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The interaction energy of electric multipoles in a plane and a point-multipole approximation for the electric field of conductors

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Contents

Abstract. A complex representation of the electric potential is applied to derive relationships describing the interaction energy of electric multipoles in a plane, which prove to be highly useful for variational calculations of the electric field in a system of parallel circular conductors.

1. Introduction

Formulation of two-dimensional electrostatic problems is dictated by the need to determine the electric field depending on only two Cartesian coordinates. For example, this is the case for a system of parallel, infinite, perfectly conducting and dielectric cylinders with electric parameters dependent on only two Cartesian coordinates in a plane normal to the elements of the cylinders. In this case, the areas on the plane produced by intersections between the plane and the cylinders correspond to the cylinders themselves, which are the models of real physical bodies. The physical parameters of these areas are characteristic of the materials of the cylinders and are invariable along the cylinders. Hereinafter, we will consider cylindrical conductors whereto conducting areas in a plane (i.e., planar conductors) correspond.

The concept of multipoles is associated with the decomposition of the electric potential of a spatially bounded system of charges into components at distances far exceeding the size of the system [1 – 3]. In this case, the applied expansion in a Taylor series in the Cartesian coordinate system is not quite convenient, since it contains many essentially similar terms that should be collected by moving to an expansion into spherical functions in a spherical coordinate system. The problem of collecting similar terms also emerges as the electric potential is decomposed in a planar Cartesian coordinate system. It can be solved by expanding the complex potential of a finite system of charges in a Laurent series [4].

In approximating an electric field [5], the relationships determining the interaction energy of the approximating systems of charge are important, in particular, expressions for the interaction energy of point multipoles. The electrodynamics literature presents only expressions for the lower-order interaction energy of point multipoles [1 – 3]. We will obtain below formulas for the arbitrary-order interaction energy of multipoles in a plane; on this basis, we will construct variational schemes approximating the electric field of a system of parallel circular conductors.

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2. Multipolar expansions in two-dimensional electrostatics

For two-dimensional electrostatics problems, it is convenient in many cases to fall back on the concept of complex multipoles that naturally appear in the expansion of the complex potential $\Pi_1(z)$ of a charge system $\mathrm{d}\lambda^{(1)}(z)$ in powers of the complex variable z [6]:

$$\Pi_1(z) = -\frac{1}{2\pi\varepsilon_0}\int \ln\left(\frac{z-\tilde{z}}{R}\right)\mathrm{d}\lambda^{(1)}(\tilde{z})\,.$$

If $|\tilde{z}-z_1| < |z-z_1|$, then

$$\ln\left(\frac{z-\tilde{z}}{R}\right) = \ln\left(\frac{z-z_1}{R}\right) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\tilde{z}-z_1}{z-z_1}\right)^n,$$

and the following expansion applies outside a certain circle, centered at the point z_1, in which all charges are located:

$$\Pi_1(z) = -\frac{\lambda_0^{(1)}}{2\pi\varepsilon_0}\ln\left(\frac{z-z_1}{R}\right) + \frac{1}{2\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{\lambda_n^{(1)}}{n(z-z_1)^n}\,, \quad (1)$$

where

$$\lambda_n^{(1)} = \int (z-z_1)^n \,\mathrm{d}\lambda^{(1)}(z)\,. \quad (2)$$

We will describe the quantity $\lambda_n = \lambda_{n\mathrm{r}} + \mathrm{i}\lambda_{n\mathrm{i}}$ as the nth-order complex multipole moment of the charge system $\mathrm{d}\lambda^{(1)}(z)$, bounded on the plane, with respect to the point z_1. In particular, the quantity $\lambda_0^{(1)}$ determines the total charge of the system; $\mathrm{Re}\,\lambda_1^{(1)} = \lambda_{1\mathrm{r}}^{(1)}$ is the projection of the dipole moment of the charge system $\mathrm{d}\lambda^{(1)}(z)$ with respect to the point z_1 onto the x-axis, and $\mathrm{Im}\,\lambda_1^{(1)} = \lambda_{1\mathrm{i}}^{(1)}$ is the projection of the dipole moment of the charge system $\mathrm{d}\lambda^{(1)}(z)$ with respect to the point z_1 onto the y-axis.

