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Thomas precession: correct and incorrect solutions

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Contents

Abstract. A wealth of different expressions for the frequency of the Thomas precession (TP) can be found in the literature, with the consequence that this issue has been discussed over a long period of time. It is shown that the correct result was obtained in the works of several authors, which were published more than forty years ago but remained unnoticed against the background of numerous erroneous works. Several TP-related physical paradoxes formulated primarily to disprove the special relativity theory are shown to be fallacious. Different techniques for deriving the correct expression are considered and the reasons for the emergence of the main incorrect expressions for the TP frequency are analyzed.

1. Introduction

The Thomas precession (TP) [1 – 3] is a relativistic kinematic effect, which consists in that the spin of an elementary particle or the rotation axis of a macroscopic mechanical gyroscope, as well as a coordinate axis of a reference frame, moves along a curvilinear trajectory rotating (precessing) about the axes of a laboratory inertial reference frame (IRF). The Thomas precession relates the angular rotation velocity of the spin of an elementary particle following a curvilinear orbit (for instance, a circumference) to the angular velocity of the orbital motion. The Thomas precession does not occur due to the action of some forces responsible for variations in the angular position of a body, and therefore has a purely kinematic origin.

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The notion of the TP originated at the dawn of the age of quantum mechanics in the description of electron motion in an atom with the inclusion of its intrinsic magnetic moment defined by the spin. The nucleus produces only the electric field, but in accordance with relativistic transformation formulas, a magnetic field also exists in the electron frame of reference, which removes degeneracy in the magnetic quantum number in the presence of an intrinsic electron magnetic moment (the spin – orbit interaction). However, the fine structure values of the atomic hydrogen spectrum calculated with the formal inclusion of the magnetic interaction turned out to be two times greater than the experimental data. As noted in Ref. [4], the rate of spin orientation change of the electron in its comoving frame of reference was earlier believed to be equal to the vector product of the electron magnetic moment and the magnitude of the magnetic field in this frame of reference. L H Thomas (1903 – 1992) arrived at the conclusion [1, 2] that this is the case only if the frame of reference in which the electron is at rest does not rotate. When the system of coordinates comoving with the electron rotates, the rate of variation of an arbitrary vector (including the coordinate axes of the system under consideration) is lower than the rate of its variation in the corresponding nonrotating frame of reference by some value known as the Thomas precession.

Thomas [1, 2] considered consecutive Lorentz transformations for a system moving along a curvilinear trajectory and showed that owing to the noncommutativity of the general Lorentz transformations, the sequence of transformations results in an additional rotation of the axes of a system moving with acceleration relative to the system at rest [1, 2]. Taking this additional rotation into account ensures that the calculated and observed values coincide. However, Thomas's prime concern in this case was with quantum mechanical problems, and for the special relativity theory (SRT), the findings in his work therefore turned out to be just one of a number of complicated and intricate questions.

We emphasize that Thomas considered the rotation of the axes of the coordinate system accompanying the electron in its motion rather than the electron spin rotation. Subsequently, this led to a misunderstanding and the emergence of incorrect work on the TP problem. It is possible to introduce three different reference frames accompanying the electron motion around a circular orbit and, in the most general case, along a curvilinear trajectory: (i) a reference frame whose coordinate axes remain parallel or retain their angular position relative to the axes of a laboratory IRF, (ii) a reference frame one of whose coordinate axes is always coincident with the electron velocity vector, and (iii) a reference frame in which the electron spin vector retains its orientation relative to the coordinate axes. It is evident that the electron spin vector precesses relative to the coordinate axes of the two first systems, but the angular velocity of its precession is different in these systems. The coordinate axes of the second and third reference frames rotate relative to the laboratory IRF axes, the rotation of the axes of the third system coinciding with the TP of the electron spin. We note that even the first system is not inertial, because its origin moves along a curvilinear rather than a straight-line trajectory. That is why the simplest and clearest way is to consider the TP of the elementary-particle spin or the axis of mechanical gyro rotation relative to the axes of a laboratory IRF (rest-frame system).

The Thomas precession determines corrections in the calculation of the effect of spin – orbit interaction on the fine structure of atomic spectra [1, 2, 4, 5], permits explaining the anomalous Zeeman effect [1, 2], and allows qualitatively explaining the nucleon interaction in a nucleus and the cause of doublet 'reversal' in a nucleus [3]. In angular velocity sensors using de Broglie waves of material particles (electrons, neutrons, atoms, etc.), whose operation relies on the Sagnac effect [6 – 9], the TP is responsible for an additional shift of the zero of the interference pattern of counter-propagating waves, unrelated to rotation [10, 11].

At present, the spin – orbit interaction terms to which the TP leads are calculated in quantum physics as an approximation to a solution of the relativistic Dirac equation, from which all SRT effects follow directly [12]. In the view of the author of Ref. [13], the TP affects the value of the Rydberg constant.

Currently, the mathematical formalism of the TP is used to describe the change in the polarization state of light in its passage through multilayer dielectric coatings or in the reflection from them [14 – 18], as well as to describe squeezed light states [19] and the change in the electron polarization state in ionizing ion – atom collisions [20]. Introducing a new elementary particle — the so-called meron, which corresponds to the TP effect and is responsible for the spin – orbit interaction — has even been proposed.

A large number of different expressions for the TP can be found in the literature. In several papers concerned with the TP, calculations are performed in the first approximation in v^2/c^2, where v is the speed of an elementary particle in the laboratory IRF and c is the speed of light. In this case, all authors arrive at the same expression first derived by Thomas [1], this being so irrespective of whether they consider the relativistic rotation of the particle spin or the relativistic rotation of the axes of the coordinate system comoving with the particle. In the most general case, however, the expressions for the TP obtained by different authors are radically different. As noted above, the problem is complicated by the fact that different authors assign different meaning to this expression: some imply the relativistic rotation of the particle spin in the laboratory IRF, some in the comoving reference frame (in this case, as noted above, the rotation law for the axes of this system may be defined in three ways), while others refer to the relativistic rotation of the axes of the coordinate system accompanying the particle in motion.

We note that owing to the relativistic effect of time dilation in the reference frame that moves relative to the rest-frame (laboratory) system, the angular TP frequency in the reference frame comoving with an elementary particle with spin or with a mechanical gyro is always γ times higher $[\gamma = (1 - \beta^2)^{-1/2},\ \beta = v/c]$ than in the laboratory IRF. Different authors give expressions for the TP in different reference frames, which additionally complicates the situation.

To avoid such misunderstanding, henceforward all expressions for the TP are given, unless otherwise specified, for the laboratory IRF and only for the relativistic spin precession, which is, unlike the relativistic rotation of the axes of the coordinate system comoving with the particle, a real observable physical phenomenon, studying which is very important and interesting. Whenever an expression for the TP calculated in a paper being cited pertains to the comoving reference frame, it is recalculated and presented in the form corresponding to the laboratory IRF in order to avoid possible misunderstandings and facilitate comparison with other expressions for the TP. Of course, treating the TP in the comoving reference frame is also quite correct if it is remembered that the TP in this case is γ times greater than in the laboratory IRF. That is why the cases of the spin TP considered in the comoving reference frame or of the relativistic rotation of the coordinate system axes comoving with the particle are each time specially noted in what follows. We emphasize that, unfortunately, some authors provide no explanation as to the frame of reference in which their expression for the TP is written and what precession — that of the spin or the coordinate axes — they are considering, which makes it difficult to understand their results.

The aim of the present paper is to analyze the complicated situation relating to the TP and elucidate which of the expressions for the TP is correct and to which precession it applies — the precession of the particle spin or of the axes of the coordinate system that accompanies the particle in motion. Below, in the Introduction, we consider the development of the awkward situation surrounding the TP in retrospect. In Section 2, we discuss the main works concerned with the TP and give ten essentially different expressions describing the quantitative characteristics of this effect obtained by different authors. Also analyzed are the well-known experiments staged to measure the TP. Discussed in Section 3 are several TP-related physical paradoxes formulated by different authors primarily with the aim of disproving the SRT. In Section 4, we present various techniques for calculating the correct expression for the TP and analyze the most frequently occurring error committed by the majority of authors who arrive at incorrect expressions for the TP. In Section 4, we also propose a simple and clear illustration of the TP, which shows that the TP of the spin of an elementary particle or of the axis of a macroscopic gyro may be regarded as a manifestation of relativistic time dilation for an object that follows a circular orbit and has a preferred axis preserving its orientation in space for $v \ll c$.

The TP is shown to have no effect on the particle energy or radiation. In Section 5, we formulate the main results of the paper and cite Thomas's reminiscences, quite interesting in our view, about the times and circumstances in which he performed his basic works [1, 2].

As noted above, the expression for the TP in Thomas's first paper [1] was obtained in the first approximation in v^2/c^2 and is always correct when this condition is fulfilled. In his subsequent work [2], on performing calculations for an arbitrary electron velocity v, Thomas derived an expression that correctly describes the relativistic rotation of the axes of the comoving coordinate system relative to the rest-frame (laboratory) system. However, because the majority of authors use the term TP in reference to the precession of the spin of an elementary particle, this subsequently led to several errors and misunderstandings.

Prerequisites for the derivation of the correct expression for the TP of a particle spin for an arbitrary velocity of the orbital motion nevertheless emerged almost simultaneously with the publication of Thomas's papers [1, 2]. Already in 1926, Ya I Frenkel, whom Wolfgang Pauli familiarized with Thomas's work [1] prior to its publication, called attention to the following fact: specifying the three-dimensional vector of magnetic moment μ, as was done by Thomas [1, 2], is insufficient for the complete characterization of the magnetic properties of the electron, and a six-vector must be used [23].

