METHODOLOGICAL NOTES

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A derivation of the Lorentz transformation based on frequency standards

A A Denisov, E Sh Teplitsky

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Contents

Abstract. The physical meaning of the Lorentz transformations is explained by deriving them (and the corresponding pseudo-Euclidean geometry) as transformations on a set of physical observables (or 'information space') obtained by observers in inertial reference frames equipped with frequency (i.e., unified time – length) standards.

1. Introduction

In the more than century that has passed since the first publication of Einstein's article on the special relativity theory (SRT) [1], defined by Einstein as a fundamental theory, i.e., a theory based on a minimum set of postulates, not one experimental fact has been discovered that contradicts this theory. There are only two such postulates (as Einstein formulated them): (1) the equations of electrodynamics are the same (their form is invariant) for all inertial reference frames, and (2) the speed of light c is constant ($c = \text{const}$) irrespective of the observer's motion (and/or of the motion of the light source). The postulates seem to contradict each other. However, according to Einstein, "the principle of the constancy of the speed of light and the principle of relativity contradict each other only as long as the postulate of absolute time, i.e., the absolute meaning of simultaneity, is retained. If, however, we allow for the relativity of time, the two principles remain compatible; it is in this case that they produce a theory which we call the theory of relativity" [2]. In this assertion, time, or more accurately ct, whose dimension is that of a spatial coordinate, must exhibit the same properties as the other coordinates, including the dependence on the velocity of the reference frame, if the condition $c = \text{const}$ is satisfied.

Even today the question of the physical meaning of special relativity, based on Lorentz transformations, has not lost its importance. The reason for this has been explained by L Brillouin: "While Einstein's conclusions are correct, Lorentz transformations constitute a mathematical tool and cannot be observed; they are very useful, but clearly have no physical meaning" [3]. At present, it is assumed convenient to consider the second postulate as consisting of two assertions: (a) the speed of light is constant for an arbitrary direction of propagation in real physical (i.e., isotropic) space, and (b) the speed of light is independent of the velocity of the radiation source [4] (broader interpretations of this postulate, including the case of the possible anisotropic propagation of light, have also been discussed [5]).

Constructing the SRT presupposes the presence of a physically substantiated derivation of a complete set of transformations of spatial coordinates and time that reflects the transition from a reference frame at rest to another reference frame moving uniformly and rectilinearly with respect to the first. Lorentz was the first to derive such transformations (hence the name Lorentz transformation for the full set of such transformations), and later Larmor and Poincaré (e.g., see Ref. [6]) performed a 'constructive' derivation based on models of the properties of light propagation (in 'world ether,' in particular) and on certain aspects in the behavior of rulers and clocks in motion, selected by Fitzgerald such that the results of Michelson's experiments could be 'explained' (the contraction of the dimensions of physical bodies in their motion).

However, the derivation of these transformations done by Einstein, who did not resort to such hypotheses, and later its variants suggested by other physicists (e.g., see Refs [7 – 9]) are not considered sufficiently complete to serve as a basis for the system of transformations of coordinates and time, since they contain an auxiliary coefficient, determining which requires using additional ideas, such as the unimodular nature of such transformations or the correspondence

A A Denisov State Institute for Standardization, Metrology and Certification of the Scientific and Industrial Association ISARI, ul. Chargali 67, 380092 Tbilisi, Georgia
Tel. (995-32) 613 090
E Sh Teplitsky Physics Department, Dzhavakhishvili Tbilisi State University,
prosp. Chavchavadze 1, 380028 Tbilisi, Georgia
Tel. (995-32) 230 856

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between the relative scales of length standards in both reference frames. The uniqueness of the Lorentz transformations (the Lorentz group) was achieved only in the SRT formulated by H Minkowski (see Ref. [6]) as a theory of space – time with a pseudo-Euclidean structure, which is in fact another form of the postulates of special relativity. Actually, this approach is an implementation of F Klein's Erlangen program, according to which the geometry of a given space is determined by the group of transformations acting on it.

