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Feynman disentangling of noncommuting operators and group representation theory

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Contents

Abstract. Feynman's method for disentangling noncommuting operators is discussed and applied to nonstationary problems in quantum mechanics, including the excitation of a harmonic oscillator by an external force and/or by time-varying frequency; spin rotation in a time-varying magnetic field; the disentangling of an atom (ion) Hamiltonian in a laser field; a model with the hidden symmetry group of the hydrogen atom; and the theory of coherent states. The Feynman operator calculus combined with simple group-theoretical considerations allows disentangling the Hamiltonian and obtaining exact transition probabilities between the initial and final states of a quantum oscillator in analytic form without cumbersome calculations. The case of a D-dimensional oscillator is briefly discussed, in particular, in application to the problem of vacuum pair creation in an intense electric field.

1. Introduction

In the 1940s, Richard Feynman proposed several novel and original approaches to be used in quantum theory: the continual integral 'over all paths' [1, 2], the diagram technique in perturbation theory [3 – 5] (later named after Feynman), and the method of disentangling[1] expressions with noncommuting operators [6]. The first two approaches, which became universally accepted and were further developed, are now widely used in different fields of theoretical physics, from quantum field theory and statistical physics to hydrodynamics and the theory of turbulence. These methods are described in more detail in many textbooks and monographs. In contrast, the Feynman disentangling method (FDM) appears to be poorly known to physicists; it is used by few researchers and rarely mentioned in the physics literature. The present paper aims to bridge this gap, expounding the main aspects of the FDM as applied to quantum mechanics, and considering some related group-theory problems.

The organization of this paper is briefly as follows. The main notions and formulas of the FDM (including the Feynman theorem of disentangling an exponential factor, important for the subsequent discussion) are presented in Section 2. Section 3 contains an elegant solution [6] of the problem of a constant-frequency harmonic oscillator excited by an arbitrarily time-dependent external force $f(t)$ (see also Refs [7 – 9]). Section 4 treats the FDM as applied to the spin rotation problem [10] for a particle with a magnetic moment in an external magnetic field $\mathbf{H}(t)$. These examples are convenient for describing the FDM because they allow focusing on certain details of the method that are usually disregarded in the literature. In

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[1] See the beginning of Section 2 for the explanation of this term.

Sections 5–8, we are more concise; as a rule, we only state a problem and present final results, referring the reader to original papers for detailed calculations. Also considered in these sections are transition probabilities in a quantum oscillator with time-varying frequency [11, 12]; the general case of an oscillator with varying parameters $\omega(t)$ and $f(t)$; the disentanglement of the atom (ion) Hamiltonian in a strong laser field; and the application of the FDM to the perturbation theory [6], to a model Hamiltonian with the 'hidden' symmetry group of the hydrogen atom [11], to the theory of coherent states, etc. The closing Sections 9 and 10 offer conclusions and describe prospects. Additional problems and computation details are discussed in the Appendices. Appendix A deals with the so-called Fock–Bargmann representation, in which wave functions of the oscillator and the hydrogen atom have the simplest form. Appendix B presents oscillator excitation parameters v and ρ [see formulas (3.15) and (5.12) in Sections 3 and 5]. Appendix C considers the Riccati equation for spin rotation in an alternating magnetic field. The unitarity condition is considered in Appendix D.

The application of the FDM to specific physical problems is substantially facilitated by group theory, which permits markedly simplifying calculations. The plethora of relevant information from group theory can be found in books [13–18]. Elementary data on unitary representations of noncompact Lie algebras are necessary for understanding the material in Sections 5 and 6; a rather popular exposition of this theory is offered in Ref. [19].

We now give the commutation relations for generators of the unitary group SU(2) and the quasi-unitary group SU(1, 1) that are necessary for further discussion; they can be written in the unified form

$$[\hat{J}_+, \hat{J}_-] = 2\eta\hat{J}_0\,, \qquad [\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm\,, \tag{1.1}$$

where

$$\hat{J}_\pm = \hat{J}_x \pm \mathrm{i}\hat{J}_y\,, \qquad \hat{J}_0 = \hat{J}_z\,, \tag{1.2}$$

$$\eta = \begin{cases} 1 & \text{for SU(2)}\,, \\ -1 & \text{for SU(1,1)}\,, \end{cases} \tag{1.3}$$

$\hat{J}_+$ is the creation operator and $\hat{J}_-$ is the annihilation operator when acting on eigenfunctions of $\hat{J}_0$:

$$\hat{J}_0|\mu\rangle = \mu|\mu\rangle\,, \qquad \hat{J}_0\hat{J}_\pm|\mu\rangle = (\mu \pm 1)\,\hat{J}_\pm|\mu\rangle\,. \tag{1.4}$$

The generators $\hat{J}_a$ commute with the Casimir operator $\hat{\mathbf{J}}^2$ ('the square of the angular momentum' on the group), whose eigenvalue specifies the irreducible representation:

$$[\hat{\mathbf{J}}^2, \hat{J}_a] = 0\,, \qquad a = \pm, 0\,, \tag{1.5}$$

$$\hat{\mathbf{J}}^2 = \hat{J}_z^2 + \eta(\hat{J}_x^2 + \hat{J}_y^2) = \hat{J}_0^2 + \frac{1}{2}\,\eta(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)\,. \tag{1.6}$$

The SU(2) group is compact and has finite-dimensional unitary irreducible representations, which are well known from quantum mechanics and on which the operator $\hat{\mathbf{J}}^2$ reduces to a c-number:

$$\hat{\mathbf{J}}^2|\psi\rangle = j(j+1)\,|\psi\rangle\,, \qquad j = 0, \frac{1}{2}\,,\ 1,\ \frac{3}{2}\,, \ldots \tag{1.7}$$

(j is the spin or angular momentum); the noncompact SU(1, 1) group has no such representations.[2] The difference between the two groups is illustrated by the example of matrices of rotation about the x axis: for the spinor ($j = 1/2$) representation of SU(2), we have

$$g_x(\vartheta) = \exp\left(-\frac{\mathrm{i}}{2}\,\vartheta\sigma_x\right) = \begin{pmatrix} \cos\dfrac{\vartheta}{2} & -\mathrm{i}\sin\dfrac{\vartheta}{2} \\ -\mathrm{i}\sin\dfrac{\vartheta}{2} & \cos\dfrac{\vartheta}{2} \end{pmatrix}, \quad 0 \leqslant \vartheta \leqslant \pi\,, \tag{1.8}$$

and in the case of SU(1, 1), with $\vartheta \to \mathrm{i}\beta$, we have

$$g_x(\beta) = \exp\left(\frac{1}{2}\,\beta\sigma_x\right) = \begin{pmatrix} \cosh\dfrac{\beta}{2} & \sinh\dfrac{\beta}{2} \\ \sinh\dfrac{\beta}{2} & \cosh\dfrac{\beta}{2} \end{pmatrix}, \quad 0 \leqslant \beta < \infty\,. \tag{1.9}$$

Evidently, the latter representation is finite-dimensional but not unitary.

In Section 4, instead of $\hat{J}_a$, we use the spin operator $\hat{s}_a$ that satisfies commutation relations (1.1) with $\eta = 1$.

2. Principles of the Feynman disentangling method

We first explain the term 'disentangling of operators.' Expressions widely used in quantum mechanics and field theory have the form

$$\hat{H} = \exp(\alpha\hat{A} + \beta\hat{B} + \gamma\hat{C} + \ldots)\,, \tag{2.1}$$

where $\hat{A}, \hat{B}, \ldots$ are noncommuting operators (e.g., $\hat{x}$ and $\hat{p}$, $\hat{a}$ and $\hat{a}^+$), and $\alpha, \beta, \ldots$ are certain constants or numerical functions. The presence of noncommuting operators in the exponent hampers calculations. Feynman proposed a peculiar operator calculus [6] in which an ordering index s is introduced to specify the order of operator action and enable the use of standard rules of calculus. By definition, the operator with a larger value of the index acts later:

$$\hat{A}_s\hat{B}_{s'} = \begin{cases} \hat{A}\hat{B}\,, & s > s'\,, \\ \hat{B}\hat{A}\,, & s < s'\,, \end{cases} \tag{2.2}$$

where $\hat{B}\hat{A}$ can be written as either $\hat{B}_1 A_0$ and or $\hat{A}_0\hat{B}_1$ (the indices 0 and 1 may be replaced by others, not necessarily integers; e.g., $\hat{A}_{-1.5}\hat{B}_{0.3} = \hat{B}\hat{A}$, because $-1.5 < 0.3$). The index s may be a continuous variable, e.g., time (see Sections 3–6). We consider an example [6] where $\hat{A} = \exp\hat{a}$ and $\hat{B} = \exp\hat{b}$; then,

$$\begin{aligned} \hat{B}\hat{A} = \exp(\hat{a}_0 + \hat{b}_1) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\hat{a}_0 + \hat{b}_1)^n \\ &= 1 + \hat{a} + \hat{b} + \frac{1}{2}(\hat{a}^2 + 2\hat{b}\hat{a} + \hat{b}^2) \\ &\quad + \frac{1}{6}(\hat{a}^3 + 3\hat{b}\hat{a}^2 + 3\hat{b}^2\hat{a} + \hat{b}^3) + \ldots \end{aligned} \tag{2.3}$$

[2] As is known from [16, 19], noncompact groups have no finite-dimensional unitary representations, except for a trivial (unit) one.

because

$$(\hat{a}_0+\hat{b}_1)^n=\sum_{k=0}^{n}\binom{n}{k}\hat{a}_0^k\hat{b}_1^{n-k}\equiv n!\sum_{k=0}^{n}\left[k!(n-k)!\right]^{-1}\hat{b}^{n-k}\hat{a}^k$$

(in the standard notation).

By the use of FDM, expression (2.1) is transformed to the form

$$\hat{S}=\exp\left(\tilde{\alpha}\hat{A}\right)\exp\left(\tilde{\beta}\hat{B}\right)\exp\left(\tilde{\gamma}\hat{C}\right)\exp\left(\delta\hat{D}\right)\ldots, \tag{2.4}$$

called 'disentangled': here, each factor contains a single operator, and therefore functions of this factor, such as $\exp(\hat{A})$, are defined unambiguously, e.g., by a power series.

The coefficients $\tilde{\alpha},\tilde{\beta},\ldots$ are to be found from relevant equations (which are usually differential) starting from the original $\alpha,\beta,\ldots$. This may lead to the appearance of new operators in (2.4) (denoted as $\hat{D}$) that were absent in the original expression. Representing the operator $\hat{S}$ in form (2.4) sometimes permits easily calculating matrix elements S_{mn} that describe transitions between different quantum states. Specific examples of the application of this procedure are given below.

In this paper, we frequently use the Feynman theorem of disentangling an exponential factor. Let $\mathcal{F}\left[\hat{M}(s),\hat{N}(s),\ldots\right]$ be a functional of operators $\hat{M},\hat{N},\ldots$ and s be an ordering parameter. In the case of unitary transformation $\hat{M}'(s)=\hat{U}^{-1}\hat{M}(s)\hat{U}$, where $\hat{U}$ is an s-independent constant operator, the factor between the operators in the product $\hat{M}(s+\mathrm{d}s)\hat{N}(s)$ entirely cancels: $UU^{-1}\equiv 1$; hence,

$$\mathcal{F}\left[\hat{M}'(s),\hat{N}'(s),\ldots\right]=\hat{U}^{-1}\,\mathcal{F}\left[\hat{M}(s),\hat{N}(s),\ldots\right]\hat{U}. \tag{2.5}$$

The case of an s-dependent operator $\hat{U}(s)$ is much more interesting. Here,

$$\begin{aligned}\hat{U}(s+\mathrm{d}s)\hat{U}^{-1}(s)&=1+\hat{P}(s)\mathrm{d}s\\&=\exp\left[\hat{P}\left(s+\frac{1}{2}\,\mathrm{d}s\right)\mathrm{d}s\right],\qquad \mathrm{d}s\to 0,\end{aligned} \tag{2.6}$$

that is, the full compensation of the factors $\hat{U}(s+\mathrm{d}s)$ and $\hat{U}^{-1}(s)$ no longer occurs; the operator $\hat{P}(s)$ arising between them is determined from the equation

$$\frac{\mathrm{d}\hat{U}(s)}{\mathrm{d}s}=\hat{P}(s)\,\hat{U}(s). \tag{2.7}$$

The argument $s+(1/2)\mathrm{d}s$ of $\hat{P}$ automatically sets its correct position in the product of operators entering the functional $\mathcal{F}$; therefore, multiplication yields the factor $\exp\left[\int_{s_0}^{s_1}\hat{P}(s)\,\mathrm{d}s\right]$. As a result, there is the relation (the Feynman theorem)

$$\begin{aligned}\mathcal{F}\left[\hat{M}'(s),\hat{N}'(s),\ldots\right]&=\hat{U}^{-1}(s_1)\,\mathcal{F}\left[\hat{M}(s),\hat{N}(s),\ldots\right]\\&\times\exp\left\{\int_{s_0}^{s_1}\hat{P}(s)\,\mathrm{d}s\right\}\hat{U}(s_0),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\hat{M}'(s)&=\hat{U}^{-1}(s)\hat{M}(s)\hat{U}(s),\\\hat{U}(s)&=\exp\left\{\int_{s_0}^{s}\hat{P}(s')\,\mathrm{d}s'\right\}\hat{U}(s_0).\end{aligned} \tag{2.8$'$}$$

Specifically, if $\hat{P}(s)=\alpha(s)\hat{P}_s$ (where $\alpha(s)$ is an arbitrary numerical function), the operator $\hat{P}_s$ is not explicitly dependent on s,[3] and $U(s_0)=1$, then

$$\begin{aligned}&\exp\left\{\int_{s_0}^{s_1}\alpha(s)\hat{P}_s\,\mathrm{d}s\right\}\mathcal{F}\left[\hat{M}(s),\hat{N}(s),\ldots\right]\\&\quad=\exp\left\{\hat{P}_{s_1}\int_{s_0}^{s_1}\alpha(s)\,\mathrm{d}s\right\}\mathcal{F}\left[\hat{M}'(s),\hat{N}'(s),\ldots\right],\end{aligned} \tag{2.9}$$

$$\hat{M}'(s)=\exp\left\{-\hat{P}_s\int_{s_0}^{s}\alpha(s')\,\mathrm{d}s'\right\}\hat{M}(s)\exp\left\{\hat{P}_s\int_{s_0}^{s}\alpha(s')\,\mathrm{d}s'\right\}. \tag{2.9$'$}$$

We note that the functional $\mathcal{F}$ in formulas (2.8) and (2.9) should not be regarded as a pre-exponential. These formulas (see Eqns (16)–(20) in [6]) give the disentanglement rule for the exponential factor $\exp\{\hat{P}\int_{s_0}^{s_1}\alpha(s)\,\mathrm{d}s\}$ from expressions of type (2.1). All operators are assumed to be ordered in accordance with (2.2).

In what follows, time plays the role of s and the ordering of operators as proposed by Feynman corresponds to the T-product. For example, the operator of quantum system evolution is

$$\begin{aligned}\hat{S}(t_1,t_0)&=\lim_{N\to\infty}\prod_{i=1}^{N}\left\{1-\frac{\mathrm{i}}{\hbar}\,\hat{H}(t_i)\Delta t_i\right\}\\&=\mathrm{T}\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1}\hat{H}(t')\,\mathrm{d}t'\right\},\end{aligned} \tag{2.10}$$

where $\hat{H}$ is the Hamiltonian and $\Delta t_i=(t_1-t_0)/N\to 0$.

We finish the introductory discussion of the FDM here, referring the reader for more details to Feynman's paper [6] (see also Appendix A in [6] describing a more rigorous derivation of the above formulas). We now move to the FDM in quantum mechanics.

3. Excitation of a harmonic oscillator by an external force

We start with a problem considered by Feynman [6, 7] and described as "the main problem of quantum electrodynamics when formulated in the most general form" [6, p. 114]:

$$\begin{aligned}\hat{H}&=\frac{1}{2}\left(\hat{p}^2+\omega^2\hat{q}^2\right)-f(t)\hat{q}\\&=\frac{1}{2}\,\omega(\hat{a}\hat{a}^{+}+\hat{a}^{+}\hat{a})-\frac{f(t)}{\sqrt{2\omega}}(\hat{a}+\hat{a}^{+}).\end{aligned} \tag{3.1}$$

Here and hereinafter, we choose the system of units where $\hbar=m=1$; in this case, $[\hat{p},\hat{q}]=-\mathrm{i}$, with $\hat{a}^{+}$ and $\hat{a}$ being the operators of creation and annihilation of oscillatory quanta:

$$\hat{a}^{+}=\frac{1}{\sqrt{2\omega}}(\omega\hat{q}-\mathrm{i}\hat{p}),\qquad \hat{a}=\frac{1}{\sqrt{2\omega}}(\omega\hat{q}+\mathrm{i}\hat{p}),\qquad [\hat{a},\hat{a}^{+}]=1, \tag{3.2}$$

$$\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle,\qquad \hat{a}^{+}|n\rangle=\sqrt{n+1}\,|n+1\rangle, \tag{3.3}$$

$$[\hat{H},\hat{a}]=-\omega\hat{a}+(2\omega)^{-1/2}f,\qquad [\hat{H},\hat{a}^{+}]=\omega\hat{a}^{+}-(2\omega)^{-1/2}f. \tag{3.4}$$

[3] Such a case is often encountered in applications, with the index s attached to P_s defining only the order of operator action relative to other operators $(\hat{M},\hat{N},\ldots)$ in (2.8); in what follows, it is omitted for brevity.