In 1929, I E Tamm [24], based on the results of G Thomsen's mathematical work [25], proposed using a spacelike four-vector for the description of spin. More recently, Tamm's method was elaborated by H Bacry [26, 27] and A A Logunov [28 – 30]. (For a comprehensive analysis of Refs [23, 24], see reviews [31 – 33]). Tamm also showed that the TP does not lead to changes in the electron energy. However, Frenkel and Tamm, like Thomas himself, were concerned primarily with the problems of quantum mechanics, and therefore the expressions for the TP in Refs [23, 24] are given in the first approximation in v^2/c^2.

In 1931, by introducing the fourth spatial variable $x_4 = \mathrm{i}ct$ and thereby transforming the Minkowski space to the Euclidean space, A Sommerfeld [34] reduced the TP problem to the problem of adding rotations in a Euclidean space. If the calculations by Sommerfeld's method are carried to their conclusion, the resulting expression is significantly different from that derived by Thomas: the angular TP velocity turns out to be γ times lower. However, Sommerfeld stopped at the first approximation in v^2/c^2, where the approximate expressions for the TP in Ref. [34] and Refs [1, 2] coincide.

We note that the classical theory of spin was later elaborated in many papers (see, e.g., Refs [35 – 65]) in the 1930s – 1960s. However, the authors of these papers supposedly did not even set themselves the task of deriving the expression for the TP.

In 1934, H Kramers [66] (see also Ref. [67]) carried out the calculation of relativistic spin dynamics on the basis of a six-vector. But in calculating the expression for the TP, he also restricted himself to the weakly relativistic approximation.

In 1942, A D Galanin, Tamm's post-graduate student, used the parallel vector transport technique to calculate the TP, also restricting himself to the weakly relativistic approximation.

In 1939, E Wigner [69] developed a method that enabled calculating the spin orientation variation of an elementary particle under a stepwise change in its velocity vector [the so-called Wigner rotation (WR)]. The Thomas precession is a particular case of the WR corresponding to an infinitely small change in the velocity vector. However, neither in Ref. [69] nor in his subsequent works [70 – 73] (see also Ref. [74]) did Wigner derive explicit expressions for either the WR or the TP.

In the mid-1940s, V L Ginzburg [75, 76] arrived at the conclusion that there was a need to derive exact relativistic expressions for spin dynamics.[1] In 1947, Ginzburg and Tamm [77] reconsidered this problem.

In 1952, in his famous monograph [78], the Danish scientist C Møller (1904 – 1980), an acknowledged expert on the theory of relativity, derived an expression for the TP that coincides, up to a sign, with the corresponding Thomas expression [2] and is correct in the comoving frame of reference. However, it was stated in Ref. [78] that this expression was written for the laboratory IRF, which is incorrect. Møller's immense scientific prestige played a negative role in this case: since then, the majority of authors of papers, monographs, and lecture courses started using the expression for the TP from Ref. [78] or, in the derivation of suchlike expressions, tried to make them coincident with that given in Ref. [78]. To characterize the influence of the result in Ref. [78] on some researchers, by way of example we mention that Bacry's thesis, which was dedicated primarily to the TP, contained a section entitled 'The Thomas – Møller Precession.' The cause of errors committed in Refs [2] and [78] is analyzed in Section 4.5.

In 1956, H Stapp [79] obtained results that in principle permitted deriving the adequate expression for the TP; however, it was not derived in his paper.

In 1961, V I Ritus [80], using the results in Refs [69 – 73, 79], for the first time obtained the analytic expressions for the WR [69 – 74], which allowed a simple transition to the formula for the TP by making the added velocity tend to zero. Unfortunately, this was not done explicitly in Ref. [80]. However, it was shown in Ref. [80] that the direction of the spin of massless particles in their curvilinear motion always coincides with the direction of their momenta. Hence, it unambiguously follows that the spin of such a particle, including a photon, makes one turn per one turn of the particle.

In 1962, Ya A Smorodinskii (1917 – 1992) (see also Refs [82, 83]) for the first time derived and wrote an adequate expression for the TP by invoking the Lobachevski geometry.

In 1964, A Chakrabarti [84] obtained an adequate expression for the TP by invoking the mathematical formalism of the Poincaré group [85 – 88] (see also his papers [89 – 93]). (Here, we mention that many of Chakrabarti's papers [89, 90, 92 – 99] are concerned with the calculation of the polarization state of elementary particles through the solution of the Dirac equations [100, 101] in the Foldy – Wouthuysen representation [102, 102].)

However, it was the mid-1960s that saw the onset of a lively debate, which still persists today, about the physical nature of the TP, which was held on the pages of *The American Journal of Physics*, with the result that a large number of inadequate expressions for the TP were published.[2] Unfortunately, the correct results in Refs [80 – 82, 84, 89, 91, 93] were hardly mentioned in the course of that debate.

[1] From which it would be possible, in particular, to extract TP-related effects.

[2] The majority of these publications are discussed in Section 2.

In 1980–1981, the Canadian researchers W Baylis [104] and J Hamilton [105] obtained adequate expressions for the TP with the aid of six-quaternions or, equivalently, the Pauli matrix method (spinors in the Minkowski space [106–115]) and, accordingly, with the aid of the Clifford algebra [116–124] (six-vectors) [105]. In 1983, V A Bordovitsyn and S V Sorokin [125], using the method of Bargmann–Michel–Telegdi (BMT) [126],[3] derived an adequate expression for the TP in the reference frame accompanying an electron in motion. In 1986, M Strandberg (USA) obtained an expression for the TP correct both in the laboratory IRF and in the reference frame comoving with a spinning material particle [127] (see also Ref. [128]). It is noteworthy that Ref. [127] is one of the few papers that explicitly states that the angular frequency of the TP in the instantaneously comoving reference frame is γ times higher than in the laboratory IRF.

In 1998, the fifth volume of *Fizicheskaya Entsiklopediya* (*Encyclopedia of Physics*) was published; it contained an article by G B Malykin and G V Permitin [3], which gave the correct expression for the TP (without a derivation).

In 1999, the author of the present paper published two papers [129, 130] in which an adequate expression for the TP of an arbitrary axis related to a solid body following a curvilinear trajectory was obtained using the well-known Ishlinskii theorem about a solid angle [131, 132] (see also Refs [133–136]) applied to the relativistic aberration of the image of a moving body [137–142]. A similar approach had earlier been used in Ref. [143], but the calculations of that work had not been carried to the final formula. Recourse to the above technique led to an erroneous result in Ref. [144], the cause of which is analyzed in Sections 4.4 and 4.5. As noted in the foregoing, in the special case of a zero-rest-mass particle, the relation between the spin rotation and the relativistic aberration was established by V I Ritus [80] back in 1961. Interestingly, this method is equivalent to the parallel vector transport technique, which was invoked by S M Rytov [145, 146] and V V Vladimirskii [147] in passing from wave optics to geometric optics more than 60 years ago. In A D Galanin's work [68], this technique is used to pass from the wave description of electrons and mesons to the dynamical description, the expansion in v/c in the lowest order yielding the weakly relativistic approximation of the TP.

Therefore, there is a need to consider the TP in detail. In particular, in article [3] written for the *Fizicheskaya Entsiklopediya*, Malykin and Permitin give an expression for the TP coinciding with the expression by Ya A Smorodinskii [81–83], A Chakrabarti [84, 89, 91, 93], W Baylis [104], Hamilton [105], Bordovitsyn and Sorokin [125], Strandberg [127], and Malykin [129, 130], while I Yu Kobzarev [148] in the same encyclopedia gives an expression[4] coinciding with the one in Ref. [2]. It is pertinent to note that in a more recent paper [149], Smorodinskii actually abandoned his earlier result in [81–83] and obtained an expression for the TP coinciding with that in Ref. [78]. It should be noted that G V Permitin in his subsequent work written jointly with A I Smirnov also arrived at the expression for the TP coinciding with that given in Ref. [78].

The existence of numerous incorrect expressions for the TP and the general uncertain situation with this issue have led to several errors, some of which may be formulated in the form of physical paradoxes. These paradoxes are not as harmless as they might seem at first glance, because their superficial examination may have the effect that doubt is cast upon the validity of the SRT. A clear demonstration of the inconsistency of these paradoxes, which is done in Section 3, has a methodological significance of its own.

2. Review of the literature

2.1 Different expressions for the Thomas precession

The expression obtained by Thomas [2] for an infinitely small rotation angle of the axes of a coordinate system describing a curvilinear trajectory is, in the present-day notation, given by

$$\mathrm{d}\boldsymbol{\phi} = (\gamma - 1)\,\frac{[\mathbf{v}\times\mathrm{d}\mathbf{v}]}{v^2} = (\gamma - 1)\,\mathrm{d}\boldsymbol{\varphi}, \tag{1}$$

where $\mathrm{d}\mathbf{v}$ is the vector of an infinitely small velocity change due to acceleration, ϕ is the rotation angle of the spatial coordinate axes of the system comoving with an elementary particle or a gyrocompass measured relative to the laboratory IRF, and φ is the orbital angle in the laboratory IRF.

Expression (1) written in terms of the angular TP velocity $\Omega_{\mathrm{T}} = \mathrm{d}\phi/\mathrm{d}t$, where t is the time in the laboratory IRF, coincides, up to a sign, with the expression later obtained by Møller [78] (see also Refs [4, 26–30, 88, 144, 149–162][5]):

$$\boldsymbol{\Omega}_{\mathrm{T}} = -(\gamma - 1)\,\frac{[\mathbf{v}\times\dot{\mathbf{v}}]}{v^2}\,; \qquad \Omega_{\mathrm{T}} = (1-\gamma)\,\omega\,, \tag{2}$$

where $\omega = \mathrm{d}\varphi/\mathrm{d}t$ is the angular velocity of orbiting measured in the laboratory IRF. The second version in (2) is given for a body rotating with a constant angular velocity ω if the frequency of the TP Ω_{T} is also constant. We note that in several other papers (see, e.g., Refs [163–170]), the expression for the TP differs from expression (2) by a sign, and therefore coincides with Thomas's result [2].

