(t) = N_{\mathrm{ph}}(t) - N_{\mathrm{ph}\,0} , \\ &\Delta D(t) = D(t) - D_0 , \quad \Delta N_1(t) = N_1(t) - N_{10} , \end{aligned} \tag{55}$$

which characterize the departure of variables from their average stationary values.

Neglecting the terms nonlinear in small deviations gives

$$\begin{aligned} \frac{\mathrm{d}\Delta D(t)}{\mathrm{d}t} &= -\Gamma_2 \Delta D(t) + (\Gamma_1 - \Gamma_2)\, \Delta N_1(t) \\ &\quad - 2\Pi\big(D_0\, \Delta N_{\mathrm{ph}}(t) + \Delta D N_{\mathrm{ph}\,0}(t)\big) + G_2(t) - G_1(t) , \end{aligned} \tag{56}$$

$$\begin{aligned} \frac{\mathrm{d}\Delta N_1(t)}{\mathrm{d}t} &= -\Gamma_1 \Delta N_1(t) \\ &\quad + \Pi\big(D_0 \Delta N_{\mathrm{ph}}(t) + N_{\mathrm{ph}\,0}\, \Delta D(t)\big) + G_1(t) , \end{aligned} \tag{57}$$

$$\begin{aligned} \frac{\mathrm{d}\Delta N_{\mathrm{ph}}(t)}{\mathrm{d}t} &= -\gamma \Delta N_{\mathrm{ph}}(t) \\ &\quad + \Pi\big(D_0 \Delta N_{\mathrm{ph}}(t) + N_{\mathrm{ph}\,0} \Delta D(t)\big) + G_{N_{\mathrm{ph}}}(t) , \end{aligned} \tag{58}$$

$$\frac{\mathrm{d}\Delta n_q(t)}{\mathrm{d}t} = -\gamma \Delta n_q(t) + \Pi\big(D_0 \Delta n_q(t) + n_0 \Delta D(t)\big) + G_{n_q}(t) . \tag{59}$$

We next perform the standard Fourier transform for the system of equations (56)–(59):

$$\begin{aligned} &-\mathrm{i}\omega \Delta D(\omega) = -\Gamma_2 \Delta D(\omega) + (\Gamma_1 - \Gamma_2)\, \Delta N_1(\omega) \\ &- 2\Pi\big(D_0\, \Delta N_{\mathrm{ph}}(\omega) + \Delta D(\omega)\, N_{\mathrm{ph}\,0}\big) + G_2(\omega) - G_1(\omega), \end{aligned} \tag{60}$$

$$- \mathrm{i}\omega \Delta N_1(\omega) = -\Gamma_1 \Delta N_1(\omega) + \Pi\big(D_0 \Delta N_{\mathrm{ph}}(\omega) + N_{\mathrm{ph}\,0} \Delta D(\omega)\big) + G_1(\omega)\,, \tag{61}$$

$$- \mathrm{i}\omega \Delta N_{\mathrm{ph}}(\omega) = -\gamma \Delta N_{\mathrm{ph}}(\omega) + \Pi\big(D_0 \Delta N_{\mathrm{ph}}(\omega) + N_{\mathrm{ph}\,0} \Delta D(\omega)\big) + G_{N_{\mathrm{ph}}}(\omega)\,, \tag{62}$$

$$- \mathrm{i}\omega \Delta n_q(\omega) = -\gamma \Delta n_q(\omega) + \Pi\big(D_0 \Delta n_q(\omega) + n_0 \Delta D(\omega)\big) + G_{n_q}(\omega)\,. \tag{63}$$

From Eqns (60) and (61) we then obtain

$$\Delta D(\omega) = \frac{1}{\zeta(\omega)}\,\big[(\Gamma_1 - \Gamma_2)\,\Pi D_0 \Delta N_{\mathrm{ph}}(\omega) - z_2 G_1(\omega) - 2\Pi D_0 z_1 \Delta N_{\mathrm{ph}}(\omega) + z_1 G_2(\omega)\big]\,, \tag{64}$$

where

$$z_j \equiv -\mathrm{i}\omega + \Gamma_j\,, \quad j = 1, 2\,, \tag{65}$$

$$\zeta \equiv z_1 z_2 + (z_2 + z_2)\,\Pi N_{\mathrm{ph}\,0}\,. \tag{66}$$

Substituting Eqn (64) into Eqn (62) gives

$$\Delta N_{\mathrm{ph}}(\omega) = \frac{-\Pi z_2 N_{\mathrm{ph}\,0} G_1(\omega) + \Pi z_1 N_{\mathrm{ph}\,0} G_2(\omega) + \zeta G_{N_{\mathrm{ph}}}(\omega)}{\zeta_1(\omega)}\,, \tag{67}$$

$$\zeta_1 \equiv \zeta \varDelta + z \Pi Q^{-1} N_{\mathrm{ph}\,0}\,, \tag{68}$$

$$\varDelta \equiv -\mathrm{i}\omega + \gamma\left(1 - \frac{1}{Q}\right), \quad z = z_1 + z_2\,. \tag{69}$$

According to the Wiener–Khintchin theorem, the fluctuation spectrum of a stationary random process $A(t)$ satisfies the relation

$$S_A(\omega) \equiv \big\langle A^+(\omega)\,A(\omega)\big\rangle = \int_{-\infty}^{\infty} \mathrm{d}\tau \exp(-\mathrm{i}\omega\tau) \big\langle A^+(\tau)\,A(0)\big\rangle,$$

where $A(\omega)$ is the Fourier transform of $A(t)$. By setting $A(t) = \Delta N(t)$ we arrive at

$$\big\langle \Delta N^+(\omega)\,\Delta N(\omega)\big\rangle = S_{\Delta N}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega' \big\langle \Delta N^+(\omega)\,\Delta N(\omega')\big\rangle = \int_{-\infty}^{\infty} \mathrm{d}\omega' \big\langle \Delta N^2(\omega)\big\rangle\,\delta(\omega + \omega') = \big\langle \Delta N^2(\omega)\big\rangle\,,$$

where use was made of the δ-correlatedness of the Fourier transforms of Langevin random sources $\big\langle F_\alpha(\omega)\,F_\beta(\omega')\big\rangle = \langle 2D_{\alpha\beta}\rangle\,\delta(\omega + \omega')$ entering into the Fourier transforms of quasilinearized equations of motion.

