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METHODOLOGICAL NOTES

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On the difference between Wigner's and Møller's approaches to the description of Thomas precession

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Contents

Abstract. **An account is given of the Wigner concept of particle spin and velocity rotations and of the variation of the angle between them under Lorentz transformations with noncollinear velocities. It is shown that Møller's description of spin rotation can be reduced to the Wigner rotation, and Møller's formula for the angle of spin rotation in the curvilinear motion of a particle is corrected. The permutation asymmetry of the relativistic velocity addition law distinguishes the Wigner sequence of Lorentzian boosts by its applicability to the description of spin and velocity rotations in curvilinear motion.**

1. Introduction

The angle between the direction of a particle spin and its velocity is not a Lorentz invariant. For example, we consider a proton resting on the platform of the Bologoe station, with its spin pointing toward St. Petersburg. For the observer traveling past Bologoe in a train from St. Petersburg to Moscow, the angle between the spin and the velocity is then equal to zero, while for the observer going from Moscow to St. Petersburg it is equal to π. But zero-mass particles have no rest frame. For them, the angle between the spin and the velocity is either zero or π and is a Lorentz invariant, like the value of spin in the direction of particle motion.

2. Wigner rotation

The nontrivial relation between the spin rotation angle of a massive particle and its momentum rotation angle under Lorentz transformations was repeatedly considered by Wigner [1, 2]. (See the collection in Ref. [3] for Russian translations of these articles.) Here, we give an excerpt of Wigner's article [1] that is significant for the subsequent discussion; we change only the numbering of formulas and set the speed of light equal to unity. Wigner wrote:

"Consider a particle at rest and polarized in the z direction. Impart to it a velocity in the z direction by subjecting it to a Lorentz transformation with the hyperbolic angle α. Later, this angle will be assumed to be very large so as to make this particle highly relativistic. At any rate, we now have a particle which is polarized in the direction of its motion — which is in the z direction. In order to obtain a particle which is polarized in the direction of its motion, but is moving in another direction, one would first subject the particle to a rotation to bring the polarization into the direction of its projected motion and then accelerate it in the desired direction. In order to test whether the statement, that the polarization has the direction of the motion of the particle, is relativistically invariant we subject the particle which moves in the direction z and is properly polarized, to a second acceleration, in the x direction, by the hyperbolic angle ε. This angle is arbitrary but will be assumed, at the end, to be much smaller than α. The particle could have achieved the same state of motion by being accelerated by the hyperbolic angle α' in the direction which includes an angle ϑ with the z axis where

$$\cosh\alpha' = \cosh\alpha\cosh\varepsilon\,, \quad \sin\vartheta = \frac{\cosh\alpha\sinh\varepsilon}{\sinh\alpha'}\,. \tag{1}$$

However, the direction of polarization would not be the same in the second case as in the first case. In order to make it the same, one has to rotate the system, before accelerating it in the ϑ direction, by an angle $\vartheta - \delta$ where δ is given by

$$\sin\delta = \frac{\sinh\varepsilon}{\sinh\alpha'} = \frac{\sinh\varepsilon}{\sqrt{\cosh^2\alpha\,\cosh^2\varepsilon - 1}}\,. \tag{2}$$

This follows, simply, from the identity for Lorentz transformations

$$A\left(\frac{\pi}{2},\,\varepsilon\right)A(0,\alpha) = A(\vartheta,\alpha')\,R\,(\vartheta-\delta)\,, \tag{3}$$

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where α, ε are arbitrary while α', ϑ and δ are defined by the last two equations [i.e., Eqns (1) and (2) — Translator's comment]. $A(\vartheta, \alpha)$ is the acceleration by a hyperbolic angle α in that direction in the xz plane which includes an angle ϑ with the z axis; $R(\varphi)$ is a rotation by φ in the xz plane. If δ were zero, the particle which was polarized in the direction of its motion after the acceleration α, would have remained polarized in the direction of its new motion (i.e., the ϑ direction) after the second acceleration, by ε. This is not the case, as δ is finite. However, δ is very small if $\varepsilon \ll \alpha$, i.e., if the second acceleration is by a much smaller hyperbolic angle than the first, and if $\alpha \gg 1$".

The matrix $A(0, \alpha)$ of the Lorentz transformation (a boost) that imparts the velocity $v = \tanh\alpha$ to a system of coordinates S' along the z axis relative to a system S is given by

$$A(0, \alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{pmatrix}. \qquad (4)$$

The matrix $R(\vartheta)$ of the clockwise rotation of the S' system through an angle ϑ in the x, z plane of the system S is

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (5)$$

Three rows and three columns of these matrices pertain to the axes x, z, t and x', z', t' of the respective systems S and S'. It is therefore natural to use the notation $A_{SS'}(\vartheta, \alpha)$ and $R_{SS'}(\vartheta)$ for the matrix elements, where the subscripts S and S' take the values x, z, t and x', z', t'. The direction in the x, z plane between the x and z axes that makes an angle ϑ with the z axis is called the 'ϑ direction' in what follows. Then, the Lorentz transformation

$$A_{SS'}(\vartheta, \alpha) = \big(R(\vartheta)\, A(0, \alpha)\, R(-\vartheta)\big)_{SS'}$$
$$= \begin{pmatrix} \cos^2\vartheta + \sin^2\vartheta\cosh\alpha & \sin\vartheta\cos\vartheta\,(\cosh\alpha - 1) & \sin\vartheta\sinh\alpha \\ \sin\vartheta\cos\vartheta\,(\cosh\alpha - 1) & \sin^2\vartheta + \cos^2\vartheta\cosh\alpha & \cos\vartheta\sinh\alpha \\ \sin\vartheta\sinh\alpha & \cos\vartheta\sinh\alpha & \cosh\alpha \end{pmatrix} \qquad (6)$$

imparts the velocity $v = \tanh\alpha$ to the system of coordinates S' in the ϑ direction relative to the system S.

The above expressions for the matrices $A(\vartheta, \alpha)$ and $R(\vartheta)$ allow verifying Eqn (3) using relations (1) and (2). This is done most conveniently when equality (3) is represented as

$$A(\vartheta, -\alpha')\, A\left(\frac{\pi}{2}, \varepsilon\right) A(0, \alpha) = R(\vartheta - \delta)\,, \qquad (7)$$

using that the matrix $A(\vartheta, -\alpha)$ is the inverse of $A(\vartheta, \alpha)$.

