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First-order relativistic effects in the electrodynamics of media moving with a nonuniform velocity

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Abstract. A systematic theory of first-order electrodynamic relativistic effects in media moving with a spatially nonuniform velocity is presented. A geometrical optics approach is developed and used to calculate the bending of rays and the change of polarization characteristics of radiation propagating through a continuous medium moving with a nonuniform velocity. Nonreciprocal (i.e., propagation direction-dependent) waveguides and lenses in media moving with a transversely nonuniform velocity are demonstrated. Radiation scattering (diffraction) by localized velocity nonuniformities is studied. Peculiarities of propagation of short-wave (X-ray) radiation in a moving medium are discussed, and some prospects for experimentation are reviewed.

1. Introduction and basic relationships

The electrodynamics of moving continuous media is an important part of the theory of relativity, the main postulates of which were formulated by A Einstein exactly a century ago [1]. It is worthwhile to note here that one of the most striking effects in this field — the partial light entrainment by a moving medium — had been intuitively predicted by Fresnel (in 1818) long before the theory of relativity was forwarded; the prediction was based on the model concept of ether and its condensation in the pores of a medium, which has been rejected by modern science [2].

Nevertheless, Fresnel's hypothesis was experimentally verified by Fizeau (in 1851) and deduced from the prerelativistic Lorentz electron theory corrected for frequency dispersion (1895) detected in Zeeman's experiments (1914) [2]. Only after that, Fresnel's result was consistently derived from the relativistic formula of velocity summation in the first approximation in the ratio of a moving medium's velocity to the speed of light in vacuum.

Apart from the Fresnel – Fizeau effect, much more attention within the theory of relativity was given to the problems of sharp interfaces between moving media. In the very first paper [1], Einstein considered light reflection from a mirror uniformly moving along the normal to its (flat) surface. This work was continued and led to the formulation of conditions for fields at moving interfaces and the derivation of the Minkowski constitutive equations [3].

An important contribution to the further development of the theory was the discovery of the Vavilov – Cherenkov effect (the emission of a charge traveling through a medium with a velocity exceeding the phase velocity of light in this medium [4, 5]) and of transient radiation effect (the emission of a charge intersecting an interface [6]). The growing interest in these problems in the 1950s was due to the development of acceleration equipment and plasma physics; investigations of that period were reviewed in Refs [7, 8].

Today, the electrodynamics of moving continuous media based on the Maxwell differential equations and the Minkowski constitutive equations is a self-sufficient theory expounded in a number of monographs and textbooks [9 – 11]. At the same time, it is not free from debatable questions, such as the forms of the energy – momentum tensor and the pondermotive force that is different in the Minkowski and Abraham approaches [9]. More important, however, is the fact that theoretical studies were largely confined to the

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spatially uniform velocity of medium motion, whereas the influence of the velocity nonuniformity on the propagation of electromagnetic radiation was outside the main stream of research [11 – 13].

In the meantime, taking into consideration velocity nonuniformity brings about a number of new important effects. For example, it is assumed in the standard description of the classical Fizeau experiment that the velocity of a fluid (water) moving through a tube is spatially uniform. However, the velocity of water laminar motion shows parabolic dependence on the distance from the tube axis; this observation implies that the phase velocity of light exhibits a similar dependence and the radiation wave fronts for the two said counter directions bent in an opposite manner. Light is focused or defocused depending on the direction of propagation; in other words, the nonuniformly moving water creates a nonreciprocal lens (with different signs of the focal distance for the opposite directions) [14].

Another example is a cylinder or a sphere that rotates about its symmetry axis in the bulk of the same transparent medium (having the same refractive index). We are interested in the characteristics of incident electromagnetic radiation scattered by a rotating body [15]. In this case, the velocity of motion is apparent 'in the pure form' because, in the absence of rotation, the entire space is uniform and no scattering takes place. In other words, relativistic effects lead to a new scattering mechanism.

In this paper, we expound a systematic electrodynamic theory of media moving with a spatially nonuniform velocity; we confine our consideration to the first-order effects in the ratio of the moving medium velocity v to the speed of light in vacuum c. This theory describes a number of nontrivial effects, whereas the reported experiments are few. Although these effects are rather small, they are quite possible to detect and even employ by virtue of modern high-precision optical experimental techniques, including lasers [16, 17].