Now, to express $\langle(\Delta N)^2\rangle$ in terms of $\langle \Delta N^2(\omega)\rangle$ we take advantage of the Wiener–Khintchin theorem:

$$\big\langle \Delta N^2(\omega)\big\rangle = \int_{-\infty}^{\infty} \mathrm{d}\tau \exp(-\mathrm{i}\omega\tau) \big\langle \Delta N^+(\tau)\,\Delta N(0)\big\rangle\,.$$

By integrating with respect to ω we obtain

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \big\langle \Delta N^2(\omega)\big\rangle = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \exp(-\mathrm{i}\omega\tau)\,\mathrm{d}\omega\right) \mathrm{d}\tau \big\langle \Delta N^+(\tau)\,\Delta N(0)\big\rangle = \int_{-\infty}^{\infty} \mathrm{d}\tau\, 2\pi\delta(\tau) \big\langle \Delta N^+(\tau)\,\Delta N(0)\big\rangle = 2\pi \big\langle (\Delta N(0))^2\big\rangle\,,$$

i.e., for stationary photon-number fluctuations one has

$$\big\langle (\Delta N)^2\big\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \big\langle \Delta N^2(\omega)\big\rangle\,.$$

By using expression (67), for stationary fluctuations of the number of photons in the resonator we find

$$\big\langle (\Delta N_{\mathrm{ph}})^2\big\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \big\langle \Delta N_{\mathrm{ph}}^+(\omega)\,\Delta N_{\mathrm{ph}}(\omega)\big\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, |\,\zeta_1|^{-2} \big\{ |\Pi N_{\mathrm{ph}\,0}\, z_2|^2 \langle 2D_{11}\rangle + |\Pi N_{\mathrm{ph}\,0}\, z_1|^2 \times \langle 2D_{22}\rangle + |\,\zeta|^2 \langle 2D_{N_{\mathrm{ph}} N_{\mathrm{ph}}}\rangle - \Pi N_{\mathrm{ph}\,0}\, 2\,\mathrm{Re}\,(z_2^* \zeta) \langle 2D_{1N_{\mathrm{ph}}}\rangle + \Pi N_{\mathrm{ph}\,0}\, 2\,\mathrm{Re}\,(z_1^* \zeta) \langle 2D_{2N_{\mathrm{ph}}}\rangle - (\Pi N_{\mathrm{ph}\,0})^2\, 2\,\mathrm{Re}\,(z_2^* z_1) \langle D_{12}\rangle \big\}\,, \tag{70}$$

$$\langle 2D_{1N_{\mathrm{ph}}}\rangle = Q\langle 2D_{1q}\rangle\,, \quad \langle 2D_{2N_{\mathrm{ph}}}\rangle = Q\langle 2D_{2q}\rangle\,,$$

$$\langle 2D_{N_{\mathrm{ph}} N_{\mathrm{ph}}}\rangle = Q\langle 2D_{n_q n_q}\rangle + Q(Q-1)\langle 2D_{n_q n_{q'}}\rangle\,.$$

The diffusion coefficients which appear in expressions (70) may be obtained from Eqns (16)–(24) by substituting the average stationary values of N_0, b_0, N_{10}, and D_0; the formulas for them were given in Section 2.

The stationary diffusion coefficients are found from Eqns (16)–(24) by substituting the average stationary quantities $N_{\mathrm{ph}\,0}$, n_0, N_{10}, D_0, and $\langle a_{q'}^+ a_{q''}\rangle_0$, which pertain to the corresponding working point. As is evident from Eqns (23) and (24), the diffusion coefficients depend on the cross field terms; to calculate them, we take advantage of relationship (49). By neglecting small terms we find

$$\langle 2D_{11}\rangle = \Gamma_1 N_{10} + \Pi S_0 N_{\mathrm{ph}\,0} Q\,, \tag{71}$$

$$\langle 2D_{22}\rangle = \Gamma_2 N_{20} + R + \Pi S_0 N_{\mathrm{ph}\,0} Q\,, \tag{72}$$

$$\langle 2D_{12}\rangle = -\Pi S_0 N_{\mathrm{ph}\,0} Q\,, \tag{73}$$

$$\langle 2D_{1n_q}\rangle = -\langle 2D_{2n_q}\rangle = \Pi S_0 n_0 Q\,, \tag{74}$$

$$\langle 2D_{n_q n_{q'}}\rangle = \gamma n_0 \delta_{qq'} + \Pi S_0 n_0 Q\,, \tag{75}$$

where $\delta_{qq'}$ is the Kronecker delta. The small terms $Q\Pi N_{20}$ and $Q^2 \Pi N_{20}$ are neglected when the generation threshold is substantially exceeded.

For the Fourier transform of the departures from the photon-number average in an individual mode, in view of Eqn (63) we obtain [50]

$$\Delta n_q(\omega) = \frac{\alpha G_1(\omega) + \beta G_2(\omega) + \mu G_q(\omega) + \nu \sum_{q' \neq q} G_{q'}(\omega)}{A(\omega) + (Q-1)\,B(\omega)}\,, \tag{76}$$

where

$$\alpha \equiv -z_2 \Pi n_0\,,$$

$$\beta \equiv z_1 \Pi n_0\,,$$

$$\mu \equiv \frac{\zeta}{A-B}\left[A + B(Q-2)\right],$$

$$\nu \equiv -\frac{B\zeta}{A-B}\,,$$

$$A(\omega) \equiv \zeta\Delta + \frac{z\gamma\Pi n_0}{Q}\,,$$

$$B(\omega) \equiv \frac{\Pi n_0 \gamma z}{Q}\,. \qquad (77)$$

Then, for photon-number fluctuations in an individual field mode one finds

$$\left\langle (\Delta n_q)^2 \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \left|A(\omega) + (Q-1)\,B(\omega)\right|^{-2}$$
$$\times \Bigg\{ |\alpha|^2 \langle 2D_{11}\rangle + |\beta|^2 \langle 2D_{22}\rangle + |\nu|^2 \sum_{q',\,q''\neq q} \langle 2D_{q'q''}\rangle$$
$$+ |\mu|^2 \langle 2D_{qq}\rangle + 2\,\mathrm{Re}(\nu^*\mu) \sum_{q'\neq q} \langle 2D_{q'q}\rangle$$
$$+ 2\,\mathrm{Re}(\alpha^*\beta)\langle 2D_{12}\rangle + 2\,\mathrm{Re}(\beta^*\mu)\langle 2D_{2q}\rangle$$
$$+ 2\,\mathrm{Re}(\alpha^*\nu) \sum_{q'\neq q} \langle 2D_{q'1}\rangle + 2\,\mathrm{Re}(\beta^*\nu) \sum_{q'\neq q} \langle 2D_{q'2}\rangle \Bigg\}\,. \qquad (78)$$