Equality (3) signifies that the product of two pure Lorentz transformations with nonparallel velocities does not reduce to a pure Lorentzian transformation, i.e., such transformations do not form a group. But pure Lorentzian transformations and spatial rotations do form a group — the homogeneous Lorentz group.

When the left-hand side of (3) is written in the form $A_{SS_1}(\pi/2, \varepsilon) \times A_{S_1S_2}(0, \alpha)$, it can be interpreted as follows. The boost $A_{S_1S_2}(0, \alpha)$ imparts the velocity $v = \tanh\alpha$ to the system S_2 along the z_1 axis relative to the system S_1, and the boost $A_{SS_1}(\pi/2, \varepsilon)$ imparts the velocity $v_1 = \tanh\varepsilon$ to the system S_1 along the x axis relative to the system S. If the

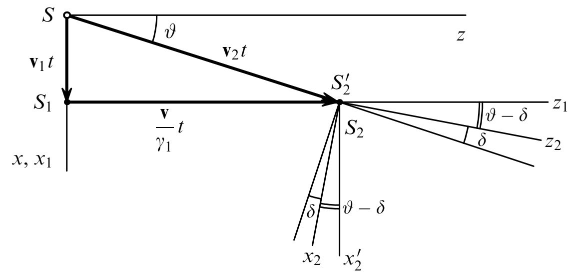

Figure 1.

origins of the three inertial systems S, S_1, and S_2 were at a common point at the instant $t = 0$, then at the laboratory time instant t they are at the points of the laboratory system S indicated by S, S_1, and S_2 in Fig. 1.

It is evident that at the instant t in laboratory time, the origin S_1 is at the distance $v_1 t$ from the origin S. At the same instant, in the laboratory system, the origin S_2 is at the distance $(v/\gamma_1)\, t$, $\gamma_1 = 1/(1 - v_1^2)^{1/2}$, from the origin S_1. Indeed, the origin S_2 is at the distance $v t_1$ from the origin S_1 at the instant t_1 of the proper time of the system S_1, and this instant is related to the instant t in laboratory time as $t_1 = t/\gamma_1$ due to the time dilation for the moving clock in comparison with the laboratory one. Therefore,

$$v t_1 = \frac{v}{\gamma_1}\, t\,. \qquad (8)$$

The velocity of the origin S_2 in the laboratory system S is then given by

$$\mathbf{v}_2 = \mathbf{v}_1 + \frac{\mathbf{v}}{\gamma_1}\,, \qquad (9)$$

and its absolute value is

$$v_2 = \sqrt{v_1^2 + \frac{v^2}{\gamma_1^2}} = \tanh\alpha'\,, \qquad (10)$$

where, instead of the velocities v and v_1, we use the hyperbolic angles α and ε,

$$v = \tanh\alpha\,, \quad v_1 = \tanh\varepsilon\,, \qquad (11)$$

and their relation (1) to the hyperbolic angle α'. Therefore, the velocity of the Lorentz transformation $A(\vartheta, \alpha')$ in the right-hand side of (3) is related to the hyperbolic angle α' by the standard relation (10). Figure 1 also confirms formula (1) for the angle ϑ between the velocity $\mathbf{v}_2$ and the z axis of the laboratory system S:

$$\sin\vartheta = \frac{v_1}{v_2} = \frac{\tanh\varepsilon}{\tanh\alpha'} = \frac{\sinh\varepsilon\cosh\alpha}{\sinh\alpha'}\,. \qquad (12)$$

The right-hand side of (3) represented as $A_{SS_2'}(\vartheta, \alpha')\, R_{S_2'S_2}(\vartheta - \delta)$ can be interpreted as follows. The transformation $R_{S_2'S_2}(\vartheta - \delta)$ rotates the system S_2 clockwise through the angle $\vartheta - \delta$ relative to S_2', and the boost $A_{SS_2'}(\vartheta, \alpha')$ imparts the velocity v_2 in the ϑ direction to the system S_2'.

Thus, if a particle is initially at rest in the laboratory system S and its spin is directed along the z axis, then two Lorentz boosts involved in the left-hand side of (3) first rotate its spin clockwise by the angle $\vartheta - \delta$ and then impart the velocity $\mathbf{v}_2$ to the particle in the ϑ direction. Wigner's formulas (1) and (2) for the spin rotation angle imply the formula

$$\omega \equiv \vartheta - \delta = \arcsin \frac{\sinh\alpha \, \sinh\varepsilon}{1 + \cosh\alpha \, \cosh\varepsilon} \, . \tag{13}$$

We note an important limit case: when the first boost $A(0, \alpha)$ imparts a velocity v arbitrarily close to the speed of light to the particle, i.e., $\alpha \to \infty$, $v \to 1$, the second boost $A(\pi/2, \varepsilon)$ rotates this velocity by a finite angle ϑ, leaving this velocity arbitrarily close to 1. In fact, writing formula (10) as

$$v_2 = \sqrt{v_1^2 + v^2(1 - v_1^2)} = \sqrt{v^2 + v_1^2(1 - v^2)} \tag{14}$$

and letting v tend to 1, we obtain $v_2 \to 1$. In this case, it follows from formula (12) that $\sin\vartheta \to v_1$. For $v = 1$, formula (12) becomes the special case of formula (5.6) in [4] for the aberration of light.

Formulas (1) and (2) imply simple relations for the angle δ:

$$\tan\delta = \frac{\tanh\varepsilon}{\sinh\alpha} = \frac{\tan\vartheta}{\gamma_1\gamma} \, , \quad \sin\delta = \frac{\sin\vartheta}{\cosh\alpha} = \frac{\sin\vartheta}{\gamma} \, . \tag{15}$$

The first of these formulas coincides, in another notation, with formula (1.7) in Ref. [2]. As $v \to 1$ ($\gamma \to \infty$), the angle $\delta \to 0$, i.e., the angle between the spin and the velocity vanishes for ultrarelativistic particles.