The relativity theory emerged in mechanics (Galileo) and used the concepts and methods of mechanics (rulers and clocks). The modern relativity theory (Einstein) focused on electrodynamics, but operated with the same, albeit not characteristic, objects (rulers and clocks), which leads to certain difficulties in interpreting the results (paradoxes). Since the time of Einstein, metrology has moved from the old metric units to new ones, based on quantum standards of frequency, which have become universal for all areas of physics and are now commonly used. Understandably, today there is no need to use the concepts from the past to explain modern physics. To put it bluntly, wave frequency and the Doppler shift are no longer corollaries of the adopted theory — they are observables that can be used in the original definition of the Lorentz transformation and of its physical meaning, including the need to introduce a pseudo-Euclidean space – time with which the observer has to deal.

The present article is an attempt to build the Lorentz transformations on the basis of Einstein's postulates: the constancy of the speed of light irrespective of the state of motion of the sources of radiation and observers in inertial reference frames, and the principle of relativity in its original form, according to which all electromagnetic process are described in inertial reference frames in the same way. This form presupposes that equations and the phenomena they describe are compared only for one area of physics (only for electrodynamics, only for mechanics, etc.) and that the ways they are compared must be based on the means of analysis in these areas of physics: rules and clocks for mechanics, electromagnetic (optical) standards for electromagnetic phenomena, etc. Within such an approach, it is assumed that the types of standards of measures and means of observation are different in different areas of physics, but are determined by the type of observation or information carriers, while the Galilean and Lorentz transformations are means of recalculating the information received by one observer from another (or his or her sensing devices) or even without the explicit participation of the moving observer. Here, the initial physical processes in each inertial reference frame are detected by the comoving observer without distortions of the observed quantities and phenomena. Relativity theory (or Minkowski geometry) as a general principle of physics is a postulate, which, being confirmed in one area of physics, may require broadening and altering in other areas of science containing mechanics and electrodynamics as constituents, but in terms appropriate to the standards and concepts of this area.

In what follows, we limit ourselves to the two-dimensional (x, t) world, because the choice of an inertial reference frame presupposes the selection of a certain single direction of motion in space, which is chosen as the x axis, while space proper is assumed to be isotropic.

2. Conditions of propagation of electromagnetic information between two moving reference frames

The postulate of the constancy of the speed of light (or electromagnetic radiation) presupposes the constancy of the speed with which information propagates in a vacuum, i.e., the propagation speed of an amplitude-modulated light wave with its wave packet envelope being the carrier of information propagating with the group velocity u_{g}:

$$u_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} \leqslant c\,, \qquad \omega = 2\pi\nu = \frac{2\pi}{T}\,, \tag{1}$$

where ω is the cyclic frequency, ν is the frequency, T is the period of electromagnetic oscillations, and k is the wave number.

The phase velocity u_{p} of propagation of a point on the constant-phase surface $\omega t - kx = \mathrm{const}$,

$$u_{\mathrm{p}} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k}\,, \tag{2}$$

may exceed the speed of light in general (e.g., see Ref. [10]), but only if the condition

$$u_{\mathrm{g}} u_{\mathrm{p}} = c^2$$

is satisfied.

However, u_{p} is not registered by the observer. For empty space (a 'vacuum'),

$$u_{\mathrm{g}} = u_{\mathrm{p}} = c\,. \tag{3}$$

At present, there are highly stable frequency standards (in the radio-frequency region and in the optical range), i.e., devices operating on the basis of laws of electrodynamics and quantum physics. Hence, their use as standards of time T_0 and wavelength $\lambda_0 = c/\nu_0$ in each inertial reference frame resolves the question of comparing different clocks and rulers (i.e., devices used in mechanics) as they move uniformly and reduces the length 'contraction' and time 'dilation' problems to a single one, which is the problem of how one observer comprehends the information content of the interrelation and transformation of this information content when one reference frame moves with respect to another reference frame. The possibility of 'contradictions' of the two SRT postulates may emerge only if this interrelation is ignored.

We assume that in the reference frame K at rest, the length x and the time t are measured in the standard units λ_0 and T_0,

$$x = n_x \lambda_0\,, \qquad t = n_t T_0\,, \tag{4}$$

while the moving observer, whose reference frame is K', measures the quantities

$$x' = n'_x \lambda'_0\,, \qquad t' = n'_t T'_0\,, \tag{5}$$

which remain independent as long as there is no exchange of information between the two observers. The first postulate of the SRT requires that

$$\lambda_0 = \lambda'_0\,, \qquad T_0 = T'_0\,, \tag{6}$$

from the standpoint of both observers. This means that the processes occurring in equivalent electromagnetic devices (quantum standards of frequency) run in exactly the same way, and the radiation parameters of these devices remain independent and do not react to the uniform motion of their reference frames. Thus, what may depend on the relative velocity u of the frames K and K' are numerical quantities measured by one observer from the signals sent by the other, moving, observer (a manifestation of the Doppler effect). In this way, an observer has two sets of observations that he compares: one set originating in the observer's reference frame and the other received by the observer in the form of signals ('distorted by the relative motion') emitted by the source of radiation in the second (moving) reference frame (below, we call it the method of single-path radiolocation, or passive location).