5. Photon-noise spectrum of radiation emanating through a resonator mirror

The electromagnetic field outside a resonator comprises the field emanating through the resonator mirror and the field of the electromagnetic thermal reservoir of the continuous mode spectrum of the empty space outside the resonator. For every mode q of laser-generated radiation, the field outside the resonator is represented as the following superposition [80]:

$$a_q^{(\mathrm{out})}(t) = \sqrt{\gamma}\, a_q(t) - b_q^{(\mathrm{in})}(t)\,, \qquad (79)$$

where $a_q(t)$ is the line-spectrum operator for the field inside the resonator, and $b_q^{(\mathrm{in})}(t)$ is the continuous-spectrum operator for the field outside the resonator. The subsequent calculations will allow us to find the form of this operator.

The field relaxation (the resonator loss) stems from the interaction of the discrete modes of the field inside the resonator and the reservoir of the continuous spectrum of the electromagnetic field outside the resonator. The Hamiltonian of the interaction responsible for the field damping in the resonator (the loss in transit through the mirrors) may be written as

$$V_{\mathrm{F-R}} = \hbar \sum_{j,\,q} g_{j,\,q}\left(b_j a_q^+ + a_q b_j^+\right),$$

where $g_{j,q}$ is the coupling constant which may be expressed in terms of the loss rates for the resonator mirrors: $g_{j,q} = |g_{j,q}| \exp(\mathrm{i}\phi_{j,q})$. The coupling constant phase $\phi_{j,q}$ is arbitrary and is defined by the geometry of the problem.

The Heisenberg equations for the field operators $a_q(t)$ and $b_j(t)$ are derived in the form

$$\dot{a}_q(t) = -\frac{\mathrm{i}}{\hbar}\left[a_q(t), H_{\mathrm{S}}\right] - \mathrm{i}\sum_j g_{j,\,q}\, b_j(t) \exp(\mathrm{i}\Delta_{qj}\, t)\,, \qquad (80)$$

$$\dot{b}_j(t) = -\mathrm{i} g_{j,\,q}^*\, a_q(t) \exp(-\mathrm{i}\Delta_{gj}\, t)\,.$$

The last term in the former equation corresponds to the random source in the Langevin equations for resonator modes. It is easily seen that the random sources are expressed in terms of reservoir operators as $F_q(t) = -\mathrm{i}\sum_j g_{j,q}\, b_j(0) \exp(\mathrm{i}\Delta_{q,j}\, t)$, $\Delta_{q,j} = \omega_q - \omega_j$. The resonator loss rates γ_q may be expressed in terms of the coupling constant and the density of field states: $\gamma_q = 2\pi |g(\omega_q)|^2 \rho(\omega_q) \approx \gamma$, where $g(\omega_q) = g_{j,q}$ for $\omega_k = \omega_q$, and $\rho(\omega)$ is the density of states of the empty space [81]. For the coupling constant we now have

$$\left|g(\omega_q)\right| = \sqrt{\frac{\gamma}{2\pi\rho(\omega_q)}}\,.$$

Let us perform the formal integration of the Heisenberg equation for $b_j(t)$ and substitute the so-obtained solutions, which contain dependences on the field operators a_q, into the following expression for the operator of the total field beyond the resonator:

$$a_q^{(\mathrm{out})}(t) = \mathrm{i}\exp(\mathrm{i}\phi_q) \sum_j \left(\frac{2\pi\omega_j}{\rho(\omega_q)\,\omega_q}\right)^{1/2} b_j(t) \exp(-\mathrm{i}\Delta_{q,j}\, t)\,,$$

which we write down in photon flux units. In the resultant expression we replace the summation over the reservoir modes j with integration and apply the Markovian approximation [81] to obtain the following relations:

$$a_q^{(\mathrm{out})}(t) \equiv E_{\mathrm{free}}^{(+)}(t) + E_{\mathrm{source}}^{(+)}(t)$$
$$= \exp(\mathrm{i}\phi_q) \sum_j \left(\frac{2\pi\omega_j}{\rho(\omega_j)\,\omega_q}\right)^{1/2} b_j(0) \exp(-\mathrm{i}\Delta_{q,j}\, t)$$
$$+ \sqrt{\gamma}\, a_q(t)\,. \qquad (79\mathrm{a})$$

Here, the operators $E_{\mathrm{free}}^{(+)}(t)$ and $E_{\mathrm{source}}^{(+)}(t)$ describe the free evolution of the reservoir field and the evolution of the field of a laser source, which is transmitted through the mirror. Since the radiation measured at the laser output contains the component of laser radiation transmitted through the mirror and the component of reservoir field reflected from the mirror, for the phase of the coupling parameter $g(\omega_q) = |g_q| \exp(\mathrm{i}\phi_q)$ we assume the value of $\phi_q = \pi$.

Therefore, we have determined the total external field operator (79a) which comprises both the reservoir field operators and the operator of the laser field of the qth resonator mode, emanating through the mirrors. It is easily seen that relation (79a) coincides with relation (79) when

$$b_q^{(\mathrm{in})}(t) = \sum_j \left(\frac{2\pi\omega_j}{\rho(\omega_j)\omega_q}\right)^{1/2} b_j(0) \exp(-\mathrm{i}\Delta_{q,j}\, t)\,.$$

The Heisenberg operator $a^{(\mathrm{out})+}(t)\, a^{(\mathrm{out})}(t)$ represents the operator of the number of photons emanating through a mirror per unit time. The operators $a_q^{(\mathrm{out})}$ and $b_q^{(\mathrm{in})}$ satisfy the commutation relations for continuous-spectrum Bose operators of the form $\left[b_q^{(\mathrm{in})}(t), b_{q'}^{+(\mathrm{in})}(u)\right] = \delta(t-u)\,\delta_{q,q'}$. For the operators of a reservoir which is in equilibrium at a temperature T we have [80, 81]

$$\left\langle b_q^{(\mathrm{in})+}(t)\, b_{q'}^{(\mathrm{in})}(u)\right\rangle = \bar{n}_{qT}\delta(t-u)\,\delta_{qq'}\,,$$

$$\left\langle b_q^{(\mathrm{in})}(t)\, b_{q'}^{(\mathrm{in})+}(u)\right\rangle = (\bar{n}_{qT}+1)\,\delta(t-u)\,\delta_{qq'}\,. \qquad (81)$$

It may be shown that the operators $b_q^{(\mathrm{in})}(t)$ are related to Langevin random source operators $F_q(t)$, which appear in Eqn (1), by the relation $F_q(t)=\sqrt{\gamma}\, b_q^{(\mathrm{in})}(t)$.