As noted in the Introduction, expression (2) is correct in the frame of reference comoving with the particle or, to be more precise, would be correct had it been written in this frame of reference. However, Møller specified quite clearly [78] that expression (2) applied to the TP of the particle spin in the laboratory IRF. In particular, it then follows from expression (2) that the spin of a photon, as of any massless particle, in the laboratory IRF would execute an infinite number of rotations per one rotation of the particle, which is in evident contradiction to the results of V I Ritus's work [80].

At the same time, there are papers [3, 81, 82, 84, 89–93, 104, 105, 125, 127, 129, 130] in which the angular TP velocity is represented as

$$\mathrm{d}\boldsymbol{\phi} = \left(1 - \frac{1}{\gamma}\right)\frac{[\mathbf{v}\times\mathrm{d}\mathbf{v}]}{v^2}\,; \qquad \Omega_{\mathrm{T}} = \left(1 - \frac{1}{\gamma}\right)\omega\,. \tag{3}$$

[3] As shown by Bagrov and Bordovitsyn [31], in the special case where a charged particle with a spin travels in uniform electric and magnetic fields, the Bargmann–Michel–Telegdi equation become the Tamm–Good equation [24, 62], which reduces to the Tamm equation [24] in the absence of the particle magnetic moment anomaly.

[4] Unfortunately, in Ref. [148], Kobzarev did not specify in what frame of reference he wrote his expression for the TP, which hinders the interpretation of his results.

[5] We note that Farago [154], despite the incorrect result, was supposedly the first to point out that the angular TP velocity in the reference frame instantaneously comoving with a spinning material particle is γ times higher than in the laboratory IRF.

We note that expression (3) may be obtained by simple transformations of the results in Ref. [80]. Clearly, expressions (2) and (3) differ both in sign and in magnitude.

This brings up the question: Which of expressions (2) and (3) is correct? With an adequate understanding of the problem, both are. Expression (3) describes the TP precession of an elementary particle spin or the axis of a mechanical gyroscope in the laboratory IRF and expression (2) in the comoving reference frame. When we consider a comoving reference frame such that the direction of spin relative to the coordinate axes remains invariable, the TP of its axes of coordinates in the laboratory IRF coincides with the TP of the spin and is described by expression (3). Conversely, when the TP of the axes of the laboratory IRF is observed from the comoving coordinate system, the TP is described by expression (2). Unfortunately, not nearly all authors precisely specify what precesses, the spin or the coordinate system axes, how this system is defined, or precisely in which frame of reference the TP is recorded. In this way, the majority of misunderstandings related to the TP emerge.

As emphasized earlier, to avoid suchlike misunderstandings, all expressions for the TP given in the present paper apply to the laboratory IRF and only to the relativistic spin precession. Unlike the relativistic rotation of the axes of the comoving coordinate system, the relativistic spin precession is a real observable physical phenomenon and its consideration is most important and interesting. In this case, expression (2) is incorrect and expression (3) is correct. We prove this by a simple special example.

According to expression (2), when the velocity of an elementary particle following its orbit tends to the speed of light, the particle spin begins to make an infinite number of turns per one turn of the particle in its orbit. This contradicts both common sense and the well-known fact that the spin of a massless particle, for instance a photon (in a pure state) propagating through empty space is always either aligned with the direction of its motion or is opposite to it. Were expression (2) valid in the laboratory IRF, in the motion of light along a curvilinear (for instance, circular) trajectory, the photon spin would execute an infinite number of turns per turn of the photon in its orbit. Meanwhile, the photon spin makes only one turn per orbital turn, and as this takes place, its direction always coincides with the direction of photon propagation, which is exactly described by formula (3). Already in 1961, Ritus showed in his famous work [80], which has now become classic, that the photon spin direction is always coincident with the direction of its velocity vector. (Chakrabarti [84] was the first to draw attention to this highly significant result [80].) It was also shown in Ref. [80] that the change in the spin direction of a nonzero-mass particle always 'lags behind' the changes in its velocity direction, which is consistent with expression (3). Furthermore, a simple rearrangement of the expressions derived in Ref. [80] permits obtaining formula (3) in explicit form. Therefore, the findings in Ritus's work directly confirm expression (3) for the TP of the particle spin in the laboratory IRF.

It is pertinent to note that in several problems of practical significance, the condition $v \ll c$ $(\gamma \sim 1)$ is certainly satisfied and expressions (2) and (3) differ only in sign, which quite often has no effect on the final result. This is supposedly the reason why until recently no researchers have noticed, strange though it may seem, so serious a discrepancy between the results of different works. It was not until the publication of Ref. [129] that V F Chub drew attention to this fact. Indeed, in the ultrarelativistic case, where $\gamma \gg 1$, expressions (2) and (3) differ substantially in magnitude, by the factor γ.

The correct expression (3) was in fact obtained independently and almost simultaneously by Ritus [80], Smorodinskii [81, 82], and Chakrabarti [84].[6] However, as noted in the Introduction, Smorodinskii subsequently accepted the correctness of expression (2) [149]. We are therefore in the right to refer to the correct expression (3) as the Ritus–Chakrabarti expression.[7]

An opinion is sometimes expressed that both expressions (2) and (3) are correct in the same frame of reference with the understanding that they are correctly interpreted. We emphasize once again: in no way can the two different expressions for the TP, expressions (2) and (3), lead to the same (correct or incorrect) result in the same frame of reference. Expression (2) is valid in the comoving reference frame and expression (3) in the laboratory IRF.

However, expressions (2) and (3) are by no means the only ways of representing the relation between the TP and the angular rotation velocity. In several papers (e.g., in Refs [23, 24, 34, 66–68, 171–177]), the expression under consideration is brought to the form

$$\Omega_{\mathrm{T}} = \frac{[\mathbf{v} \times \dot{\mathbf{v}}]}{2c^2}\,; \qquad \Omega_{\mathrm{T}} = 0.5\left(1 - \frac{1}{\gamma^2}\right)\omega\,. \tag{4}$$

There are papers [178–181] in which the corresponding expression may be brought to a form similar to expression (4) but with the opposite sign or differing from expression (4) by a factor of 2 [182]. It should be noted that expression (4) was first obtained by Frenkel under the assumption that $v \ll c$ in his famous work [23] published during the period between the first and second of Thomas's publications [1] and [2]. In this case, expressions (3) and (4) lead to the same result in the first order in v^2/c^2. The reservation that expression (4) is valid for $v \ll c$ was also made by Tamm [24], Sommerfeld [34], Kramers [66], and Galanin [68]. However, the majority of authors omitted this reservation at a later time, which gave rise to additional misunderstandings.

Using the results in Ref. [183] for circular motion, the expression considered therein may be brought to the form

$$\Omega_{\mathrm{T}} = -\frac{(\gamma^2 - 1)(\gamma^2 - \gamma + 1)\omega}{(\gamma^2 + 1)\gamma}\,. \tag{5}$$

In Ref. [184], the expression for the TP is given as

$$\Omega_{\mathrm{T}} = -\frac{1}{2}\,(\gamma^2 - 1)\omega\,. \tag{6}$$

Finally, the expression considered in Ref. [185] may be represented in the form

$$\Omega_{\mathrm{T}} = 0.5\left(1 - \frac{1}{\gamma^2}\right)\left(\frac{\operatorname{arctanh}\beta}{\beta}\right)^2\omega\,. \tag{7}$$

An expression close to formula (7) was obtained in an appendix [186] written by G A Zaitsev to the Russian

[6] In Section 4, expression (3) is derived by several methods and, furthermore, we discuss the reasons underlying the emergence of mistakes committed by Thomas [2] and Møller [78] and accordingly of errors in expressions (1) and (2).

[7] In Chakrabarti's work [84], which was published three years later than Ritus's work [80], expression (3) is written in explicit form.

translation of M A Tonnelat's monograph Ref. [187]:

$$\mathrm{d}\boldsymbol{\phi} = \frac{\operatorname{arctanh}\beta}{\beta}\left(1-\frac{1}{\gamma}\right)\frac{[\mathrm{d}\mathbf{v}\times\mathbf{v}]}{v^2}\,;$$
$$\Omega_{\mathrm{T}} = \left(1-\frac{1}{\gamma}\right)\frac{\operatorname{arctanh}\beta}{\beta}\,\omega\,. \tag{8}$$

This expression, which differs from (3) by a factor containing a hyperbolic arctangent, results from an incorrectly chosen parameterization.

The authors of some papers believe that the TP frequency varies in time in accordance with some law even for $\omega = \mathrm{const}$. In particular, the following expression for the TP is given in Ref. [188]:

$$\Omega_{\mathrm{T}} = \beta^2\cos^2(\xi)\,\omega\,, \tag{9}$$

where ξ is the initial angle between the gyroscope and the axis perpendicular to the gyroscope orbit plane. Because the angle ξ periodically changes in time under the influence of the TP (the form of the $\xi(t)$ dependence is not considered in Ref. [188]), the TP frequency does not remain invariable, according to expression (9), even for $\omega = \mathrm{const}$. Next, the author of Ref. [188] averages the value of the TP in the case of gyroscope motion along a circular orbit ($\omega = \mathrm{const}$) and, in his view, the exact (and not approximate) mean TP value corresponds to expression (4).