We suppose that the observer in the reference frame K discovers that a signal originating in the reference frame K' in the standard units λ_0 and T_0 has the parameters

$$\overline{n_x} = k_x(u) n_x\,, \qquad \overline{n_t} = k_t(u) n_t \tag{7}$$

(the first postulate is used here). But according to the second postulate,

$$c = \frac{n_x \lambda_0}{n_t T_0} = \frac{\overline{n_x}\,\lambda_0}{\overline{n_t}\,T_0} = \text{const}\,, \tag{8}$$

i.e., the temporal measure of the signal $k_t(u)$ used to match (7) and (8) must vary with u in the same way as the spatial measure $k_x(u)$ does. However, knowing the above relations is not enough to determine $k(u)$.

The two postulates do not contradict each other if a relation more general than (7) holds:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a_1(u) & a_2(u) \\ b_1(u) & b_2(u) \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \tag{9}$$

or

$$\begin{aligned} x' &= a_1(u)x + a_2(u)\,t\,, \\ t' &= b_1(u)x + b_2(u)\,t\,, \end{aligned} \tag{10}$$

with

$$a_1(0) = b_2(0) = 1\,, \qquad a_2(0) = b_1(0) = 0\,.$$

According to the postulate of the constancy of the speed of light, for the reference frames K and K', we have (for any value of u)

$$\frac{x'}{t'} = \frac{x}{t} = c\,,$$

i.e.,

$$c = c\,\frac{a_1(u)}{b_2(u)}\,\frac{1 + a_2(u)/\big(a_1(u)c\big)}{1 + \big(b_1(u)/b_2(u)\big)c}\,. \tag{11}$$

This condition leads to two variants of possible relations between the parameters of transformation (10):

$$\text{I.} \quad a_1(u) = b_2(u)\,, \qquad a_2(u) = b_1(u)\,c^2\,; \tag{12}$$

$$\text{II.} \quad \frac{a_1(u)}{1 + \big(b_1(u)/b_2(u)\big)c} = \frac{b_2(u)}{1 + a_2(u)/\big(a_1(u)c\big)}\,. \tag{13}$$

Clearly, variant I is a particular case of variant II and, as is shown in Section 3, corresponds to the Lorentz transformations. The more general variant II leads to a broader type of transformations, known as projective transformations, which were often used in the early days of the SRT but were classified by Wolfgang Pauli as being of little interest to physics [11]. Today, we see a revival of interest in these transformations.

In what follows, we examine only the first variant and derive the Lorentz transformations.

3. The single-path Doppler effect

We first examine the direct exchange of information between observers in the reference frames K and K'. It appears that the most rational way to exchange information between the observers is to transmit a frequency of oscillations whose period T determines a time interval and wavelength λ a spatial interval. We equip each reference frame with a frequency standard, a transmitter of signals of the frequency standard, and a receiver with a frequency meter, which measures the frequency, period, and wavelength of the signal received from the other reference frame.