Now assume that two systems of charges are located on a plane inside nonintersecting circles centered at the points z_1 and z_2. The interaction energy of these systems of charges can be represented, using the Green reciprocity formulas [2], as

$$W_{12} = \mathrm{Re}\int \Pi_2(z)\,\mathrm{d}\lambda^{(1)}(z) = \mathrm{Re}\int \Pi_1(z)\,\mathrm{d}\lambda^{(2)}(z)\,,$$

where $\Pi_2(z)$ is the complex potential of the second system of charges. The potential $\Pi_1(z)$ is an analytical function within a certain circle centered at the point z_2 and bounding the system of charges $\mathrm{d}\lambda^{(2)}(z)$; therefore, it can be written as the Taylor series

$$\Pi_1(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\,\partial_z^k\Pi_1(z)\Big|_{z=z_2}(z-z_2)^k.$$

The substitution of this expansion of $\Pi_1(z)$ into the expression for the interaction energy W_{12} yields

$$W_{12} = \mathrm{Re}\sum_{k=0}^{\infty}\frac{1}{k!}\,\partial_z^k\,\Pi_1(z)\Big|_{z=z_2}\lambda_k^{(2)}\,. \quad (3)$$

In view of representation (1), we rewrite this relationship as

$$W_{12} = \mathrm{Re}\Bigg(-\frac{\lambda_0^{(1)}\lambda_0^{(2)}}{2\pi\varepsilon_0}\ln\left(\frac{z_2-z_1}{R}\right) + \sum_{k=1}^{\infty}\frac{1}{2\pi\varepsilon_0 k}\frac{(-1)^k\lambda_0^{(1)}\lambda_k^{(2)}}{(z_2-z_1)^k} + \sum_{n=1}^{\infty}\frac{1}{2\pi\varepsilon_0 n}\frac{\lambda_n^{(1)}\lambda_0^{(2)}}{(z_2-z_1)^n} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(n+k-1)!}{2\pi\varepsilon_0 n!k!}\frac{(-1)^k\lambda_n^{(1)}\lambda_k^{(2)}}{(z_2-z_1)^{n+k}}\Bigg)\,. \quad (4)$$

The quantity

$$W_{00}(z_1,z_2) = -\mathrm{Re}\frac{\lambda_0^{(1)}\lambda_0^{(2)}}{2\pi\varepsilon_0}\ln\left(\frac{z_2-z_1}{R}\right)$$

can be interpreted as the interaction energy of two point charges $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$ located at the points z_1 and z_2, respectively, while the quantity

$$W_{nk}(z_1,z_2) = \mathrm{Re}\frac{(n+k-1)!}{2\pi\varepsilon_0 n!k!}\frac{(-1)^k\lambda_n^{(1)}\lambda_k^{(2)}}{(z_2-z_1)^{n+k}} \quad (5)$$

can be regarded as the interaction energy of the two point multipoles $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ located at the points z_1 and z_2.

It is interesting that the force exerted by the first system of charges on the second one can be found by differentiating the complex quantity W_{12} with respect to z_1:

$$F_x - \mathrm{i}F_y = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k)!}{2\pi\varepsilon_0 n!k!}\frac{(-1)^k\lambda_n^{(1)}\lambda_k^{(2)}}{(z_2-z_1)^{n+k+1}}\,. \quad (6)$$

An individual term in this sum can be treated as the force exerted by the first multipole on the second one.

If the second system of charges is rotated as a whole through the angle α, the multipole moments

$$\tilde{\lambda}_k^{(2)} = \exp(\mathrm{i}k\alpha)\,\lambda_k^{(2)}\,,$$

along with the energy defined in formula (4), will change according to definition (2). We determine the derivative of the energy with respect to α for an angle of $\alpha = 0$ and change its sign to obtain the moment of forces exerted by the first system of charges on the second one:

$$M = \mathrm{Im}\Bigg(\sum_{k=1}^{\infty}\frac{1}{2\pi\varepsilon_0}\frac{(-1)^k\lambda_0^{(1)}\lambda_k^{(2)}}{(z_2-z_1)^k} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(n+k-1)!}{2\pi\varepsilon_0 n!(k-1)!}\frac{(-1)^k\lambda_n^{(1)}\lambda_k^{(2)}}{(z_2-z_1)^{n+k}}\Bigg)\,. \quad (7)$$

3. Approximating the electric fields of circular conductors by the fields of their characteristic multipoles

Let us consider a system of conducting nonintersecting circles $|z-z_p| < a_p$ in a complex plane. Kazantsev [4] has shown that any region in the complex plane possesses its specific system of characteristic multipoles — basis charge distributions over the boundary of the region. In particular, for the pth circle [4] we find

$$\sigma_0^{(p)}(z) = \frac{1}{2\pi a_p}\,; \quad \sigma_{k\mathrm{r}}^{(p)}(z) = \frac{\cos k\theta_p}{\pi a_p^{k+1}}\,;$$
$$\sigma_{k\mathrm{i}}^{(p)}(z) = \frac{\sin k\theta_p}{\pi a_p^{k+1}}\,; \quad \theta_p = \arg(z-z_p)\,. \quad (8)$$