The authors of a collection of exercises [189] also, in fact, arrive at the conclusion that the TP magnitude varies periodically in time. The final expression for the TP is not given in Ref. [189]; however, by carrying the calculations in Ref. [189] to their conclusion, one may obtain

$$\Omega_{\mathrm{T}} = -\frac{\omega(\gamma^2-1)\cos^2(\gamma\omega t+v)}{\gamma^2+(\gamma^2-1)\cos^2(\gamma\omega t+v)}\,, \tag{10}$$

where v is some initial angle of the electron spin orientation. The physical meaning of v is not discussed in Ref. [189]. The authors of Ref. [189] hypothesize that the expression for the TP for $\gamma - 1 \ll 1$ should coincide, up to a sign, with expression (2), although this does not follow from (10). We note that even if such a coincidence did occur, the result for the TP in the collection of exercises [189], which was specially assembled to facilitate mastering the lecture course [156], would differ in sign from the corresponding result in Ref. [156].

We thus have ten different expressions for the angular TP frequency, and under no circumstances can the last six of them be considered correct.

We note that similar questions also arise in the WR problem [69 – 74].

2.2 Experiments in which the Thomas precession manifests itself

First of all, we mention that the TP cannot be observed in its pure form: to make a macroscopic body or a material particle following a curvilinear trajectory requires the presence of a centripetal force, which may emerge due to a mechanical bond or a gravitational field or, when the body or particle is charged, due to an electric or magnetic field. As a consequence, in real cases, the TP — a kinematical effect — is accompanied by different dynamical effects. For instance, when an electron orbits in the Coulomb field of an atomic nucleus, in its comoving frame of reference there exists a magnetic field that interacts with the magnetic moment of the electron. When the electron velocity has a component parallel to the magnetic field lines, it also follows a circular trajectory; in this case, added to the TP are other effects related to the interaction of the electron magnetic moment with the magnetic field arising from the electric field of the nucleus in the reference frame of the electron [126, 190 – 193]. In the case where the electron moves in arbitrary fields [194] or a macroscopic rotating body with electric conduction travels in a nonuniform magnetic field [193], the situation is even more complicated. In the motion of a mechanical gyroscope in a gimbal suspension that is rotated about some center with the aid of a rod, the magnitude of its precession depends on the speed of sound in the material of the gyroscope. Finally, in the case where the probe body with a mechanical moment rotates in a gravitational field produced by a large mass, even more complicated additional effects arising from the curvature of space around the gravitating mass emerge [172, 173, 195, 196]. The last-mentioned case, as well as the effect of mechanical moment on the trajectory of a probe body [197], are considered in detail in Ref. [198]. When the gravitating mass rotates relative to the IRF, additional terms appear in the expression for the TP [172, 173, 196, 199] due to the Lense – Thirring effect [200 – 203] of the general relativity theory (GRT). Hence, it follows that the kinematic approach leads to sufficiently accurate results, in particular, for the motion of a small-size mechanical gyroscope rotating about a relatively small nonrotating gravitating mass, when the nonuniformity of the gravitational field and the curvature of space may be neglected.

2.2.1 On the feasibility of recording the Thomas precession with the aid of mechanical gyroscopes in their orbital motion. As far as we know, direct observations of the TP have never been undertaken. More than forty years ago, L Schiff (Stanford University, USA) proposed an experiment [172, 173] to discover the TP, as well as two GRT effects — geodetic precession [204] and the Lense – Thirring effect [200 – 203] — using a single-axis mechanical gyroscope mounted aboard an artificial earth satellite that follows a drift-free polar orbit (these issues are discussed in detail in Ref. [196]). Work on the fabrication and improvement of a mechanical gyroscope with the corresponding precision has uninterruptedly been pursued at Stanford ever since 1964 [196, 205]. We note that two other effects have already been recorded: the geodetic precession was discovered (to be more precise, calculated during the data processing of astronomical observations over a long period of time) by the French astronomer U Leverrier back in 1859 through the example of the precession of Mercury's perihelion [206] and was more recently interpreted in the GRT [203], while the Lense – Thirring effect was recorded by NASA and European researchers in 1998 from the orbit excursion of two spacecrafts by 2 m per year in the direction of Earth's rotation [210, 211]. [8]

2.2.2 Recording of the Thomas precession in the motion of charged elementary spinning particles in the storage ring of an accelerator. The possibility exists to indirectly record the TP in the course of experiments on the measurement of the

[8] The idea of this experiment was proposed by Ginzburg [207] in 1956, and much later by L Miller and I Shapiro [208] (1971) and R Van Patten and C Everitt [209] (1976).

anomalous part of the magnetic moment of elementary particles — leptons (electrons, positrons, and muons), which are conducted in the storage ring of an accelerator (see review Ref. [212] and the review part of Ref. [213]).[9] The idea underlying these experiments is as follows: when an elementary particle whose velocity is orthogonal to the magnetic field lines follows a circular trajectory, two effects occur — the Larmor precession (LP) and the TP. Since the particle has an anomalous magnetic moment, characterized by the Landé factor [the so-called g-factor, $g = 2(1 + a_e)$, where $a_e \ll 1$ is the anomalous part of the g-factor], this results in some additional LP of the particle spin, proportional to a_e, which is recorded in the course of measurements. In this case, it is possible to record the variation of the resonance frequency of transitions from one orbital level to another, which is related to the anomalous part of the magnetic moment (the spin–resonance technique), or directly measure the additional spin rotation angle (the precessional technique) [213].

The first experiments with electrons were carried out at the University of Michigan (USA) back in 1953 [215]; in this case, the TP effect on the total precession of the electron spin was disregarded although the magnitude of the TP was estimated at 40% of the LP. In more recent experiments, which were performed in Michigan [216, 217] and at CERN (Geneva, Switzerland) (see also Refs [212, 218]), the particle velocity was ultrarelativistic and the TP was taken into account. In the subsequent discussion, we follow Ref. [213], where this question was considered in the greatest detail. The circular LP frequency is given by

$$\Omega_L = \frac{g}{2}\,\omega_0\,; \qquad \omega_0 = \frac{eH}{m_0 c}\,, \tag{11}$$

where H is the magnitude of the magnetic field in the laboratory IRF, m_0 is the rest mass, e is the electron charge, and ω_0 is the frequency difference between the neighboring Landau (Rabi–Landau) levels [219–221] with different orbital numbers. The frequency ω_0 is independent of the particle velocity, while the cyclotron frequency of the particle (the frequency of its orbital circular motion) is inversely proportional to the γ-factor,

$$\omega_c = \frac{\omega_0}{\gamma}\,. \tag{12}$$

Relation (12) has the following physical meaning. The cyclotron frequency ω_c is the classical frequency of motion of a charged finite-mass body along a circular trajectory in a magnetic field, when the Lorentz force is the centripetal force. As $v \to c$, the mass of the particle and its orbit radius increase indefinitely, while its angular velocity accordingly decreases such that the Lorentz force is equal to the centripetal force as before. By contrast, the ω_0 frequency is the quantum frequency defined by the external magnetic field; it characterizes some internal motion of the elementary particle. In the general case, the particle is 'spread' (i.e., resides with equal probability) not only over the entire circular orbit (with the consequence that it does not radiate an electromagnetic field at the frequency ω_c) but also between different orbits with different energy levels. Increasing the particle energy results only in the particle beginning to occupy progressively higher energy levels with a higher probability, i.e., higher orbits, while the separations between the neighboring levels are determined by the energy value $h\omega_0/2\pi$ and the levels remain approximately equidistant.[10]

Therefore, it is valid to say that at relativistic velocities, a charged spinning particle moves in an external magnetic field with the effective angular velocity ω_0, while its real angular velocity ω_c has no effect on either the magnitude of the LP or (as shown below) on the magnitude of the TP. This is a consequence of the quantum mechanical nature of the effects under consideration, the LP and TP for elementary particles. It is well known, for instance, that the mechanical moment of the electron cannot result from its rotation because, as estimates show, for this to be the case, the linear velocity of the electron surface must exceed the speed of light by several orders of magnitude. Also, the orbital motion of the electron supposedly cannot be identified with its real rotation, with the consequence that the frequency ω_0 should be considered in lieu of the cyclotron frequency ω_c in this case. Were similar experiments conducted for a miniature macroscopic mechanical gyroscope moving in an external magnetic field and possessing an appreciable electrical charge and hence a magnetic moment, suchlike questions would not arise: in this case, there would be only one real angular velocity equal to the cyclotron frequency. Of course, such a device would be impossible to accelerate to relativistic velocities, and recording the TP would require accumulating the TP-related turn of the gyroscope axis for a very long time. Furthermore, were it possible to find an elementary particle with a charge and a mechanical moment but without a magnetic moment, the LP would not occur in its motion in an external magnetic field and the TP would therefore be possible to record in its pure form.

The expression for the angular velocity (or, equivalently, for the circular frequency) of the TP of the electron spin is [213]

$$\Omega_T = \frac{\gamma - 1}{\gamma}\,\omega_0\,. \tag{13}$$

Expression (13) is in complete agreement with formula (3) and relates the TP to the intrinsic particle frequency ω_0. This is natural, because the LP is also related to the ω_0 frequency in this case. It should be noted that expressions for the LP and TP in Refs [89, 91, 93] also coincide with expressions (11) and (13). Next, following Ref. [213], to calculate the spin precessional frequency, we subtract the TP frequency from the LP frequency:

$$\Omega_s = \Omega_L - \Omega_T = \left(a_e + \frac{1}{\gamma}\right)\omega_0 = a_e\omega_0 + \omega_c\,. \tag{14}$$

We mention that expression (14) without the separation into the Larmor and Thomas parts was first obtained by H Mendlowitz and K Case [226] in 1955 and more recently by M Carrassi [227] (see also Refs [228, 229]). From expression (14), in particular, it follows that if the magnetic moment of a charged particle has no anomaly, then the total action of the LP and TP has the effect that the particle spin rotates with the cyclotron frequency.

[9] There are also other methods, for instance spectroscopic, for measuring the anomalous part of the magnetic moment of electrons [214].