As regards the external force $f(t)$ acting on the oscillator, it is only assumed that $f(t) \to 0$ as $t \to \pm\infty$. The problem is to find transition probabilities between the initial ($|m\rangle$) and final ($|n\rangle$) states of the oscillator with given numbers of quanta.

Applying the Feynman theorem to the oscillator evolution operator

$$\hat{S}(t, t_0) = \mathrm{T} \exp\left(-\mathrm{i}\int_{t_0}^{t} \hat{H}(t')\,\mathrm{d}t'\right), \tag{3.5}$$

we first disentangle the operator $\hat{H}_0 = (1/2)\omega(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$ associated with the free oscillator:

$$\begin{aligned} &\hat{a}(t) = \exp(\mathrm{i}\hat{H}_0 t)\,\hat{a}\exp(-\mathrm{i}\hat{H}_0 t) = \hat{a}\exp(-\mathrm{i}\omega t)\,, \\ &\hat{a}^+(t) = \hat{a}^+\exp(\mathrm{i}\omega t)\,; \end{aligned} \tag{3.6}$$

this operation corresponds to the transition to the interaction representation. Hence,

$$\begin{aligned} \hat{S}(t, t_0) &= \exp\left[-\mathrm{i}\hat{H}_0(t - t_0)\right] \\ &\times \mathrm{T}\exp\left\{\frac{\mathrm{i}}{\sqrt{2\omega}}\int_{t_0}^{t} f(t')\left[\hat{a}\exp(-\mathrm{i}\omega t') + \hat{a}^+\exp(\mathrm{i}\omega t')\right]\mathrm{d}t'\right\}. \end{aligned} \tag{3.7}$$

Now, the FDM can be applied to disentangle the term with $\hat{a}^+$. The relations

$$\begin{aligned} &\hat{a}' = \exp(-\mathrm{i}\beta\hat{a})\,\hat{a}\exp(\mathrm{i}\beta\hat{a}^+) = \hat{a} + \mathrm{i}\beta\,, \\ &\hat{a}^{+\prime} = \exp(-\mathrm{i}\beta\hat{a})\,\hat{a}^+\exp(\mathrm{i}\beta\hat{a}^+) = \hat{a}^+ - \mathrm{i}\beta\,, \end{aligned} \tag{3.8}$$

which can be verified by differentiating them with respect to the parameter β taking the commutation relations and the initial condition $\hat{a}'(\beta = 0) = \hat{a}$ into account,[4] allow finding the amplitude of transition between the initial $|m\rangle$ and final $|n\rangle$ states of an oscillator with a given number of quanta as

$$A_{nm} = \frac{1}{\sqrt{m!\,n!}}\left\langle 0\middle|\hat{a}^n\exp(\mathrm{i}\beta^*\hat{a}^+)\exp(\mathrm{i}\beta\hat{a})\,\hat{a}^{+m}\middle|0\right\rangle, \tag{3.9}$$

where, in accordance with (3.7),

$$\beta(t, t_0) = \frac{1}{\sqrt{2\omega}}\int_{t_0}^{t} f(t')\exp(-\mathrm{i}\omega t')\,\mathrm{d}t'\,. \tag{3.10}$$

We next assume that $t_0 \to -\infty$, $t \to \infty$, and $\beta \equiv \beta(\infty, -\infty)$ is the parameter determining the degree of oscillator excitation during the entire action time of the external force $f(t)$. Permuting the factors $\exp(\mathrm{i}\beta\hat{a})$ and $\hat{a}^{+m}$, $\hat{a}^n$, and $\exp(\mathrm{i}\beta^*\hat{a}^+)$, and using that

$$\begin{aligned} &|n\rangle = (n!)^{-1/2}\hat{a}^{+n}|0\rangle\,, \qquad \hat{a}|0\rangle = \langle 0|\hat{a}^+ \equiv 0\,, \\ &\langle 0|\hat{a}^{+n}\hat{a}^m|0\rangle = n!\,\delta_{mn}\,, \qquad \exp(\mathrm{i}\beta\hat{a})\,|0\rangle = |0\rangle\,, \end{aligned} \tag{3.11}$$

we finally obtain [6]

$$\begin{aligned} A_{nm} &= \frac{1}{\sqrt{m!\,n!}}\left\langle 0\middle|(\hat{a} + \mathrm{i}\beta^*)^n(\hat{a}^+ - \mathrm{i}\beta)^m\middle|0\right\rangle A_{00} \\ &= A_{00}\sum_r \frac{\sqrt{m!\,n!}}{r!(n-r)!(m-r)!}(\mathrm{i}\beta^*)^{n-r}(-\mathrm{i}\beta)^{m-r}\,, \end{aligned} \tag{3.12}$$

where $0 \leqslant r \leqslant \min(m, n)$ and A_{00} is the vacuum – vacuum transition amplitude [7],

$$\begin{aligned} |A_{00}|^2 &= \exp\left\{-\frac{1}{2\omega}\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}\omega|t - t'|\right) f(t)\,f(t')\,\mathrm{d}t\,\mathrm{d}t'\right\} \\ &= \exp\left(-|\beta|^2\right). \end{aligned} \tag{3.13}$$

Feynman first derived formulas (3.12) and (3.13) by another method (using the Lagrangian form of quantum mechanics) in Ref. [7]. He noticed that the sum entering (3.12) could be expressed via a Laguerre polynomial, but "such representation has no advantage whatever" [7, p. 451]. This last statement does not seem altogether self-evident. Soon after Refs [6, 7] appeared, Schwinger obtained (in a different way) a more compact expression for the transition probabilities [9]:

$$w_{nm} = |A_{nm}|^2 = \frac{n_<!}{n_>!}\exp(-v)\,v^k\left[L_{n_<}^{(k)}(v)\right]^2 \tag{3.14}$$

(a relatively simple derivation of this formula based on the Fock – Bargmann representation is given in Appendix A). Here,

$$\begin{aligned} &n_< = \min(m, n)\,, \quad n_> = \max(m, n)\,, \\ &k = n_> - n_< = |m - n|\,, \quad v = |\beta|^2\,, \end{aligned} \tag{3.15}$$

and the generalized Laguerre polynomials are defined following [20, 21]:

$$\begin{aligned} &L_0^{(k)}(x) = 1\,, \quad L_1^{(k)}(x) = k + 1 - x\,, \\ &L_2^{(k)}(x) = \frac{1}{2}(k+1)(k+2) - (k+2)x + \frac{1}{2}\,x^2\,, \\ &L_n^{(k)}(x) = \frac{(n+k)!}{n!\,k!} - \frac{(n+k)!}{(n-1)!\,(k+1)!}\,x + \ldots + \frac{(-1)^n}{n!}\,x^n\,. \end{aligned}$$

Schwinger formula (3.14) is equivalent to (3.12) but is more convenient for large quantum numbers because it permits using the known asymptotic expressions for the Laguerre polynomials. For the 'diagonal' ($m = n$) transitions, we have

$$w_{nn} = 1 - (2n+1)v + \frac{3}{2}\left(n^2 + n + \frac{1}{3}\right)v^2 - \ldots, \quad v \to 0\,; \tag{3.16}$$

for $n \gg 1$ and any v,

$$w_{nn} = \left[J_0\left(\sqrt{2(2n+1)v}\right)\right]^2 + O(n^{-3/4})\,, \tag{3.16'}$$

where J_0 is the Bessel function (a Hilb-type formula, see formula 10.15.2 in [20]). As the excitation parameter v increases, the transition probabilities change nonmonotonically (except in the case of small quantum numbers m, n), as can be seen in Fig. 1. We note that asymptotic expression (3.16′) as $v \to 0$ differs from the exact expansion (3.16) by the coefficient at v^2 containing $n^2 + n + 1/4$ instead of the correct factor $n^2 + n + 1/3$. In the limit $v \gg 1$ (a strongly excited oscillator),

$$w_{nm} = \exp(-v)\,\frac{v^{m+n}}{m!\,n!}\left[1 - 2mnv^{-1} + O(v^{-2})\right]. \tag{3.17}$$

In certain cases, formulas (3.12) and (3.14) are simplified. For example, they reduce to the Poisson distribution for

[4] It is implied here that the parameter β is independent of $\hat{a}$ and $\hat{a}^+$.

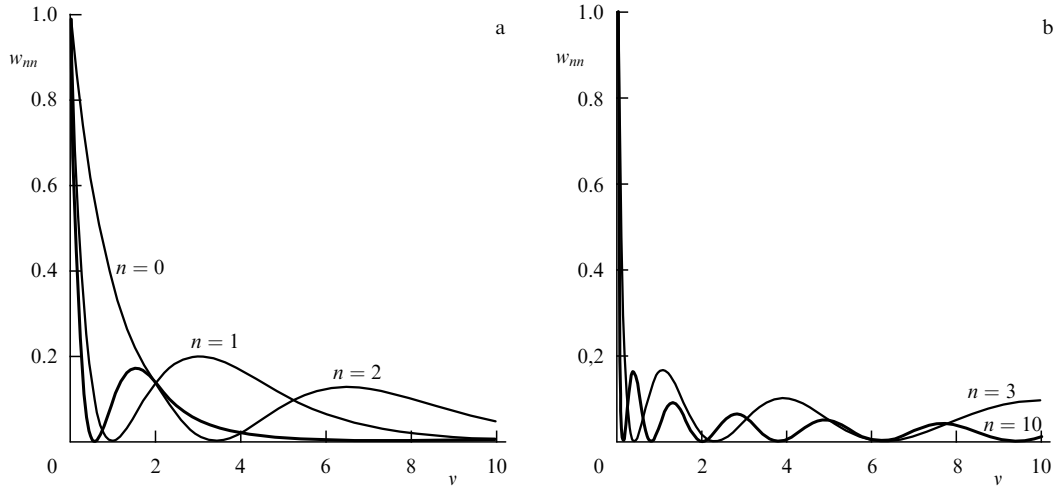

Figure 1. Excitation of a harmonic oscillator by an external force: 'diagonal' ($m = n$) transition probabilities depending on v for $n = 0$, 1, and 2 (a); $n = 3$ and 10 (b). Figures at the curves denote values of the quantum number n.

transitions either from or to the ground level:

$$w_{n0}(v) = w_{0n}(v) = \exp(-v)\,\frac{v^n}{n!}\,, \qquad n = 0, 1, 2, \ldots\,. \quad (3.18)$$

For transitions from the first excited level of the oscillator,

$$w_{01} = \exp(-v)v\,, \qquad w_{n1}(v) = \exp(-v)\,\frac{v^{n-1}(n-v)^2}{n!}\,, \quad (3.18')$$

and for diagonal transitions,

$$w_{nn}(v) = \exp(-v)\left[\sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!^2}(-v)^k\right]^2. \quad (3.18'')$$

It follows from (3.10) that the harmonic oscillator is excited only by the spectral component of the external force whose frequency coincides with the oscillator eigenfrequency ω, due to the linearity of this system.

To conclude this section, a few remarks are in order.

(a) Schwinger derived formula (3.14) when solving the following quantum electrodynamics (QED) problem. Let the states of an isolated (free) electromagnetic field be specified on space-like surfaces σ_1 and σ_2, one obtainable from the other by parallel transition. In the region between σ_1 and σ_2, the field interacts with an external current $J_\mu(x)$ that depends on time but vanishes on σ_1 and σ_2. The problem is to find the probability of excitation of different states of the electromagnetic field. The solution has the form (see formula (39) in [9])

$$p(n, n') = \prod_{\lambda k}\left\{\frac{n_<!}{n_>!}\left(|J|^2\right)^{n_>-n_<}\left[L_{n_<}^{(n_>-n_<)}\left(|J|^2\right)\right]^2\exp\left(-|J|^2\right)\right\}, \quad (3.19)$$

where n and n' are the numbers of quanta with momentum k and polarization λ in the initial and final states of the field. Formula (3.14) is a special case of (3.19) for a one-oscillator model. Schwinger used these expressions to consider the semiclassical limit, in which $n, n' \gg 1$ and $\Delta n = |n - n'| \ll n, n'$.

Reference [9] belongs to the known series of Schwinger's works devoted to the construction of quantum field theory based on a common dynamic principle; the series also includes Refs [22, 23]. Surprisingly, neither [9] nor other publications in this series contain references to any earlier works by Feynman [6, 7].

(b) Using (3.18) and (3.18′), it is easy to show that for $m = 0$ and 1,

$$\sum_{n=0}^{\infty} w_{nm}(v) = 1\,. \quad (3.20)$$

Certainly, unitarity condition (3.20) must be satisfied for an arbitrary initial state $|m\rangle$; however, this is not easy to verify directly from expressions (3.12) and (3.14).

(c) Transition probabilities w_{nm} depend on a single parameter v given by formulas (3.10) and (3.15) as $t \to +\infty$. When the characteristic force action time is greater than the oscillator period $T = 2\pi/\omega$ and the force $f(t)$ varies smoothly (the adiabatic limit), the excitation parameter v is small (see Appendix B) and transitions are restricted to the neighboring levels:

$$\begin{aligned} w_{n+1,n} &= (n+1)v + O(v^2)\,,\\ w_{nn} &= 1 - (2n+1)v + \ldots\,,\\ w_{n-1,n} &= nv + \ldots\,, \end{aligned} \quad (3.21)$$

and, in the general case,

$$\begin{aligned} w_{n+k,n}(v) &= \frac{(n+k)!}{n!\,k!^2}\,v^k + \ldots\,,\\ w_{n-k,n}(v) &= \frac{n!}{(n-k)!\,k!^2}\,v^k\,, \qquad k = 1, 2, \ldots\,. \end{aligned} \quad (3.22)$$

(d) The Hamiltonian

$$\hat{H} = \frac{1}{2}\,\hat{p}^2 + \frac{1}{2}\,\omega^2\left(\hat{q} - \xi(t)\right)^2 \quad (3.23)$$

corresponds to an oscillator whose equilibrium point shifts (oscillates) in an arbitrary mode. In this case,

$$\hat{S}(\infty, -\infty) = \exp(-\mathrm{i}\varphi) \times \mathrm{T}\exp\left\{-\mathrm{i}\int_{-\infty}^{\infty}\left[\frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) - \omega^2\xi(t)\hat{q}\right]\mathrm{d}t\right\}, \quad (3.24)$$

where φ is a phase that has no effect on transition probabilities. Therefore, all the preceding formulas now hold with

$$v = \frac{1}{2}\,\omega^3 \left| \int_{-\infty}^{\infty} \xi(t) \exp(-\mathrm{i}\omega t)\,\mathrm{d}t \right|^2 . \tag{3.25}$$

Thus, the FDM naturally leads to the exact solution of the problem of excitation of a harmonic oscillator in the case of an arbitrary form of the external force $f(t)$. An essential point is that Hamiltonian (3.1) contains only two operators, $\hat{a}$ and $\hat{a}^+$, whose commutator is a c-number. In other cases, application of the FDM encounters more difficulties.

4. Particle with an arbitrary spin in an external magnetic field

The Schrödinger equation for the spin wave function [24, 25]

$$\mathrm{i}\dot{\psi} = -\gamma\big(\mathbf{H}(t)\hat{\mathbf{s}}\big)\psi\,, \qquad \psi = \begin{pmatrix} \psi_s \\ \psi_{s-1} \\ \vdots \\ \psi_{-s} \end{pmatrix} \tag{4.1}$$

(γ is the gyromagnetic ratio, $\hbar = 1$) with the explicit form of spin matrix elements $\hat{\mathbf{s}}_{\sigma\sigma'}$ taken into account reduces to a system of $2s+1$ rather cumbersome coupled equations for the components ψ_σ. The FDM may be used to find the solution of this system with an arbitrary time dependence of the magnetic field $\mathbf{H}(t)$.