It would be well to express the field operators inside and outside the resonator in terms of amplitude and phase operators:

$$a_q^+(t) = \left(r_{q0}(t)+\Delta r_q(t)\right)\exp\left\{-\mathrm{i}\left[\phi_{q0}(t)+\Delta\phi_q(t)\right]\right\},$$

$$a_q^{(\mathrm{out})+}(t) = \left(r_{q0}^{(\mathrm{out})}(t)+\Delta r_q^{(\mathrm{out})}(t)\right)\exp\left\{-\mathrm{i}\left[\psi_{q0}(t)+\Delta\psi_q(t)\right]\right\},$$

$$a_q(t) = \exp\left\{\mathrm{i}\left[\phi_{q0}(t)+\Delta\phi_q(t)\right]\right\}\left(r_{q0}(t)+\Delta r_q(t)\right), \qquad (82)$$

$$a_q^{(\mathrm{out})}(t) = \exp\left\{\mathrm{i}\left[\psi_{q0}(t)+\Delta\psi_q(t)\right]\right\}\left(r_{q0}^{(\mathrm{out})}(t)+\Delta r_q^{(\mathrm{out})}(t)\right).$$

Here, r_{q0}, $r_{q0}^{(\mathrm{out})}$, ϕ_{q0}, and ψ_{q0} are the average values of field amplitudes and phases, Δr_{q0}, $\Delta r_q^{(\mathrm{out})}$, $\Delta\phi_q$, and $\Delta\psi_q$ are the operators of the fluctuations in field amplitudes and phases; the amplitude and phase operators are expressed in the form $r_q(t)=r_{q0}(t)+\Delta r_q(t)$, $\phi_q(t)=\phi_{q0}(t)+\Delta\phi_q(t)$, etc. The Hermitian operators of the amplitude and phase of a given form may be correctly defined in the framework of the approach introduced by Pegg and Barnett (see, for instance, Ref. [76]). In this case, the following relations

$$n_q(t) = \left(r_{q0}(t)+\Delta r_q(t)\right)^2,$$

$$n_q^{(\mathrm{out})}(t) = \left(r_{q0}^{(\mathrm{out})}(t)+\Delta r_q^{(\mathrm{out})}(t)\right)^2 \qquad (83)$$

are fulfilled. We substitute expression (82) into relation (79) to find

$$\Delta r_q^{(\mathrm{out})}(t) = \sqrt{\gamma}\,\Delta r_q(t) - \frac{1}{2}\left[\exp\left(\mathrm{i}\psi_q(t)\right) b_q^{(\mathrm{in})}(t) + b_q^{+(\mathrm{in})}(t)\exp\left(-\mathrm{i}\psi_q(t)\right)\right]. \quad (84)$$

In the derivation of expression (84), use was made of the relationships $\cos(\Delta\psi_q-\Delta\phi_q)\approx 1$, $\langle\Delta\phi_q\rangle=\langle\Delta\psi_q\rangle=0$, and $\phi_{q0}=\psi_{q0}$, the last of which follows from Eqn (79a).

Since it follows from relations (83) that

$$\Delta r_q(t) = \frac{\Delta n_q(t)}{2r_{q0}(t)}\,, \quad \Delta r_q^{(\mathrm{out})}(t) = \frac{\Delta n_q^{(\mathrm{out})}(t)}{2r_{q0}^{(\mathrm{out})}(t)}\,,$$

$$n_{q0}^{(\mathrm{out})}(t) = \gamma n_{q0}(t)\,,$$

we find from expression (84) that

$$\Delta n_q^{(\mathrm{out})}(t) = \gamma\Delta n_q(t) - F_{rq}(t)\,, \qquad (85)$$

where

$$F_{rq}(t) = 2\sqrt{\gamma n_{q0}}\,f_{rq}(t)\,,$$

$$f_{rq}(t) \equiv \frac{1}{2}\Big(\exp\left(\mathrm{i}\psi_q(t)\right) b_q^{(\mathrm{in})}(t) + b_q^{+(\mathrm{in})}(t)\exp\left(-\mathrm{i}\psi_q(t)\right)\Big). \qquad (86)$$

Taking into account expressions (81), it is easy to verify that

$$\left\langle f_{rq}(t) f_{rq'}^+(u)\right\rangle = \frac{1}{2}\left(\bar{n}_{qT}+\frac{1}{2}\right)\delta(t-u)\,\delta_{qq'}\,, \qquad (87)$$

$$\left\langle F_{rq}(t)\, F_{rq'}^+(u)\right\rangle = 2\gamma n_{q0}\left(\bar{n}_{qT}+\frac{1}{2}\right)\delta(t-u)\,\delta_{qq'}\,. \qquad (88)$$

For the total-photon-number operator $N=\sum_{q=1}^{Q} n_q$ we have

$$\Delta N^{(\mathrm{out})}(t) = \gamma\Delta N(t) - F_r(t),$$

$$F_r(t) = 2\sqrt{\gamma}\sum_q \sqrt{n_{q0}}\,f_{rq}(t), \qquad (89)$$

$$\left\langle F_r(t)\, F_r^+(u)\right\rangle = 2\gamma N_0\,\delta(t-u)\,,$$

$$\bar{n}_{qT} \ll 1\,, \quad \forall q\,. \qquad (90)$$

The quantity that characterizes the statistics of laser radiation as it passes through the output mirror of a laser resonator is the stationary photon-number fluctuation spectrum (spectral density) of the following form

$$S_{\Delta x}^{(\mathrm{out})}(\omega) \equiv \left\langle\left(\Delta x^{(\mathrm{out})}(\omega)\right)^2\right\rangle = \int_{-\infty}^{\infty}\left\langle\Delta x^{(\mathrm{out})+}(\omega)\,\Delta x^{(\mathrm{out})}(\omega')\right\rangle \mathrm{d}\omega' = \int_{-\infty}^{\infty}\left\langle\left(\Delta x^{(\mathrm{out})}(\omega)\right)^2\right\rangle\delta(\omega-\omega')\,\mathrm{d}\omega'\,, \quad x=\Delta N,\ \Delta n_q\,. \qquad (91)$$