We equip the reference frames K and K' with an orthogonal Cartesian coordinate system. From the origin of the ordinate (vertical) axis, as the unit 'vector,' we plot a spatial segment whose length is equal to the wavelength λ_1 of a wave with the standard frequency ν_1, for which the 'temporal' unit vector is the period T_1 of the same wave of standard frequency, i.e., the time taken by the front of the plane light (electromagnetic) wave to travel the distance $\lambda_1 = cT_1$ along the ordinate axis. Measurements of the time and frequency along the ordinate axis are then reduced to measurement of the path $\lambda = cT$ traveled by the light signal. The second 'label' on the ordinate axis corresponds to the wavelength $\lambda_2 = cT_2 = 2\lambda_1$, which corresponds to the period $T_2 = 2T_1$ and the frequency $\nu_2 = \nu_1/2$. Accordingly, for the third 'label,' we take $\lambda_3 = cT_3 = 3\lambda_1$, $T_3 = 3T_1$, $\nu_3 = \nu_1/3$, and so on. The ordinate axis $\lambda = cT$ defines the world line of an observer, on which each observer labels the value of the wavelength λ' corresponding to the received frequency ν' (which differs from the standard frequency ν_1 emitted from the other reference frame because of the Doppler effect). The detected values of the frequency ν', wavelength λ', and period T' are then compared with the respective standards ν_1, λ_1, and T_1. The values of spatial displacements are marked on the spatial abscissa axis on the same scale equal to λ_1. The straight line 0L in Fig. 1 is the limit corresponding to the relative motion of a reference frame at the speed of light and is at the angle of 45° to the coordinate axes. All relative motions of the reference frames K and K' with velocities u below the speed of light form a set of straight lines whose slopes α in relation to the observer's world line depend on the value of the relative velocity u.

We suppose that the observers whose reference frames move with a relative velocity u away from each other measure the frequency (and, hence, the period and wavelength) of the received signal (taking the Doppler effect into account) of the frequency standard transmitted by another reference frame. They then obtain a new frequency value (period and wavelength). We mark the resulting wavelength $\lambda' = cT'$ on the observer's world axis and label it C. The question is at what value of the relative velocity u the observer measures this particular value of the wavelength.

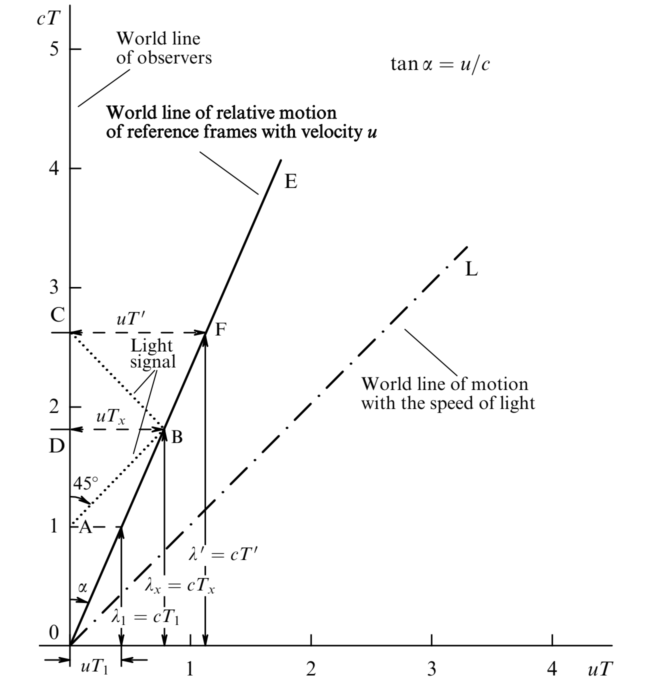

Figure 1. The single-path Doppler effect.

To answer this question, we must establish how the standard frequency (period and wavelength) depends on the relative velocity u of the reference frame K and K'. For this, in Fig. 1, we connect the point C ($\lambda' = cT'$) and the point A (corresponding to the value standard wavelength $\lambda_1 = cT_1$) by light rays. The intersection of the light rays is at a point B. From the vertex B of the right angle, we drop the straight line BD (the altitude) on the observer's world line $\lambda = cT$. Then $\mathrm{BD} = \mathrm{DA} = \mathrm{DC}$, and

$$\mathrm{0D} = \frac{\mathrm{0C} + \mathrm{0A}}{2} = \lambda_x = cT_x\,.$$

In such a construction, $\lambda_x = cT_x$ is the average wavelength between the standard wavelength λ_1 and the detected wavelength λ'. If the average wavelength is equal to the wavelength of the frequency standard, $\lambda_x = \lambda_1$, the relative velocity of the reference frames K and K' is zero. When the reference frames move away from each other, $\lambda_x > \lambda_1$; when they move toward each other, $\lambda_x < \lambda_1$. The straight line 0E connecting the origin with the point B determines the world line of the motion of the reference frames with the relative velocity u and its slope α in relation to the observer's world line for the value of point B ($\lambda_x = cT_x$).