[10] In the weakly relativistic case, the last-mentioned is true only for levels with sufficiently low energy, but for them, as shown in Refs [222, 223], a hardly noticeable nonequidistance permits producing the level population inversion, which is required, for instance, for the operation of a gyrotron — a cyclotron-resonance maser [224, 225].

Therefore, expression (14) given in the classic work [213] testifies to the fact that experiments in a storage ring for particles without the magnetic moment anomaly confirm the validity of expression (3); in particular, it follows from expression (3) that the particle spin makes one turn per particle rotation as $v \to c$.

Because the cyclotron frequency is easy to determine, expression (14) permits high-precision calculations of the anomalous part of the magnetic moment of electrons [212, 213, 216, 217] and muons [213] from experimental data.

Therefore, it is valid to say that experiments on leptons following a circular path in an orthogonal magnetic field [212, 213, 216, 217] indirectly confirm the validity of expression (3) for the TP, although not with respect to the real frequency of particle circular orbital revolution, the cyclotron frequency ω_c, but with respect to the intrinsic particle frequency ω_0, because in this case radiation arising from the quantum transitions between the neighboring Landau levels occurs rather than synchrotron radiation. Also related to the ω_0 frequency in this case is the LP of the particle spin.

At the same time, because the experimentally observed particle spin precession is caused by the sum of two effects, the TP and the LP, it is possible to choose different expressions for each of these effects, only provided that these expressions add up to correspond to expression (14). However, it is not necessary to give these expressions: they are discussed in Ref. [230]. We note that only the total spin precession is described in several papers (see, e.g., Ref. [231]).

In 1959, the famous paper by V Bargmann, L Michel, and V Telegdi [126] (see also Ref. [232]) was published, which was concerned with the motion of elementary charged spinning particles with an anomalous magnetic moment in uniform electric and magnetic fields. The results in Ref. [126] were obtained by the method of semiclassical approximation of the Dirac equation with the anomalous Pauli interaction [233]. The special feature of the solutions in Ref. [126] [the so-called Bargmann – Michel – Telegdi (BMT) equation] is that the values of the electric and magnetic fields in Ref. [126] are written in the laboratory IRF, while the spin precession is expressed in the reference frame comoving with the particle, which in some papers (see, e.g., Ref. [234]) is referred to as the Michel reference frame. The Thomas precession was not considered in Ref. [126] at all, because the Dirac equations allow obtaining the solution for the total particle spin precession without splitting it into the Larmor and Thomas parts. In 1962, Bacry [26] (see also Refs [27, 235 – 237]), using the results in Ref. [126], completely passed to the laboratory IRF to derive the expressions for the LP and the TP (which are widely used at present, e.g., in Refs [154, 238]), which differ from expressions (11) and (13). It comes as no surprise: in Ref. [26], Bacry adhered to expression (2) and therefore had to abandon the description of spin precession by expressions (11) and (13), because otherwise he would have been forced to admit the validity of expression (3).

At the same time, Chakrabarti [89, 91, 93] used the results in Ref. [126] to derive the expressions for the TP and the LP coincident with expressions (11) and (13) and thus confirmed the correctness of expression (3). In the recently published monograph Ref. [239] (see also the substantially earlier paper [31] by some of the authors of Ref. [239]), this question was considered at great length. In Ref. [239], the BMT method was used to obtain formulas for the LP and the TP, the expression for the LP coinciding with expression (11) and the expression for the TP, when brought to the laboratory IRF, coinciding with expression (3).

Thus, the results of experiments involving spin precession measurements are treated differently. Those authors who erroneously associate the resonance frequency of transitions between the orbital electron levels in a magnetic field with the cyclotron frequency ω_c describe the TP by expression (2) (see, e.g., Refs [27, 213, 226, 227, 235 – 237]). Other authors who associate the resonance transition frequency with the quantum frequency $\omega_0 = \omega_c/\gamma$ [89, 91, 93] or correctly use the BMT equation [125, 239] describe the TP by expression (3).

It is pertinent to note that the anomalous part of the magnetic moment of muons was also measured in their motion along a helical path in a uniform magnetic field (over a path length of 6 m) [234]. In this case, the effect of the TP on the spin precession was also taken into account.

To summarize this section, we conclude that the preferred method is the above-discussed method of recording the TP with the aid of mechanical gyroscopes in their orbital motion because quantum mechanical effects in experiments on charged elementary particles partly complicate the interpretation of experimental data.

3. On alleged paradoxes and misconceptions related to the consideration of the Thomas precession

Consideration of some questions related to the TP (or the WR) sometimes leads to serious misunderstanding, which is actually due to the inadequate understanding of several complicated aspects of the SRT, among which is the TP. Because these questions are routinely addressed in the literature and during discussions, we believe there is good reason from the methodological standpoint to consider the main mistakes and provide correct physical explanations. Some of these mistakes have been formulated (or may be easily formulated on the basis of original papers) in the form of physical 'paradoxes,' which at first sight cast doubt on the validity of the SRT. As shown below, these paradoxes are in fact pseudoparadoxes. We reveal the basic errors responsible for the formulation of several of them. Here, we give three.[11]

(1) The Bacry paradox (formulated in Ref. [26]).

In an IRF S, let a body execute a rectilinear motion with an acceleration $\mathbf{a}$, which is constant in magnitude and direction. Naturally, no TP is observed in the system S. However, in some other inertial system S' that moves with a velocity $\mathbf{v}$ relative to S, $\mathbf{a}$ and $\mathbf{v}$ being mutually orthogonal or at least nonparallel, the TP should be observed in accordance with expression (2). Proceeding from this paradox, Bacry draws the conclusion that expression (2) is invalid for rectilinear motion but is valid for curvilinear motion.

(2) The Neganov paradox.

This paradox, which is related to the WR, is easily formulated on the basis of B S Neganov's works [240 – 243]. We consider an arbitrary velocity triangle, whereby a body moves relative to some IRF first with a velocity $\mathbf{v}_1$, then with a velocity $\mathbf{v}_2$, and then with a velocity $\mathbf{v}_3$, which form a triangle

[11] We note that although all these pseudoparadoxes are associated with the use of incorrect expression (2), they would have qualitatively, in essence, remained unaltered had their authors proceeded from other expressions for the TP.

(see Refs [79, 80]). On completion of the cycle, the body returns to the starting point and in doing so turns by some angle, i.e., precession occurs (in this case, the WR). The magnitude of the net turn of the body is invariant with respect to the arbitrary choice of IRF; however, the values of local body turns at the points where the body velocity changes depend on the velocity of the inertial system of the observer. Hence, the author of Refs [240 – 243] draws the conclusion that the relativity principle is violated and there exists some preferential IRF related to the 'world ether' or physical vacuum, in which all local turns are the same. The validity of the SRT is therefore cast into doubt. Since Neganov considers a step-wise change of the body velocity rather than a smooth one, strictly speaking, this paradox applies not only to the TP but also to the more general case of the WR.

(3) The Mocanu paradox.

This paradox was formulated by A Ungar [244] (see also his papers [245 – 251]) on the basis of C Mocanu's paper [252] as follows: the net result of two successive Lorentz transformations from a system S to a system S' and then to a system S'' is no longer a Lorentz transformation when the velocity of S'' relative to S is not parallel to the velocity of S'' relative to S'.[12] From this fact, Mocanu draws the conclusion that the SRT is invalid as is, in particular, Maxwell's electrodynamics.

We now turn to the consideration of the hidden errors that lie in the very formulation of these paradoxes.

The Bacry paradox may be disproved as follows. As shown in Refs [3, 129, 130], the TP is due to the relativity of the notion of curvilinear translational motion of a system of material points. If in one IRF, S, the velocities of all body points at a time instant t are equal, in another IRF S', at an instant t', they are different for accelerated motion of the body. The existence of the above effect shows that there exists no curvilinear translational motion in the SRT, unlike in classical mechanics. Bacry's mistake consists in precisely the fact that he seeks to ascribe the curvilinear translational motion to a solid body in an arbitrary IRF.

We note that in the case of curvilinear motion, the velocity of a solid body constantly changes its direction, and by the appropriate selection of the IRF, it is possible to eliminate the TP only at some time instant. Here, we are facing a relativistic analog of the Zeno paradox considered in our work [9]. We recall that if a noninertial rotating reference frame is replaced with an instantaneously comoving IRF, the results of the corresponding calculations demonstrate the absence of the Sagnac effect [6 – 9], which is contrary to common sense. The fallacy of this approach lies in the fact that the above replacement is incorrect: one should consider the set of IRFs accompanying the rotating reference frame at different time instants [9]. By directing the difference of the temporal intervals between the neighboring IRFs to zero and performing a continuous transition from one IRF to another, it is possible to correctly calculate the magnitude of the Sagnac effect. Similarly, the instantaneous vanishing of the TP by choosing an appropriate IRF leads to incorrect conclusions: in this case, one should also consider the set of IRFs accompanying the curvilinear motion of the solid at different time instants. Herein lies the analogy between the Bacry paradox and the relativistic Zeno paradox.

It is possible to provide another explanation for the Bacry paradox. The angular turn of a body arising from the TP may be unambiguously calculated only when the body returns to the initial point as a result of circular or, in general, elliptic motion (the so-called cyclic evolution). When the body trajectory is not closed, the value of the angular turn under consideration is calculated up to a constant quantity. In the absence of cyclic evolution, some uncertainty is in general characteristic of geometrical (topological) phases [136, 255 – 263], among which is the TP [129, 130]. A similar situation occurs for the Ishlinskii effect in classical mechanics: the angle of an additional gyrocompass turn resulting from a conic motion of an object is rigorously calculated in the case of return to the starting point [131, 132]. In the case considered by Bacry, the uniformly accelerated body never returns to the starting point.