We note that

$$(\mathbf{H}\hat{\mathbf{s}}) = \sum_{\mu=-1}^{1} H_\mu(t)\hat{s}_\mu\,, \qquad H_\pm = \frac{1}{2}(H_x \mp \mathrm{i}H_y)\,, \qquad H_0 = H_z\,, \tag{4.2}$$

where the operators $\hat{s}_\mu$ are defined as in (1.2). The evolution operator $\hat{S}$ can be written in the form of a T-exponential, although in this case the operators $\mathbf{H}\hat{\mathbf{s}}$ are noncommuting at different time instants.[5] We try to represent $\hat{S}$ in the 'disentangled' form

$$\hat{S}(t, t_0) = \exp(a\hat{s}_+)\exp(b\hat{s}_0)\exp(c\hat{s}_-)\,, \tag{4.3}$$

where a, b, and c are certain functions of time. Because the expressions $\exp(c\hat{s}_-)\,|\sigma'\rangle$ and $\langle\sigma|\exp(a\hat{s}_+)$ contain only a few nonzero first terms of the series expansion of the exponential, it is easy to use (4.3) to deduce from [10] a formula resembling (3.12) for probabilities of transition from one spin state to another ($\sigma' \to \sigma$):

$$w_{\sigma\sigma'} = \big|S_{\sigma\sigma'}(t, t_0)\big|^2\,, \quad -s \leqslant \sigma\,, \quad \sigma' \leqslant s\,. \tag{4.4}$$

The solution can be presented in a more compact form. Following [10], we write

$$\mathbf{H}\hat{\mathbf{s}} = \chi\hat{s}_+ + (H_+ - \chi)\hat{s}_+ + H_0\hat{s}_0 + H_-\hat{s}_- \tag{4.5}$$

without fixing the function $\chi(t)$ a priori, and apply the Feynman theorem on disentangling exponential factors to the first term,

$$\hat{S}(t, t_0) = \exp\big(a(t)\hat{s}_+\big) \times \mathrm{T}\exp\left\{\mathrm{i}\gamma\int_{t_0}^{t}\big[(H_+ - \chi)\hat{s}'_+ + H_0\hat{s}'_0 + H_-\hat{s}'_-\big]\,\mathrm{d}t'\right\}, \tag{4.6}$$

where, in accordance with (2.8′),

$$a(t) = -\mathrm{i}\gamma\int_{t_0}^{t}\chi(t')\,\mathrm{d}t'\,, \qquad \hat{s}'_\mu(t) = \exp(-a\hat{s}_+)\,\hat{s}_\mu\exp(a\hat{s}_+)\,. \tag{4.6′}$$

Differentiation of (4.6′) with respect to a and recalling commutators (1.1) with $\eta = 1$, we obtain

$$\hat{s}'_+ = \hat{s}_+\,, \quad \hat{s}'_0 = \hat{s}_0 + a\hat{s}_+\,, \quad \hat{s}'_- = \hat{s}_- - 2a\hat{s}_0 - a^2\hat{s}_+\,. \tag{4.7}$$

Substituting (4.7) in (4.6), we require that the operator $\hat{s}_+$ be disentangled completely; this yields the Riccati equation for $a(t)$:

$$\dot{a} = \mathrm{i}\gamma(H_+ + H_0 a - H_- a^2)\,, \qquad a(t_0) = 0\,. \tag{4.8}$$

Proceeding further, we arrive at explicit (even if rather cumbersome) [10] expressions for the functions $b(t)$ and $c(t)$. There is no need to do this, however, as we see shortly.

Because Eqn (4.1) involves $\mathbf{s}^2$, the transformation of the wave function is a spatial rotation:

$$S_{\sigma\sigma'}(t, t_0) = D^{(s)}_{\sigma\sigma'}(\varphi, \vartheta, \psi) = \exp\big[-\mathrm{i}(\sigma\varphi + \sigma'\psi)\big]d^s_{\sigma\sigma'}(\vartheta)\,, \tag{4.9}$$

where ψ, ϑ, and φ are t-dependent Euler angles and $D^{(s)}_{\sigma\sigma'}$ is a finite rotation matrix in the SU(2) group or a Wigner D-function [13, 18, 26]. The probability of changing the particle spin projection on the z axis is expressed as

$$w_{\sigma\sigma'} = |S_{\sigma\sigma'}|^2 = \big|d^s_{\sigma\sigma'}\big(\vartheta(t)\big)\big|^2\,. \tag{4.10}$$

The functions $d^s_{\sigma\sigma'}(\vartheta)$, corresponding to the rotation through an angle ϑ about the Ox axis are well known and tabulated [26], e.g.,

$$d^j_{jj}(\vartheta) = d^j_{-j,\,-j}(\vartheta) = \left(\cos\frac{\vartheta}{2}\right)^{2j}, \qquad d^j_{j,\,-j}(\vartheta) = \mathrm{i}^{2j}\left(\sin\frac{\vartheta}{2}\right)^{2j};$$

in the general case, they are expressed in terms of Jacobi polynomials [18].

It remains to specify the relation of the angle $\vartheta(t)$ to $a(t)$. Equation (4.8) contains γ and the external magnetic field but, unlike Eqn (4.1), does not explicitly involve the particle spin s. Therefore, the sought relation can be found for any spin s, i.e., for any irreducible representation of the unitary group SU(2). It is natural to choose the simplest representation in which $\hat{s}_\pm = (1/2)(\sigma_x \pm \mathrm{i}\sigma_y)$, $\hat{s}_0 = (1/2)\sigma_z$, and σ_i are the Pauli matrices:

$$\hat{s}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \exp(a\hat{s}_+) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

$$\hat{s}_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \exp(b\hat{s}_0) = \begin{pmatrix} \exp\left(\frac{b}{2}\right) & 0 \\ 0 & \exp\left(-\frac{b}{2}\right) \end{pmatrix},$$

$$\hat{s}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \exp(c\hat{s}_-) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad \hat{s}_\pm^2 = 0\,.$$

[5] Except in the trivial case where the magnetic field retains its direction in space: $[\mathbf{H}_1\hat{\mathbf{s}}, \mathbf{H}_2\hat{\mathbf{s}}] = \mathrm{i}(\mathbf{H}_1 \times \mathbf{H}_2)\hat{\mathbf{s}}$.

Multiplication of the matrices in (4.3) gives

$$\hat{S}(t,t_0)=\begin{pmatrix}\exp\left(\frac{b}{2}\right)+ac\exp\left(-\frac{b}{2}\right) & a\exp\left(-\frac{b}{2}\right)\\ c\exp\left(-\frac{b}{2}\right) & \exp\left(-\frac{b}{2}\right)\end{pmatrix} \tag{4.11}$$

and comparison with the standard expression [13, 18] for the Wigner functions

$$D^{(1/2)}(\psi,\vartheta,\varphi)=\begin{pmatrix}\alpha & -\beta^*\\ \beta & \alpha^*\end{pmatrix},$$

$$\alpha=\cos\frac{\vartheta}{2}\exp\left[\frac{\mathrm{i}(\psi+\varphi)}{2}\right],\qquad \beta=\mathrm{i}\sin\frac{\vartheta}{2}\exp\left[\frac{\mathrm{i}(\psi-\varphi)}{2}\right],$$

finally yields $c=a\exp\left[\mathrm{i}(\psi-\varphi)\right]$ and

$$\tan^2\frac{\vartheta}{2}=\left|a(t)\right|^2. \tag{4.12}$$

Thus, a single function $a(t)$, which is determined from Riccati equation (4.8), is needed [6] to calculate the transition probabilities $w_{\sigma\sigma'}$ in the case of a particle with an arbitrary spin s.

As an illustration, we give spin transition probabilities for the lower spins $s=1/2$ and 1:

$$\begin{aligned}&w_{1/2,1/2}=\left(1+|a|^2\right)^{-1},\qquad w_{-1/2,1/2}=|a|^2\left(1+|a|^2\right)^{-1},\\ &w_{11}=\left(1+|a|^2\right)^{-2},\qquad w_{01}=2|a|^2\left(1+|a|^2\right)^{-2},\\ &w_{-1,1}=|a|^4\left(1+|a|^2\right)^{-2},\end{aligned} \tag{4.13}$$

and for a particle with any spin s:

$$w_{\sigma s}=\frac{(2s)!}{(s+\sigma)!(s-\sigma)!}\,|a|^{2(s-\sigma)}\left(1+|a|^2\right)^{-2s}, \tag{4.14}$$

assuming the maximum spin projection, $\sigma'=s$, at the initial instant.

It may seem that this approach holds only until the instant $t=t_1$ at which $a(t)$ becomes infinite [which corresponds to complete reorientation of the particle spin, as follows from (4.13)]. This is not the case, however; it suffices to pass from (4.8) to the equation for the function $\tilde{a}(t)=1/a(t)$, i.e., to the Riccati equation

$$\dot{\tilde{a}}=\mathrm{i}\gamma(H_--H_0\tilde{a}-H_+\tilde{a}^2), \tag{4.15}$$

in which $\tilde{a}(t_1)\to 0$. Using these two equations alternately, it is possible to continue the solution of $a(t)$ to arbitrarily large times t. Examples of the application of the Riccati equation are given in Appendix C.

It is worthwhile to note that Eqn (4.8) was derived by Majorana (naturally, without application of the FDM) as early as 1932 [25], but it was not used in concrete calculations. The dynamics of spin $s=1/2$ in a magnetic field of the form $\mathbf{B}(t)=\{B_x,0,B_zt\}$, where B_x and B_z are some constants, was considered in [25]. In this case, the z component of the field vanishes at $t=0$ and spin reorientation is possible even if the magnetic field changes slowly (at macroscopic distances) as a result of broken adiabaticity. Majorana proposed the solution of Schrödinger equation (4.1) in the form of a contour integral on the complex plane and computed the probability of spin reorientation; the Riccati equation corresponding to this example is presented in Appendix B [see formula (B.7)].

Paper [27] contains interesting historical comments on Majorana's work to the effect that it has a direct bearing on the theory of nonadiabatic transitions between quasi-intersecting levels, e.g., for slow atomic collisions. [7] This problem was considered (using different methods) by Landau [30], Zener [31], and Stückelberg [32] in the same year of 1932. These results are well known to atomic physicists, although their relation to Majorana's work [25] was noticed only recently [27].

It is worth noting that Majorana first used an elegant technique to consider a particle with an arbitrary spin s as a set of $2s$ 'particles' with spins $1/2$ assumed to be parallel and rotating independently in the magnetic field. This approach was many times used in later studies [33, 34] (see also [24, Section 114]) and is presently described in textbooks.

Equations (4.1) and (4.8) represent the nonrelativistic case. If a particle with a magnetic moment travels with a relativistic speed in an electromagnetic field $F_{\mu\nu}$ and its orbital movement may be regarded as classical, a change in the spin (or in the particle polarization vector) can be found from the Bargmann – Michel – Telegdi equation (see [35, 36] and also [37, Section 41]):

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau}=\frac{e}{2mc}\left[gF^{\mu\nu}s_\nu+(g-2)\,u^\mu F^{\alpha\beta}u_\alpha s_\beta\right], \tag{4.16}$$

where g is the gyromagnetic ratio, $u^\mu=\mathrm{d}x^\mu/\mathrm{d}\tau$ is the 4-velocity of the particle, and τ is the proper time. Some solutions of this equation are presented in [37]. Reference [38] reports the angular velocity of spin rotation [found based on (4.16)] in the rest frame of a particle rigidly linked to its trajectory. [8]

5. Quantum oscillator with varying frequency

In this case, the Hamiltonian has form (3.1) with $f(t)\equiv 0$, but the frequency ω now varies with time. The limits

$$\omega(t)\to\begin{cases}\omega_-, & t\to-\infty,\\ \omega_+, & t\to+\infty\end{cases} \tag{5.1}$$

are assumed to exist, which allows introducing the initial and final states of the oscillator with a definite number of quanta (m and n, respectively) and exploring transitions between them. It is assumed that $\omega_\pm>0$, although $\omega^2(t)<0$ at finite t; this situation corresponds to an unstable oscillator (or subbarrier region in quantum mechanics, $t\to x$).

We represent the evolution operator

$$\hat{S}(t,t_0)=\mathrm{T}\exp\left\{-\mathrm{i}\int_{t_0}^{t}\left(\frac{1}{2}\,\hat{p}^2+\frac{1}{2}\,\omega^2(t')\hat{q}^2\right)\mathrm{d}t'\right\} \tag{5.2}$$

[6] This fact was overlooked in [10], which somewhat complicated the calculation.

[7] Güttinger [28] appears to have been the first to discuss computation of probabilities of nonadiabatic transitions in quantum mechanics. Some inaccuracies in his work were corrected by Schwinger [29].

[8] An analog of the Frenet trihedron for a four-dimensional trajectory $x^\mu(\tau)$ in Minkowski space.

in the disentangled form. The commutation relations

$$[\hat{p}^2, \hat{q}^2] = -2\mathrm{i}(\hat{p}\hat{q} + \hat{q}\hat{p}),$$
$$[\hat{p}^2, \hat{p}\hat{q} + \hat{q}\hat{p}] = -4\mathrm{i}\hat{p}^2, \tag{5.3}$$
$$[\hat{q}^2, \hat{p}\hat{q} + \hat{q}\hat{p}] = 4\mathrm{i}\hat{q}^2$$

indicate that the operator algebra closes, which allows using the FDM. Setting

$$\hat{A} = \frac{1}{2}\,\hat{q}^2, \qquad \hat{B} = \frac{1}{2}\,\hat{p}^2, \qquad \hat{C} = \frac{1}{4}(\hat{p}\hat{q} + \hat{q}\hat{p}), \tag{5.4}$$

we have

$$[\hat{A}, \hat{B}] = 2\mathrm{i}\hat{C}, \qquad [\hat{B}, \hat{C}] = -\mathrm{i}\hat{B}, \qquad [\hat{C}, \hat{A}] = -\mathrm{i}\hat{A}. \tag{5.5}$$

Using the same approach as in (4.5) and (4.6), we set $\hat{H} = \chi\hat{A} + (\omega^2 - \chi)\hat{A} + \hat{B}$, where $\chi(t)$ is to be determined in the process of disentanglement. Taking the relations

$$\hat{A}' = \hat{A}, \qquad \hat{B}' = \hat{B} + 2\mathrm{i}a\hat{C} - a^2\hat{A}, \qquad \hat{C}' = \hat{C} + \mathrm{i}a\hat{A},$$
$$a(t) = \mathrm{i}\int_{t_0}^{t} \chi(t')\,\mathrm{d}t'$$

into account, with the operators $\hat{A}'(t)$ and others defined as in (2.8′), leads to [11]

$$\hat{S}(t, t_0) = \exp\left(-\frac{1}{2}\,a\hat{q}^2\right)\exp\left[\frac{c}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})\right]\exp\left(-\frac{1}{2}\,b\hat{p}^2\right), \tag{5.6}$$

where the operators are completely disentangled, the T-product sign may be omitted, and the function $a(t)$ is to be found from the Riccati equation

$$\dot{a} = \mathrm{i}\left[\omega^2(t) - a^2\right], \tag{5.7}$$

with $b(t)$ and $c(t)$ expressed through it in quadratures:

$$b(t) = \mathrm{i}\int_{t_0}^{t} \exp\left(-2\mathrm{i}c(t')\right)\mathrm{d}t',$$
$$c(t) = \int_{t_0}^{t} a(t')\,\mathrm{d}t'. \tag{5.7′}$$

In calculating the transition amplitude

$$A_{nm} = \left\langle n, \omega_+ \middle| \hat{S}(+\infty, -\infty) \middle| m, \omega_- \right\rangle, \tag{5.8}$$

it is convenient to use the wave function of the initial state $|m, \omega_-\rangle$ in the p-representation and the wave function of the final state $|n, \omega_+\rangle$ in the q-representation, using the explicit form of eigenfunctions of the $\hat{C}$ operator. This accomplished, the problem may be considered solved in principle, even though the arising integrals (especially for arbitrary quantum numbers m and n) are rather cumbersome.

Computations are simplified if the FDM is supplemented by group-theory considerations. The operators

$$\hat{J}_\pm = \hat{C} \pm \frac{\mathrm{i}}{2}(\hat{A} - \hat{B}) = \pm\frac{\mathrm{i}}{4}(\hat{q} \mp \mathrm{i}\hat{p})^2,$$
$$\hat{J}_0 = \frac{1}{2}(\hat{A} + \hat{B}) = \frac{1}{4}(\hat{p}^2 + \hat{q}^2) \tag{5.9}$$

satisfy commutation relations (1.1) with $\eta = -1$. Direct calculations using the explicit form of operators (5.9) give

$$\hat{\mathbf{J}}^2 = \hat{J}_0^2 - \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) = -\frac{3}{16}. \tag{5.9′}$$

Because the Casimir operator $\hat{\mathbf{J}}^2$ reduces to a c-number, we here have an irreducible [14–17] representation of the SU(1, 1) group (or algebra, to be precise). If $\mathbf{J}^2 = j(j+1)$, then $j = -1/4$ or $-3/4$, which means that two irreducible representations (with specified 'weights' j) are realized on the oscillator wave functions; these representations are unitary (because $S^+S = 1$ for the evolution operator in quantum mechanics) and infinite-dimensional.[9]

It is clear from (5.9) that the operator $\hat{J}_0$ has the form $\lambda_n = (2n+1)/4$ c $n = 0, 1, 2, \ldots$. For even $n = 2k$, $\lambda_n = k - j$ with $j = -1/4$, and for odd $n = 2l + 1$, $\lambda_n = l - j$ with $j = -3/4$, where $k, l = 0, 1, 2, \ldots$ are integers. Transitions occur between states $|m, \omega_-\rangle$ and $|n, \omega_+\rangle$ with equal parity, in compliance with the parity of the oscillator potential. The transition amplitude A_{nm} is expressed through the Wigner functions, i.e., finite rotation matrices for the above representations:

$$A_{nm} = D^j_{\mu\nu}(\psi, \beta, \varphi) = \exp\left[-\mathrm{i}(\mu\psi + \nu\varphi)\right] f^{(j)}_{\mu\nu}(\beta),$$
$$w_{nm} = |A_{nm}|^2 = \left| f^{(j)}_{k-j,\,l-j}(\beta) \right|, \qquad 0 \leqslant \beta < \infty, \tag{5.10}$$

with

$$j = -\frac{1}{4}, \qquad k = \frac{n}{2}, \qquad l = \frac{m}{2}, \tag{5.10′}$$

for even oscillator levels and

$$j = -\frac{3}{4}, \qquad k = \frac{n-1}{2}, \qquad l = \frac{m-1}{2} \tag{5.10″}$$

for odd levels; in either case, $k - j = (1/2)(n + 1/2)$ and $l - j = (1/2)(m + 1/2)$. The quantum transition probabilities are expressed (omitting technical details of computation) as [11]

$$w_{nm} = \frac{L!}{K!^2\,S!}\,\frac{\Gamma(L - 2j)}{\Gamma(S - 2j)} \times \rho^K(1 - \rho)^{-2j}\left[{}_2F_1(-S, L - 2j; K + 1; \rho)\right]^2, \tag{5.11}$$

where $L = \max(k, l)$, $S = \min(k, l)$, $K = L - S = |m - n|/2$, $\rho = \tanh^2(\beta/2)$ is the parameter describing the degree of oscillator excitation ($0 \leqslant \rho < 1$), ${}_2F_1(\ldots)$ is the Gauss hypergeometric function, and β is the angle of hyperbolic rotation in the group SU(1, 1) that can be found from the classical equation of motion $\ddot{x} + \omega^2(t)x = 0$ or from Riccati equation (5.7):

$$a(-\infty) = \omega_-, \qquad \rho = \lim_{t \to +\infty}\left|\frac{\omega_+ - a(t)}{\omega_+ + a(t)}\right|^2,$$
$$\beta = \ln\frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}}. \tag{5.12}$$

[9] This is not surprising because SU(1, 1) is a noncompact group (as can already be seen from the fact that matrix elements (1.9) are unbounded). Such a situation is more familiar to physicists from the example of the homogeneous (or proper) Lorentz group [39–41]. The mathematical theory of Lorentz group representations is considered at length in [42, 43]; the theory of unitary (infinite-dimensional) representations of noncompact Lie algebras is expounded in [19]. In [18], the SU(1, 1) group is denoted as QU(2).