Figure 1 shows that

$$\frac{\mathrm{DB}}{\mathrm{0D}} = \tan\alpha = \frac{u}{c}\,, \tag{14}$$

$$\begin{aligned} \mathrm{0C} &= \mathrm{0D} + \mathrm{DC} = cT_x + uT_x = cT'\,, \\ \mathrm{0A} &= \mathrm{0D} - \mathrm{DA} = cT_x - uT_x = cT_1\,. \end{aligned} \tag{15}$$

The ratio 0C/0A is the Doppler factor

$$k_{\mathrm{D}}(u) = \frac{\lambda'}{\lambda_1} = \frac{T'}{T_1} = \frac{1 + u/c}{1 - u/c}\,, \tag{16}$$

or

$$\frac{u}{c} = \frac{k_{\mathrm{D}}(u) - 1}{k_{\mathrm{D}}(u) + 1}\,. \tag{17}$$

This result shows that observers equipped with the quantum frequency standard and a frequency meter that measures the frequency received from another observer (with a different reference frame) are able to determine the speed (and direction) of his or her relative motion.

We suppose that the observer in the reference frame K' wants to establish the type of transformation of the spatial coordinates and time of the light signals he receives and the way in which scales (measurable quantities) of the known standards of length and time vary (he takes Einstein's two postulates into account). Combining (10) and (12), he obtains

$$\begin{aligned} x' &= a_1(u)x + b_1(u)\,c^2 t\,, \\ t' &= b_1(u)x + a_1(u)t\,. \end{aligned} \tag{18}$$

The other observer, in the reference frame K', knows that if the signals were to originate in her reference frame while the receiver is in the reference frame K, the following relations would hold:

$$\begin{aligned} x &= a_1(-u)x' + b_1(-u)\,c^2 t'\,, \\ t &= b_1(-u)x' + a_1(-u)\,t'\,. \end{aligned} \tag{19}$$

Substituting (19) in (18), we obtain a system of equations for the coefficients of the transformation between coordinates and time:

$$\begin{aligned} & a_1(u)\,a_1(-u) + b_1(u)\,b_1(-u)\,c^2 = 1\,, \\ & a_1(-u)\,b_1(u) + b_1(-u)\,a_1(u) = 0\,. \end{aligned} \tag{20}$$

The solution of this system can be written in two ways:

$$\text{(i)}\qquad b_1(u) = \frac{u}{c^2}\,a_1(u)\,, \qquad a_1(u) = \frac{1}{1 - u/c}\,, \tag{21}$$

i.e.,

$$x' = \frac{x + ut}{1 - u/c}\,, \qquad t' = \frac{t + ux/c^2}{1 - u/c}\,. \tag{22}$$

These relations can be defined as single-path Doppler transformations (in the sense of a frequency transformation) or single-path Lorentz transformations, in terms of spatial and temporal transformations of lengths and times in the transition from one inertial reference frame to another when the reference frames move with respect to each other with the relative velocity u. Relations (22) can also be written in the symmetric form

$$x' = x\,\frac{1 + u/c}{1 - u/c} = k_{\mathrm{D}}(u)x\,, \qquad t' = t\,\frac{1 + u/c}{1 - u/c} = k_{\mathrm{D}}(u)t\,, \tag{23}$$

and we thus obtain Eqns (16), which were derived on the basis of geometric ideas.

$$\text{(ii)}\qquad b_1(u) = \frac{u}{c^2}\,a_1(u)\,, \qquad a_1(u) = \frac{1}{\sqrt{1 - u^2/c^2}}\,. \tag{24}$$

In this case,

$$x' = \frac{x + ut}{\sqrt{1 - u^2/c^2}} = x\sqrt{\frac{1 + u/c}{1 - u/c}}\,,$$
$$t' = \frac{t + ux/c^2}{\sqrt{1 - u^2/c^2}} = t\sqrt{\frac{1 + u/c}{1 - u/c}}\,, \tag{25}$$

i.e.,

$$k_D(u) = \sqrt{\frac{1 + u/c}{1 - u/c}}\,. \tag{26}$$

Thus, the single-path variant of the Lorentz transformation is still ambiguous. Moreover, this approach does not solve the main problem: which transformations, (22) or (25), should the observer (e.g., the one in K) use to check that his equations remain the same in the other reference frame K'?