For the velocity $v_1 \ll v$, the angles ϑ and δ are small and $\gamma_1 \approx 1$. In this case, $\delta = \vartheta/\gamma$ and hence the spin rotation angle $\omega \equiv \vartheta - \delta$ is related to the velocity rotation angle ϑ by the simple formula

$$\omega \equiv \vartheta - \delta = \left(1 - \frac{1}{\gamma}\right)\vartheta, \quad \gamma = (1 - v^2)^{-1/2}, \quad \vartheta \ll 1 \, . \tag{16}$$

For ultrarelativistic particles, the spin and velocity rotation angles coincide.

3. Three-parameter formulas for the spin rotation angle

The spin rotation was considered in the more general case, where the velocities $\mathbf{v}$ and $\mathbf{v}_1$ of two successive Lorentz transformations are not orthogonal, by Stapp [5], the author [6], and several others. In this case, the velocity $\mathbf{v}_2$ of the origin of the system S_2 relative to the laboratory system S is given by the vector sum

$$\mathbf{v}_2 = \frac{1}{1 + \mathbf{v}\mathbf{v}_1} \left\{ \mathbf{v}_1 \left[\frac{(\mathbf{v}\mathbf{v}_1)}{v_1^2} \left(1 - \frac{1}{\gamma_1}\right) + 1 \right] + \frac{\mathbf{v}}{\gamma_1} \right\} = \mathbf{v}_1 \oplus \mathbf{v} \tag{17}$$

of the velocity $\mathbf{v}$ of the system S_2 relative to S_1 and of the velocity $\mathbf{v}_1$ of the system S_1 relative to S. Formula (17) expresses the relativistic summation law for velocities $\mathbf{v}_1$ and $\mathbf{v}$ and coincides with formula (5.1) in Ref. [4] if the velocities $\mathbf{v}_1$, $\mathbf{v}$, and $\mathbf{v}_2$ are replaced by the respective velocities $\mathbf{V}$, $\mathbf{v}'$, and $\mathbf{v}$ in Ref. [4].

In (17), we also introduce the notation for the sum of two velocities with coefficients depending on the absolute values of these velocities and the angle between them. The succession of velocities in this notation is made clear when the velocities are endowed with physical meaning by assigning the indices of the reference systems to each of them. Then, if $\mathbf{v}_{S_1S_2}$ denotes the velocity $\mathbf{v}$ of the system S_2 relative to S_1 and so forth, expression (17) takes the form

$$\mathbf{v}_{2SS_2} = \mathbf{v}_{1SS_1} \oplus \mathbf{v}_{S_1S_2} \, .$$

We note that if velocities $\mathbf{v}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ satisfy summation law (17), then they also satisfy the inverse summation law $\mathbf{v} = (-\mathbf{v}_1) \oplus \mathbf{v}_2$.

For the spin rotation angle ω, Stapp [5] derived the formula

$$\mathbf{n}\sin\omega = [\mathbf{v}\mathbf{v}_1]\,\gamma\gamma_1 \frac{1 + \gamma + \gamma_1 + \gamma_2}{(1+\gamma)(1+\gamma_1)(1+\gamma_2)} \, ,$$
$$\gamma_2 = \gamma\gamma_1(1 + \mathbf{v}\mathbf{v}_1) \, , \tag{18}$$

where $\mathbf{n}$ is the unit vector along the direction of the vector product $[\mathbf{v}\mathbf{v}_1]$. In this formula, ω is expressed in terms of three independent parameters, not two as in Wigner's formulas: in terms of the absolute values v, v_1 of the velocities $\mathbf{v}$, $\mathbf{v}_1$ and the angle between them, because γ_2 is also expressed in terms of these three quantities.

When the velocities $\mathbf{v}$ and $\mathbf{v}_1$ are orthogonal, the velocity $\mathbf{v}_2$ takes simple form (9), and we obtain the spin rotation angle ω as

$$\sin\omega = \frac{vv_1\gamma\gamma_1}{1 + \gamma\gamma_1} \, , \tag{19}$$

which is Wigner's formula (13).

In an earlier work by the author [6], the spin rotation angle ω was explicitly expressed in terms of the absolute values of the velocities $\mathbf{v}$ and $\mathbf{v}_1$ and the angle θ between them:

$$\omega(u, u_1, \theta) = 2\arctan \frac{uu_1 \sin\theta}{uu_1\cos\theta + (\gamma+1)(\gamma_1+1)} \, . \tag{20}$$

Hereinafter, $\mathbf{u} = \mathbf{v}\gamma$, $\mathbf{u}_1 = \mathbf{v}_1\gamma_1$, and $\mathbf{u}_2 = \mathbf{v}_2\gamma_2$ are the spatial parts of four-velocities and γ, γ_1, and γ_2 are their temporal components. Formula (20) is related to Stapp's formula (18) because of the identity

$$\sin\omega = \frac{2\tan(\omega/2)}{1 + \tan^2(\omega/2)} \, .$$

Because of a certain symmetry expressed by formulas (37) and (38) in Ref. [6], the angle ω can also be expressed in terms of the absolute values of the velocities $\mathbf{v}_2$ and $-\mathbf{v}_1$ and the angle θ' between them:

$$\omega(u_2, u_1, \theta') = 2\arctan \frac{u_2u_1 \sin\theta'}{u_2u_1\cos\theta' + (\gamma_2+1)(\gamma_1+1)} \, , \tag{21}$$

as well as in terms of the absolute values of $\mathbf{v}$ and $\mathbf{v}_2$ and the angle ϑ between them:

$$\omega(u, u_2, \vartheta) = 2\arctan \frac{uu_2 \sin\vartheta}{uu_2\cos\vartheta + (\gamma+1)(\gamma_2+1)} \, . \tag{22}$$

When v and v_2 coincide with the speed of light, $\omega = \vartheta$. Otherwise, the spin rotation angle is smaller than the velocity rotation angle, $\omega < \vartheta$; this is the central qualitative statement in Ref. [6].

For $\theta = \pi/2$, formula (20) becomes

$$\omega\left(u, u_1, \frac{\pi}{2}\right) = \arctan\frac{uu_1}{\gamma+\gamma_1} = \arcsin\frac{uu_1}{1+\gamma\gamma_1} \tag{23}$$

and is consistent with Wigner's expression (19).