The basic equations used throughout this paper (except in Section 6) are the Maxwell differential equations for the strengths $\mathbf{E}$ and $\mathbf{H}$ of electric and magnetic fields, respectively, as well as for electric and magnetic inductions $\mathbf{D}$ and $\mathbf{B}$ of the medium:

$$\begin{aligned} &\operatorname{div}\mathbf{B} = 0\,, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\,, \\ &\operatorname{div}\mathbf{D} = 0\,, \qquad \operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\,, \end{aligned} \tag{1.1}$$

and the Minkowski constitutive equations written in the first order in the parameter v/c:

$$\begin{aligned} &\mathbf{D} = \varepsilon\mathbf{E} + \frac{\varepsilon\mu - 1}{c}\,[\mathbf{vH}]\,, \\ &\mathbf{B} = \mu\mathbf{H} + \frac{\varepsilon\mu - 1}{c}\,[\mathbf{Ev}]\,. \end{aligned} \tag{1.2}$$

Here, t is the time; ε and μ are the permittivity and permeability of a continuous medium, respectively, and $\mathbf{v}$ is its velocity. Medium dispersion is considered to be negligibly small. To be more precise, Eqns (1.2) should be written down as

$$\begin{aligned} &\mathbf{D} = \varepsilon_0\mathbf{E} + \frac{\varepsilon_0\,\mu_0 - 1}{c}\,[\mathbf{vH}] + \delta\mathbf{D}\,, \\ &\mathbf{B} = \mu_0\mathbf{H} + \frac{\varepsilon_0\,\mu_0 - 1}{c}\,[\mathbf{Ev}] + \delta\mathbf{B}\,, \end{aligned} \tag{1.3}$$

where ε_0 and μ_0 are the respective permittivity and permeability of the homogeneous medium at rest (at $\mathbf{v} = 0$).

Quantities $\delta\mathbf{D}$ and $\delta\mathbf{B}$ describe dynamo-optical and gyromagnetic phenomena, respectively, and also other minor perturbations such as ordinary scattering by medium nonuniformities. For example, in solids with small mechanical deformations caused by their motions with a nonuniform velocity, components of the vector $\delta\mathbf{D}$ have the form [11]

$$\delta D_i = a_{iklm}\,u_{lm}E_k\,. \tag{1.4}$$

Here, a_{iklm} is the fourth-rank tensor associated with elastooptical constants, u_{ik} is the strain tensor, E_k are the components of vector $\mathbf{E}$, indices run from 1 to 2, 3 corresponding to the Cartesian coordinates x, y, z, respectively, and the repeating indices imply summation. The relationship for $\delta\mathbf{B}$ and the corrections for scattering by medium nonuniformities are written down in analogy with formula (1.4) [11].

The solution of the problem being considered is markedly facilitated by the smallness of the parameter v/c. To begin with (no matter how surprising it may seem), the smallness of v/c makes it possible to get rid of parasitic nonrelativistic effects. Indeed, when radiation propagation is considered, say, in a rotating solid dielectric, the rotation causes mechanical stress that in turn leads to a change in its optical properties (photoelastic effects). However, these effects are quadratic in the velocity of motion — that is, they change the effective refractive index of the medium by $\sim (v/v_s)^2$, where v_s is the speed of sound in the medium [11, 18]. On the other hand, the magnitude of relativistic effects is of the first order in velocity ($\sim v/c$, where c is the speed of light).

Thus, the relativistic effects of interest may be regarded as the dominant ones if the velocity of motion of the medium is small [15]:

$$v < v_0 = \frac{v_s^2}{c}\,.$$

To be more precise, velocity v in this relationships should be replaced by its step.

Methodically, the smallness of the parameter v/c makes it possible to obviate the technically difficult problem of accounting for the continuity conditions at the medium interface (see also Ref. [8]). By way of demonstrating this approach, we present in Section 2 the analysis of the influence of dielectric rotational motion on the static electric and magnetic fields. Thereafter, the propagation of electromagnetic waves is discussed. Section 3 deals with the case of high-frequency fields, opposite to the static one; specifically, the basic equations of geometrical optics of moving media are derived, which permits us to consistently calculate the shifts and bending of light rays and polarization changes caused by the nonuniform motion of a given medium along relatively short trajectories (shorter than the diffraction length).

In Section 4, the above nonreciprocal lens and waveguide effects are analyzed in the same approximation and also with the use of the wave approach. Radiation diffraction (scattering) on velocity distribution nonuniformities is considered in Section 5. Propagation of X-ray radiation in moving media is beyond the scope of electrodynamics of continuous media because the macroscopic description of the medium for such short wavelengths is not justified; the relevant analysis is presented in Section 6.

Finally, the concluding Section 7 summarizes the results reviewed, compares them with the propagation of sound

waves in moving media, and discusses prospects for the arrangement of further experiments.