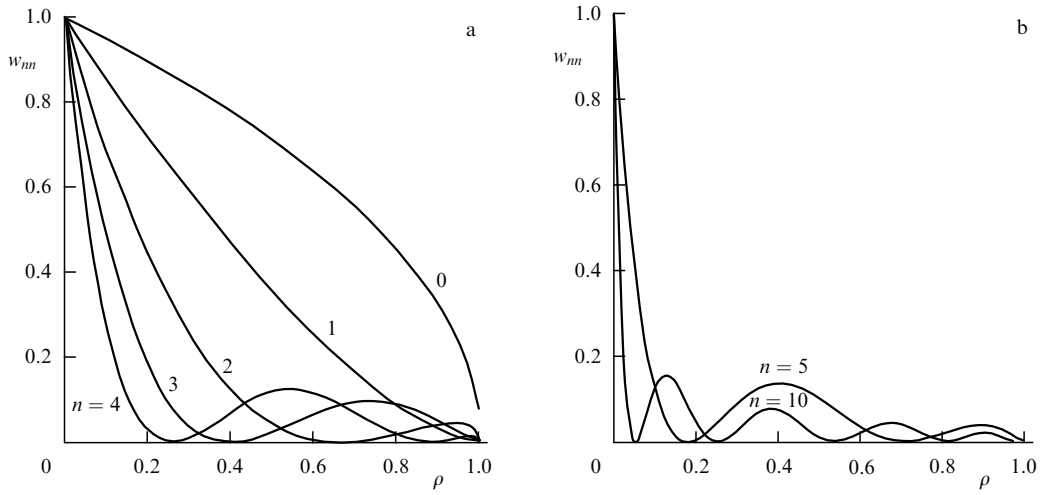

Figure 2. Oscillator with varying frequency. Transition probabilities $w_{nn}(\rho)$ at (a) $0 \leqslant n \leqslant 4$ and (b) $n = 5$ and 10.

For example, if $\omega(t) = \omega_- \theta(-t) + \omega_+ \theta(t)$ [where $\theta(t)$ is the Heaviside step function] describes an instantaneous jump in the oscillator frequency from ω_- to ω_+ [see also formula (B.6)], then

$$\beta = \ln \frac{\omega_+}{\omega_-}\,, \qquad \rho = \tanh^2 \frac{\beta}{2} = \left[\frac{\omega_+ - \omega_-}{\omega_+ + \omega_-}\right]^2. \tag{5.13}$$

Because $S \geqslant 0$ in (5.11) is an integer, the hypergeometric function always reduces to a polynomial. We consider a few specific cases.

The formulas are markedly simplified for transitions from the two lower oscillator levels ($m = 0$ and 1):

$$w_{n_0} = \frac{\Gamma\big((n+1)/2\big)}{(n/2)!\,\Gamma(1/2)}\,\rho^{n/2}\sqrt{1-\rho}\,, \qquad n = 0, 2, 4, \ldots, \tag{5.14}$$

$$w_{n1} = \frac{\Gamma\big((n+2)/2\big)}{\Gamma\big((n+1)/2\big)\,\Gamma(3/2)}\,\rho^{(n-1)/2}(1-\rho)^{3/2}, \qquad n = 1, 3, 5, \ldots .$$

Expressions (5.14) coincide with those obtained in Refs [44–47]. In the case of 'diagonal' ($m = n$) transitions, the probabilities w_{nn} can be expressed in terms of the generalized hypergeometric series [11]:

$$w_{nn} = \sqrt{1-\rho}\;{}_3F_2\left(-n, n+1, \frac{1}{2}\,; 1, 1; \rho\right), \tag{5.15}$$

which is convenient, for example, in the adiabatic case, where $\rho \ll 1$:

$$w_{nn}(\rho) = 1 - \frac{1}{2}\,N\rho + \frac{1}{32}(3N^2 - 4N - 3)\,\rho^2 - \frac{1}{576}(5N^3 - 28N^2 + 11N + 48)\,\rho^3 + \ldots, \tag{5.16}$$

with $N = n^2 + n + 1$. The small parameter is here given by $n^2\rho$, the coefficient at ρ^k being a kth-degree polynomial in N. Transition probabilities with $m \neq n$ at $\rho \to 0$ rapidly decrease with increasing $|m - n| = 2K$:

$$w_{nm} = \frac{n_>!}{(2^K K!)^2 n_<!}\left[\rho^K - \frac{mn + (m+n)/2 + 1}{2(K+1)}\,\rho^{K+1} + \ldots\right]. \tag{5.17}$$

As the excitation parameter ρ increases, the transition probabilities vary in a rather peculiar manner, as can be seen in Fig. 2 (as compared to Fig. 1) and Figs 3 and 4. Figure 4 depicts probabilities of upward transitions from the mth level $W_m^{(+)} = \sum_{n>m} w_{nm}$. For small ρ, upward transitions predominate over downward ones:

$$W_m^{(+)} = \frac{1}{4}(m+1)(m+2)\left[\rho - \frac{1}{16}(3m^2 + 5m - 4)\,\rho^2 + \ldots\right], \tag{5.18}$$

$$\frac{W_m^{(-)}}{W_m^{(+)}} = \frac{m(m-1)}{(m+1)(m+2)}\big[1 + O(\rho)\big]\,; \tag{5.19}$$

the same is true at $\rho \sim 1$, as follows from numerical calculations in [11]. With the exception of the two lower levels, the dependence of the transition probabilities w_{nm} on ρ is nonmonotonic and undergoes oscillations. Specifically, transition probabilities vanish at ρ values corresponding to zeros of the Legendre functions.

In the semiclassical limit $m, n \gg 1$, averaging the w_{mn} over fast oscillations characteristic of quantum mechanics leads to a representation corresponding to the classical oscillator:

$$\langle w_{nm}\rangle = \frac{2}{\pi\sqrt{(n-n_1)(n_2-n)}}\,, \qquad n_{1,2} = m\exp(\mp\beta)\,, \tag{5.20}$$

as is apparent from Fig. 3d, e. Here, $n_{1,2}$ are the turning points, and transition probabilities decrease exponentially for $n < n_1$ and $n > n_2$ [46].

In the case where $m = n$, Eqn (5.11) can be represented as

$$w_{nn} = \sqrt{1-\rho}\,\left[P_n\big(\sqrt{1-\rho}\big)\right]^2, \tag{5.21}$$

where P_n is the Legendre polynomial. A similar expression for arbitrary quantum numbers m and n was deduced (using a more complicated method) by directly solving the Schrödinger equation in [46].

In the adiabatic case, expression (5.11) is preferred to (5.21). It is worth noting that the transition probabilities w_{nm} were computed in the adiabatic approximation by Dykhne [48]; his results were somewhat refined in Ref. [11] [see formulas (4.6) and (4.7) in that work].

The main result in this section is formula (5.11), which was derived in Ref. [11] by means of a heuristic analytic

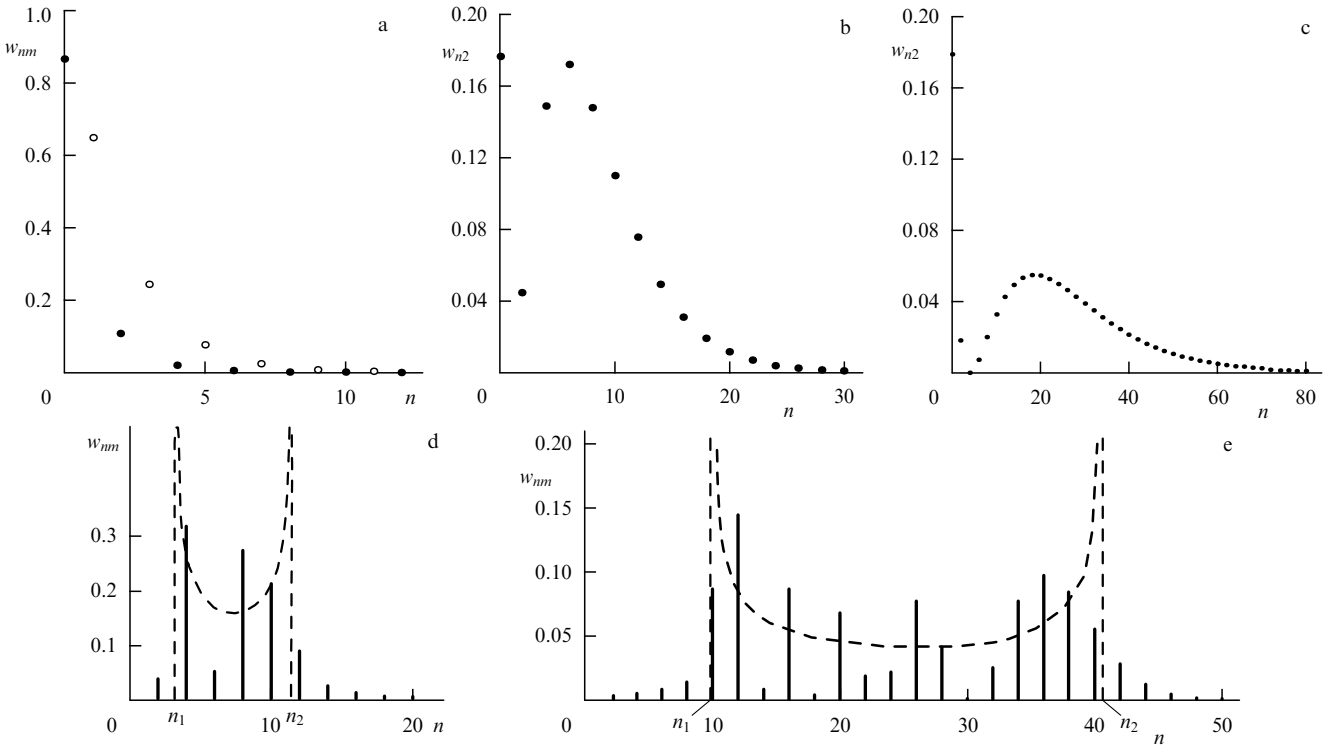

Figure 3. Distribution of transition probabilities w_{nm} from the mth level of the oscillator: (a) $\rho = 0.25$, $m = 0$(•) and $m = 1$(○); (b) $\rho = 0.5$, $m = 2$; (c) $\rho = 0.8$, $m = 2$; (d) $m = 6$, $\rho = 0.096$; (e) $m = 20$, $\rho = 0.12$. Dashed curves in Figs d and e correspond to distribution (5.20) in classical mechanics.

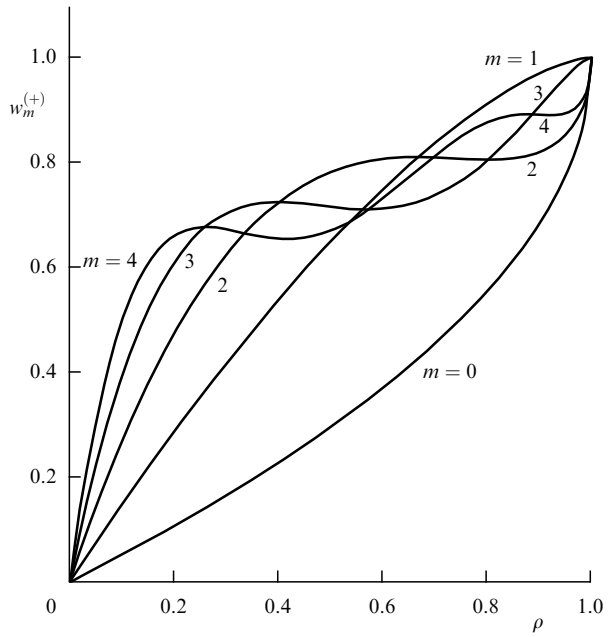

Figure 4. Probabilities of upward transitions ($n > m$) from the mth level of the oscillator depending on the excitation parameter ρ.

continuation of Wigner function (4.9) from the group SU(2) to the quasi-unitary group SU(1, 1). In this case, the unitarity for finite-dimensional representations is broken, as follows from (1.9). Nevertheless, the unitarity condition is satisfied in the specific case of finite-dimensional representations with weights $j = -1/4$ and $-3/4$:

$$\sum_{n=0}^{\infty} w_{nm}(\rho) = 1\,, \qquad m = 0, 1, 2, \ldots\,, \tag{5.22}$$

which substantiates the above results (see also Appendix D in connection with this).

6. Oscillator with varying parameters

We consider Hamiltonian (3.1) in which both the frequency ω and the force f are arbitrarily time-dependent. Calculation of commutators of the operators entering the Hamiltonian,

$$[\hat{p}^2, \hat{q}^2] = -2\mathrm{i}(\hat{p}\hat{q} + \hat{q}\hat{p})\,, \qquad [\hat{p}^2, \hat{p}\hat{q} + \hat{q}\hat{p}] = -4\mathrm{i}\hat{p}^2\,,$$
$$[\hat{q}^2, \hat{p}\hat{q} + \hat{q}\hat{p}] = 4\mathrm{i}\hat{q}^2\,, \qquad [\hat{p}^2, \hat{q}] = [\hat{p}, \hat{p}\hat{q} + \hat{q}\hat{p}] = -2\mathrm{i}\hat{p}\,,$$
$$[\hat{q}^2, \hat{p}] = [\hat{q}, \hat{p}\hat{q} + \hat{q}\hat{p}] = 2\mathrm{i}\hat{q}\,, \qquad [\hat{p}, \hat{q}] = -\mathrm{i}\,, \tag{6.1}$$

shows that the operator algebra closes, which suggests the possibility of applying the FDM to the evolution operator $\hat{S}$. The procedure described in Section 5 leads to

$$\hat{S}(t, t_0) = \exp\left(-\frac{a}{2}\,\hat{q}^2\right) \exp\left[\frac{c}{2}\,(\hat{p}\hat{q} + \hat{q}\hat{p})\right] \exp\left(-\frac{b}{2}\,\hat{p}^2\right) \hat{U}\,, \tag{6.2}$$

where $a(t)$, $b(t)$, and $c(t)$ are the same functions as in (5.6),

$$\hat{U}(t, t_0) = \mathrm{T}\,\exp\left\{\mathrm{i}\int_{t_0}^{t} \left[u(t')\hat{q} + v(t')\,\hat{p}\right] \mathrm{d}t'\right\},$$
$$u(t) = \int_{t_0}^{t} f(t') \exp\left[\mathrm{i}c(t')\right] \mathrm{d}t'\,, \tag{6.3}$$
$$v(t) = -\mathrm{i}\int_{t_0}^{t} f(t')b(t') \exp\left(\mathrm{i}c(t')\right] \mathrm{d}t'\,.$$

It now remains to disentangle the operator $\hat{U}$, i.e., represent it as a product $\exp(iu_1\hat{q})\exp(iv_1\hat{p})$; but it is more convenient to act in a different way. We note that the first three factors in (6.2) coincide with the right-hand side of (5.6), and the operator $\hat{U}$ alone depends on the external force. Using the completeness condition, we can write the transition matrix element as

$$S_{mn} = \sum_{m'} A_{nm'} U_{m'n}\,, \tag{6.4}$$

where the factor $A_{nm'}$ is the same as in (5.8) and (5.10); the second factor, related to the oscillator with a constant frequency ω_-,

$$U_{m'n}(t) = \left\langle m', \omega_- \middle| \mathrm{T}\exp\left\{\mathrm{i}\int_{t_0}^{t}\left[u(t')\hat{q} + v(t')\hat{p}\right]\mathrm{d}t'\right\}\middle| m, \omega_- \right\rangle, \tag{6.5}$$

is easily calculated by passing to the operators $\hat{a}$ and $\hat{a}^+$. Thus, the problem is solved in principle: the evolution operator is completely disentangled, although determination of the probabilities w_{nm} for arbitrary quantum numbers m and n requires very cumbersome calculations.