Under the approximation in which the fields of the circles are replaced with the fields of a finite number of their characteristic multipoles, we can choose

$$\sigma^{(p)}(z) = \lambda_0^{(p)}\sigma_0^{(p)}(z) + \sum_{k=1}^{N_p}\left[\lambda_{k\mathrm{r}}^{(p)}\sigma_{k\mathrm{r}}^{(p)}(z) + \lambda_{k\mathrm{i}}^{(p)}\sigma_{k\mathrm{i}}^{(p)}(z)\right]. \quad (9)$$

Then, the proper energy of the charges of the pth circle can be found using the relationship [4]

$$W_{pp} = \frac{\lambda_0^{(p)\,2}}{4\pi\varepsilon_0}\ln\frac{R}{a_p} + \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N_p}\frac{1}{ka_p^{2k}}(\lambda_{kr}^{(p)\,2} + \lambda_{ki}^{(p)\,2})\,. \quad (10)$$

We calculate the interaction energy between the charges on the pth and qth circles using formula (4) with the subscripts 1 and 2 substituted by p and q, and with the summation over n and k restricted to N_p and N_q numbers. We also obtain the charge energy of all circles as the sum of the corresponding

energies:

$$W = \sum_{p=1}^{P} W_{pp} + \sum_{p=1}^{P} \sum_{q>p}^{P} W_{pq}\,, \tag{11}$$

where P is the total number of circles.

The solution to the fundamental electrostatics problem for a system of circular conductors in an external electric field can be obtained based on the Thomson variational principle [7], according to which the true charge distributions over the circle boundaries correspond to the minimum electrostatic energy. If the term corresponding to the constant energy of the charges that produce the external field is excluded from this energy, the procedure of solving the problem will require the minimization of the energy functional

$$L = W + W_{\text{int}}, \tag{12}$$

where W_{int} is the interaction energy between the approximating charges and the external field. If we denote the complex potential of the external field as $\Pi(z)$, we can write down that

$$W_{\text{int}} = \operatorname{Re} \sum_{p=1}^{P} \sum_{k=0}^{N_p} \frac{1}{k!} \lambda_k^{(p)} \partial_z^k \Pi(z) \Big|_{z=z_p}. \tag{13}$$

The energy functional L can be minimized with respect to $\lambda_k^{(p)}$ under various conditions depending on the formulation of the problem in question. In particular, if we seek the capacitance coefficients of a system of parallel circular conductors, we should assume

$$\operatorname{Re} \Pi(z) \Big|_{|z-z_p| \leqslant a_p} = U_p\,, \tag{14}$$

where the real U_p are the constant potentials of the conductors; in this case, the energy functional should be minimized over all multipole moments.

To solve the problem of the dipolar polarizabilities of a system of circles, we should set

$$\Pi(z) \Big|_{|z-z_p| \leqslant a_p} = E_p^* z\,, \tag{15}$$

where the constant complex quantities E_p^* specify the uniform electric fields $E_p^* = E_{px} - \mathrm{i}E_{py}$ in which individual circular conductors are located. The energy functional should be minimized for $\lambda_k^{(p)}$ under the condition that the circles are neutral: $\lambda_0^{(p)} = 0$.

In the problem of the system of uncharged circular conductors in an external electric field, we have to assume that $\lambda_0^{(p)} = 0$ and carry out minimization for $\lambda_k^{(p)}$ multipoles of orders higher than zero. Systems of linear equations in multipole moments of the circles will appear in all these problems, and the minimized values L_{m} of the energy functional will correspond to the solutions of these problems. The quality of the approximation can be judged by the convergence of these quantities. Let us illustrate our variational procedure of solving the fundamental electrostatics problem for a system of circular conductors by some particular examples.