Because

$$\mathbf{u}_1 = (\mathbf{u}_2 - \mathbf{u})\,C\,, \quad C = \frac{\gamma+\gamma_2}{\gamma\gamma_2 + \mathbf{u}\mathbf{u}_2 + 1} = \frac{\gamma_1+1}{\gamma+\gamma\gamma_1+\mathbf{u}\mathbf{u}_1}\,, \tag{24}$$

the single rotation axis in each of the three representations (20)–(22) can be expressed in terms of the corresponding vector product, since

$$[\mathbf{u}\mathbf{u}_1] = [\mathbf{u}_2\mathbf{u}_1] = [\mathbf{u}\mathbf{u}_2]\,C\,. \tag{25}$$

In formulas (20)–(22), the argument of arctan involves the lengths of coincident vectors (25) in the numerators and the coincident quantities

$$\begin{aligned}\mathbf{u}\mathbf{u}_1 + (\gamma+1)(\gamma_1+1) &= -\mathbf{u}_2\mathbf{u}_1 + (\gamma_2+1)(\gamma_1+1)\\ &= C\big(\mathbf{u}\mathbf{u}_2 + (\gamma+1)(\gamma_2+1)\big)\end{aligned} \tag{26}$$

in the denominators, where $\mathbf{u}$ and $\mathbf{u}_2$ are related by the Lorentz transformation with the velocity $\mathbf{v}_1$.

The angle ω and the rotation direction $\mathbf{n}$ are characteristics of this Lorentz transformation. The inverse transformation of the vector $\mathbf{u}_2$ to $\mathbf{u}$ by a boost with the velocity $-\mathbf{v}_1$ is characterized by the same angle ω but with the opposite direction of rotation.

4. Møller's approach and its relation to Wigner's approach

In § 2.8 of his book [7], Møller considers a chain of inertial reference systems S, S', and S'' moving relative to one another, with the system S'' moving with a velocity $\mathbf{u}'$ relative to S' and the system S' moving with a velocity $\mathbf{v}$ relative to S. Then, the velocity $\mathbf{w}$ of the system S'' relative to S is given by formula (17), with the roles of $\mathbf{v}_1$ and $\mathbf{v}$ played by Møller's velocities $\mathbf{v}$ and $\mathbf{u}'$, i.e.,

$$\mathbf{w} = \frac{1}{1+\mathbf{v}\mathbf{u}'}\left\{\mathbf{v}\left[\frac{(\mathbf{v}\mathbf{u}')}{v^2}\left(1-\frac{1}{\gamma}\right)+1\right]+\frac{\mathbf{u}'}{\gamma}\right\} = \mathbf{v}\oplus\mathbf{u}'. \tag{27}$$

This formula coincides with Møller's formula (2.59). Møller also gives expression (2.59′) for the velocity $\mathbf{w}''$ of the system S relative to S'':

$$\begin{aligned}\mathbf{w}'' &= -\frac{1}{1+\mathbf{u}'\mathbf{v}}\left\{\mathbf{u}'\left[\frac{(\mathbf{u}'\mathbf{v})}{u'^2}\left(1-\frac{1}{\gamma_{u'}}\right)+1\right]+\frac{\mathbf{v}}{\gamma_{u'}}\right\}\\ &= (-\mathbf{u}')\oplus(-\mathbf{v}) = -(\mathbf{u}'\oplus\mathbf{v})\,.\end{aligned} \tag{28}$$

It differs from expression (27) by velocity permutation and by a sign. This follows from the fact that $\mathbf{w}''$ is the relativistic sum of the velocity $(-\mathbf{v})$ of the system S relative to S' and the velocity $(-\mathbf{u}')$ of the system S' relative to S''. Assigning the indices of the systems to all velocities in expressions (27) and (28), we can write

$$\mathbf{w}_{SS''} = \mathbf{v}_{SS'}\oplus\mathbf{u}'_{S'S''}\,, \quad \mathbf{w}''_{S''S} = (-\mathbf{u}')_{S''S'}\oplus(-\mathbf{v})_{S'S}\,. \tag{29}$$

The relativistic sum of two noncollinear velocities $\mathbf{u}'$ and $\mathbf{v}$ is asymmetric with respect to their permutation [4], and therefore velocities $\mathbf{w}$ and $\mathbf{w}''$ are not opposite to each other, although they are equal in magnitude:

$$\mathbf{w} = \mathbf{v}\oplus\mathbf{u}' \neq \mathbf{u}'\oplus\mathbf{v} = -\mathbf{w}''\,, \quad \mathbf{w}^2 = \mathbf{w}''^2\,. \tag{30}$$

Comparing formulas (17) and (28) shows that if the velocity $\mathbf{v}_1$ is set equal to $\mathbf{u}'$, $\mathbf{v}_1 = \mathbf{u}'$, then $\mathbf{v}_2 = -\mathbf{w}''$. This signifies that $-\mathbf{w}''$ has the physical meaning of the velocity of the system S_2 relative to laboratory system S and is represented by the relativistic sum of the velocity $\mathbf{v}$ of the system S_2 relative to S_1 and the velocity $\mathbf{u}'$ of the system S_1 relative to S:

$$\mathbf{v}_{2SS_2} = (-\mathbf{w}'')_{SS_2} = \mathbf{u}'_{SS_1}\oplus\mathbf{v}_{S_1S_2}\,. \tag{31}$$

We next show that the angle ω between the vectors $\mathbf{w}$ and $\mathbf{v}_2 = -\mathbf{w}''$ coincides with the spin rotation angle of a particle when its velocity changes from the value $\mathbf{v}$ to the value $\mathbf{v}_2 = -\mathbf{w}''$. In this case, ω is given by the left-hand side of the formula

$$\mathbf{n}\sin\omega = \frac{[\mathbf{w}, -\mathbf{w}'']}{w^2} = [\mathbf{v}\mathbf{v}_1]\,\gamma\gamma_1\,\frac{1+\gamma+\gamma_1+\gamma_2}{(\gamma+1)(\gamma_1+1)(\gamma_2+1)}\,, \tag{32}$$

where the unit vector $\mathbf{n}$ is aligned with all vector products encountered hereinafter. The right-hand side of (32) follows from the calculation of the vector product with expressions (27) and (28) for $\mathbf{w}$ and $-\mathbf{w}''$ taken into account and coincides with Stapp's formula (18). For brevity, we have reverted to the notation

$$\mathbf{v}_1 = \mathbf{u}'\,, \quad \gamma_1 = \gamma_{u'}\,, \quad \gamma_2 = \gamma_{w''} = \gamma_w = \gamma\gamma_{u'}(1+\mathbf{v}\mathbf{u}')\,. \tag{33}$$

Therefore, the statement about a double meaning of the angle ω is proven.