With the help of Eqns (85) and (89) we find, in view of formulas (67) and (76) for the Fourier transforms $\Delta N(\omega)$ and $\Delta n_q(\omega)$, as well as of the relationship $\langle G_{x,\omega}^+ G_{y,\omega'}\rangle = \langle 2D_{xy}\rangle\,\delta(\omega-\omega')$, $x,y=1,2,q,q'$, the total-photon-number fluctuation spectrum (spectral noise density) at the resonator output:

$$S_{\Delta N}^{(\mathrm{out})}(\omega) = \gamma^2\left\langle\Delta N^2(\omega)\right\rangle + \gamma N_0 - \gamma\left[\left\langle\Delta N^+(\omega)\, F_{\mathrm{r}}(\omega)\right\rangle + \left\langle F_r^+(\omega)\,\Delta N(\omega)\right\rangle\right]. \qquad (92\mathrm{a})$$

In much the same way we obtain the photon-number fluctuation spectrum for an individual mode:

$$S_{\Delta n_q}^{(\mathrm{out})}(\omega) = \gamma^2\left\langle\Delta n_q^2(\omega)\right\rangle + \gamma n_0 - \gamma\left[\left\langle\Delta n_q^+(\omega)\, F_{\mathrm{r}}(\omega)\right\rangle + \left\langle F_r^+(\omega)\,\Delta n_q(\omega)\right\rangle\right]. \qquad (92\mathrm{b})$$

Let us calculate the spectral correlation functions which enter into the fluctuation spectra of the form (92). To do this, we take advantage of expression (67) for the Fourier component of the operator $\Delta N(\omega)$ of total-photon-number fluctuations inside the resonator. Substituting Eqn (67) into expression (92a) gives the following expression for the spectral Fano factor defined as $V_{\Delta N}^{(\mathrm{out})}=S_{\Delta N}^{(\mathrm{out})}(\omega)/(\gamma N_0)$:

$$V_{\Delta N}^{(\mathrm{out})}(\omega) = \frac{S_{\Delta N}^{(\mathrm{out})}(\omega)}{\gamma N_0} = \frac{1}{\gamma N_0}\Big\{|\zeta_1|^{-2}\big[|\Pi N_0 z_2|^2\langle 2D_{11}\rangle + |\Pi N_0 z_1|^2\langle 2D_{22}\rangle + |\zeta|^2\langle 2D_{NN}\rangle - \Pi N_0\, 2\,\mathrm{Re}\,(z_2^*\zeta)\langle 2D_{1N}\rangle + \Pi N_0\, 2\,\mathrm{Re}\,(z_1^*\zeta)\langle 2D_{2N}\rangle - (\Pi N_0)^2\, 2\,\mathrm{Re}\,(z_2^* z_1)\langle 2D_{12}\rangle\big]\Big\} - 2\gamma^2\,\mathrm{Re}\left(\frac{\zeta_1}{\zeta}\right) + 1\,, \quad (93)$$

where

$$\langle 2D_{1N}\rangle = Q\langle 2D_{1n_q}\rangle, \quad \langle 2D_{2N}\rangle = Q\langle 2D_{2n_q}\rangle,$$

$$\langle 2D_{NN}\rangle = Q\langle 2D_{n_q n_q}\rangle + (Q-1)\,Q\langle 2D_{n_q n_{q'}}\rangle, \quad q' \neq q\,. \quad (94)$$

Along similar lines, the spectral Fano factor $S^{(\text{out})}_{\Delta n}(\omega)/(\gamma n_0)$ for laser radiation intensity in an individual mode is determined from expression (92b). The spectral photon-noise density $V(\omega)$ normalized to the shot-noise level [the spectral Fano factor (93)] assumes a value smaller than 1 in the squeezing at a fixed frequency in the case of inhomogeneous squeezing of laser radiation, or turns out to be less than 1 at all frequencies for a homogeneous squeezing.

6. Calculated results for the photon-noise spectrum of multimode laser radiation

Among known three-level multimode lasers are neodymium-glass and Nd:YAG lasers. Experiments with the neodymium-glass laser have demonstrated the feasibility of lasing for a large number of modes with intensities close in magnitude [82]. This oscillation regime of a laser with a three-level scheme of atomic energy levels in the active medium is closest to the theoretical model under consideration. In this connection, we carried out calculations of the quantum fluctuations in laser radiation with the use of parameters typical for the solid-state lasers indicated above. Typical values of parameters for these lasers are as follows [82–85]: resonator loss rate $\gamma = 10^8$ s^{-1}, gain parameter $\Pi = 10^{-5}$ s^{-1}, and laser-level relaxation rates $\Gamma_1 = 10^7$ s^{-1}, $\Gamma_2 = 10^3$ s^{-1}.

For a noisy pumping with standard (Poissonian) noise characteristics, as shown in Fig. 1, the maximum total-photon-number squeezing of the field emanating from the resonator is achieved for the optimal value of the excess ξ_Q over threshold for every number $Q > 1$ of modes. For frequencies in the $\omega < \gamma$ range, the spectral Fano factor assumes values below 1. For the values of $\xi > \xi_Q$, the magnitude of squeezing lowers and the fluctuations reach, upon a further increase in ξ, the shot-noise level (Poissonian photon distribution). For the accepted values of laser parameters γ, Γ_1, Γ_2, and Π, appreciable squeezing is possible for greater Q and ξ. The maximum squeezing $V_{\Delta N} = 0.5$ is reached at $Q \approx 101$ and $\xi_Q \approx 100$. For $Q > 101$, the magnitude of squeezing decreases, and at $Q \approx \Gamma_1/\Gamma_2$ the noise level assumes a value of $V_N = 1$ (shot noise) for all ξ.