We confine ourselves to the physically most interesting case, where the oscillator is initially in the ground (non-excited) state:

$$\psi_0(x,t) = \langle x,t|0,\omega_-\rangle = \left(\frac{\omega_-}{\pi}\right)^{1/4}\exp\left\{-\frac{1}{2}\,\omega_-(x^2+\mathrm{i}t)\right\}.$$

It follows from the Schrödinger equation that the wave function remains a Gaussian packet at any instant t,

$$\psi(x,t) = \exp\left\{-\frac{1}{2}\left[a(t)x^2 + 2b(t)x + c(t)\right]\right\}, \tag{6.6}$$

if [44]

$$\mathrm{i}\dot{a} = a^2 - \omega^2\,, \qquad \mathrm{i}\dot{b} = ab + 2f\,, \qquad \dot{c} = \mathrm{i}(a - b^2) \tag{6.7}$$

with the initial conditions $a = \omega_-$, $b = 0$, and $c = \mathrm{i}\omega_- t - (1/2)\ln(\omega_-/\pi)$ as $t \to -\infty$. The transition matrix element

$$\langle n,\omega_+|\hat{S}|0,\omega_-\rangle \propto \int_{t_0}^{\infty}\exp\left\{-\frac{1}{2}\left[(a+\omega_+)x^2 + 2bx\right]\right\}H_n(\sqrt{\omega_+}x)\,\mathrm{d}x$$

is calculated analytically (Gaussian transformation, see formula 10.13.30 in Ref. [20]). As a result [46, 47],

$$w_{n0} = w_{00}\,\frac{\rho^{n/2}}{2^n n!}\left|H_n\left(\sqrt{\frac{v}{\sinh\beta}}\,\exp(-\mathrm{i}\varphi)\right)\right|^2, \quad n = 0,1,2,\ldots, \tag{6.8}$$

where $\sinh\beta = 2\sqrt{\rho}/(1-\rho)$. Specifically, for the vacuum–vacuum transition and transition to the first level,

$$w_{00} = \sqrt{1-\rho}\,\exp\left[-v(1-\sqrt{\rho}\,\cos 2\varphi)\right], \qquad w_{10} = v(1-\rho)w_{00}\,, \tag{6.9}$$

and in two limit cases,

$$w_{00} = \begin{cases} \exp(-v)\left[1 + \rho^{1/2}v\cos 2\varphi + \dfrac{1}{2}\,\rho(v^2\cos^2 2\varphi - 1) + \ldots\right], & \rho \to 0\,, \\ \sqrt{1-\rho}\,\exp(-2v\sin^2\varphi)\,, & \rho \to 1\,. \end{cases}$$

Here, $\rho = \tanh^2\beta/2$ and the parameter v describes oscillator excitation by an external force: $v = |d|^2$, where $d(t)$ is the displacement of a point depicting the classical oscillator state on the phase plane under the effect of the force $f(t)$ and H_n is the Hermit polynomial. The quantities d and v and the phase φ are exactly defined in Refs [44, 49].

We consider limit cases. If the frequency $\omega(t)$ changes slowly (adiabatic case), i.e., if $\rho \ll 1$, then

$$w_{n0} = \exp(-v)\,\frac{v^n}{n!} \times \left\{1 + \sqrt{\rho}\left[v - (n^2-n)v^{-1}\right]\cos 2\varphi + O(\rho)\right\}, \tag{6.10}$$

for $n^2\rho^{1/2} \ll v \ll \rho^{-1/2}$. As $\rho \to 0$, this formula becomes Poisson distribution (3.18), and the first correction to it has the order $\sqrt{\rho}$ (rather than ρ).

When $v \ll 1$, distribution (6.8) has different forms for even and odd n. For $n = 0,2,4,\ldots$, it turns into (5.14), and for $n = 1,3,5,\ldots$, into

$$w_{n0} = \frac{8\,\Gamma(n/2+1)}{\sqrt{\pi}\,\Gamma\big((n+1)/2\big)}\,\rho^{(n-1)/2}(1-\rho)^{3/2}v + O(v^2)\,. \tag{6.11}$$

The formulas become more complicated for arbitrary m and n: in the general case, the transition probabilities w_{nm} can be expressed [49] through generalized Hermit polynomials in two complex variables. The semiclassical limit for w_{nm} is considered in Refs [47, 49].

To conclude, we note that relations (6.1) acquire a more symmetric form when passing to creation and annihilation operators:

$$J_0 = \frac{1}{4}(aa^+ + a^+a)\,, \quad J_+ = \frac{\mathrm{i}}{2}\,a^{+2}\,, \quad J_- = -\frac{\mathrm{i}}{2}\,a^2\,. \tag{6.12}$$

In this case,

$$[J_+, J_-] = -2J_0\,, \qquad [J_0, J_\pm] = \pm J_\pm\,, \tag{6.13}$$

$$\mathbf{J}^2 = \frac{1}{16}\left\{(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})^2 - 2(\hat{a}^2\hat{a}^{+2} + \hat{a}^{+2}\hat{a}^2)\right\} = -\frac{3}{16}\,, \tag{6.14}$$

$$[J_0, a] = \frac{1}{2\mathrm{i}}\,[J_-, a^+] = -\frac{1}{2}\,a\,, \qquad [J_0, a^+] = \frac{\mathrm{i}}{2}\,[J_+, a] = \frac{1}{2}\,a^+\,, \tag{6.15}$$

$$[J_+, a^+] = [J_-, a] = 0\,, \qquad [a, a^+] = 1\,,$$

where $\hbar = m = \omega = 1$. Equations (6.13) fully correspond to the SU(1,1) generators and the relation $[a, a^+] = 1$ corresponds to the Heisenberg (nilpotent) algebra. The whole set of ten commutation relations defines a certain Lie algebra, which is no longer semisimple, however.

7. Multidimensional case

We briefly consider generalization of the above results to the case of a D-dimensional oscillator.[10] Let

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{D}\{\hat{p}_i^2 + \omega_i^2(t)\hat{q}_i^2\}\,, \qquad [\hat{p}_j, \hat{q}_k] = -\mathrm{i}\delta_{jk}\,. \tag{7.1}$$

Using the FDM, the evolution operator is found as a product D of operators of type (5.6) that commute for different i. The vacuum – vacuum transition probability is

$$w_{00}^{(D)} = \sqrt{(1-\rho_1)(1-\rho_2)\ldots(1-\rho_D)}\,; \tag{7.2}$$

in particular, for an isotropic oscillator,

$$w_{00}^{(D)} = (1-\rho)^{D/2} = \left(\cosh\frac{\beta}{2}\right)^{-D}, \tag{7.3}$$

where β is the hyperbolic rotation angle in the SU(1, 1) group. The weight of the irreducible representation is $j = -D/4$; hence, the probability of the transition $|0, \omega_-\rangle \to |2n, \omega_+\rangle$ is expressed as

$$w_{2n,0}^{(D)} = \frac{\Gamma(n + D/2)}{n!\,\Gamma(D/2)}\,\rho^n(1-\rho)^{D/2}\,, \qquad \sum_{n=0}^{\infty} w_{2n,0}^{(D)} = 1 \tag{7.4}$$

(only even levels are excited in the case of the oscillator ground state). At $D = 2$, formula (7.4) assumes the form

$$w_{2n,0} = (1-\rho)\,\rho^n\,, \qquad n = 0, 1, 2, \ldots, \tag{7.5}$$

and describes the distribution with respect to the number of pairs n of charged scalar (spinless) bosons created from the vacuum under the effect of a uniform electric field $\mathcal{E}(t)$. This assertion is conveniently clarified using group theory. In this case, the Klein – Gordon equation reduces [53] to the equation $\ddot{\xi} + \omega^2(t)\xi = 0$ for field oscillators, where

$$\omega^2(t) = m^2 + \mathbf{p}^2(t)\,, \qquad \mathbf{p} = \mathbf{p}_- + e\int_{-\infty}^{t}\mathcal{E}(t')\,\mathrm{d}t'\,, \tag{7.6}$$

and $\mathbf{p}(t)$ is the momentum of a classical particle in the external field $\mathcal{E}(t)$. We consider the Heisenberg operators $\hat{a}_{\mathbf{p}}(t)$ and $\hat{b}_{-\mathbf{p}}(t)$ for particles and antiparticles. The equations of motion for them have solutions in the form of Bogolyubov's canonical transformation

$$\begin{aligned}\hat{a}_{\mathbf{p}}(t) &= u(t)\hat{a}_{\mathbf{p}}(t_0) + v(t)\hat{b}^+_{-\mathbf{p}}(t_0)\,,\\ \hat{b}^+_{-\mathbf{p}}(t) &= v^*(t)\hat{a}_{\mathbf{p}}(t_0) + u^*(t)\hat{b}^+_{-\mathbf{p}}(t_0)\,,\end{aligned} \tag{7.7}$$

where $|u(t)|^2 - |v(t)|^2 = 1$, i.e., transformation (7.7) belongs to the group SU(1, 1). The mean number of bosonic pairs created in the state with this momentum is

$$\langle n_{\mathbf{p}}\rangle = \lim_{t\to+\infty}\langle 0|\hat{a}^+_{\mathbf{p}}(t)\hat{a}_{\mathbf{p}}(t)|0\rangle = |v(\infty)|^2 = \frac{\rho}{1-\rho}\,, \tag{7.8}$$

where $|0\rangle$ is the initial vacuum vector and the distribution over pair numbers has form (7.5). In other words, the problem of vacuum pair creation under the effect of a uniform field $\boldsymbol{\mathcal{E}}(t)$ can be reduced to the problem of excitation of a two-dimensional varying-frequency oscillator, where the parameter ρ coincides with the coefficient of reflection from the barrier given by the function $\omega^2(x), t \to x$ (see examples in Appendix B).

We now assume (see Ref. [53]) that

$$J_+ = a^+b^+\,, \qquad J_0 = \frac{1}{2}(a^+a + b^+b + 1)\,, \qquad J_- = ba = ab\,. \tag{7.9}$$

For brevity, we omit the index $\mathbf{p}$ (in the fermionic case, $\mathbf{p}\sigma$, where σ is the spin projection) labeling the creation and annihilation operators. Because $[a, a^+] = [b, b^+] = 1$, operators (7.9) satisfy the same commutation relations as the SU(1, 1) generators. Similarly, for fermions ($s = 1/2$), $\{a, a^+\} = \{b, b^+\} = 1$, where $\{\ldots\}$ is the anicommutator and the operators

$$J_+ = a^+b^+\,, \qquad J_0 = \frac{1}{2}(a^+a + b^+b - 1)\,, \qquad J_- = ba = -ab \tag{7.10}$$

are generators of SU(2). The Casimir operator is given by

$$\begin{aligned}\mathbf{J}^2 &= j(j+1)\left[1 - (a^+a - b^+b)^2\right],\\ j &= \begin{cases} -\dfrac{1}{2}\,, & s = 0\,,\\[2mm] \dfrac{1}{2}\,, & s = \dfrac{1}{2}\end{cases}\end{aligned} \tag{7.11}$$

(for the vacuum state, $a|0\rangle = b|0\rangle = 0$ and $J^2 = j(j+1) = -1/4$ or $3/4$). The probability of creating n pairs in the state with a momentum $\mathbf{p}$ is given by the squared Wigner function $d^{(j)}_{n-j,-j}(\theta)$ for the corresponding representation. In the bosonic case, it is the infinite-dimensional unitary representation of SU(1, 1) with weight $j = -1/2$; for fermions, it is the spinor ($j = 1/2$) representation of SU(2). In the latter case, only two values are possible: $n = 0$ (no pairs are created) and $n = 1$, with the respective probabilities $\cos^2\theta/2$ and $\sin^2\theta/2$. This accounts for the difference between the cases where $s = 0$ and $s = 1/2$.[11]

For a pulsed field of the special form $\mathcal{E}(t) = \mathcal{E}_0/\cosh^2\omega t$, expression (7.6) looks like the Eckart potential [64] if the parameters $\omega_\pm$ and ω_0 are properly chosen (see formula (B.7) in Appendix B). This immediately gives formulas [53] for the parameter ρ at spin values $s = 0$ and $1/2$, which entirely coincide with the results in Ref. [65], where they were deduced directly from the Klein – Gordon – Dirac equations (allowing an exact solution in this case).

The uniform electric field $\mathcal{E}(t)$ is an idealization overestimating the number of created pairs N. An electromagnetic wave always has a magnetic field that decreases N (as known, pairs are not created from the vacuum at all in a purely magnetic field or in a plane wave of an arbitrary intensity and

[10] The problem of a D-dimensional oscillator with time-dependent parameters was also considered in Refs [50 – 52]; the last reference contains an extensive bibliography.

[11] But we note that for $\mathcal{E} \ll \mathcal{E}_{\mathrm{cr}} = m^2c^3/e\hbar$ (which can be realistically expected in experiment [54 – 60]), the excitation parameter of field oscillators is exponentially small, $\rho \propto \exp(-\pi\mathcal{E}_{\mathrm{cr}}/\mathcal{E})$, and just a single pair can be created from the vacuum. In this situation, the difference between bosonic and fermionic cases disappears. Here, $\mathcal{E}_{\mathrm{cr}}$ is the so-called critical field in QED [61 – 63]: $\mathcal{E}_{\mathrm{cr}} = 1.3\times10^{16}$ V cm^{-1} for e^+e^-, $\mathcal{E}_{\mathrm{cr}} \sim 10^{21}$ V cm^{-1} for $\pi^+\pi^-$.

spectral composition) [63]. The authors of Refs [58, 59] considered a realistic three-dimensional model of a focused laser pulse based on the exact solution of Maxwell equations in the vacuum. Numerical integration over a momentum 4-volume permitted exploring the dependence of the number N of created e^+e^- pairs on the parameters in this problem (the focal spot radius R, the diffraction length $L = R/\Delta$, and the focusing parameter $\Delta = c/\omega R = \lambda/(2\pi R)$ differentiating a laser pulse from a plane wave) and on the radiation polarization. Physical aspects of the e^+e^- pair creation from the vacuum after focusing superpowerful laser radiation are beyond the scope of this paper. They are paramount in view of rapid progress in laser technologies and are considered in many other publications (see, e.g., Refs [54–60] and the references therein).

We also note that the problem of a quantum oscillator with varying frequency pertains not only to the QED in strong fields but also to the theory of particle creation in the gravitational field near the cosmological singularity [66–68].

8. Further applications of the Feynman disentangling method

We here consider several more problems where the use of the FDM may be helpful.

(1) The first example is borrowed from Feynman's work [6]. The introduction of an ordering index s for arbitrary operators $\hat{\alpha}$ and $\hat{\beta}$ gives

$$\exp(\hat{\alpha}+\hat{\beta}) = \exp\left\{\int_0^1 (\hat{\alpha}+\hat{\beta})\,\mathrm{d}s\right\} = \exp\hat{\alpha}\exp\left\{\int_0^1 \hat{\beta}'(s)\,\mathrm{d}s\right\}, \tag{8.1}$$

where, in accordance with (2.8′),

$$\hat{\beta}'(s) = \exp\left(-\int_0^s \hat{\alpha}\,\mathrm{d}s'\right)\hat{\beta}\exp\left(\int_0^s \hat{\alpha}\,\mathrm{d}s'\right) = \exp(-s\hat{\alpha})\,\hat{\beta}\exp(s\hat{\alpha})\,. \tag{8.2}$$

This expression is formally exact, but the integral over s in (8.1) is hardly possible to calculate in the explicit form, for all its apparent simplicity, because $[\hat{\alpha},\hat{\beta}] \neq 0$. We suppose that the operator $\hat{\beta}$ is small in a certain sense and use this in the perturbation theory:

$$\exp(\hat{\alpha}+\hat{\beta}) = \exp\hat{\alpha} + \int_0^1 \exp\left[(1-s)\hat{\alpha}\right]\hat{\beta}\exp(s\hat{\alpha})\,\mathrm{d}s + \ldots\,. \tag{8.3}$$

In the representation where the operator $\hat{\alpha}$ is diagonal and has eigenvalues α_n,

$$\hat{\alpha}|n\rangle = \alpha_n|n\rangle\,, \qquad \langle m|n\rangle = \delta_{mn}\,, \tag{8.4}$$

it is easy to show that (8.3) implies a perturbation-theory formula for matrix elements,

$$\left\langle \exp(\hat{\alpha}+\hat{\beta})\right\rangle_{mn} = \delta_{mn}\exp\alpha_n + \frac{\exp\alpha_m - \exp\alpha_n}{\alpha_m - \alpha_n}\,\beta_{mn} + O(\beta^2) \tag{8.5}$$

[cf. Eqn (2) in Ref. [6], where it is demonstrated that the use of the FDM permits obtaining standard results of the theory of time-dependent perturbations].