We introduce a vector $\mathbf{\Omega}$ using the relation

$$\mathbf{n}\sin\omega = \frac{[\mathbf{w}, -\mathbf{w}'']}{w^2} = -\mathbf{\Omega}\,. \tag{34}$$

The minus sign in front of $\mathbf{\Omega}$ is placed solely to make it coincident in sign with the vector of Møller. Then, the rotation operator D defined in Møller's book by formula (2.60), specifically, $D\mathbf{w}'' = -\mathbf{w}$, can be expressed in terms of $\mathbf{\Omega}$ as

$$D\mathbf{w}'' = -\mathbf{w} = \mathbf{w}''\sqrt{1-\mathbf{\Omega}^2} + [\mathbf{\Omega}\,\mathbf{w}'']\,. \tag{35}$$

Beginning with formula (2.61), Møller restricts himself to the approximation where the velocity $\mathbf{u}'$ is small in comparison with $\mathbf{v}$. Then the spin rotation angle ω and the modulus of $\mathbf{\Omega}$ are small in comparison with unity. In this approximation, Møller obtains expression (2.64) for $\mathbf{\Omega}$:

$$-\mathbf{\Omega} = (\gamma-1)\,\frac{[\mathbf{v}\,\mathrm{d}\mathbf{v}]}{v^2}\,, \tag{36}$$

where $\mathrm{d}\mathbf{v} = \mathbf{w} - \mathbf{v}$ is merely the notation for the difference $\mathbf{w} - \mathbf{v}$, according to Møller's formula (2.63).

My statement is as follows: in (36), i.e., in Møller's formula (2.64), we first use the difference $\mathbf{w} - \mathbf{v}$ instead of $\mathrm{d}\mathbf{v} = \mathbf{w} - \mathbf{v}$, then use its expression

$$\mathbf{w} - \mathbf{v} = \frac{1}{\gamma}\left[\mathbf{u}' + \mathbf{v}\,\frac{(\mathbf{v}\mathbf{u}')}{v^2}\left(\frac{1}{\gamma}-1\right)\right]\,, \quad u' \ll v\,, \tag{37}$$

from the top line of Møller's formula (2.62), and, finally, use the middle line

$$\mathbf{w}'' = -\left[\mathbf{v} + \mathbf{u}' - \mathbf{v}\,(\mathbf{v}\mathbf{u}')\right], \quad u' \ll v, \tag{38}$$

of the same formula (2.62) for $\mathbf{u}'$ in the occurring product $[\mathbf{v}\mathbf{u}']$; then we obtain three identical expressions for $\mathbf{\Omega}$:

$$\begin{aligned} -\mathbf{\Omega} &= (\gamma - 1)\,\frac{[\mathbf{v}\mathbf{w}]}{v^2} = \left(1 - \frac{1}{\gamma}\right)\frac{[\mathbf{v}\mathbf{u}']}{v^2} \\ &= \left(1 - \frac{1}{\gamma}\right)\frac{[\mathbf{v}, -\mathbf{w}'']}{v^2}, \quad u' \ll v. \end{aligned} \tag{39}$$

To calculate the vector products in expressions (39), we can conveniently represent the velocities $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}'$, and $-\mathbf{w}''$ appearing therein in one laboratory reference system.

If the origins of Møller's reference systems S, S', and S'' coincide at the instant $t = 0$, then at the instant $t = 1$ s in laboratory time they are at the points of laboratory system S marked by S, S', and S'' in Fig. 2.

Also indicated in Fig. 2 are the positions of the origins of Wigner's reference systems S, S_1, and S_2 at the same laboratory time instant $t = 1$ s. For simplicity, we restrict ourself to the case where the velocities $\mathbf{v}$ and $\mathbf{v}_1 = \mathbf{u}'$ are orthogonal, and therefore

$$\mathbf{w} = \mathbf{v} + \frac{\mathbf{u}'}{\gamma}, \quad \mathbf{v}_2 = -\mathbf{w}'' = \mathbf{v}_1 + \frac{\mathbf{v}}{\gamma_1}, \quad \mathbf{v}_1 = \mathbf{u}'. \tag{40}$$

Depicted in Fig. 2 is the situation where $v = 2v_1 = 0.94$, and hence $\gamma = 3$ and $\gamma_1 = 1.14$. But in calculating the vector products in expression (39), we are interested in the case where $v_1 = u' \ll v$. The angles ϑ and δ are then small, and therefore

$$\frac{[\mathbf{v}\mathbf{w}]}{v^2} = \mathbf{n}\,\delta, \quad \frac{[\mathbf{v}\mathbf{u}']}{v^2} = \mathbf{n}\,\vartheta, \quad \frac{[\mathbf{v}, -\mathbf{w}'']}{v^2} = \mathbf{n}\,\vartheta. \tag{41}$$

For $u' \ll v$, the small velocity $\mathbf{u}' \perp \mathbf{v}$ rotates the velocity $\mathbf{v}$ towards $\mathbf{v}_2 \equiv -\mathbf{w}''$ through the small angle $\vartheta = u'/v$, which is γ times greater than the angle of $\mathbf{v}$ rotation towards $\mathbf{w}$ when the velocity summation is effected in reverse order. This is because, in accordance with the $\mathbf{v}_2 \equiv -\mathbf{w}'' = \mathbf{u}' \oplus \mathbf{v}$ law, the small $\mathbf{u}'$ velocity is referenced to the laboratory system, while the small $\mathbf{u}'$ velocity in the sum $\mathbf{w} = \mathbf{v} \oplus \mathbf{u}'$ is referenced to the system S', which rapidly moves with the velocity $\mathbf{v}$ relative to the laboratory system. Owing to the time dilation in the system S' relative to the time in the laboratory system S, the departure $\Delta x_{S'S''}$ of the origin of S'' from the origin of S' in the transverse direction to $\mathbf{v}$ therefore proceeds in the laboratory system with the velocity u'/γ, i.e., γ times slower than in the system S':

$$\Delta x_{S'S''} = u'\Delta t' = u'\sqrt{1 - v^2}\,\Delta t = \frac{u'}{\gamma}\,\Delta t = \frac{1}{\gamma}\,\Delta x_{SS_1}, \tag{42}$$

where Δt and $\Delta t' = \Delta t/\gamma$ are the time intervals taken to depart for the same distance $\Delta x_{S'S''}$ in systems S and S'. On the other hand, the departure Δx_{SS_1} of the origin of S_1 from the origin of S, which occurs in the laboratory system with a velocity $\mathbf{u}' \perp \mathbf{v}$, in the same time Δt is γ times greater than the departure $\Delta x_{S'S''}$.