4. A circular conductor in an external electric field

It is quite natural to consider first, as a very simple example, the problem of a conducting uncharged circle $|z| < a$ in an electric field with a complex potential $\Pi(z)$. In this case, the energy functional (12) assumes the form

$$L^{(N)} = \sum_{k=1}^{N} \left\{ \frac{1}{4\pi\varepsilon_0 k a^{2k}} \lambda_k \lambda_k^* + \frac{1}{2k!} \left[\lambda_k \partial_z^k \Pi(z) \Big|_{z=0} + \lambda_k^* \left(\partial_z^k \Pi(z) \big|_{z=0} \right)^* \right] \right\}. \tag{16}$$

The minimization of $L^{(N)}$ yields

$$\lambda_k = -\frac{2\pi\varepsilon_0 a_p^{2k}}{(k-1)!} \left(\partial_z^k \Pi(z) \big|_{z=0} \right)^*. \tag{17}$$

We denote the sequence of approximating potentials of the charges in the circle as $\tilde{\Pi}^{(N)}(z)$. Based on expansion (1), we may write

$$\tilde{\Pi}^{(N)}(z) = -\sum_{k=1}^{N} \frac{a_p^{2k}}{k! z^k} \left(\partial_z^k \Pi(z) \big|_{z=0} \right)^*. \tag{18}$$

The exact solution of the problem can be obtained in the limit $N \to \infty$ as

$$\tilde{\Pi}(z) = -\sum_{k=1}^{\infty} \frac{a_p^{2k}}{k! z^k} \left(\partial_z^k \Pi(z) \big|_{z=0} \right)^* = \left[\Pi(0) - \Pi\left(\frac{a^2}{z^*} \right) \right]^*. \tag{19}$$

The second equality here follows from the power-series expansion of the complex potential of the external field in the circle $|z| < a$. This relationship is very useful in solving the particular problems, since $\Pi(z)$ is an arbitrary analytical function.

For example, let a neutral conducting circle $|z| < a$ be in an external field of a point multipole Λ_n located at the point $\tilde{z}$ ($|\tilde{z}| > a$). Using relationship (19), we then obtain

$$\Pi(z) = \frac{1}{2\pi\varepsilon_0 n} \frac{\Lambda_n}{(z-\tilde{z})^n}; \qquad \tilde{\Pi}(z) = \frac{(-1)^n}{2\pi\varepsilon_0 n} \frac{\Lambda_n^*}{\tilde{z}^{*n}} \left(1 - \frac{z^n}{(z - a^2/\tilde{z}^*)^n} \right). \tag{20}$$

Similarly, if the circle is found in an external electric field of a point charge, then

$$\Pi(z) = -\frac{\Lambda_0}{2\pi\varepsilon_0} \ln\left(\frac{z-\tilde{z}}{R} \right); \quad \tilde{\Pi}(z) = \frac{\Lambda_0}{2\pi\varepsilon_0} \ln\left(\frac{a^2 - \tilde{z}^* z}{\tilde{z}^* z} \right). \tag{21}$$

We note that the problem of a charged circle in an external electric field no longer raises any difficulties; thus, we move to a more sophisticated example.

5. The problem of two identical circular charged conductors

Let two conducting circles of the same radius a, centered at the points $z_1 = l/2$ and $z_2 = -l/2$ of the x-axis, have equal charges $\lambda_0^{(1)} = \lambda_0^{(2)} = \lambda_0^{(+)}/2$. Such conditions correspond to the problem of the capacitance of a conductor formed by connected circles; its solution gives one of the basis potentials $\Pi^{(+)}(z)$. The second basis potential $\Pi^{(-)}(z)$ can be obtained by solving the problem of the capacitance of the system of two circles as capacitor plates. In this case, we should set $\lambda_0^{(1)} = -\lambda_0^{(2)} = \lambda_0^{(-)}$. The general solution to the problem of two similar conducting circles that carry the charges $\lambda_0^{(1)}$ and

$\lambda_0^{(2)}$ is found as the superposition of the basis potentials:

$$\Pi(z) = (\lambda_0^{(1)} + \lambda_0^{(2)})\,\Pi^{(+)}(z) + \frac{1}{2}(\lambda_0^{(1)} - \lambda_0^{(2)})\,\Pi^{(-)}(z)\,.$$

As for the capacitance of two circles forming a capacitor, the solution to the problem has long been known [6, 8] and can be obtained using relationship (21); it can be concluded from its form that $\Pi^{(-)}(z)$ is a superposition of the potentials of point charges equal to 1 and -1, and located at conjugate points with respect to the inversion transformation for both circles at once [6], namely

$$\Pi^{(-)}(z) = -\frac{1}{2\pi\varepsilon_0}\ln\left(\frac{z - l/2 + b}{z + l/2 - b}\right);$$
$$b = \frac{l}{2} - \sqrt{\left(\frac{l}{2}\right)^2 - a^2}.$$