(2) The Schrödinger equation for an atom's electron in the field of an electromagnetic wave has the form

$$\mathrm{i}\,\frac{\partial\psi}{\partial t} = \hat{H}\psi\,, \qquad \hat{H} = \frac{1}{2}\,\mathbf{p}^2 + V(\mathbf{r},t) + U(r)\,, \tag{8.6}$$

where $V(\mathbf{r},t) = -e\boldsymbol{\mathcal{E}}(t)\mathbf{r}$ is the dipole approximation, $\boldsymbol{\mathcal{E}}(t) = -c^{-1}\,\partial\mathbf{A}/\partial t$ is the electric field of a plane light wave, and $U(r)$ is the potential describing the electron interaction with the atomic backbone. Equation (8.6) corresponds to the so-called 'length gauge' used by Keldysh in [69] and by many authors after him.

Setting $\hat{A} = (1/2)\mathbf{p}^2$, $\hat{B} = \boldsymbol{\mathcal{E}}\mathbf{r}$, and $\hat{C} = \boldsymbol{\mathcal{E}}\mathbf{p}$, we have

$$[\hat{A},\hat{B}] = -\mathrm{i}\hat{C}\,, \qquad [\hat{B},\hat{C}] = \mathrm{i}\boldsymbol{\mathcal{E}}^2\hat{I}\,, \qquad [\hat{C},\hat{A}] = 0\,, \tag{8.7}$$

where $\hat{I}$ is the unit operator (commuting with all the others). The operator algebra thus closes, and it is possible to use the FDM:

$$\hat{S}(t,t_0) = \mathrm{T}\exp\left\{-\mathrm{i}\int_{t_0}^t \hat{H}(t')\,\mathrm{d}t'\right\} = \exp\left(\mathrm{i}\,\frac{e}{c}\,\mathbf{A}(t)\mathbf{r}\right)\exp\left\{-\frac{\mathrm{i}}{2}\int_{t_0}^t \mathbf{P}^2(t')\,\mathrm{d}t'\right\}. \tag{8.8}$$

The Wolkow wave function [37, 70]

$$\psi_{\mathbf{p}}(\mathbf{r},t) = \hat{S}\exp(\mathrm{i}\mathbf{p}\mathbf{r}) = (2\pi)^{-3/2}\exp\left\{\mathrm{i}\left[\mathbf{P}(t)\mathbf{r} - \frac{1}{2}\int_{t_0}^t \mathbf{P}^2(t')\,\mathrm{d}t'\right]\right\} \tag{8.9}$$

is frequently used in the theory of atomic ionization and excitation by laser light. Here, $\mathbf{P} = \mathbf{p} - (e/c)\mathbf{A}(t)$ is the generalized momentum.

Expression (8.8) corresponds to the interaction between the atom and the external field chosen in the form

$$V(\mathbf{r},t) = \frac{1}{c}\,\mathbf{A}(t)\mathbf{p} + \frac{1}{2c^2}\,\mathbf{A}^2(t)\,, \tag{8.6′}$$

and is referred to in the literature as the 'velocity gauge.' These two gauges are used in numerous works on the theory of atomic ionization and excitation by intense laser radiation (see, e.g., reviews [60, 71]). Our aim is to demonstrate that using the FDM for disentangling operators contained in the Hamiltonian makes it easy to pass from one wave function to another. When disentangling operators in $\hat{S}(t,t_0)$, we assume $U(r)$ in (8.6) to be a short-range potential, which is a good approximation in the case of ionization of one-charge negative ions like H^-, Li^-, or Na^- (see, e.g., Refs [72, 73]) — the limiting case represented by the model of the zero-radius potential, well-known in atomic and nuclear physics [47].

(3) We consider a model with the hidden symmetry group:

$$\hat{H} = \hat{H}_0 + \boldsymbol{\omega}_1(t)\mathbf{L} + \boldsymbol{\omega}_2(t)\mathbf{A}\,, \qquad \hat{H}_0 = \frac{1}{2}\,\mathbf{p}^2 - \frac{1}{r}\,, \tag{8.10}$$

where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is the orbital momentum,

$$\mathbf{A} = \frac{1}{2}\left\{(\mathbf{L}\times\mathbf{p}) - (\mathbf{p}\times\mathbf{L})\right\} + \frac{\mathbf{r}}{r}$$

is the Laplace–Runge–Lenz vector, and the frequencies ω_1 and ω_2 are the given functions of time. The operator $\hat{H}$ may be regarded as the projection of the hydrogen atom Hamiltonian in alternating electric and magnetic fields on the subspace of fixed-energy states.

As is known from Refs [17, 24], the hydrogen atom is characterized by the so-called 'accidental' degeneracy of levels with given $n = (-2E)^{-1/2}$ and orbital momenta $l = 0, 1, \ldots, n-1$. This degeneracy is explained by the Hamiltonian $\hat{H}_0$ having not only the manifest geometric symmetry group SO(3) but also the hidden symmetry group of the Coulomb potential, i.e., the group SO(4) for discrete spectrum states and the Lorentz group SO(3, 1) for the continuous spectrum [17, 74–78].

In the former case, setting $\mathbf{M} = (-2H_0)^{-1/2}\mathbf{A}$, we obtain

$$[\mathbf{L}, \mathbf{L}] = [\mathbf{M}, \mathbf{M}] = \mathrm{i}\mathbf{L}\,, \qquad [\mathbf{L}, \mathbf{M}] = \mathrm{i}\mathbf{M}\,,$$
$$\mathbf{L}^2 + \mathbf{M}^2 = (-2H_0)^{-1} - 1 = n^2 - 1\,, \tag{8.11}$$

in correspondence with commutation relations for generators of the four-dimensional rotation group $\mathrm{SO}(4) = \mathrm{SO}(3) \otimes \mathrm{SO}(3)$. Passing to the commuting operators $\mathbf{I}_{1,2} = (\mathbf{L} \pm \mathbf{M})/2$, we have

$$\hat{S}(t, 0) = \exp(-\mathrm{i}E_n t) \exp(-\mathrm{i}\boldsymbol{\Omega}_1\mathbf{I}_1) \exp(-\mathrm{i}\boldsymbol{\Omega}_2\mathbf{I}_2)\,,$$
$$\boldsymbol{\Omega}_{1,2}(t) = \int_{t_0}^{t} \left[\boldsymbol{\omega}_1(t') \pm \frac{1}{n}\,\boldsymbol{\omega}_2(t')\right] \mathrm{d}t' \tag{8.12}$$

for the evolution operator.

Further disentanglement of the operators I_+, I_0, and I_- is performed by the FDM exactly as in the spin rotation problem (see Section 4).

In the case of continuous-spectrum states with an energy $E = k^2/2 > 0$, the operator $\mathbf{M}$ becomes non-Hermitian and must be replaced with the operator $\mathbf{N} = (2H_0)^{-1/2}\mathbf{A}$. This leads to the commutation relations

$$[\mathbf{L}, \mathbf{L}] = \mathrm{i}\mathbf{L}\,, \qquad [\mathbf{L}, \mathbf{N}] = \mathrm{i}\mathbf{N}\,, \qquad [\mathbf{N}, \mathbf{N}] = -\mathrm{i}\mathbf{L}\,, \tag{8.13}$$

corresponding to the Lorentz group generators.[12] Further computations are analogous to (4.12). The following notation was used in the foregoing discussion: $[\mathbf{A}, \mathbf{B}] = \mathrm{i}\mathbf{C}$ means that $[\hat{A}_j, \hat{B}_k] = \mathrm{i}\varepsilon_{jkl}\hat{C}_l$, where ε_{jkl} is the totally antisymmetric 3-tensor ($\varepsilon_{123} = 1$) and the indices j, k, and l take values 1, 2, and 3. Most of these equations correspond to the known commutation rule [24]: $[\mathbf{L}, \mathbf{V}] = \mathrm{i}\mathbf{V}$, where $\mathbf{L}$ is the angular momentum operator in the SO(3) group and $\mathbf{V}$ is an arbitrary three-dimensional vector; the last of relations (8.13), differing in sign from the two others, corresponds, in terms of group theory, to the effect of Thomas precession in relativistic quantum mechanics [79–82].

In group theory, it is possible to write matrix elements of transitions between states $|n, l, m\rangle$ of the hydrogen atom with a fixed principal quantum number n. For $E_n = -1/2n^2 < 0$, these are Wigner functions for the representation $D(j_1, j_2)$ of the SO(4) group with $j_1 = j_2 = (n-1)/2$; for the energy $E = k^2/2 > 0$, they relate to the infinite-dimensional unitary representation $D(0, \rho)$ of the Lorentz group of the so-called principal series [43]. The eigenvalue of the scalar $\hat{F}$ is then given by

$$\hat{F} = \frac{1}{2}\,M^{\mu\nu}M_{\mu\nu} = \mathbf{L}^2 - \mathbf{N}^2 = -\left(1 + \frac{\rho^2}{4}\right), \tag{8.14}$$

[12] In the traditional realization of the Lorentz group, $\hat{N}_i$ is the boost operator (i.e., operator of a proper Lorentz transformation without axis rotation) along the ith axis of the reference frame [39, 42].

where $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ is the orbital 4-momentum operator. This expression is derived from (8.11) by the replacement $n \to \mathrm{i}\rho/2$, whence $\rho = 2/k$. The second invariant of the Lorentz group (a pseudoscalar) is zero [it differs from zero only when the particle has a spin [39–42], but the electron spin is neglected in (8.10)].

(4) One more example has a mathematical slant. Let $\hat{A}$ and $\hat{B}$ be operators commuting with $[\hat{A}, \hat{B}]$:

$$\left[\hat{A}, [\hat{A}, \hat{B}]\right] = \left[\hat{B}, [\hat{A}, \hat{B}]\right] = 0\,. \tag{8.15}$$

Formally introducing a parameter t (on which operators $\hat{A}$ and $\hat{B}$ do not depend), we have

$$\hat{S}(t) \equiv \exp\left[(\hat{A} + \hat{B})t\right] = \exp\left\{\int_0^t (\hat{A} + \hat{B})\,\mathrm{d}t'\right\}.$$

Using the FDM, we then disentangle the first factor as

$$\hat{S}(t) = \exp(\hat{A}t) \exp\left(\int_0^t \hat{B}'\,\mathrm{d}t'\right), \tag{8.16}$$

where, in accordance with (2.8′),

$$\hat{B}'(t') = \exp(-\hat{A}t')\,\hat{B}\exp(\hat{A}t') = \hat{B} - [A, \hat{B}]\,t'\,.$$

Using the FDM procedure further and taking condition (8.15) into account, we arrive at the Glauber formula [83]:

$$\exp\left\{(\hat{A} + \hat{B})t\right\} = \exp(\hat{A}t) \exp(\hat{B}t) \exp\left(-\frac{1}{2}\,[\hat{A}, \hat{B}]t^2\right)$$
$$\equiv \exp(\hat{B}t) \exp(\hat{A}t) \exp\left(\frac{1}{2}\left([\hat{A}, \hat{B}]t^2\right)\right). \tag{8.17}$$

Clearly, for noncommuting operators, $\exp(\hat{A} + \hat{B}) \neq \exp\hat{A}\exp\hat{B}$; this is the main difficulty in noncommutative analysis to be overcome with the aid of the FDM.

Operators satisfying condition (8.15) are fairly frequent in theoretical physics [47, 83, 89]; therefore, identity (8.17) is quite useful. It is a special case of the Baker–Hausdorff series known from group theory:

$$\exp\hat{A}\exp\hat{B} = \exp\left\{\hat{A} + \hat{B} + \frac{1}{2}\,[\hat{A}, \hat{B}]\right.$$
$$\left. - \frac{1}{12}\left(\left[\hat{A}, [\hat{A}, \hat{B}]\right] + \left[[\hat{A}, \hat{B}], \hat{B}\right]\right) + \ldots\right\}. \tag{8.18}$$

The exponent in the right-hand side involves a formal power series whose convergence is not discussed. We refer the reader to Refs [84, 85] for the next terms of this series.

(5) We consider the operator $U(\alpha) = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a})$, which is unitary for any complex α:

$$U^+(\alpha) = \exp(\alpha^*\hat{a} - \alpha\hat{a}^+) = U(-\alpha) = U^{-1}(\alpha)\,. \tag{8.19}$$

It can be easily disentangled using relation (8.17):

$$U(\alpha) = \exp\left(-\frac{1}{2}\,|\alpha|^2\right) \exp(\alpha\hat{a}^+) \exp(-\alpha^*\hat{a})$$
$$= \exp\left(\frac{1}{2}\,|\alpha|^2\right) \exp(-\alpha^*\hat{a}) \exp(\alpha\hat{a}^+)\,. \tag{8.20}$$

Hence,

$$U(\alpha)|0\rangle = \exp\left(-\frac{1}{2}\,|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \equiv |\alpha\rangle\,, \tag{8.21}$$

i.e., the operator $U(\alpha)$ maps the vacuum state $|0\rangle$ into the coherent state $|\alpha\rangle$ preserving the normalization: $\langle\alpha|\alpha\rangle = \langle 0|0\rangle = 1$.

Here are more relations:

$$\begin{aligned} &U^+(\alpha)\,\hat a\,U(\alpha) = \exp\left(-\alpha\hat a^+\right)\hat a\exp\left(\alpha\hat a^+\right) = \hat a + \alpha\,,\\ &U^+(\alpha)\,\hat a^+\,U(\alpha) = \exp\left(\alpha^*\hat a\right)\hat a^+\exp\left(-\alpha^*\hat a\right) = \hat a^+ + \alpha^*\,, \end{aligned} \tag{8.22}$$

$$U(\alpha)U(\beta) = U(\alpha+\beta)\exp\left(\mathrm{i}\varphi\right), \qquad \varphi = \mathrm{Im}\left(\alpha\beta^*\right), \tag{8.23}$$

by virtue of which the operator $U(\alpha)$ is sometimes called the displacement (shift) operator.

These relations, as well as the coherent states $|\alpha\rangle$ themselves first introduced by Schrödinger [86, 88], are still extensively used in quantum optics to describe coherent properties of laser light (see, e.g., Refs [87–89]).

(6) The natural question whether the operators $\hat a$ and $\hat a^+$ have eigenfunctions is most easily answered in terms of the Fock–Bargmann representation for oscillators [90–92], to be considered in Appendix A. It turns out that the spectrum of the annihilation operator $\hat a$ continuously fills the entire complex plane, whereas the operator $\hat a^+$ has no eigenfunctions (see also Ref. [47]). This shows that non-Hermitian (non-self-adjoint, to be precise) operators, such as $\hat a$ and $\hat a^+$, may have very unusual properties.

(7) One more formula is worth noting [8]:

$$\exp\hat A\exp\hat B = \sum_{n=0}^{\infty}\frac{1}{n!}\,\hat B_n\,, \tag{8.24}$$

where $\hat B_0 = \hat B$ and $\hat B_{n+1} = [\hat A, \hat B_n]$; its use typically results in an infinite sequence of operators. Sometimes, however, the process terminates:

$$\text{if}\quad [\hat A, \hat B_N] = 0\,,\quad \text{then}\quad \hat B_{N+1} = \hat B_{N+2} = \ldots = 0\,, \tag{8.25}$$

and the system of a finite number of operators becomes closed with respect to commutation relations. In this case, the FDM may also be helpful.

The above examples give some idea of the advantages of the FDM when used in quantum mechanics and mathematical physics.

9. Counterexamples

There is a natural question concerning the possibility of using the FDM in other problems. We start from the anharmonic oscillator[13]

$$\hat H = \frac{1}{2}\,p^2 + \frac{g}{4}\,x^4\,. \tag{9.1}$$

[13] A popular model in quantum mechanics and field theory examined from various perspectives by many authors (see, e.g., Refs [93–97]); however, the FDM appears to have never been applied to this model. Here, we omit the hats ($\hat{}$) over the operators for brevity.

Consecutive computation of commutators using the relations

$$\begin{aligned} &[p, x^n] = -\mathrm{i}nx^{n-1}\,, \qquad [p^n, x] = -\mathrm{i}np^{n-1}\,,\\ &[p^2, x^n] = -\mathrm{i}n(px^{n-1} + x^{n-1}p)\,,\\ &[p^2, px^n + x^np] = -2\mathrm{i}n(p^2x^{n-1} + x^{n-1}p^2)\\ &\quad - \mathrm{i}n(n-1)(n-2)x^{n-3} \end{aligned} \tag{9.2}$$

yields

$$\begin{aligned} &[p^2, x^4] = -4\mathrm{i}(px^3 + x^3p)\,,\\ &[p^2, px^3 + x^3p] = -6\mathrm{i}(p^2x^2 + x^2p^2 + 1)\,,\\ &[p^2, p^2x^2 + x^2p^2] = -4\mathrm{i}(p^3x + xp^3)\,,\\ &[p^2, p^3x + xp^3] = -4\mathrm{i}p^4\,, \qquad [p^2, p^4] = 0\,; \end{aligned} \tag{9.3}$$

at this point, the chain of commutators starting from p^2 terminates. But we also have

$$[px^3 + x^3p,\, x^4] = -8\mathrm{i}x^6\,, \qquad [px^3 + x^3p,\, x^6] = -12\mathrm{i}x^8\,,$$

and in the general case,

$$[px^3 + x^3p,\, x^{2k}] = -4\mathrm{i}kx^{2k+2}\,, \qquad k = 1, 2, 3, \ldots\,. \tag{9.4}$$

In commutation, the powers of x become arbitrarily large, and therefore the FDM does not lead, in this case, to a closed operator algebra with a finite number of generators, and disentanglement of the evolution operator $\hat S$ gives rise to an infinite process. The same situation occurs when x^4 in Hamiltonian (9.1) is replaced with anharmonism of the general form x^{2N}, $N \geqslant 2$.