Therefore, for $u' \ll v$, the angle $\delta = \vartheta/\gamma$ and formulas (34), (39), and (41) yield the following relation between the spin rotation angle and the velocity rotation angle:

$$\omega = \left(1 - \frac{1}{\gamma}\right)\vartheta, \tag{43}$$

which coincides with limit expression (16) obtained from more general formulas by Wigner, Stapp, and the author.

Thus, using formulas in § 2.8 of Møller's book and Wigner's definition of the spin rotation angle under successive Lorentz transformations with noncollinear velocities [which coincides with Møller's definition up to a sign; see relation (34)], we obtain the same result in the limit $u' \ll v$.

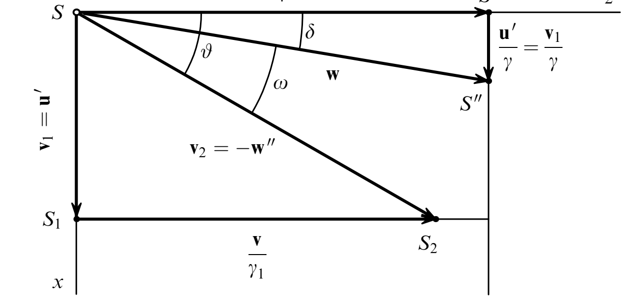

Figure 2.

5. Spin rotation in the curvilinear motion of a particle

We now turn to the discussion of Møller's formula (36). The notation $\mathrm{d}\mathbf{v} = \mathbf{w} - \mathbf{v}$ is not accidental. Møller intends to apply the formula for spin rotation in passing from one inertial system to another to the description of spin rotation in the particle motion along a curvilinear trajectory. In this case, the velocity $\mathbf{v}(t)$ is time-dependent and its values at close instants $t = 0$ and $t = \delta t$ are related by the formula

$$\mathbf{v}(\delta t) = \mathbf{v}(0) + \dot{\mathbf{v}}(0)\,\delta t, \tag{44}$$

where the velocity and acceleration at $t = 0$ are hereinafter denoted simply by $\mathbf{v}$ and $\dot{\mathbf{v}}$ and satisfy the condition $|\dot{\mathbf{v}}\delta t| \ll v$.

With the difference $\mathbf{w} - \mathbf{v}$ denoted by $\mathrm{d}\mathbf{v}$, Møller introduces the acceleration $\dot{\mathbf{v}} = \mathrm{d}\mathbf{v}/\mathrm{d}t$, where the interval $\mathrm{d}t$ is evidently equal to the laboratory time of velocity variation from $\mathbf{v}$ to $\mathbf{w}$. We restrict ourself to the case where $u' \ll v$ and compare formula (44) for $\delta t = \mathrm{d}t$ with formula (37) for the difference $\mathbf{w} - \mathbf{v}$ to obtain the condition for the coincidence of the velocities $\mathbf{v}(\mathrm{d}t)$ and $\mathbf{w}$:

$$\dot{\mathbf{v}}\,\mathrm{d}t = \frac{1}{\gamma}\left[\mathbf{u}' + \mathbf{v}\,\frac{(\mathbf{v}\mathbf{u}')}{v^2}\left(\frac{1}{\gamma} - 1\right)\right]. \tag{45}$$

Clearly, the magnitude of u' must be taken proportional to $\mathrm{d}t$.

The formal solution of Eqn (45) is given by

$$\mathbf{u}' = \left[\dot{\mathbf{v}} + \mathbf{v}\,\frac{(\mathbf{v}\dot{\mathbf{v}})}{v^2}\,(\gamma - 1)\right]\gamma\,\mathrm{d}t. \tag{46}$$

But such a solution for any nonzero acceleration $\dot{\mathbf{v}}$ satisfying the condition $|\dot{\mathbf{v}}\,\mathrm{d}t| \ll v$ does not satisfy the condition $u' \ll v$ in the ultrarelativistic limit, i.e., for $\gamma \gg 1$, even if $\mathbf{v}\dot{\mathbf{v}} = 0$. The condition for the smallness of u' imposes a substantially more stringent requirement on the interval $\mathrm{d}t$: $|\dot{\mathbf{v}}\,\gamma\,\mathrm{d}t| \ll v$. This interval becomes dependent not only on the acceleration but also on the velocity. This condition actually defines another velocity-independent interval $\Delta t = \gamma\, \mathrm{d}t$ of the laboratory time in which the velocity changes from $\mathbf{v}$ to $\mathbf{v}_2 \equiv -\mathbf{w}''$ by rotating through the angle ϑ with the spin rotating through the angle ω, while the condition $|\dot{\mathbf{v}}\Delta t| \ll v$ becomes the condition for the smallness of these angles. Møller, however, disregards this circumstance and uses formula (36) to obtain the formula for the variation rate of the spin rotation angle with laboratory time as

$$-\boldsymbol{\omega}_{\mathrm{M}} \equiv -\frac{\boldsymbol{\Omega}}{\mathrm{d}t} = (\gamma - 1)\,\frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2}\,, \tag{47}$$

which he labeled (2.65). (We indicate Møller's spin rotation rate with the subscript M to distinguish it from our spin rotation angle ω. The vector $\boldsymbol{\omega}_{\mathrm{M}}$ has the dimensionality rad s^{-1}, while ω is measured in radians.)