Then, the capacitance $C^{(-)}$ of the system in question will equal

$$C^{(-)} = \frac{\pi\varepsilon_0}{\ln(a/b)}\,.$$

To obtain the complete solution to the problem of two similar conducting charged circles, we should find $\Pi^{(+)}(z)$; to this end, we apply the above-described variational approach based on the Thomson principle. We approximate $\Pi^{(+)}(z)$ by a superposition of the potentials of characteristic multipoles (8) of the circles. In formula (9), taking into account the symmetry of the problem, we choose $N_1 = N_2 = N$, $\lambda_0^{(1)} = \lambda_0^{(2)} = \lambda_0^{(+)}/2$, and

$$\lambda_{k\mathrm{r}}^{(1)} = (-1)^k\lambda_{k\mathrm{r}}^{(2)} = \lambda_0^{(+)}a^k\Lambda_k; \quad \lambda_{k\mathrm{i}}^{(1)} = \lambda_{k\mathrm{i}}^{(2)} = 0\,. \tag{22}$$

To determine Λ_k, we minimize the electrostatic energy (11) for these parameters, writing it down in the form

$$W = \frac{\lambda_0^2}{4\pi\varepsilon_0}(b + 2\mathbf{b}\cdot\mathbf{\Lambda} + \mathbf{\Lambda}\cdot\hat{B}\cdot\mathbf{\Lambda})\,, \tag{23}$$

where

$$b = \frac{1}{2}\ln\left(\frac{R^2}{al}\right), \quad b_m = \frac{(-1)^m}{m}\,\delta^m\,,$$
$$B_{mm} = \frac{2}{m} + 2\,\frac{(2m-1)!}{m!^2}\,\delta^{2m}\,,$$
$$B_{mn} = 2(-1)^{n+m}\,\frac{(n+m-1)!}{n!m!}\,\delta^{n+m}, \quad \delta = \frac{a}{l}\,. \tag{24}$$

At any $\mathbf{\Lambda}$, according to the Thomson variational principle, the quantity W will be larger than the true energy W_{tr} expressed in terms of the capacitance $C^{(+)}$ of the connected cylinders and the outer conformal radius A [7] of the exterior to the two circles as

$$W_{\mathrm{tr}} = \frac{\lambda_0^2}{2C^{(+)}} = \frac{\lambda_0^2}{4\pi\varepsilon_0}\ln\frac{R}{A}\,.$$

With the use of the inequality

$$W_{\mathrm{tr}} < \min W = \frac{\lambda_0^2}{4\pi\varepsilon_0}\,(b - \mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b})\,,$$

we obtain for the outer conformal radius the following estimate

$$A > \frac{a}{\sqrt{\delta}}\exp(\mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b})\,. \tag{25}$$

We will judge the accuracy to which the trial field approximates the true field by the proximity of the right-hand side of inequality (25) to the true value of A.

For a few lower-order approximations, the inequality (25) can be written down as follows:

$$A > A_0 = \frac{a}{\sqrt{\delta}}\,, \quad A > A_1 = \frac{a}{\sqrt{\delta}}\exp\left(\frac{\delta^2}{2(1+\delta^2)}\right), \tag{26}$$

$$A > A_2 = \frac{a}{\sqrt{\delta}}\exp\left(\frac{2\delta^2 + \delta^4 + 3\delta^6}{4(1+\delta^2+3\delta^4+\delta^6)}\right), \tag{27}$$

$$A > A_3 = \frac{a}{\sqrt{\delta}}\exp\left(\frac{\delta^2(11\delta^{10} + 12\delta^8 + 50\delta^6 + 11\delta^4 + 3\delta^2 + 6)}{12(\delta^{12} + 6\delta^{10} + 7\delta^8 + 11\delta^6 + 3\delta^4 + \delta^2 + 1)}\right). \tag{28}$$

The values of

$$\mathbf{\Lambda} = -\hat{B}^{-1}\cdot\mathbf{b}\,, \tag{29}$$

corresponding to these inequalities, are equal to

$$\mathbf{\Lambda}_0 = 0\,, \quad \mathbf{\Lambda}_1 = -\left(\frac{\delta}{2(1+\delta^2)}\right),$$
$$\mathbf{\Lambda}_2 = \frac{(\delta(1+2\delta^4);\ -\delta^2(1-\delta^2))}{2(\delta^6 + 3\delta^4 + \delta^2 + 1)}\,, \tag{30}$$
$$\mathbf{\Lambda}_3 = \frac{1}{2D(\delta)}\begin{pmatrix} \delta + 2\delta^5 + 9\delta^7 + 3\delta^{11} \\ -\delta^2(1 - \delta^2 + 6\delta^6 - 3\delta^8) \\ \delta^3(1 - 2\delta^2 - 3\delta^4 + \delta^6) \end{pmatrix},$$

$$D(\delta) = \delta^{12} + 6\delta^{10} + 7\delta^8 + 11\delta^6 + 3\delta^4 + \delta^2 + 1\,. \tag{31}$$