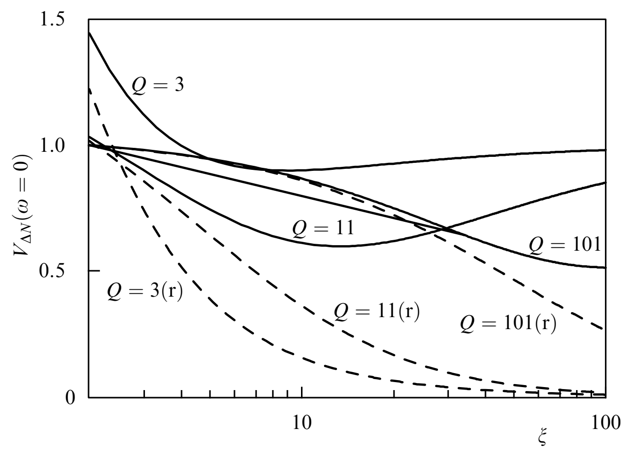

Figure 1. Spectral density of fluctuations at a frequency $\omega \approx 0$ for the total number of photons of laser radiation as a function of excess ξ over lasing threshold for a different number Q of modes. Laser parameters are as follows: $\gamma = 10^9$ s^{-1}, $\Pi = 10^{-5}$ s^{-1}, $\Gamma_1 = 10^7$ s^{-1}, and $\Gamma_2 = 10^3$ s^{-1}. Solid curves show the results for a noisy pumping, and the dashed lines for a regular (r) pumping.

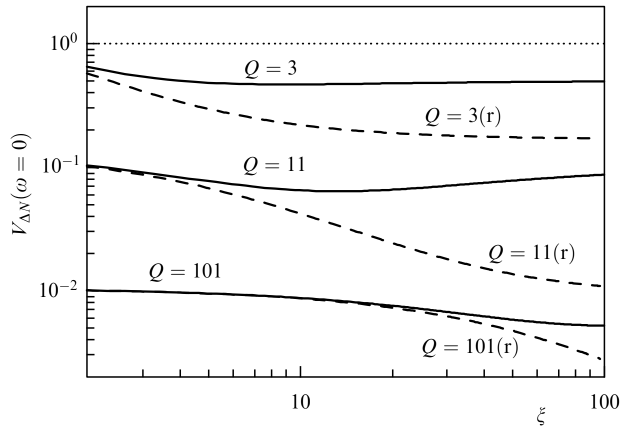

Figure 2. Spectral density of fluctuations at a frequency $\omega \approx 0$ for the number of photons in an individual laser-radiation mode as a function of excess ξ over lasing threshold for a different number Q of modes. Laser parameters are as follows: $\gamma = 10^9$ s^{-1}, $\Pi = 10^{-5}$ s^{-1}, $\Gamma_1 = 10^7$ s^{-1}, and $\Gamma_2 = 10^3$ s^{-1}. Solid curves show the results for a noisy pumping, and the dashed lines for a regular pumping.

An unrestrictedly large photon-number squeezing for the total laser radiation is possible with the use of a regular (noiseless) pump. As shown in Fig. 1, a ten-fold increase in squeezing is reached at $\xi = 70$ for $Q = 3$. We emphasize that a substantial pump-over-threshold excess is required to achieve strong squeezing in the case of a regular pump and a large number of modes. An important prerequisite for the squeezing of the total output laser radiation is the fulfillment of the strong inequality $\Gamma_1 \gg \Gamma_2$ for the decay rates of the upper and lower atomic levels involved in lasing transition.

Laser radiation in an individual mode at the resonator output also exhibits pronounced nonclassical properties. As shown in Fig. 2, in the excess-over-threshold range $2 < \xi < 100$ considered, sub-Poissonian photon statistics in an individual field mode are observed for any number of modes $Q > 1$ participating in lasing. In this case, the noise level monotonically lowers with increasing the number Q of modes. Employing a regular pumping improves the radiation noise characteristics; in this case, the efficiency of applying a regular pumping rises as the excess over lasing threshold increases and as Q decreases. With an increase in the number of modes, the use of a regular pumping does not lead to an appreciable lowering of the quantum noise in an individual laser mode.

As with the total radiation of a multimode laser, the magnitude of squeezing depends on the ratio between the decay rates of atomic levels involved in laser action. A higher decay rate of the lower laser level compared with the upper laser level, $\Gamma_1 \gg \Gamma_2$, is an important prerequisite for the attainment of strong squeezing of the total laser radiation. However, the magnitude of squeezing in an individual field mode depends only slightly on the above ratio. For instance, as laser parameters change ($\Gamma_1 = 10^3$ s^{-1}, $\Gamma_2 = 10^4$ s^{-1}) there is no squeezing of the total radiation at $Q = 3$, while there is almost a two-fold increase in squeezing of individual-mode radiation for all ξ. With retention of the inequality $\Gamma_1 \gg \Gamma_2$ and an invariable γ value, the photon-number fluctuations both inside and outside the resonator depend only slightly on the magnitude of Π.

7. Comparison of theoretical and experimental data on quantum photon-number fluctuations in multimode laser radiation

The problem of generating photon-number-squeezed (sub-Poissonian) radiation by multimode lasers has attracted considerable attention from researchers since the mid-1990s. In Ref. [52], where the phenomenon of multimode laser radiation squeezing was experimentally examined for the first time, it was ascertained that the radiation of a semiconductor laser (laser diode) in the multimode oscillation regime exhibited sub-Poissonian photon statistics, being integrated over all modes. In this work, use was made of a regular pumping of the laser diode, whereby there were no current fluctuations. The theory employed in Ref. [52] to interpret the experimental data also predicted the settling of sub-Poissonian photon-noise level in an individual laser-radiation mode for a high excess above the lasing threshold. Under the oscillation conditions typical for a semiconductor laser, the radiation spectrum comprised the dominant central mode and two side modes with substantially lower intensities in comparison with the central mode intensity. The effect of photon-number fluctuation anticorrelation for different modes, which was experimentally established in Ref. [52], was considered by the authors of the work as the reason for the squeezing of the laser radiation integrated over all modes.

Subsequent research performed by several authors [53–62] for laser diodes confirmed the conclusions reached in Ref. [52]. Presented in Ref. [52], where measurements were made of the integrated radiation of a laser diode with regular pumping, was the dependence of the Fano factor of total laser radiation, which is qualitatively consistent with the results of our calculations. It is noteworthy that the effective two-level theory describing the oscillation of a semiconductor laser corresponds to the theory of a three-level laser, which we consider in the context of adiabatic elimination of active-medium polarization and under the assumption that the decay rate of the lower laser level is far greater than the decay rate of the upper laser level, viz. $\Gamma_1 \gg \Gamma_2$. Our calculations suggest that this ratio between the laser level relaxation rates is optimal for the production of the squeezed laser-radiation state.