4. The Lorentz transformation (the double-path Doppler effect)

By its very physical meaning, the factor $k_D(u)$ determines the single-path Doppler effect, which forms the basis for passive location. The method allows determining not only the change in the frequency that is received (and recorded) by an observer in another reference frame when the two are in relative motion but also (at $c = \text{const}$) the changes in the observed values of the wavelength and period:

$$\nu' = \frac{\nu}{k_D(u)}\,, \qquad T' = k_D(u)T\,, \qquad \lambda' = k_D(u)\lambda\,.$$

If the reference frames continue to move apart (with the same relative velocity u) but emit the received frequency ν' into the vacuum, the value of the new frequency detected by the observer is changed by the factor $k_D(u)$:

$$\nu'' = \frac{\nu'}{k_D(u)} = \frac{\nu}{k_D^2(u)}\,, \qquad T'' = k_D^2(u)T\,, \qquad \lambda'' = k_D^2(u)\lambda\,. \tag{27}$$

We illustrate this with Fig. 2 using the constructions made in Fig. 1. From the value of the detected wavelength λ' (point C), we draw a light ray to its intersection point P with the world line of relative motion with the velocity u. The ray, refracted at point P through the angle of 90°, returns to the observer's world line $\lambda = cT$ and intersects it at a point R, which determines the value of the new wavelength, $\lambda'' = cT''$, corresponding to the detected ν'' on the observer's world line. But the single-path Doppler effect is the primary effect for the observer. Hence, we must connect the points R (the obtained value $\lambda'' = \nu T''$) and A (corresponding to the value $\lambda_1 = cT_1$ of the standard's wavelength) by light rays. The extensions of the light rays from the points B and P determine the point M, and the straight line 0N drawn from the origin to M is the new world line of motion with a relative velocity w.

According to (27),

$$k_D(w) = k_D(u)\,k_D(u)\,,$$

where w is the relative velocity of motion of the reference frame, which the observer determines (calculates) according to the rules of the single-path Doppler effect, the primary

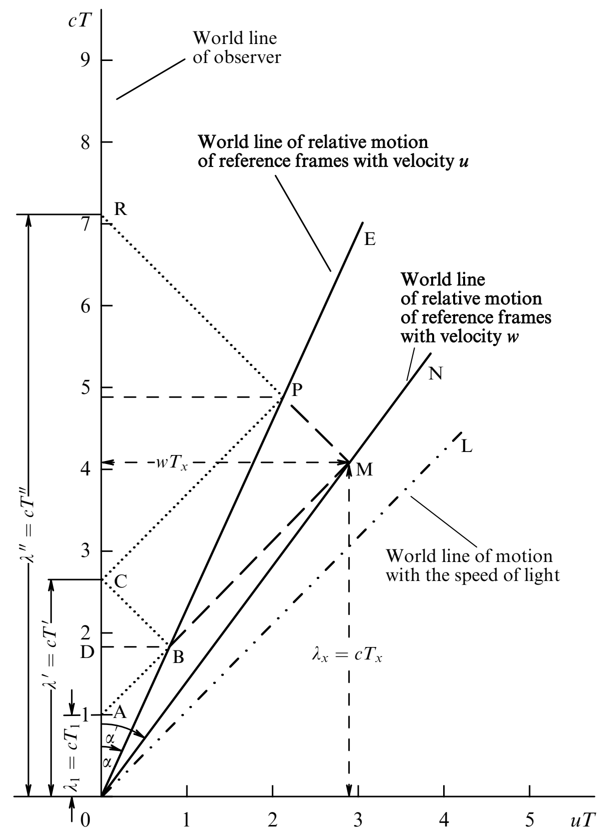

Figure 2. The double-path Doppler effect.

effect for him. This yields [variant (23), or (i)]

$$k_D(w) = \frac{1 + w/c}{1 - w/c} = \left(\frac{1 + u/c}{1 - u/c}\right)^2\,, \tag{28}$$

whence

$$\frac{w}{c} = \frac{2u/c}{1 + u^2/c^2}\,, \tag{29}$$

where w/c is the relative velocity of the reference frames K and K' expressed in fractions of the speed of light from the standpoint of the observer using the double-path Doppler effect.