An exact solution of the Schrödinger equation exists for the Coulomb potential, besides the anharmonic oscillator. In this case, the use of the FDM encounters the same difficulties as above. As an example, for a one-dimensional model of a 'hydrogen atom,' $H = p^2 + x^{-1}$, we have

$$\begin{aligned} &[p^2, x^{-1}] = \mathrm{i}(px^{-2} + x^{-2}p)\,,\\ &[p^2, px^{-2} + x^{-2}p] = 2\mathrm{i}(p^2x^{-3} + 2px^{-3}p + x^{-3}p^2)\,,\\ &[px^{-2} + x^{-2}p, x^{-1}] = 2\mathrm{i}x^{-4}\,,\\ &[px^{-2} + x^{-2}p, x^{-4}] = 8\mathrm{i}x^{-7}\,, \end{aligned} \tag{9.5}$$

and so on. The chain of commutators does not terminate, and progressively more singular (at $x \to 0$) operators arise at each new step. The situation is further aggravated with the transition to the three-dimensional hydrogen atom:

$$\begin{aligned} &[\mathbf{p}^2, r^{-1}] = \mathrm{i}\{(\mathbf{pr})r^{-3} + r^{-3}(\mathbf{rp})\}\,,\\ &[(\mathbf{pr})r^{-3} + r^{-3}(\mathbf{rp}), r^{-1}] = 2\mathrm{i}r^{-4}\,,\\ &[\mathbf{p}^2, r^{-n}] = \mathrm{i}n\{(\mathbf{pr})r^{-(n+2)} + r^{-(n+2)}(\mathbf{rp})\}, \ldots\,. \end{aligned} \tag{9.6}$$

The above examples indicate that application of the FDM in nonrelativistic quantum mechanics may be restricted. It would be appropriate to assess, parallel to the search for new exact solutions, whether the FDM can be used with approximate and qualitative methods of quantum theory [98, 99].

10. Conclusion

We have considered several quantum mechanical problems in which the FDM permits us to completely disentangle operators entering a Hamiltonian and obtain the exact (analytic) solution of the time-dependent Schrödinger equation. However, the FDM is far from being a universal method, in contrast, for example, to the Feynman diagram technique in the perturbation theory. The application of the FDM is hard to standardize and it requires ingenuity. Nevertheless, if realized successfully, it permits having a solution in the case of arbitrary time-dependent changes of Hamiltonian parameters, such as the force $f(t)$ and the frequency $\omega(t)$, which is interesting in itself. A few remarks are in order.

(1) The problem of oscillator excitation by an external force involves only two operators, $\hat{a}$ and $\hat{a}^+$, whose commutator is a c-number; therefore, the use of the FDM leads directly to the desired result [6]. The authors of Refs [10–12] considered cases in which operators of a Hamiltonian gave rise to Lie algebra containing three generators ($\hat{J}_+$, $\hat{J}_0$, and $\hat{J}_-$). The evolution operator $\hat{S}$ is disentangled in the framework of the FDM by the so-called 'incomplete disentangling' technique [12] as described in [10], the essence of which is apparent from formulas (4.5)–(4.8). The Riccati equation emerging from the use of the FDM fixes the amplitudes A_{nm} and the quantum transition probabilities $w_{nm} = |A_{nm}|^2$. The amplitudes A_{nm} are expressed via matrix elements of irreducible unitary representations of the SU(2) group or its noncompact analog SU(1, 1); this simplifies solving the problem.

(2) As regards field theory, Feynman [6] discussed the possibility of applying the method in question to certain QED problems, such as the proof of equivalence of various QED formulations (by Dirac, Fock, Dyson, and Feynman himself), and the application of the FDM to the Dirac equation in a nonquantized external field. However, we are unaware of any further continuation of these studies. It would be very interesting to look for new applications of the FDM not only in quantum mechanics but also in field theory.

(3) It should be mentioned that the FDM is also considered in the mathematical literature under the guise of noncommutative analysis (see [100] and the references therein). The authors of these works use complicated mathematical constructions and terminology that are not familiar to physicists (filter of sections, operator symbol, poly-Banach algebras, etc.). To my knowledge, no new physical results have been obtained in these studies.

(4) A function of noncommuting variables is a much more complicated mathematical object than an ordinary function and requires developing appropriate tools. It is worth noting an alternative approach to this problem developed in Refs [101–103]. We confine our consideration to the exponential and set

$$\exp\left\{(\hat{a}+\hat{b})\tau\right\} = \exp(\hat{b}\tau)\,\hat{K}(\tau)\exp(\hat{a}\tau)\,, \tag{10.1}$$

where τ is a parameter (either real or imaginary). The factor $\hat{K}$ takes the noncommutativity of the operators into account (if $[\hat{a},\hat{b}] = 0$, then $\hat{K}(\tau) \equiv 1$) and is to be determined from the differential equation

$$\frac{\partial \hat{K}}{\partial \tau} = \exp(\hat{b}\tau)\,\hat{a}\exp(-\hat{b}\tau)\hat{K} - \hat{K}\hat{a} \tag{10.2}$$

with the initial condition $\hat{K}(0) = 1$,

$$\exp(-\hat{b}\tau)\,\hat{a}\exp(\hat{b}\tau) = \hat{a} + \sum_{n=1}^{\infty}\frac{(-\tau)^n}{n!}\underbrace{[\hat{b}[\hat{b},\ldots[\hat{b},\hat{a}]\ldots]]}_{n\text{ times}}\,. \tag{10.3}$$

As a rule, all commutators here differ from zero, and solving the problem is extremely difficult. But if the operators $\hat{a}$ and $\hat{b}$ are such that the majority of commutators vanish, the equations may have a simple solution. Hence, it is easy to deduce Glauber formula (8.17) and consider cases such as $[\hat{a},\hat{b}] = -\lambda\hat{a}$, where

$$\exp\left\{(\hat{a}+\hat{b})\tau\right\} = \exp(\hat{b}\tau)\exp\left\{\frac{\hat{a}\left[1-\exp(-\lambda\tau)\right]}{\lambda}\right\} \tag{10.4}$$

(λ is a c-number) or $[\hat{a},\hat{b}] = \lambda\hat{b}$, where

$$\exp\left\{(\hat{a}+\hat{b})\tau\right\} = \exp\left\{\frac{\hat{b}\left[\exp(\lambda\tau)-1\right]}{\lambda}\right\}\exp(\hat{a}\tau)\,. \tag{10.4$'$}$$

The FDM and the method used in Refs [101–103] are mutually complementary, although the author considers the FDM to be a more general technique. For example, for arbitrary operators $\hat{a}$ and $\hat{b}$, the FDM gives

$$\begin{aligned}\exp\left\{(\hat{a}+\hat{b})\tau\right\} &\equiv \exp\left\{\int_0^\tau(\hat{a}+\hat{b})\,\mathrm{d}t\right\}\\ &= \exp(\hat{b}\tau)\exp\left\{\int_0^\tau\exp(-\hat{b}t)\,\hat{a}\exp(\hat{b}t)\,\mathrm{d}t\right\}\\ &= \exp\left\{\int_0^\tau\exp(\hat{a}t)\,\hat{b}\exp(-\hat{a}t)\,\mathrm{d}t\right\}\exp(\hat{a}\tau)\,,\end{aligned} \tag{10.5}$$

whence formulas (10.4), (10.4′), and some others follow immediately.

The present work originates from a question put to the author by I Ya Pomeranchuk many years ago: What quantum mechanics problems, besides those considered in [6, 10], can be solved with the help of the FDM? The author is indebted to K G Boreskov, L B Okun, V A Rubakov, Yu A Simonov, and M A Trusov for reading the manuscript and for their useful comments. Thanks are also due to V D Mur for the discussion of the results and for bringing Refs [101–103] to the author's attention. The author thanks S G Pozdnyakov for numerical calculations and N S Libova and M N Markina for assistance in preparing this paper for publication.

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11. Appendices

A. Fock – Bargmann representation

We recall the main formulas for this representation [90–92].

(1) Realization of the creation and annihilation operators:

$$\hat{a}^+ = z\,, \qquad \hat{a} = \frac{\mathrm{d}}{\mathrm{d}z}\,, \tag{A.1}$$

where $z \in C$ is an auxiliary complex variable. The Hilbert space of oscillator states is realized in analytic functions of z.

(2) The scalar product of wave functions:

$$\langle\psi|\varphi\rangle = \int \psi^*(z)\varphi(z)\,\mathrm{d}\mu(z)\,, \qquad \mathrm{d}\mu(z) = \frac{1}{\pi}\exp\left(-\rho^2\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,,$$

where

$$z = \rho\exp\left(\mathrm{i}\varphi\right), \quad 0 \leqslant \rho < \infty\,, \quad 0 \leqslant \varphi \leqslant 2\pi\,, \quad \int \mathrm{d}\mu(z) = 1\,. \tag{A.2}$$

(3) The Hamiltonian of a harmonic oscillator and other operators:

$$\begin{aligned} &\hat{H} = 2\hat{J}_0 = \frac{1}{2}\,\omega(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) = \omega\left(z\,\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\right), \\ &\hat{x} = \frac{1}{\sqrt{2\omega}}\left(z + \frac{\mathrm{d}}{\mathrm{d}z}\right), \qquad \hat{p} = \mathrm{i}\sqrt{\frac{\omega}{2}}\left(z - \frac{\mathrm{d}}{\mathrm{d}z}\right), \\ &\hat{p}\hat{x} + \hat{x}\hat{p} = 4\hat{J}_1 = \frac{1}{2\mathrm{i}}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - z^2\right). \end{aligned} \tag{A.3}$$

(4) Normalized wave functions of stationary (n-quantum) states:[14]

$$\langle z|n\rangle = \frac{z^n}{\sqrt{n!}}\,, \qquad \langle n|n'\rangle = \int\langle n|z\rangle\langle z|n'\rangle\,\mathrm{d}\mu(z) = \delta_{nn'}\,. \tag{A.4}$$

For comparison, in the coordinate representation [24, 47], they become

$$\langle x|n\rangle = \left(2^n n!\right)^{-1/2}\left(\frac{\omega}{\pi}\right)^{1/4} H_n\left(\sqrt{\omega}x\right)\exp\left(-\frac{1}{2}\,\omega x^2\right), \tag{A.4$'$}$$

where H_n is the Hermit polynomial.

(5) The following equality is satisfied:

$$\sum_{n=0}^{\infty}\langle z|n\rangle\langle n|z'\rangle = \sum_{n=0}^{\infty}\frac{\left(zz'^*\right)^n}{n!} = \exp\left(zz'^*\right) \equiv \delta(z, z')\,,$$

where $\delta(z, z')$ plays the role of the δ-function in integration with the measure $\mathrm{d}\mu(z)$: $\psi(z) = \int\delta(z, z')\psi(z')\,\mathrm{d}\mu(z')$.

(6) Coherent states:

$$\begin{aligned} &|\alpha\rangle = \exp\left(-\frac{1}{2}\,|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \\ &\langle z|\alpha\rangle = \exp\left(-\frac{1}{2}\,|\alpha|^2 + \alpha z\right); \end{aligned} \tag{A.5}$$

these functions are not mutually orthogonal,

$$\begin{aligned} &\langle\alpha|\beta\rangle = \left\langle 0\middle|U(-\alpha)U(\beta)\middle|0\right\rangle \propto \langle\alpha - \beta|0\rangle\,, \\ &\left|\langle\alpha|\beta\rangle\right|^2 = \exp\left(-|\alpha - \beta|^2\right) \neq 0\,, \end{aligned} \tag{A.6}$$

and form an overcomplete system [47, 105]. Time evolution of the coherent state:

$$|\alpha, t\rangle = \exp\left(-\frac{\mathrm{i}\omega t}{2}\right)\left|\alpha\exp\left(-\mathrm{i}\omega t\right)\right\rangle. \tag{A.7}$$

(7) The problem of eigenvalues of the operators $\hat{a}$ and $\hat{a}^+$ is easily solved in the Fock–Bargmann representation. Taking (A.1) into account leads to the equations

$$\hat{a}\psi_\alpha(z) \equiv \frac{\mathrm{d}\psi_\alpha}{\mathrm{d}z} = \alpha\psi_\alpha(z)\,, \qquad \hat{a}^+\psi_\lambda(z) \equiv z\psi_\lambda = \lambda\psi_\lambda(z)\,, \tag{A.8}$$

the first of which has the obvious solution

$$\psi_\alpha(z) = N\exp\left(\alpha z\right) = N\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,\langle z|n\rangle \tag{A.9}$$

[see (A.4)] with $|N| = \exp\left[-(1/2)|\alpha|^2\right]$ from the normalization condition. Similarly, the eigenvalue of the annihilation operator $\hat{a}$ may be any complex number α and its eigenfunctions coincide with coherent states.

On the other hand, the second equation in (A.8) has the form $(z - \lambda)\psi_\lambda(z) = 0$, which is satisfied only under the condition $\psi_\lambda(z) \equiv 0$; such a function is not an eigenfunction.

(8) Following [106], we now show how the problem of evolution of a constant-frequency oscillator is solved in the Fock–Bargmann representation. The Schrödinger equation

$$\mathrm{i}\dot{\psi} = \left\{\omega\left(z\,\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\right) - \frac{f(t)}{\sqrt{2\omega}}\left(\frac{\mathrm{d}}{\mathrm{d}z} + z\right)\right\}\psi \tag{A.10}$$

has the solution $\psi(z, t) = \exp\left(az + b\right)\varphi(z - c, t)$, where a, b, and c are certain functions of time. Substituting this solution in (A.10), we obtain the equation

$$\begin{aligned} \mathrm{i}\left[\dot{\varphi} + (\dot{a}z + \dot{b})\varphi - \dot{c}\,\frac{\partial\varphi}{\partial z}\right] &= \omega\left(z\,\frac{\partial\varphi}{\partial z} + \left(az + \frac{1}{2}\right)\varphi\right) \\ &- \frac{f(t)}{\sqrt{2\omega}}\left(\frac{\partial\varphi}{\partial z} + (z + a)\varphi\right). \end{aligned} \tag{A.10$'$}$$

Let the function $\varphi(z, t)$ satisfy the free (without an external force) equation

$$\mathrm{i}\dot{\varphi} = \omega\left(z\,\frac{\partial}{\partial z} + \frac{1}{2}\right)\varphi\,.$$

Equating the coefficients at φ, $z\varphi$, and $\partial\varphi/\partial z$, we obtain the equations

$$\begin{aligned} &\mathrm{i}\dot{a} = \omega a - (2\omega)^{-1/2}f(t)\,, \\ &\mathrm{i}\dot{b} = -(2\omega)^{-1/2}f(t)a\,, \\ &\mathrm{i}\dot{c} = \omega c - (2\omega)^{-1/2}f(t)\,, \end{aligned} \tag{A.11}$$

whence

$$\begin{aligned} a(t) = c^*(t) &= \frac{\mathrm{i}}{\sqrt{2\omega}}\int_0^t f(t')\exp\left[-\mathrm{i}\omega(t - t')\right]\mathrm{d}t' \\ &= \mathrm{i}\exp\left(-\mathrm{i}\omega t\right)\beta^*(t)\,, \\ b(t) = b_1 + \mathrm{i}b_2 &= \frac{\mathrm{i}}{\sqrt{2\omega}}\int_0^t f(t')a(t')\,\mathrm{d}t'\,, \end{aligned} \tag{A.12}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|a(t)|^2 = \sqrt{\frac{2}{\omega}}f(t)a_2(t)\,, \tag{A.12$'$}$$

$$b_1(t) = -\frac{1}{\sqrt{2\omega}}\int_0^t f(t')a_2(t')\,\mathrm{d}t' = -\frac{1}{2}\,|a(t)|^2 \tag{A.12$''$}$$

[14] We use Dirac's notations [104] $\hbar = m = 1$.

[see $\beta(t)$ in (3.10); we assume here that $t_0 = 0$]. Because $a = b = c = 0$ at $t = t_0$, $\varphi(z, 0)$ coincides with the wave function of the initial oscillator state $\psi_0(z)$. Hence, for $t > t_0 = 0$,

$$\psi(z,t) = \exp\left\{-\frac{1}{2}\left|a(t)\right|^2 + a(t)z + \mathrm{i}\gamma\right\}\psi_0\bigl(z - a^*(t), t\bigr)\,, \tag{A.13}$$

which solves the problem of evolution of an arbitrary initial state. We consider two examples.

(a) If the initial oscillator state is a coherent state $|\alpha\rangle$, then

$$\psi(z,t) = \exp\left\{-\frac{1}{2}\left|a+\alpha\right|^2 + (a+\alpha)z + \mathrm{i}\gamma'\right\}, \tag{A.14}$$

where γ and γ' are t-dependent complex phases and $a(t)$ is defined in (A.12). Therefore, the oscillator is at any time in the coherent state $|\alpha\rangle$ up to an inessential phase factor:

$$\alpha(t) = \alpha(t_0) + a(t)\,. \tag{A.15}$$

Specifically, if $\alpha(t_0) = 0$, then $\psi_0(z) = \langle z|0\rangle \equiv 1$ and $\psi(z,t) = \exp(az + b_1)$. The Taylor series expansion of this function leads directly to (3.18) with the parameter $v = |a|^2$. Similarly, in the case of transitions from the first level, $\psi(z,t) = (z - a^*)\exp(az + b)$, which gives formula (3.18′).