In several experimental laser-diode investigations, regular pumping turned out to be insufficient for the generation of squeezed-state light [53, 55–57, 59, 61]. In these cases, the technique of phase and frequency mode locking, which enabled attaining a nearly two-fold photon-number squeezing for the light generated by multimode lasers, was efficient [49, 53].

Photon-number fluctuation measurements were performed in Refs [54, 58, 61] for the total radiation of a laser diode in the absence of regular pumping, for a pump current with Poissonian fluctuations. The investigations showed that the integrated-radiation squeezing is also present in the case of external mode locking. For strengthening the squeezing effect use was made of the feedback technique and the injection of an external signal at the frequency of the central laser mode. Under conditions inherent in a laser diode, when lasing is quasisingle-mode in character, employing the feedback technique, whereby a part of the radiation emanating through the resonator mirror is fed back into the resonator with the aid of a diffraction grating, resulted in the suppression of the side modes participating in the lasing. At the same time, the lasing was effected in the quasisingle-mode regime and the photon-number distribution squeezing in this case was observed in the field integrated over all modes, including the wealth of side modes with intensities that were low in comparison with the central-mode intensity. As a result of locking in the presence of an external feedback signal, the intensities of a large number (~ 100) of the side modes were almost equalized. It was determined in Refs [54–57] that the presence of a large number of laser modes in the measured radiation spectrum is, in the case under consideration, a necessary condition for photon-number squeezing of light, even though the side mode intensities may be much lower than the central-mode intensity. This effect stems from the nonclassical quantum anticorrelation between different laser field modes.

Figure 3 demonstrates the data calculated for the spectrum of noise produced by the total number of laser radiation photons for parameter values typical of a laser diode: $\gamma = 10^{12}$ s^{-1}, $\Pi = 3 \times 10^3$ s^{-1}, $\Gamma_1 = 10^{12}$ s^{-1}, and $\Gamma_2 = 10^9$ s^{-1} [52–58]. Shown for comparison are the experimental data of Ref. [54], in which the feedback technique was employed for diode-laser mode locking. Under the experimental conditions, whereby a small fraction of the output laser radiation at the frequency of the dominant central mode was fed back to the resonator with the help of a diffraction grating, the intensities of a large number of side modes were equalized. The intensity of the feedback field fed to the resonator was rather low — $I \approx 10^{-4} I_{\text{out}}$, where I_{out} is the intensity of the output radiation at the central-mode frequency. As is evident from Fig. 3, the experimental data obtained with an excess $\xi < 10$ over lasing threshold are close to the theoretical values. At $\xi = 2$, the experimental value is nearly equal to the theoretical one for a number of modes equal to $Q = 3$. As the pumping becomes stronger, a constantly increasing number of modes come to participate in multimode lasing. In particular, the experimental data at $\xi = 3$ are close to the theoretical data obtained for $Q = 11$. When ξ increases from 3 to 10, the experimental data are

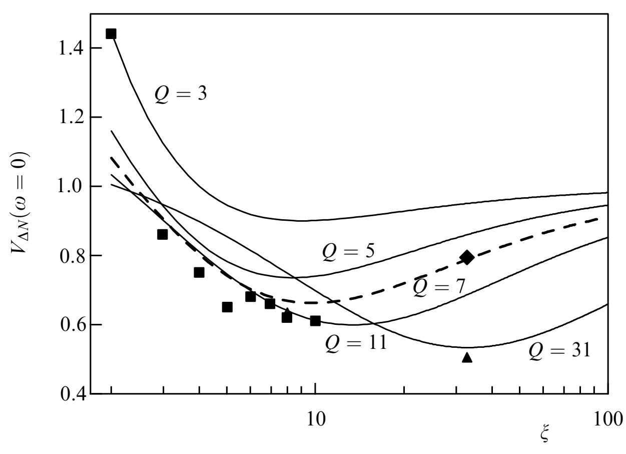

Figure 3. Comparison of the data calculated for the spectral density of fluctuations at a frequency $\omega \approx 0$ in the total number of laser-diode radiation photons for parameter values $\gamma = 10^{12}$ s^{-1}, $\Pi = 3 \times 10^3$ s^{-1}, $\Gamma_1 = 10^{12}$ s^{-1}, and $\Gamma_2 = 10^9$ s^{-1} (curves) with the experimental data of Ref. [58] (squares) and Ref. [54] (triangles and a diamond).

quantitatively consistent with the theoretical ones for a fixed number of modes equal to $Q = 11$.

As shown in Refs [53 – 57], injecting a weak signal at the central-mode frequency leads to the locking of side modes without their suppression, giving rise to total-laser-radiation squeezing, now in the multimode regime, as observed in Refs [55 – 57].

Figure 3 gives the experimental data obtained in Refs [54, 58], where laser-diode mode locking was achieved by injecting a weak signal at the central-mode frequency from another laser. As in the experiments of Refs [55 – 57], a large number of the weak side modes generated by the laser were also equalized in intensity. The experimental data obtained under these conditions in Ref. [58] at $\xi = 8$ agree well with the experimental data of Ref. [54], as well as with our calculated data. Referring to Fig. 3, the noise spectrum is quite close to the theoretical one for $Q = 31$.

It is pertinent to note that a significant photon-number fluctuation squeezing for the total laser radiation was experimentally discovered also in free-running lasing without recourse to an external signal [54]. The experimental value $V_{\Delta N}(\omega = 0) = 0.8$ given in Fig. 3, which was obtained in the free-running lasing regime in Ref. [54], at $\xi = 33$ coincides quantitatively with the theoretical result for $Q = 7$.

Our calculated results are compared in Fig. 4 with the experimental data obtained in Ref. [52] using a regular (noiseless) pumping of a laser diode. In Ref. [52], a direct measurement of the noise spectrum of the laser diode with a regular pumping for different threshold values of the pump current at different temperatures was performed. Referring to Fig. 4, the theoretical findings are in quantitative agreement with both groups of experimental data obtained for different values of the threshold pump current. As in the case of employing an external signal discussed above, with increasing ξ the experimental data come to agree with the theoretical data as the number of modes participating in the lasing increases. In particular, the experimental data at $\xi \approx 2.5$ are close to the theoretical ones for $Q = 1$ (single-mode lasing), at $\xi \approx 3$ an agreement with the experiment is attained for $Q = 3$, and at $\xi \approx 10$ the theoretical and experimental data agree for $Q = 41$. For a group of experimental results obtained for a lower threshold value of the pump current (in these conditions, experimenters managed to achieve a greater excess over the pump threshold — $\xi > 30$), a good agreement with the theory is also reached for $Q = 41$.