In our notation, the above expression should be equal to $\mathbf{n}\,(\omega/\mathrm{d}t)$. But this quantity is not equal to the angular velocity of spin rotation, i.e., to the derivative of the spin rotation angle with respect to the laboratory time, because ω is the spin rotation angle as the velocity changes from $\mathbf{v}$ to $\mathbf{v}_2 \equiv -\mathbf{w}''$, and $\mathrm{d}t$ is the laboratory time during which the velocity changes from $\mathbf{v}$ to $\mathbf{w}$. Because the angle ϑ between the velocities $\mathbf{v}$ and $\mathbf{v}_2$ is γ times greater than the angle δ between $\mathbf{v}$ and $\mathbf{w}$ (see Fig. 2), for an angular velocity $[\mathbf{v}\dot{\mathbf{v}}]/v^2$, which is given at the instant $t = 0$, the time Δt during which the velocity changes from $\mathbf{v}$ to $\mathbf{v}_2$ is γ times longer than the time $\mathrm{d}t$ during which it changes from $\mathbf{v}$ to $\mathbf{w}$, $\Delta t = \gamma\, \mathrm{d}t$. Therefore, the correct expression for the angular velocity of spin rotation is

$$\mathbf{n}\,\frac{\omega}{\Delta t} = -\frac{\boldsymbol{\omega}_{\mathrm{M}}}{\gamma} = -\frac{\boldsymbol{\Omega}}{\Delta t} = \left(1 - \frac{1}{\gamma}\right)\frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2}\,. \tag{48}$$

The left- and right-hand sides of this equality and the equality itself can be written as

$$\mathbf{n}\,\dot{\omega}(0) = \left(1 - \frac{1}{\gamma}\right)\dot{\vartheta}(0)\,\mathbf{n}\,, \tag{49}$$

where $\omega(t) = \dot{\omega}(0)\,t$ and $\vartheta(t) = \dot{\vartheta}(0)\,t$ are the spin and velocity rotation angles, which depend linearly on the laboratory time t as long as they are small in comparison with unity.

Because the laboratory time t is related to the proper time t' of the system S' as $t = \gamma t'$ and a particle that is at rest at the origin of S'' moves at a nonrelativistic velocity $u' \ll v$ in the system S', Møller's vector $-\boldsymbol{\omega}_{\mathrm{M}}$ may be assigned the meaning of the angular velocity of spin rotation in the proper system:

$$-\boldsymbol{\omega}_{\mathrm{M}} = \mathbf{n}\,\omega'(0) = (\gamma - 1)\,\dot{\vartheta}(0)\,\mathbf{n}\,. \tag{50}$$

The prime and the dot denote the respective derivatives with respect to the proper time and the laboratory time. Then, for an ultrarelativistic velocity $\mathbf{v}(0)$ and a fixed angular velocity of motion $\dot{\vartheta}(0)$, the angular velocity of spin rotation in the proper system of the particle can be arbitrarily high simply due to the proper time dilation in comparison with the laboratory time.

Thus, formula (36), which precedes formula (47) in Møller's book, can be represented in terms of Wigner angles,

$$\mathbf{n}\,\omega = -\boldsymbol{\Omega} = (\gamma - 1)\,\frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2}\,\mathrm{d}t = (\gamma - 1)\,\delta\,\mathbf{n}\,, \tag{51}$$

using that $[\mathbf{v}\dot{\mathbf{v}}]/v^2 = \mathbf{n}\,\dot{\vartheta}(0)$ is the angular velocity of motion at the instant $t = 0$ and $\mathrm{d}t$ is the time of velocity rotation from $\mathbf{v}$ to $\mathbf{w}$, i.e., of the rotation through the angle $\delta = \dot{\vartheta}(0)\,\mathrm{d}t$ (see Fig. 2). This formula is equivalent to

$$\omega \equiv \vartheta - \delta = (\gamma - 1)\,\delta \tag{52}$$

and leads to the previously encountered relations $\vartheta = \gamma\delta$ and $\omega = (1 - 1/\gamma)\,\vartheta$ between the angles ϑ, δ, and ω. Therefore, Møller's mistake lies in the very last step, when he divides the correct expression (51) by $\mathrm{d}t$ and states that expression (47) is the angular velocity of spin rotation in the laboratory system. This is not correct, because the spin rotates by the angle ω in the proper system during the laboratory time $\Delta t = \gamma\, \mathrm{d}t$ and during the same interval the velocity rotates in the laboratory system by the angle ϑ. The applicability condition $|\,\dot{\mathbf{v}}\,\Delta t| \ll v$ for the above formulas is equivalent to the smallness of the angle ϑ. In particular, for a given acceleration $\dot{\mathbf{v}}$ and γ tending to infinity, this condition does not permit fixing the angle δ. In this case, the interval $\mathrm{d}t$ tends to zero, and with it the angle

$$\delta = \dot{\vartheta}(0)\,\mathrm{d}t = \dot{\vartheta}(0)\,\frac{\Delta t}{\gamma}\,, \tag{53}$$

because the velocity $\mathbf{w}$ approaches $\mathbf{v}$ (see Fig. 2).

The essential dependence of the interval $\mathrm{d}t$ on the velocity $\mathbf{v}$ in the relativistic domain and the coincidence of its magnitude with the magnitude of the interval $\Delta t'$ of the proper time corresponding to the interval $\Delta t = \gamma \Delta t'$ of the laboratory time during which the particle velocity changes from the value $\mathbf{v}$ to the value $\mathbf{v}_2 = \mathbf{v}\,(\Delta t)$,

$$\mathrm{d}t = \Delta t' = \frac{\Delta t}{\gamma}\,, \tag{54}$$

lead to some complications in considering spin rotation in curvilinear motion with the use of the Møller sequence of Lorentzian boosts.