A clear explanation of Neganov's mistake was provided by A A Logunov [28] (see also his monograph Refs [29, 30]), who considered the dependence of the TP value on the angle between the force acting on a particle and the particle velocity, the direction of the latter depending on the choice of the IRF of the observer: "...When the force is directed in some IRF along the particle velocity, the spin precession is absent. But the parallelism of the vectors of force **F** and velocity **v** is violated even under Galilean transformations, to say nothing of Lorentz transformations from one IRF to another. That is why the effect of precession, which is equal to zero for the observer in one IRF, is nonzero for the observer in some other IRF. Does this circumstance testify to the inequivalence of different IRFs? Of course, not. The matter is that due to the very formulation of the physical problem, we have already fixed the class of IRFs by directing the action of the force **F** along the direction of the particle velocity. The equivalence of IRFs did take place until the instant of selection of the IRF in which we applied the force **F** and aligned it with the particle velocity."

It is noteworthy that the Bacry and Neganov paradoxes are intimately related: both rely on ignoring the dependence of local measurement data on the IRF of the observer.

The Mocanu paradox is attributable to its author's incorrect understanding of Lorentz transformations, to which he assigned only boosts. As already shown by Thomas [1, 2], a sequence of nonparallel boosts is not a pure boost but also involves a rotation, which, however, is an element of the Lorentz group. But Mocanu ascribes only boosts to Lorentz transformations and draws the wrong conclusion that the set of Lorentz transformations is open and hence the relativity theory is incorrect.

The main conclusion drawn from the review of the literature is as follows. The situation with theoretical calculations of the expression for the TP, as well as with the interpretation of experimental data on the measurements of spin precession of a charged particle with an anomalous magnetic moment, is complicated and knotty. This was bound to affect the attitude of some of researchers towards this problem. Some of them take advantage of the rather uncertain situation and endeavor to derive their original, as a rule, inadequate expressions for the TP, which complicates the situation still further. Others used this situation to develop paradoxes, which at first sight seized the imagination and were actually intended to discredit the SRT. The third, more cautious, group avoids the issue of a quantitative expression for the TP and instead provides a large number of cumbersome formulas, which bear no direct relation to this issue, and

[12] A similar reasoning is also given in Mocanu's works [253, 254].

lengthy qualitative reasoning (see, e.g., Refs [118, 244–251, 264–268]). In particular, in seven of A Ungar's papers [244–250], which are written at a rather high scientific level and occupy 112 journal pages, and in his recently published monograph [251], which summarizes his longstanding TP-related research, the expression for the TP is never given. Only one [177] of seven of N Salingaros's papers [120–123, 177, 265–267] gives the approximate expression (4) for the TP, which is valid for any method of considering this issue, as noted in Section 2.1.

Meanwhile, it has been more than forty years since the publication of several papers that gave the correct expression for the TP. Several techniques of calculation that enable deriving the correct expression for the TP were elaborated even earlier — more than seventy years ago. In Section 4, we briefly consider the works whose authors derived, explicitly or implicitly (or could have derived on abandoning the assumption that $v \ll c$), the correct expression (3).

4. Derivation of the correct expression for the Thomas precession

As noted in Sections 1 and 2, the correct expression for the TP was first obtained by V I Ritus [80]. In explicit form, formula (3) was given in the works of Smorodinskii [82], who used the Lobachevski geometry in the velocity space, and Chakrabarti [84]. Earlier, Sommerfeld [34], Kramers [66], and Galanin [68] had actually come quite close to this expression. Conceivably, on seeing its disagreement with Thomas formula (1) (also, we must not rule out the possibility that their concern was only with quantum-mechanical problems), they restricted themselves to the first approximation in v^2/c^2. It should be noted that Chakrabarti in his series of works [84, 89–93] virtually realized Kramers's approach without any approximations. All these approaches explicitly or implicitly rely on the vector (bivector) nature of rotation angles and boost in the SRT. The author of the present paper arrived at formula (3) by using the Ishlinskii theorem.

It is also important to elucidate the reason why Thomas [2] and Møller [78] obtained incorrect expressions (1) and (2), because this gave rise to a large number of fallacious works.

4.1 The Wigner–Stapp–Ritus' method

In his works [69–73] written between 1939 and 1957 (part of these papers were translated into Russian [74]), E Wigner considered an effect that later came to be known as Wigner rotation. The effect consists in the following. When a spinning particle with a nonzero velocity acquires an additional momentum at the right angle to the former velocity, then not only the direction of its velocity but also the spin orientation changes.

In 1956, H Stapp used Wigner's results to obtain an important equation describing the spin rotation of an elementary particle when its initial velocity $\mathbf{v}_1$ relative to the laboratory IRF experiences two stepwise changes — $\mathbf{v}_2$ and $\mathbf{v}_3$ — but after the second change becomes equal to $\mathbf{v}_1$ once again. Then, of course, unlike in the case of orbital motion, the particle coordinate does not return to the initial state in the general case, but this is of no consequence as regards the spin rotation. Subsequently, Stapp's equation [79] came to be known as the 'velocity triangle equation,' but it would be more properly referred to as 'the equation for a triangle in the velocity space.' We give it in the notation adopted in the present paper:

$$\sin|\Phi| = \big|[\mathbf{v}_1 \times \mathbf{v}_2]\big| \, \frac{1+\gamma_1+\gamma_2+\gamma_3}{(1+\gamma_1)(1+\gamma_2)(1+\gamma_3)}\,, \qquad (15)$$

where Φ is the spin rotation angle of the particle when its velocity returns to the initial state and γ_1, γ_2, and γ_3 are the Lorentz factors corresponding to the velocities $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

From Eqn (15), it is possible to derive the correct expression (3) for the TP, but Stapp supposedly did not set himself this task. However, he found the relations between the vertex angles of the triangle in the velocity space and the local spin rotations of the particle at each change in its velocity. These relations subsequently caused several misunderstandings. As noted in Section 3, only the net angle of spin rotation upon the return of the particle velocity to its initial state is invariant in passing from one IRF to another, the magnitude of the local rotation angle depending on the IRF in which the observer is located. Ignoring this fact has brought several TP-related paradoxes into existence (see, e.g., Neganov's paradox in Section 3).

A more general case of this effect, where the angle between the initial particle velocity and the additional momentum is arbitrary, was considered in the famous paper by Ritus [80] (1961). As noted in the Introduction, the TP is a special case of the WR. Therefore, as the magnitude of additional momentum tends to zero and its direction is orthogonal to the previous velocity, the corresponding expression for the WR in this limit case should evidently turn into the correct expression for the TP. Unfortunately, neither Wigner, nor Stapp, nor Ritus considered this special case at one time, because the issue of TP did not enter the wide scope of their work. The authors of several papers (see, e.g., Ref. [269]) do not relate the WR to the effect of TP.

Recently, on the author's request, Ritus familiarized himself with the manuscript of this paper and expressed an interest in this issue. In particular, he showed that the correct expression for the TP may be obtained from expressions (32) and (36) in Ref. [80].[13] We note that Ref. [80] has not lost its significance until now. In particular, its results are used in Ref. [271] to calculate the relativistic rotation of the total spin of two elementary particles.

4.2 Smorodinskii's method (Lobachevski geometry)

The calculation leading to expression (30) was first completed by Smorodinskii [82], who considered the velocity space and showed (following Sommerfeld) that it is a Lobachevski space in the SRT. That is why the triangles turn out to be pseudospherical. In considering a spinning top, Smorodinskii showed that the turn of the axis of the top is, in the first approximation, equal to the area of the triangle formed by the top velocities at two instants, t and $t + \mathrm{d}t$, which, as follows from the Lobachevski geometry, is

$$\mathbf{S} = \frac{\gamma}{1+\gamma}\,[\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}]\,\mathrm{d}t\,. \qquad (16)$$

Therefore, the top precesses under the action of an acceleration $\dot{\boldsymbol{\beta}}$ with the angular velocity

$$\boldsymbol{\Omega}_{\mathrm{T}} = -\frac{\gamma}{1+\gamma}\,[\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}]\,, \qquad (17)$$

whence expression (3) follows.

[13] This issue is considered at length in Ritus's paper [270].

4.3 Chakrabarti's method

The representation of Lorentz transformations by a six-vector in Chakrabarti's works is in fact similar to the Kramers method. However, Chakrabarti begins with finite Lorentz transformations resulting in a *Wigner rotation*:

$$R = \Lambda_{(\Lambda p)}\,\Lambda\,\Lambda^{-1}_{(p)}\,,$$

where p is the four-dimensional particle momentum, Λ is the Lorentz transformation, and $\Lambda_{(p)}$ is the proper Lorentz transformation:

$$\Lambda_{(p)}\,p = (m, 0)\,.$$

To parameterize Lorentz transformations, Chakrabarti introduced a bivector $J_i = \varepsilon_{ijk}\,M_{jk}$, $K_i = M_{0i}$ and considered the action of Lorentz transformations on the vector of the intrinsic moment $\mathbf{S}$. For infinitely small transformations (due to acceleration), he obtained the formula

$$\delta\mathbf{S} = \frac{\gamma}{1+\gamma}\,[\mathbf{v}\times\dot{\mathbf{v}}]\times\mathbf{S}\,, \qquad (18)$$

which corresponds to formula (3).

It is noteworthy that a similar derivation of the correct expression for the TP was made by W Baylis [104] and J Hamilton [105].

4.4 Thomas precession as a corollary of the Ishlinskii theorem

The method for calculating the expression for the TP proposed by the present author [129, 130, 272] relies on the theorem about a solid angle [131, 132] (see also Refs [133–136]), which was proved by A Yu Ishlinskii in the early 1950s. This theorem may be formulated as follows [135]. If some axis selected in a solid body with three degrees of freedom spans a closed conic surface in the course of body motion and the projection of the angular body velocity on this axis is zero, then after the axis returns to the initial position, the body becomes rotated about it by the angle equal to the solid angle of the cone. The translational motion of the axis is irrelevant in this case.