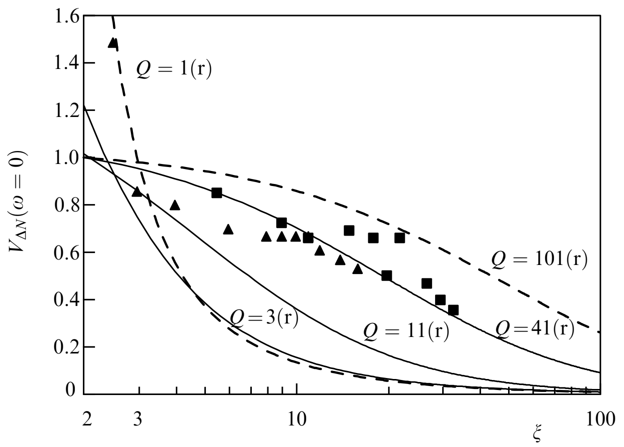

Figure 4. Spectral density of total-photon-number fluctuations at a frequency $\omega \approx 0$ for the radiation of a laser diode with a regular pumping for the parameters $\gamma = 10^{12}$ s^{-1}, $\Pi = 3 \times 10^3$ s^{-1}, $\Gamma_1 = 10^{12}$ s^{-1}, and $\Gamma_2 = 10^9$ s^{-1} (curves). The squares and triangles represent the experimental data of Ref. [52], which were obtained for different temperatures and threshold values of the pump current.

Notice that both the theoretical data obtained by other authors [62] and the experimental measurements of Refs [54, 60] have demonstrated the feasibility of suppressing the photon noise in individual modes (in the dominant central mode, in particular), when the oscillation threshold is substantially exceeded.

The photon-number radiation fluctuations in a vertical-cavity surface-emitting semiconductor laser oscillating in a two-mode regime were measured in Refs [59, 61]. Under the experimental conditions considered, the laser generated two transverse modes with close intensities. As with an ordinary laser diode, the total-photon-number radiation fluctuations for regular pumping were found to be substantially below the Poissonian level. It was experimentally established in Ref. [61] that producing the photon-number-squeezed state of light was possible in two cases: first, for perfect single-mode lasing, and, second, for perfect two-mode lasing whereby the intensities of both modes were equal.

A squeezed state of the radiation of a quantum-well laser diode was experimentally discovered in Ref. [60]. In that paper, the total-photon-number laser-diode radiation squeezing was observed in free-running lasing by two dominant longitudinal resonator modes. An analysis of the experimental data allowed the authors of Ref. [60] to draw a conclusion about the occurrence of light squeezing in both of these modes as well. These experimental data are in qualitative agreement with the predictions of Refs [50, 51].

The data of most recent *ab initio* calculations [62] performed for the specific case of the three-mode oscillation of a semiconductor laser with external mode locking are also in good agreement with experimental data. Described in Ref. [62] is the regime of squeezed-field-state production for the individual central mode in the case of lasing with a fluctuating pumping in the presence of a weak external signal. In this case, it was noted that total-radiation squeezing was possible even for a low intensity of the external signal, while the central-mode squeezing required the higher degree of locking, attainable upon increasing the signal intensity. Photon fluctuations in an individual side field mode are abruptly reduced in the case of three-mode lasing with complete mode locking, considered by the authors of Ref. [62].

We emphasize that a lowering of photon-number fluctuations in an individual field mode was discovered in one of the early papers [86] concerned with measurements of the fluctuation level of a quasisingle-mode semiconductor laser. On exceeding the oscillation threshold, the fluctuation suppression with increasing pumping level was found to occur both for the dominant central mode and for weak side modes.

8. Conclusions

Experimental research into the photon statistics of the radiation from multimode lasers, performed during the last 15 years, unambiguously points to the feasibility of using them as sub-Poissonian nonclassical light sources. The squeezing of light is possible both in the total laser radiation integrated over all modes and in an individual field mode. Our consistent quantum-mechanical calculations of the statistical properties of multimode laser radiation, discussed in this paper, suggest that photon-noise suppression (its two-fold reduction in comparison with the shot-noise level) which is substantial and limited in magnitude is feasible for the total laser radiation for the optimal number of generated modes. The number of modes optimal for producing the squeezed state of light is determined by the parameters of the active medium and laser resonator. The magnitude of squeezing of total radiation, integrated over all modes, is limited by the occurrence of additional quantum noise of the laser pumping. Lowering the pump noise or completely eliminating it (regular pumping) permits an unbounded reduction of the photon-number fluctuations in total laser radiation for the optimal selection of relaxation parameters for the active medium.

The squeezing of radiation in an individual laser mode may be quite significant (the noise spectrum near the zero frequency $V_{\Delta n} \ll 1$) even for a noisy (Poissonian) pumping at a large number of generated modes. However, like for total radiation, the optimal ratio between the relaxation rates of atomic laser states is the necessary condition for achieving the nonclassical character of the radiation in an individual mode: the lower-level relaxation rate must exceed the upper-level relaxation rate ($\Gamma_1 \gg \Gamma_2$). In this case, it turns out that the pump noise, which makes a large contribution to the photon noise of the total laser radiation, is completely suppressed in an individual mode due to the large number of generated modes. Under these conditions, the pump noise is divided between the numerous modes of equal intensity, with the result that its contribution to the noise of each field mode turns out to be insignificant. This mechanism of lowering the noise level below the shot-noise level is characteristic of a laser model with a homogeneously broadened line, when all field modes interact with the same ensemble of active atoms.

The multimode lasing regime with a homogeneous line broadening for the spectrum of equivalent modes considered in our work is a clear demonstration of the significant potentialities of a laser for producing high-intensity squeezed light. Qualitative agreement with experimental data confirms the validity of approximations employed in our calculations.

Examples of the practical application of squeezed (sub-Poissonian) light in different areas of spectroscopy and quantum informatics are well known today, the area of application of the light with reduced noise level becoming progressively broader. In this connection, the investigation into the theoretically predicted generation of sub-Poissonian photon-number-squeezed light by means of multimode lasers remains a topical problem for both theorists and experimenters in many laboratories in a number of countries.

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