As one would expect, the analytical formulas determining $\mathbf{\Lambda}_N$ and $(\mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b})_N$ become more complex as the order of approximation is increased. Thus, in particular, they can be written down in one line at $N = 3$, whereas more space is needed for them at $N = 4$. For example, one finds

$$(\mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b})_4 = \frac{f_4(\delta)}{g_4(\delta)}\,,$$
$$f_4(\delta) = \delta^2(25\delta^{18} + 58\delta^{16} + 409\delta^{14} + 311\delta^{12} + 181\delta^{10} + 423\delta^8 + 103\delta^6 + 22\delta^4 + 6\delta^2 + 12)\,,$$
$$g_4(\delta) = 24(\delta^{20} + 10\delta^{18} + 25\delta^{16} + 63\delta^{14} + 56\delta^{12} + 37\delta^{10} + 42\delta^8 + 11\delta^6 + 3\delta^4 + \delta^2 + 1)\,.$$

Obviously, the accuracy of the approximations A_N and, accordingly, the accuracy to which the trial fields approximate the true electric-field distribution will grow with a decrease in δ. For this reason, we analyze the convergence of A_N for the least favorable case of $\delta = 0.5$, in which the circles are tangent to each other at the origin of coordinates. The substitution of $\delta = 0.5$ into formulas (26)–(31) yields

$$\frac{A_0}{a} = \sqrt{2} = 1.41421356\,,$$
$$\frac{A_1}{a} = \sqrt{2}\exp\left(\frac{1}{10}\right) = 1.56294770\,,$$
$$\frac{A_2}{a} = \sqrt{2}\exp\left(\frac{13}{124}\right) = 1.57052868\,,$$
$$\frac{A_3}{a} = \sqrt{2}\exp\left(\frac{2825}{26919}\right) = 1.57071312\,,$$

$$\frac{A_4}{a} = \sqrt{2}\exp\left(\frac{4827473}{45990744}\right) = 1.570728896\,,$$
$$\frac{A_5}{a} = \sqrt{2}\exp\left(\frac{27199086917}{259063704600}\right) = 1.57076623\,,$$
$$\frac{A_6}{a} = \sqrt{2}\exp\left(\frac{40833357818741}{388865710509000}\right) = 1.570791939\,.$$

A comparison of the obtained approximate values with the exact value

$$\frac{A}{a} = \frac{\pi}{2} = 1.57079633$$

indicates that the error in calculating the outer conformal radius based on the octupole approximation is

$$\Delta = \frac{A - A_3}{A} = 5.30 \times 10^{-5} = 0.00530\,\%\,,$$

i.e., it amounts to a few thousandths of a percent. The quantity

$$\sqrt{\Delta} = 0.73\,\%$$

can be regarded as the rms error in approximating the true field with the trial field. As the distance between the centers of the circles is increased (i.e., δ is decreased), these errors decrease.

If the obtained accuracy is sufficient for practical calculations, the approximating complex potential can be found using formula (1) with the substitutions $N_p = N = 3$ and

$$\mathbf{\Lambda}_3 = \frac{(120;\,-426;\,84)}{6729}\,.$$

If the mutual capacitance of the two conducting circles and the approximate value of

$$C_N^{(+)} = \frac{2\pi\varepsilon_0}{\ln(R/A_N)}$$

are known, we can arrive at a lower-bound estimate for the matrix of the capacitance coefficients for the system of two identical conducting circles:

$$\hat{C}_N^{(i)} = \begin{pmatrix} C^{(-)} + \frac{1}{4}\,C_N^{(+)} & \frac{1}{4}\,C_N^{(+)} - C^{(-)} \\ \frac{1}{4}\,C_N^{(+)} - C^{(-)} & C^{(-)} + \frac{1}{4}\,C_N^{(+)} \end{pmatrix}.$$

Thus, the variational procedure of solving the problem of the capacitance coefficients of two identical circular conductors yields very simple and accurate approximations which can be represented analytically to the point of tangency of the circles.

Let us consider now a particular case of the three-body problem as a more sophisticated example.