Penrose [138] and Terrell [137] showed in 1959 that light quanta that arrive simultaneously at the observer were emitted by different points of the object at different time instants — the points located further from the observer emitted photons earlier than the points located closer to him. For this reason, the Lorentz contraction effect is compensated and, when the object dimensions are much smaller than the distance to the object, the image of the object appears to be simply rotated by some angle rather than distorted.

When an object moves rectilinearly with a velocity $\mathbf{v}$, the angle Θ' that defines some direction in the object in the reference frame associated with the object and the angle Θ at which this direction is observed in the laboratory IRF are related by the well-known formulas describing the relativistic aberration [141, 273]:

$$\sin\Theta = \frac{\sqrt{1-v^2/c^2}\,\sin\Theta'}{1+(\cos\Theta')(v/c)}\,. \qquad (19)$$

An observer at rest sees the object turned by the aberration angle $\Delta\Theta = \Theta - \Theta'$. Let the object move rectilinearly in the plane orthogonal to the straight line that connects the object and the observer and let the axis that defines the direction also lie in this plane, i.e., $\Theta' = 90°$. From (19), we then obtain

$$\Delta\Theta = \Theta - 90°\,, \qquad \cos(\Delta\Theta) = \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma}\,. \qquad (20)$$

Let the object move along a circular trajectory in the plane under consideration and let $\Theta' = 90°$ as before. Then, the direction of the axis seen by the observer changes, the object image becomes rotated, and the axis spans a cone with the vertex angle $2\Delta\Theta$ per turn of the object. The solid angle confined by the cone is equal to the area bounded on the unit-radius sphere by the generatrix of the cone whose apex coincides with the center of the sphere [274]. Hence, it is easy to obtain the expression relating the solid angle Ξ to the cone vertex angle:

$$\Xi = 4\pi\sin^2\left(\frac{\Delta\Theta}{2}\right) = 2\pi\big[1-\cos(\Delta\Theta)\big] = 2\pi\left(1-\frac{1}{\gamma}\right). \qquad (21)$$

It follows from expression (3) that the angular turn of the body due to the TP upon one circular revolution is

$$\alpha = 2\pi\,\frac{\Omega_{\mathrm{T}}}{\omega} = 2\pi\left(1-\frac{1}{\gamma}\right). \qquad (22)$$

Comparing expressions (21) and (22), we conclude that $\alpha \equiv \Xi$, i.e., the angular turn of a body caused by the TP is equal to the solid angle described, according to Ishlinskii's theorem, by the axis related to a solid body in its motion along a circular path when the real change in its orientation angle is equal to the change in the rotation angle, observed in the laboratory IRF, of a body relativistically moving along a curvilinear trajectory. Therefore, the TP may be interpreted as a consequence of the formal application of Ishlinskii's theorem to the solid angle corresponding to the change in the observed turn of the image of the solid body in its motion along a curvilinear trajectory relative to an observer at rest. This is supposedly the simplest and most lucid way of calculating the TP.

We see that the angular velocity of intrinsic rotation $\omega - \Omega_{\mathrm{T}}$ decreases as γ tends to infinity.

4.5 Analysis of the reasons why Thomas and Møller obtained an incorrect expression for the Thomas precession

In Section 2.1, we gave ten different expressions for the TP, only one of them being correct. A detailed analysis of the reasons that underlie the derivation of all the incorrect expressions would require a significant increase in the volume of the paper (the reasons of some of the most serious mistakes are briefly mentioned in Section 2.1). What matters is to reveal the reasons why Thomas [2] and Møller [78] arrived at incorrect expressions (1) and (2), because this gave rise to the overwhelming majority of erroneous works on the TP [as noted in Section 2.1, some papers give expression (2) with the opposite sign, which corresponds to expression (1)].

First of all, we emphasize that the application of the solid-angle theorem by itself does not ensure the correctness of calculating the expression for the TP. In particular, in Refs [144, 150], this approach led to the incorrect expression (1) for the TP. The reason of the error in Refs [144, 150] lies in the fact that the authors applied the Ishlinskii theorem (which is sometimes referred to as the Hamilton–Ishlinskii theorem [136]) for calculations in the four-dimensional Minkowski space and not in the real three-dimensional space. Supposedly for the same reason, there occurred a mistake in Thomas's basic work [2], in which he stated, without expounding intermediate calculations, that he resorted to the Rodrigues – Hamilton theorem in the four-dimensional Minkowski space. As shown in our work [136], this theorem is close in meaning to the solid-angle Ishlinskii theorem [131, 132], and it is therefore likely that Thomas, like the authors of Refs [144, 150], had overrated the magnitude of the TP by the factor γ to arrive at the incorrect expression (1).

It is conceivable that Møller, who, 25 years after the publication of Thomas's work [2], made an attempt for the first time to obtain the exact expression for the TP [78] not involving an expansion in the small parameter v^2/c^2, endeavored to obtain a close result and arrived at expression (2), which differs from expression (1) only in sign. However, the reason for Møller's mistake was different, as shown below.

This issue, in particular, is the focus of Ritus's work [270], which is now prepared for publication. There is no point in outlining all the results in Ref. [270],[14] because readers will be able to familiarize themselves with it on the pages of *Physics – Uspekhi* in the near future. However, with the kind permission of Ritus, briefly formulated below is the result in Ref. [270] that concerns our problem. First of all, we note that Møller, when calculating the TP in Ref. [78], used three IRFs: the laboratory IRF S, the instantaneously accompanying IRF S' in which the spinning particle is at rest at the time instant t, and the instantaneously accompanying IRF S'' in which the spinning particle is at rest at $t + \mathrm{d}t$. As shown in Ref. [270], Møller committed no errors almost until the end of his calculations, i.e., right up to expression (2.64) in his monograph [78]:

$$\mathbf{\Omega} = -(\gamma - 1)\,\frac{\mathbf{v} \times \mathrm{d}\mathbf{v}}{v^2}\,, \tag{23}$$

where, in the notation of Ref. [78], $\mathbf{\Omega}$ is some vector whose direction coincides with the direction of a point-like compass — a material spinning particle, $|\mathbf{\Omega}|$ being the spin rotation angle, and $\mathbf{v}$ is the particle velocity vector relative to the laboratory IRF S. Expression (23) involves the differential $\mathrm{d}\mathbf{v}$ of the particle velocity vector, which was defined by Møller in his monograph [78] as $\mathrm{d}\mathbf{v} = \mathbf{w} - \mathbf{v}$, where $\mathbf{w}$ is the velocity vector of the IRF S'' relative to the IRF S'. As shown in Ref. [270], had Møller substituted this value of $\mathrm{d}\mathbf{v}$ in expression (23) (formula (2.64) in Ref. [78]), he would have obtained the correct expression (3). However, subsequently, Møller used another expression for the differential of the velocity vector: $\mathrm{d}\mathbf{v} = \dot{\mathbf{v}}\,\mathrm{d}t$, consequently arriving at the incorrect expression (2). It might seem that Møller made a glaring computational mistake in Ref. [78]. However, as shown in Ref. [270], the mistake is conceptual in nature: the sequence of passing from one IRF to another was violated in Ref. [78].

It is sometimes pointed out that Møller in Ref. [78] implied the precession of the coordinate axes of the frame of reference associated with the moving particles and not the spin precession. However, in his monograph [78, p. 45], it was clearly stated several times that he bore in mind the precession of the axis of a point-like compass relative to the IRF.

[14] The most interesting result in Ref. [270] is, in our view, the derivation of the direct relation between the parameters of the WR and TP.

4.6 Illustrations of the kinematics of the Thomas precession

We now consider a purely illustrative example corresponding to expression (3). It clearly shows that the TP may be considered a manifestation of the effect of relativistic time dilation in the circular orbital motion of an object with a preferred axis, which retains its orientation in space for $v \ll c$. Let a clock attached to a rod of length R, which is pivoted at a point O, follow a circular orbit of radius R with an angular velocity ω in the counterclockwise direction (see Fig. 1). The clock size is small in comparison with R. That is why the clock base always faces the point O. The clock is adjusted such that the hour hand executes a complete revolution in a time $T = 2\pi/\omega$, i.e., the hour hand rotates with an angular velocity ω.

When the orbital clock velocity $v = R\omega$ is much smaller than the speed of light, $\gamma - 1 \ll 1$, clearly, the hour hand indicates a constant direction in space at each time instant, and it may therefore be regarded, in principle, as a spatial compass.

Figure 1 represents the orbital clock motion for $\gamma = 6/5$, when the value $v = R\omega$ is comparable to the speed of light. Owing to the relativistic time dilation, the clock is 'slow,' with the result that in the view of the observer in the laboratory IRF, the hour hand turns forward (in the direction of clock motion) as the clock follows its circular orbit. The hour hand deviates from the initial position by the angle $2\pi(1 - 1/\gamma)$ per revolution. Initially, the hour hand pointed at 12 (see Fig. 1), i.e., was in the vertical position. Upon one clock revolution, the clock is 2 hours 'slow' due to the relativistic time dilation and the hour hand points at 10, i.e., the hand has turned by 60° in the direction of the orbital clock motion.