(b) In the case of the m-quantum initial state,

$$\psi(z,t) = \frac{(z-a^*)^m}{\sqrt{m!}}\exp\left\{-\frac{1}{2}\left|a(t)\right|^2 + az + \mathrm{i}\gamma''\right\}. \tag{A.16}$$

The known formulas [20, 21]

$$\exp(az)\,(z-a^*)^m = \sum_{k=0}^{\infty} L_k^{m-k}\bigl(|a|^2\bigr)(-a^*)^{m-k}z^k\,,$$

$$L_k^{m-k}(x) = \frac{m!}{k!}\,(-x)^{k-m}L_m^{k-m}(x)\,, \qquad k \geqslant m\,,$$

allow expanding the wave function (A.16) in the basis functions $\langle z|n\rangle$, which immediately gives (3.14), where $v = \left|a(t)\right|^2$ as $t \to +\infty$.

(9) The Fock–Bargmann method is generalized in Ref. [107] for the hydrogen atom, whose wave functions of the discrete spectrum take a very simple form. In the parabolic basis $|n_1, n_2, m\rangle$,

$$\langle \mathbf{z}|n_1 n_2 m\rangle = \frac{(n-1)!}{\left[n_1!\,n_2!\,\bigl(n_1+|m|\bigr)!\,\bigl(n_2+|m|\bigr)!\right]^{1/2}}\,z_1^{n_1+\mu}z_2^{n_2+\mu}\,, \tag{A.17}$$

where $\mathbf{z} = (z_1, z_2)$, $z_{1,2}$ are two independent complex variables, n_1 and n_2 are parabolic quantum numbers, $n = n_1 + n_2 + |m| + 1$ [24], $\mu = \bigl(m + |m|\bigr)/2$, and wave functions in the spherical basis $|nlm\rangle$ are homogeneous polynomials in z_1 and z_2 of degree $n + m - 1$. For *ns* and *np* states,

$$\langle \mathbf{z}|n,0,0\rangle = a_0(z_1 - z_2)^{n-1}\,, \qquad a_0 = n^{-1/2}\,, \tag{A.18}$$

$$\begin{aligned}
&\langle \mathbf{z}|n,1,1\rangle = b_1 z_1 z_2 (z_1 - z_2)^{n-2}\,,\\
&\langle \mathbf{z}|n,1,0\rangle = b_0(z_1 + z_2)(z_1 - z_2)^{n-2}\,,\\
&\langle \mathbf{z}|n,1,-1\rangle = b_{-1}(z_1 - z_2)^{n-2}\,;
\end{aligned} \tag{A.19}$$

in the general case, for states with the maximum projection $m = l$,

$$\langle \mathbf{z}|n,l,l\rangle = A_{nl}(z_1 z_2)^l\,(z_1 - z_2)^{n-l-1}\,; \tag{A.20}$$

next, there is a descent along m to $m = -l$:

$$\langle \mathbf{z}|n,l,m-1\rangle = \sqrt{\frac{l+m}{l-m+1}}\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)\langle \mathbf{z}|n,l,m\rangle\,, \tag{A.21}$$

with coefficients b_m and A_{nl} defined in Ref. [107]. By expanding the polynomial $\langle \mathbf{z}|nlm\rangle$ in powers of z_1 and z_2 and taking (A.17) into account, we find [107] the relation between the wave functions in these two bases [108, 109]:

$$|nlm\rangle = \sum_{m_1 + m_2 = m} C^{lm}_{jm_1,\,jm_2}|n_1 n_2 m\rangle\,, \tag{A.22}$$

where $0 \leqslant l \leqslant n-1$, $j = (n-1)/2$, $m_1 = (m + n_1 - n_2)/2$, $m_2 = (m - n_1 + n_2)/2$, and $C^{lm}_{jm_1,jm_2}$ is the Clebsch–Gordan coefficient. The very possibility of such a relation is due to the 'hidden' symmetry group of the Coulomb field allowing the decomposition $\mathrm{SO}(4) \sim \mathrm{SO}(3) \otimes \mathrm{SO}(3)$ and the Clebsch–Gordan coefficients being related to the summation of two orbital momenta $j = (n-1)/2$ in the SO(3) group.

B. About the quantum oscillator excitation parameter

We start from Hamiltonian (3.1), assuming that $f(t) = F_0\varphi(t/\tau)$ in (3.10), where φ is a dimensionless function. Then,

$$\begin{aligned}
v &= v_0(\omega\tau)^2\left|\int_{-\infty}^{\infty}\varphi(s)\exp(-\mathrm{i}\omega\tau s)\,\mathrm{d}s\right|^2\\
&= v_0\left|\int_{-\infty}^{\infty}\dot{\varphi}(s)\exp(-\mathrm{i}\omega\tau s)\,\mathrm{d}s\right|^2,
\end{aligned} \tag{B.1}$$

where

$$v_0 = \frac{F_0^2}{2m\hbar\omega^3} = \frac{1}{2}\left(\frac{d}{x_0}\right)^2 = -\frac{\Delta E}{\hbar\omega}\,, \tag{B.2}$$

$d = F_0/m\omega^2$ is the displacement of the oscillator equilibrium position, $x_0 = \sqrt{\hbar/m\omega}$, and $\Delta E = -F_0^2/2m\omega^2$ is the shift of the oscillator energy in the presence of a uniform external field F_0. The last equality in (B.1) holds when $f(t)$ does not vanish as $t \to \pm\infty$ but tends to constant limits $f_\pm$.

If $\tau \gg 1/\omega$, the excitation parameter v is exponentially small (adiabatic), as shown in examples below.

1. $\varphi = \left[1 + \exp\left(-\dfrac{t}{\tau}\right)\right]^{-1}, \qquad v = v_0\left(\dfrac{\pi\omega\tau}{\sinh\pi\omega\tau}\right)^2.$

If $\tau \to 0$, then $\varphi = \theta(t)$ and $v = v_0$; in other words, the parameter v_0 corresponds to oscillator excitation under an instantaneous jump of the external force $f(t)$ from 0 to F_0. Further on,

2. $\varphi = \left(\cosh\dfrac{t}{\tau}\right)^{-2}, \qquad v = \dfrac{v_0(\omega\tau)^4}{\sinh^2(\pi\omega\tau/2)}\,,$

3. $\varphi = \exp\left(-\dfrac{t^2}{2\tau^2}\right), \qquad v = 2\pi v_0(\omega\tau)^2\exp\left\{-(\omega\tau)^2\right\}.$

In the adiabatic case, i.e., for $\omega\tau \gg 1$, the value of v is determined by the contribution of a singular point of the function φ closest to the real axis, which accounts for the exponential smallness of this parameter.

The same situation holds for the oscillator with varying frequency (5.2). A characteristic example is

$$\omega(t) = \left\{\frac{1}{2}\left[\omega_+^2 + \omega_-^2 + (\omega_+^2 - \omega_-^2)\tanh\omega t\right]\right\}^{1/2}, \quad \text{(B.3)}$$

with [24, 110]

$$\rho = \left\{\frac{\sinh\left[\pi(\omega_+ - \omega_-)/2\omega\right]}{\sinh\left[\pi(\omega_+ + \omega_-)/2\omega\right]}\right\}^2 \quad \text{(B.4)}$$

and $\omega \to 0$ (adiabatic case); then,

$$\rho \approx \exp\left(-\frac{2\pi\omega_<}{\omega}\right) + O\left(\exp\left[-\frac{\pi(\omega_+ + \omega_-)}{\omega}\right]\right), \quad \text{(B.5)}$$

where $\omega_< = \min(\omega_+, \omega_-)$. But if the oscillator frequency changes abruptly ($\omega \gg \omega_\pm$), then

$$\rho = \left(\frac{\omega_+ - \omega_-}{\omega_+ + \omega_-}\right)^2\left(1 - \frac{\pi^2}{3}\frac{\omega_+\omega_-}{\omega^2} + \ldots\right), \quad \text{(B.6)}$$

which becomes (5.13) in the limit as $\omega \to \infty$. The adiabatic approximation is inapplicable here, and the excitation parameter ρ is not small in general.

Generalization of formulas (B.3) and (B.4) corresponding to the exactly solvable Eckart potential [64] in quantum mechanics,

$$\omega^2(t) = \frac{\omega_+^2}{1 + \exp(-2\omega t)} + \frac{\omega_-^2}{1 + \exp(2\omega t)} + \frac{\omega_0^2 - \omega^2}{\left[\exp(\omega t) + \exp(-\omega t)\right]^2}, \quad \text{(B.7)}$$

is presented in Ref. [11]. Exponential asymptotic form (B.5) remains valid in the adiabatic region.

C. Spin rotation in a magnetic field and the Riccati equation

We consider an alternating magnetic field

$$H_x = H_1\cos\omega t, \quad H_y = H_1\sin\omega t, \quad H_z = H_0, \quad \text{(C.1)}$$

related to the experimental measurement of magnetic moments of atomic nuclei [111]. In this case, Eqn (4.8) becomes

$$\mathrm{i}\frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{1}{2}\sin\theta\left[\exp(-\mathrm{i}\lambda\tau) - a^2\exp(\mathrm{i}\lambda\tau)\right] + a\cos\theta, \quad \text{(C.2)}$$

where

$$\tau = \omega_0 t, \quad \omega_0 = -\gamma\sqrt{H_0^2 + H_1^2} = -\frac{\gamma H_0}{\cos\theta},$$
$$\frac{H_1}{H_0} = \tan\theta, \quad \lambda = \frac{\omega}{\omega_0},$$

with $\lambda\tau = \omega t$. It is easy to see that it has a special solution of the form $a(\tau) = N\exp(-\mathrm{i}\lambda\tau)$ (the constant N is found from the quadratic equation). This suffices [112] for obtaining the general solution of the Riccati equation, which allows satisfying the initial condition $a(t_0) = 0$. As a result,[15]

$$|a(t)|^2 = \tan^2\frac{\vartheta}{2} = \frac{\sin^2\delta}{\Delta + \cos^2\delta}, \quad \Delta = \left(\frac{\lambda - \cos\theta}{\sin\theta}\right)^2 \quad \text{(C.3)}$$

[cf. (4.12)]. According to (4.13), the probability of a flip of spin $s = 1/2$ in a magnetic field of form (C.1) is

$$w_{-1/2,\,1/2} = \frac{\sin^2\delta}{1 + \Delta} = \frac{\sin^2\theta}{1 - 2\lambda\cos\theta + \lambda^2}\sin^2\delta \leqslant \frac{1}{1 + \Delta}, \quad \text{(C.4)}$$

where $\delta = (1/2)\omega_0(t - t_0)\sqrt{1 - 2\lambda\cos\theta + \lambda^2}$. For a particle with an arbitrary spin s,

$$w_{-s,\,s}(t) = \left[w_{-1/2,\,1/2}(t)\right]^{2s}. \quad \text{(C.5)}$$

The particle spin may completely flip over if $\delta = (n + 1/2)\pi$ and $\Delta = 0$ or $\omega = \omega_0\cos\theta = \gamma H_0$ (magnetic resonance [33, 111]; see [34] for more details).

Majorana [25] considered the magnetic field

$$H_x = A, \quad H_y = 0, \quad H_z = -Ct, \quad \text{(C.6)}$$

where A and C are certain constants. In this case, Eqn (4.8) has a rather simple form,

$$\dot{a} = \mathrm{i}\gamma\left\{\frac{1}{2}A(1 - a^2) - Cta\right\}, \quad \text{(C.7)}$$

but Majorana solved Schrödinger equation (4.1) for spin $s = 1/2$ instead of (C.7). He deduced the following equation for spin-flip probability through time from $t = -\infty$ to $t = \infty$:

$$w_{-1/2,\,1/2} = \exp\left(-\frac{\pi}{2}k\right), \quad k = \frac{\gamma A^2}{C}, \quad \text{(C.8)}$$

where γ is the gyromagnetic ratio. The numerical example [25] characterizing the degree of adiabaticity violation is $w_{-1/2,\,1/2} = 21\%$ and 4.3% at $k = 1$ and 2.

To conclude, here are functions entering (4.3) in the simplest case where the magnetic field changes in strength but preserves its direction in space (the field is parallel to the z axis):

$$a(t) = c(t) = 0, \quad b(t) = \mathrm{i}\gamma\int_{t_0}^{t} H_z(t')\,\mathrm{d}t'. \quad \text{(C.9)}$$

D. Unitarity condition

The derivation of formula (5.11) for transition probabilities in Ref. [11] is based on the analytic continuation of Wigner functions from a compact to a noncompact group:

$$d_{\mu\nu}^{(j)}(\vartheta) \to f_{\mu\nu}^{(j)}(\beta), \quad \vartheta \to \mathrm{i}\beta. \quad \text{(D.1)}$$

Mathematically, such a method is not rigorous; moreover, comparison of formulas (1.8) and (1.9) indicates that unitarity

[15] See [10] for computation details. In that work, normalization of spin operators $\hat{S}_\pm$ differs from the normalization in the present paper; hence, the necessity to introduce additional factors $\mp 2^{-1/2}$ in the expressions for the functions $a(t)$ and $c(t)$ in Ref. [10]. In the notation of Ref. [10], $\delta = (\mu - \lambda)\tau/2$.

can be violated in this case.[16] It is therefore necessary to verify the unitarity for the representations with weights $j = -1/4$ and $-3/4$ that we use.

In the case of an oscillator with varying frequency for transitions from two lower levels ($m = 0$ and 1), unitarity condition (5.22) takes the form

$$\sqrt{1-\rho}\sum_{k=0}^{\infty}\frac{\Gamma(k+1/2)}{k!\,\Gamma(1/2)}\,\rho^k = (1-\rho)^{3/2}\sum_{k=0}^{\infty}\frac{\Gamma(k+3/2)}{k!\,\Gamma(3/2)}\,\rho^k = 1 \qquad (D.2)$$

and is satisfied in accordance with Newton's binomial formula. It was numerically verified with high accuracy ($\sim 10^{-12}$) for $m = 2$ and 3 [11] and analytically proved thereafter for all m and ρ based directly on (5.11). Unfortunately, it is impracticable to reproduce this proof [113] here because of its unwieldiness.

In the general case of time-dependent $\omega(t)$ and $f(t)$, expressions for w_{nm} are too complicated. Therefore, we confine the discussion to transitions from the ground state, $m = 0$. Such an approach requires certain data from the theory of special functions, unlike situations where the frequency ω is independent of t or the force $f(t)$ is absent (these cases allow elementary verification of the unitarity in transitions $|0,\omega_-\rangle \to |n,\omega_+\rangle$). We use Meler's formula (see formula 10.13.22 in [20]):

$$\sum_{n=0}^{\infty}\frac{\rho^{n/2}}{2^n n!}\left|H_n\left(\frac{z}{\sqrt{2}}\right)\right|^2 = \frac{1}{\sqrt{1-\rho}}\exp\left\{|z|^2\,\frac{\sqrt{\rho}-\rho\cos 2\varphi}{1-\rho}\right\}, \qquad (D.3)$$

where $\varphi = \arg z$ and H_n is the Hermit polynomial. In our case, $|z|^2 = 2\nu/\sinh\beta = \nu(1-\rho)/\sqrt{\rho}$ and (6.9) is taken into account. It is therefore easy to see that the probabilities w_{n0} in (6.8) satisfy the unitarity condition at arbitrary parameters ν, ρ, and φ.

We can proceed further in the case of a constant-frequency oscillator. Schwinger used the quantum action principle [22, 114] and showed [115] that

$$\sum_{m=0}^{\infty} w_{nm}(\nu)u^m = u^n L_n\left[-\frac{\nu(1-u)^2}{u}\right]\exp(\nu u - \nu) \qquad (D.4)$$

(in Ref. [115], $u = \exp(-\beta\omega)$, where ω is the oscillator frequency, and $\beta^{-1} = \theta$ can be interpreted as the thermostat temperature). Hence, with formula 22.9.15 in [21], the generating function is given by

$$G(u,v) = \sum_{m,n=0}^{\infty} w_{nm}u^m v^n = (1-uv)^{-1}\exp\left\{-\nu\,\frac{(1-u)(1-v)}{1-uv}\right\}. \qquad (D.5)$$

In this case, $G(u,1) = (1-u)^{-1} = \sum_{m=0}^{\infty} u^m$; this means that unitarity relation (3.20) is satisfied for any m. Also, the function $G(u,v)$ is symmetric with respect to u and v, and therefore $w_{nm}(\nu) = w_{mn}(\nu)$ for an arbitrary form of the force $f(t)$. Such a symmetry of the transition probabilities with respect to initial and final states is oscillator-specific and is related to the properties of the coefficient of reflection from a one-dimensional barrier (as noticed by L P Pitaevskii, see [47, p. 259]). At $u = 0$, Eqn (D.5) gives Poisson distribution (3.18).

[16] The same is true if expressions for the Wigner functions presented in Tables 4.3–4.12 in [26] are used.

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