6. The problem of the outer conformal radius of three circles identical in size and relative positions

Assume that three circles of the same radius a are centered at the vertices of a regular triangle with its side equal to $l \geqslant 2a$ and with its center located at the origin of coordinates. Let the center of the first circle be at the semiaxis of $x > 0$; in this case,

$$z_1 = \frac{l}{\sqrt{3}}\,, \quad z_2 = \frac{l}{\sqrt{3}}\exp\left(\mathrm{i}\,\frac{2\pi}{3}\right), \quad z_3 = \frac{l}{\sqrt{3}}\exp\left(-\mathrm{i}\,\frac{2\pi}{3}\right).$$

Since the problem at hand is equivalent to the problem of the capacitance of three connected circular conductors, we approximate $\Pi^{(+)}(z)$ by a superposition of the potentials of the characteristic multipoles (8) of the circles. In formula (9), in view of the symmetry of the problem, we choose $N_1 = N_2 = N_3 = N$, $\lambda_0^{(1)} = \lambda_0^{(2)} = \lambda_0^{(2)} = \lambda_0^{(+)}/3$, and

$$\lambda_n^{(1)} = \lambda_{n\mathrm{r}}^{(1)} = \lambda_0 a^n \Lambda_n\,, \quad \lambda_n^{(2)} = \exp\left(\mathrm{i}\,\frac{2n\pi}{3}\right)\lambda_{n\mathrm{r}}^{(1)}\,,$$
$$\lambda_n^{(3)} = \exp\left(-\mathrm{i}\,\frac{2n\pi}{3}\right)\lambda_{n\mathrm{r}}^{(1)}\,. \tag{32}$$

Thus, the number of parameters that can be varied reduces by a factor of six compared to the problem in which there is no symmetry in the disposition of the circles.

The electric-field energy corresponding to this approximation can be written down as

$$W = \frac{\lambda_0^2}{4\pi\varepsilon_0}\left(b + 2\mathbf{b}\cdot\mathbf{\Lambda} + \mathbf{\Lambda}\cdot\hat{B}\cdot\mathbf{\Lambda}\right), \tag{33}$$

where, as follows from the computations based on formulas (4) and (10), one has

$$b = \frac{1}{3}\ln\frac{R^3}{al^2}\,, \quad b_n = (-1)^n\frac{2}{n}\cos\left(\frac{n\pi}{6}\right)\delta^n\,,$$
$$B_{nn} = 3\left(\frac{1}{n} + 2\frac{(2n-1)!}{n!^2}\delta^{2n}\right),$$
$$B_{mn} = 6\frac{(n+m-1)!}{m!n!}\,(-1)^{n+m}\cos\left((m-n)\frac{\pi}{6}\right)\delta^{n+m}. \tag{34}$$

At any $\mathbf{\Lambda}$, the quantity W will be greater than the true energy W_{tr} expressed in terms of the capacitance $C^{(+)}$ of the connected circles and the outer conformal radius A as

$$W_{\mathrm{tr}} = \frac{\lambda_0^2}{2C^{(+)}} = \frac{\lambda_0^2}{4\pi\varepsilon_0}\ln\frac{R}{A}\,.$$

According to the inequality

$$W_{\mathrm{tr}} < \min W = \frac{\lambda_0^2}{4\pi\varepsilon_0}\left(b - \mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b}\right),$$

we find for the outer conformal radius that

$$A > a\delta^{-2/3}\exp\left(\mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b}\right). \tag{35}$$

The accuracy to which the trial field approximates the true field can be judged by the proximity of the right-hand side of inequality (35) to the true value of A.

Next, we write down inequality (35) for a few lower-order approximations:

$$A > A_0 = a\delta^{-2/3}\,, \quad A > A_1 = a\delta^{-2/3}\exp\left(\frac{\delta^2}{1+2\delta^2}\right), \tag{36}$$

$$A > A_2 = a\delta^{-2/3}\exp\left(\frac{6\delta^2 + \delta^4 + 26\delta^6}{6(1 + 2\delta^2 + 6\delta^4 + 6\delta^6)}\right), \tag{37}$$

$$A > A_3 = a\delta^{-2/3}\exp\left(\frac{f_3(\delta)}{g_3(\delta)}\right), \tag{38}$$

$$f_3(\delta) = \delta^2(157\delta^{10} + 20\delta^8 + 120\delta^6 + 26\delta^4 + \delta^2 + 6),$$
$$g_3(\delta) = 6(30\delta^{12} + 48\delta^{10} + 37\delta^8 + 26\delta^6 + 6\delta^4 + 2\delta^2 + 1)\,.$$