In the hypothetical case where $v = R\omega = c$, the clock stops in the view of the observer in the laboratory IRF. In this case, the hand executes one turn per orbital revolution of the clock and the angle between the hand and the clock trajectory remains constant. In particular, if the hand is aligned with the tangent to the orbit at the start of a the

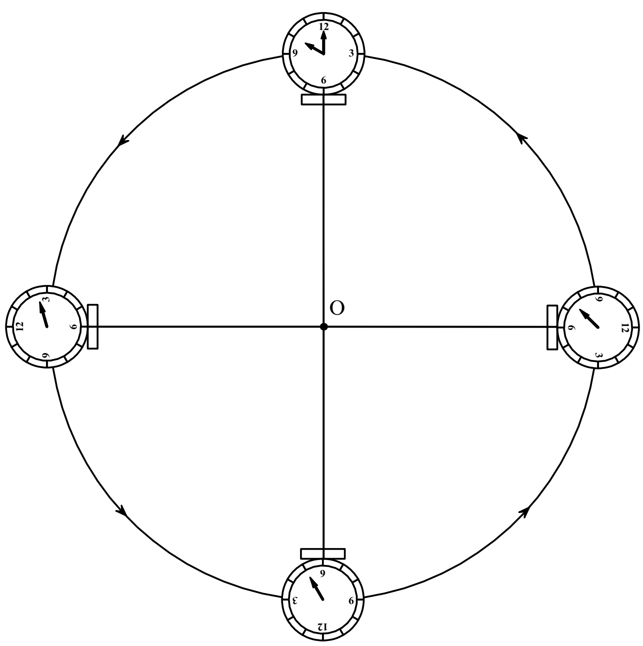

Figure 1. Representation of orbital clock motion with the velocity $v \sim 0.55c$ ($\gamma = 6/5$).

thought experiment, then it always remains in this position. The spin of a photon behaves in precisely the same way in its motion along a curvilinear trajectory. It is well known [275, 276] that the spin is always either aligned with the direction of motion (for right-circularly polarized photons) or directed opposite to it (for left-circularly polarized photons).

4.7 Effect of the Thomas precession on the energy and radiation of a material spinning particle

That the TP does not lead to a change in the electron energy was first predicted already by Tamm [24]. The question of whether the TP is the source of relativistic electromagnetic radiation was considered in a recently published monograph [239].[15] The answer to this question is not quite evident because the terms corresponding to the TP enter the total radiation power of the electron and its intrinsic magnetic moment. In Ref. [239], this problem is treated from the standpoints of both the classical radiation theory and the quantum theory of mixed radiation, both solutions being in perfect agreement. The analysis in Ref. [239] suggests that the TP cannot be regarded as the source of relativistic radiation power. Account must be taken of the fact that the existence of electron spin should lead to an insignificant correction to the electron mass [72], which should have a small effect on the radiation power. But, as shown in Refs [24, 239], the TP itself does not make any contribution to the effective electron mass and therefore the TP, being a relativistic kinematical effect, can in no way affect the radiation power.

5. Conclusion

We summarize the main results in this paper. The TP problem is by no means simple: there is a large number of different expressions for the TP [see formulas (1)–(10)]. Furthermore, the same expression may be interpreted both correctly and incorrectly, depending on the reference frame in which the TP is calculated and on precisely which precession is considered — the precession of the spin of an elementary particle (the axis of a mechanical gyroscope) or the coordinate axes of a system associated with the particle — and also on how this coordinate system is defined and relative to what this system processes. For definiteness, we proposed always considering the TP of the spin of an elementary particle or the axis of a mechanical gyroscope in the laboratory IRF. We have shown that expression (3) is correct in this case. The corresponding experiments permit confirming the correctness of expression (3) only indirectly, because measurements were made of the sum of the TP and the LP. This explains why until now it has remained unclear which of the numerous expressions for the TP is correct. A small number of correct works by V I Ritus, Ya A Smorodinskii, A Chakrabarti, W Baylis, J Hamilton, V A Bordovitsyn and S V Sorokin, M Strandberg, G B Malykin, and G V Permitin [3, 80–84, 89–93, 104, 105, 125, 127, 129, 130] simply went unnoticed against the background of several hundred incorrect works. In deriving expressions for the TP, the majority of authors were supposedly guided by the incorrect expression (2) for the TP from C Møller's monograph [78]. A negative role was also played by the fact that Smorodinskii in his more recent works [149] actually abandoned expression (3) derived in his earlier works [81–83], and arrived at the expression from Ref. [78].

The central result in this paper is that we have shown that expression (3) may be derived in four independent ways at least. We have also revealed the reason why there emerged an error in L H Thomas's basic work [2]. Moreover, the results in Ritus's work [270] reveal the cause of error in Møller's monograph [78] responsible for the incorrect expression (2), which was picked up by the majority of authors of works on the TP. Broadly speaking, the significance of Ritus's basic work [80] for the study of TP- and WR-related issues is hard to overestimate. Immediately after the publication of Ref. [80], in the 1960s, its results were used in practice by top-level experts. In particular, Chakrabarti in his notable work [84], where he derived the correct expression for the TP, used the results in Ref. [80] on the TP of the spin of massless particles. Unfortunately, many researchers either misunderstood the results in Ref. [80] or were completely unfamiliar with it. It is valid to say that this was the cause of the emergence of a large number of incorrect works on the TP.

That the TP problem is by no means simple is also confirmed by the story of its origination and solution, which was related by Thomas himself in the preface to his paper [278]:

"I am to explain how I came to do my early work on the spinning electron.

I will try to talk about what I knew and did a very long time ago. I hope that I can keep myself to that and not too much later hindsight. I want to explain what I knew at that time and how little it was.

I will set the stage by relating that I had graduated at Cambridge, England, in the summer of 1924 by obtaining a first class in the mathematical tripos examinations with distinctions in applied mathematics. I was awarded a graduate studentship on the basis of the showing in these examinations.

My director of studies during my first year of graduate studies was R H Fowler who was not technically a professor. He spent that first year at Bohr's Laboratory in Copenhagen and I spent the time on my own working on various problems in quantum theories, on the basis of Bohr's Correspondence Principle.

At the end of the year, Fowler arranged that I should go to Copenhagen the next year, 1925–1926, to work in Bohr's Laboratory. This was a very interesting time. It was the year in which Schrödinger, Heisenberg and Dirac began developing modern quantum mechanics, of which at that time I understood nothing, because nothing was published about it yet. My knowledge was confined to the older work of Bohr, especially his theory of the spectrum of the hydrogen atom postulating atomic states stationary except for interaction with radiation obeying Planck's law. The Zeeman splitting of the absorption lines in a magnetic field followed and Bohr's theory had been extended to relativity theory by Sommerfeld. Radial atomic fields to lead to the observed spectra of more complicated atoms were being obtained by Fock and Hartree.

My work on Spin was pure accident. Goudsmit and Uhlenbeck[16] had put out the idea of Spin and mailed their paper to Dr. Bohr. This suggested that the electron should have an angular momentum of its own. Bohr and Kramers were arguing just before Christmas and they said this really did not work because they required a twice as large spin

[15] This problem had earlier been considered by J Schwinger [176, 277].

[16] [279, 280] (see also Refs [281, 182]). (*Here and henceforward, G B M's comment.*)

angular momentum for the observed Zeeman effect as for the multiplets in the absence of a magnetic field. This is the anomalous Zeeman effect, and the structure of the hydrogen spectrum did not fit any value for the spin angular momentum of the electron.

I being a reasonably brash young man in the presence of Bohr said, 'Why doesn't someone work it out relativistically.' Kramers who had known of the earlier work on the motion of the moon by De Sitter [17] said to me 'It would be a very small relativistic correction. You can work it out, I won't.' Over that weekend I looked at it. I had the advantage of having attended Eddington's lectures [18] on relativity theory and I knew how to work the mathematics. I found that if you look at the change in the direction of the axis of a rotating electron, there should be a very considerable relativistic effect, in fact, a factor of two. I brought this idea back with a formula to Kramers and Bohr just after that one Christmas weekend. Bohr insisted that a letter should be written to *Nature*, [19] which had this result in it. This letter, which is my second or third original paper, was published in *Nature* in April 1926.

I was ignorant of any of the work by various people. De Sitter's formula, that I used, is for the theory of relativistic corrections for the motion of the moon, where it is indeed a very small correction as Kramers knew. De Sitter's formula is mentioned and described in Eddington's book and that is my only knowledge of it. Eddington's book was just published at that time. It was only later that I knew De Sitter had anything to do with this. The idea of the Spin is almost exactly as Goudsmit and Uhlenbeck had put it forward. When it had been put forward a year or two earlier by Kronig [20] it was poo-pooed by Pauli, who said this was all nonsense. Kronig had not pushed it any further. This indeed often happens. Older people find it difficult to change their ideas. One with any new ideas is very likely to be poo-pooed. Don't be bluffed that way yourselves. I was fortunate that I was working in Bohr's laboratory and Bohr and Kramers were not like that.

My full paper on the subject was published in January 1927 in the *Philosophical Magazine* [21] after the beginning of the work of Dirac, Heisenberg and Schrödinger had come out. I didn't understand one word of their work on Quantum Mechanics at that time. The only way I got my present poor understanding of quantum theory is because I have had to give lectures on it at 8 o'clock in the morning. The poor graduate students had no other time because most of their time was spent teaching younger students.''

L H Thomas noted that a similar effect had earlier come under study in the GRT (see, e.g., Ref. [287]).

It is pertinent to note that interest in the TP, apart from that of Thomas himself, was taken at different times by Frenkel, Tamm, Sommerfeld, Kramers, H Hönl, H Bhabha, Galanin, A Papapetrou, Ginzburg, H Goldstein, N Ramsey, Møller, D Bohm, M A Tonnelat, Ritus, J Jackson, Smorodinskii, I Yu Kobzarev, J Aharonov, C Misner, K Thorne, and J Wheeler, V B Berestetskii, E M Lifshitz and L P Pitaevskii, and A A Logunov.

[17] [283].
[18] [284].
[19] [1].
[20] Even prior to R Kronig, M Abraham [285] and A Compton [286] had come up with the idea of the electron spin.
[21] [2].

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