Comparing (28) with the transformations of coordinates and time for the single-path Doppler effect in (25), in the case of the 'double-path transformation,' we obtain the expression

$$\frac{x'}{x} = \frac{1 + u/c}{1 - u/c} = \sqrt{\frac{(1 + w/c)^2}{1 - w^2/c^2}}\,,$$

or

$$x' = \frac{x(1 + w/c)}{\sqrt{1 - w^2/c^2}} = \frac{x + wt}{\sqrt{1 - w^2/c^2}}\,, \tag{30a}$$

and

$$t' = \frac{t(1 + w/c)}{\sqrt{1 - w^2/c^2}}\,, \tag{30b}$$

which constitutes the complete set of Lorentz transformations from the standpoint of the observer that is able to measure the velocity w, which on the basis of the above derivation is involved in the Lorentz transformation, instead of the velocity u used in the single-path Doppler effect.

We now turn to variant (ii), for which we have Eqn (26). In this case, instead of (28), we have

$$\sqrt{\frac{1+w/c}{1-w/c}}=\frac{1+u/c}{1-u/c}\,, \tag{31}$$

i.e., w and u are related by expression (29) that emerged in variant (i), which means that the ambiguity in selecting a variant has been resolved.

Thus, the use of the double-path Doppler effect allows fully reproducing the Lorentz transformations and clarifying their physical and mathematical meaning.

Only these transformations form a group (the identity and inverse transformations included) and attest to the interrelation between the temporal and spatial variables and the geometry of space – time defined by them.

If the inertial reference frame K' has no observer equipped with a transponder (radar), the process of the double-path Doppler effect may be implemented by equipping the K' frame with a mirror that reflects the signal back to K.

The case with a mirror is an act of active radiolocation, excluding one of the observers.

We note that the Lorentz transformations emerge as a way of describing the measuring process of observing a moving object (using a reference frame, a physical process of the electromagnetic type, etc.) from the standpoint of only one observer rather than as a result of exchanging information between two or three observers. No condition of this type ever appeared in the case of Galilean transformations (more exactly, it was never formulated), because the velocity of physical bodies under investigation was always lower than the speed of light, while the transmission of information between reference frames by optical means did not obey the laws of mechanics and was not taken into account when the principle of relativity was formulated. Einstein's second postulate, which places a limit on the speed of light and states that this speed is independent of the speed of moving sources and detectors of radiation, at the same time places a limit on the selection and properties of one more parameter, time, which must be transformed by *almost* the same rules as spatial coordinates. Actually, the second SRT postulate 'adds' time (more exactly, ct) to the number of coordinates of the space of states of physical systems that allow motion in the entire range $0 < u < c$.

The 'almost' was added because the properties of time, and hence the geometry of the space of states, differs in some respect from those of spatial coordinates, as follows from the derivation of the rules for active radiolocation (the double-path Doppler effect), i.e., the Lorentz transformation, and from Fig. 2.

Indeed, the method of active radiolocation is based on the use of optical signals sent by an observer, reflected, and then received by the same observer, i.e., the isotropic nature of space is explicitly taken into account in the form

$$u \Leftrightarrow -u\,,$$

or, more exactly,

$$\frac{(x)}{(t)} \Leftrightarrow \frac{(-x)}{(t)}\,,$$

while the 'direction' of time remains the same and cannot be reflected in principle, although the equations of physics (linear equations at least) allow such an operation. In mathematics, this situation is described by the term orientation: oriented surfaces, oriented curves, etc.; this is clearly reflected by the properties of the geometries associated with such objects. For instance, isotropic spaces defined solely by spatial coordinates (the Galilean – Newtonian world) are Euclidean, while space – time

$$(\mathbf{x}, ct)\,,$$

in which both components have the same dimensions and are linked by a general transformation law when one of them is oriented, is a non-Euclidean space and has the properties of the Lobachevski space [12]. In particular, introducing the variables

$$\cosh\varphi=\frac{1}{\sqrt{1-u^2/c^2}}\,,\qquad \sinh\varphi=\frac{u/c}{\sqrt{1-u^2/c^2}}\,, \qquad \cosh^2\varphi-\sinh^2\varphi=1\,, \tag{32}$$

we can write the Lorentz transformations as

$$\begin{aligned} x' &= x\cosh\varphi + ct\sinh\varphi\,,\\ \tanh\varphi &= \frac{u}{c}\,,\\ ct' &= ct\cosh\varphi + x\sinh\varphi\,, \end{aligned} \tag{33}$$

for which the Minkowski (Lobachevski) interval is an invariant:

$$(x')^2-(ct')^2=(x)^2-(ct)^2=\text{inv}\,.$$

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