The following values of

$$\mathbf{\Lambda} = -\hat{B}^{-1} \cdot \mathbf{b} \tag{39}$$

correspond to these inequalities:

$$\mathbf{\Lambda}_0 = 0\,, \quad \mathbf{\Lambda}_1 = \left(\frac{\sqrt{3}\delta}{3(1+2\delta^2)} \right),$$

$$\mathbf{\Lambda}_2 = \frac{\left(\sqrt{3}\delta(1+5\delta^4);\, -\delta^2(1-4\delta^2)\right)}{3(6\delta^6+6\delta^4+2\delta^2+1)}\,, \tag{40}$$

$$\mathbf{\Lambda}_3 = \frac{1}{3D(\delta)} \begin{pmatrix} \sqrt{3}\delta(1+5\delta^4+20\delta^6+34\delta^{10}) \\ -\delta^2(1-4\delta^2+20\delta^6-47\delta^8) \\ -3\sqrt{3}\delta^5(1+2\delta^2-3\delta^4) \end{pmatrix}, \tag{41}$$

$$D(\delta) = 30\delta^{12}+48\delta^{10}+37\delta^8+26\delta^6+6\delta^4+2\delta^2+1\,.$$

As could be expected, the analytical formulas determining $\mathbf{\Lambda}_N$ and $\left(\mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b}\right)_N$ become more complex as the order of approximation is increased. One more estimate for the outer conformal radius, corresponding to $N=4$, is given by

$$\left(\mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b}\right)_4 = \frac{f_4(\delta)}{g_4(\delta)}\,,$$

$$f_4(\delta) = \delta^2(2950\delta^{18}+728\delta^{16}+5677\delta^{14}+2986\delta^{12} + 480\delta^{10}+882\delta^8+241\delta^6+52\delta^4+2\delta^2+12)\,,$$

$$g_4(\delta) = 12(250\delta^{20}+560\delta^{18}+790\delta^{16}+820\delta^{14}+400\delta^{12} + 188\delta^{10}+107\delta^8+26\delta^6+6\delta^4+2\delta^2+1)\,.$$

Obviously, the smaller δ, the higher the accuracy of the approximations A_N and, accordingly, the accuracy to which the trial fields approximate the true field. For this reason, let us analyze the convergence of A_N for the least favorable case of $\delta = 0.5$ in which the circles are tangent at the origin of coordinates. The substitution of $\delta = 0.5$ into formulas (36)–(41) yields

$$\frac{A_0}{a} = 2^{2/3} = 1.58740105\,, \quad \frac{A_1}{a} = 2^{2/3}\exp\left(\frac{1}{6}\right) = 1.87529276,$$

$$\frac{A_2}{a} = \frac{A_1}{a}\,, \quad \frac{A_3}{a} = 2^{2/3}\exp\left(\frac{3407}{20316}\right) = 1.87723219\,,$$

$$\frac{A_4}{a} = 2^{2/3}\exp\left(\frac{1071625}{6385716}\right) = 1.87744924\,,$$

$$\frac{A_5}{a} = 2^{2/3}\exp\left(\frac{6616067471}{39423215880}\right) = 1.87745987\,.$$

A comparison of the obtained values of the outer radius with the subsequent ones

$$\frac{A_6}{a} = 1.87749291\,, \quad \frac{A_7}{a} = 1.87749902\,,$$

$$\frac{A_8}{a} = 1.87749913\,, \quad \frac{A_9}{a} = 1.87750048$$

shows that a reasonable estimate of the error in the octupole-approximation-based value of the outer conformal radius is

$$\Delta = \frac{A_9 - A_3}{A_9} = 1.43\times10^{-4} = 0.0143\,\%\,,$$

amounting to about one hundredth of a percent. The quantity

$$\sqrt{\Delta} = 1.19\,\%$$

can be regarded as the rms error in the approximation of the true field by the trial field. These errors will be smaller for larger distances between the centers of the circles (for smaller δ).

If the achieved accuracy is sufficient for practical calculations, the approximating complex potential can be found from formula (1) with the substitutions of $N=3$ and $\mathbf{\Lambda}_3$ from formula (41). If necessary, for a given accuracy of approximation, we can obviously find an N value corresponding to this accuracy.

7. Conclusion

To conclude, we call attention to the fact that it is the employment of a complex representation of electrostatic relationships that enabled us to compactly formulate the concept of point multipoles in a plane and find their interaction energy. The natural combination of this approach with the variational principles made it possible to construct the complete solution to an electrostatics many-body problem. Calculations of the electric fields in the system of a large number of parallel circular wires, based on the constructed variational procedure, could be done only with computer.

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