The spin rotation in the particle motion along a curvilinear trajectory is more convenient and conceptually clear to describe by using the Wigner sequence of Lorentzian boosts rather than the Møller one. The first boost then imparts the velocity $\mathbf{v}$ to the particle along its spin and the second boost imparts the additional velocity $\mathbf{v}_1$ to the particle already in motion. For $v_1 \ll v$, the resultant particle velocity, according to expression (17), becomes

$$\mathbf{v}_2 = -\mathbf{w}'' = \mathbf{v} + \mathbf{v}_1 - \mathbf{v}(\mathbf{v}\mathbf{v}_1)\,, \tag{55}$$

and the term linear in $\mathbf{v}_1$ in this formula can be identified with the term $\dot{\mathbf{v}}\Delta t$ in formula (44), i.e.,

$$\dot{\mathbf{v}}\Delta t = \mathbf{v}_1 - \mathbf{v}(\mathbf{v}\mathbf{v}_1)\,. \tag{56}$$

For this identification, the value of v_1 must be considered proportional to the interval Δt of the laboratory time during which the velocity changes from $\mathbf{v}$ to $\mathbf{v}_2$. The formal solution of Eqn (56) is

$$\mathbf{v}_1 = \left[\dot{\mathbf{v}} + \mathbf{v}\,(\mathbf{v}\dot{\mathbf{v}})\,\gamma^2\right]\Delta t\,. \tag{57}$$

It satisfies the condition $v_1 \ll v$ for an arbitrary γ if $|\dot{\mathbf{v}}\Delta t| \ll v$ and $\mathbf{v}\dot{\mathbf{v}} = 0$. Then, either of the two last terms in expression

(39) leads to the formula

$$\mathbf{n}\,\omega = -\boldsymbol{\Omega} = \left(1 - \frac{1}{\gamma}\right)\frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2}\,\Delta t = \left(1 - \frac{1}{\gamma}\right)\dot{\vartheta}(0)\,\Delta t\,\mathbf{n}\,, \quad (58)$$

which coincides with formulas (16) or (43) because $\dot{\vartheta}(0)\,\Delta t = \vartheta$ is the angle between the velocities $\mathbf{v}$ and $\mathbf{v}_2$. Hence, relations (48) and (49) between the angular velocities immediately follow and no confusion arises.

We here note that in the nonrelativistic limit $v \ll 1$, the solutions (46) and (57) in Møller's and Wigner's approaches satisfy the condition $u' = v_1 \ll v$ if $|\dot{\mathbf{v}}\,\mathrm{d}t|$, $|\dot{\mathbf{v}}\Delta t| \ll v$, the velocity-to-acceleration orthogonality condition is dropped. In this case, the intervals $\mathrm{d}t$ and Δt and the angles δ and ϑ are practically coincident and

$$\omega = \frac{1}{2}\,v^2\vartheta\,, \quad (59)$$

which is just Thomas's formula [8].

6. More about the relation between Møller's and Wigner's approaches

Formally, the Møller sequence of Lorentzian boosts differs from the Wigner sequence by permutation of the matrices $A(0,\alpha)$ and $A(\pi/2,\varepsilon)$. This permutation is equivalent to the transposition of the left- and right-hand sides of expression (3) and leads to the matrix

$$M \equiv A(0,\alpha)\,A\left(\frac{\pi}{2}\,,\varepsilon\right) = R(\delta - \vartheta)\,A(\vartheta,\alpha') = A(\delta,\alpha')\,R(\delta - \vartheta)\,, \quad (60)$$

because the Lorentz transformation matrix remains unchanged under transposition [see transformation (6)] and the rotation matrix changes the sign of the rotation angle [see matrix (5)]. The last expression for the matrix M in formula (60) was obtained using the relations

$$A(\vartheta,\alpha') = R(\vartheta)\,A(0,\alpha')\,R(-\vartheta)\,,\ \ R(\delta - \vartheta)\,R(\vartheta) = R(\delta)\,. \quad (61)$$

Similarly, the Wigner sequence of boosts can be represented by two polar decompositions, i.e., the products of symmetric and orthogonal matrices [9],

$$W \equiv A\left(\frac{\pi}{2}\,,\varepsilon\right)A(0,\alpha) = A(\vartheta,\alpha')\,R(\vartheta - \delta) = R(\vartheta - \delta)\,A(\delta,\alpha')\,. \quad (62)$$

Therefore, the Wigner and Møller matrices are related by both transposition and the equivalence relation:

$$W = \widetilde{M} = R(\vartheta - \delta)\,M\,R(\vartheta - \delta)\,. \quad (63)$$

We refer to Refs [9, Ch. 3, § 5; Ch. 9, § 14; Ch. 11, § 2] for the polar decompositions and the equivalence of matrices.

Equivalence relation (63) can be interpreted as follows. We read from right to left. The matrix $R(\vartheta - \delta)$ rotates the spin in the particle rest frame clockwise through the angle $\vartheta - \delta$ from the z axis. The matrix $R(\delta - \vartheta)$, which enters the right-hand representation for M in expression (60), brings the spin back to the z axis. The boost $A(\delta,\alpha')$ imparts the velocity $\mathbf{w}$, $w = \tanh\alpha'$, to the particle in the direction of δ, and hence the angle between the spin and the velocity is equal to δ. The matrix $R(\vartheta - \delta)$ rotates both the velocity and the spin clockwise through the angle $\vartheta - \delta$, with the result that the particle velocity becomes $\mathbf{v}_2 = -\mathbf{w}''$, $v_2 = w'' = \tanh\alpha'$. The velocity $\mathbf{v}_2$ is diverted from the z axis by the angle ϑ, while the spin is diverted from the z axis by the angle $\vartheta - \delta$. Precisely this information is contained in representation (3) or (62) for the Wigner matrix W.

To conclude, we emphasize that Wigner considered the rotation of the spin and velocity of a particle and the variation of the angle between the spin and the velocity in passing from one inertial system to another (Wigner rotation). In Møller's book, this consideration is used (adapted) for the description of spin rotation following velocity rotation in a curvilinear particle motion in the same inertial system under the assumption that forces change the velocity direction but do not impart a torque to the spin. Thomas's formula and the term Thomas precession relate to precisely this case.

This paper is a result of the analysis of the problems discussed in G B Malykin's review submitted to *Physics – Uspekhi*, which was recently published [10].

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Added to the English translation

The expression for C in (24) can also be written as

$$C = \frac{\gamma_1 + 1}{\gamma_2 + \gamma_2\gamma_1 - \mathbf{u}_2\mathbf{u}_1}\,.$$

Taken together with the last expression in (24), this form emphasizes the permutation symmetry $\mathbf{v} \leftrightarrow \mathbf{v}_2$, $\mathbf{v}_1 \leftrightarrow -\mathbf{v}_1$.