2. Electro- and magnetostatic fields in moving dielectrics

Although the electrodynamics of moving media, based on Eqns (1.1) and (1.2), permits, in principle, solving a large variety of problems, the continuity equations at interfaces between two media are difficult to use. For this reason, such interfaces are assumed in almost all approximations to be either flat, cylindrical, or spherical [7, 8, 11, 13]. This difficulty can be obviated by taking into account that the continuity conditions at medium interfaces are not independent of the Maxwell equations themselves; just the opposite, they are corollaries from these equations. They may be made unnecessary by introducing a smooth change between characteristics of the media and thereafter reducing the width of the transition region to zero.

This methodical section is designed to demonstrate the possibility of introducing an integral approach (without applying the continuity conditions at interfaces) for the evaluation of the strengths of static electric and magnetic fields in the case of a relatively arbitrary velocity distribution of motion of a homogeneous medium. More precisely, it deals with the case of stationary (time-independent) velocity distribution of a moving medium that arises, for example, when a part of a body having an axisymmetric shape rotates about the axis of symmetry. In this case, the electric and/or magnetic field remains static in a moving medium, too.

Permittivity and permeability of the moving and stationary parts of the medium are assumed to be identical, and the analysis refers to the effects of the first order of smallness in the ratio of the moving medium velocity to the speed of light in vacuum. We shall demonstrate that, in the specific case of a rotating sphere for which the solution can also be found in a traditional way (using the continuity conditions at the interface between the moving and the stationary parts of the medium [11]), the results of these two approaches agree with each other [19].

Maxwell 'differential' equations (1.1) for static fields have the form

$$\operatorname{div}\mathbf{B} = 0\,, \qquad \operatorname{rot}\mathbf{E} = 0\,, \qquad \operatorname{div}\mathbf{D} = 0\,, \qquad \operatorname{rot}\mathbf{H} = 0\,. \tag{2.1}$$

Permittivity (ε) and permeability (μ) are assumed to be constant, as appropriate for a medium at rest. With such an approach, relativistic effects manifest themselves in the purest form. Let us consider the motion of a medium with a coordinate-dependent velocity in a uniform static electric field of strength $\mathbf{E}_0 = \text{const}$.

In the zeroth order of the perturbation theory with respect to v/c, an electric field coincides with $\mathbf{E}_0$, and magnetic field with $\mathbf{H}_0 = 0$. In the first order in v/c, the constitutive relationships (1.2) take the form

$$\mathbf{D}_1 = \varepsilon\mathbf{E}_1\,, \qquad \mathbf{B}_1 = \mu\mathbf{H}_1 + \frac{\varepsilon\mu - 1}{c}\,[\mathbf{E}_0\mathbf{v}]\,. \tag{2.2}$$

Then, it follows from the first Maxwell equation (2.1) that

$$\operatorname{div}\mathbf{H}_1 = -\frac{\varepsilon\mu - 1}{\mu c}\,\operatorname{div}[\mathbf{E}_0\mathbf{v}]\,. \tag{2.3}$$

It appears from the last equation (2.1) that the magnetic field is irrotational and possesses the potential:

$$\mathbf{H}_1 = -\operatorname{grad}\Psi\,. \tag{2.4}$$

Taking into account Eqn (2.3) and the uniformity of field $\mathbf{E}_0$, the potential assumes the form

$$\begin{aligned}\Psi(\mathbf{r}) &= -\frac{1}{4\pi}\int\frac{\operatorname{div}\mathbf{H}_1(\mathbf{r}')}{R}\,\mathrm{d}\mathbf{r}' \\ &= \frac{\varepsilon\mu - 1}{4\pi\mu c}\int\frac{\operatorname{div}\left[\mathbf{E}_0\,\mathbf{v}(\mathbf{r}')\right]}{R}\,\mathrm{d}\mathbf{r}' \\ &= -\frac{\varepsilon\mu - 1}{4\pi\mu c}\int\frac{\mathbf{E}_0\operatorname{rot}\mathbf{v}(\mathbf{r}')}{R}\,\mathrm{d}\mathbf{r}'\,,\end{aligned} \tag{2.5}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between point $\mathbf{r}$ at which the field is calculated and the vector $\mathbf{r}'$ of integration variable.

Relationship (2.5) presents the general solution of the problem being considered for an arbitrary velocity distribution of moving medium. It may be applied to the case of rotation of a sphere with radius a and angular velocity $\mathbf{\Omega}$ directed along the z-axis, i.e., $\mathbf{\Omega} = (0, 0, \Omega)$. If the external electric field is assumed to be also parallel to the z-axis: $\mathbf{E}_0 = (0, 0, E_0)$, then the magnetic potential outside the sphere is equal to

$$\Psi = -\frac{4}{15}\,\frac{\varepsilon\mu - 1}{\mu c}\,E_0\Omega\,\frac{a^5}{r^3}\left(1 - 3\,\frac{z^2}{r^2}\right). \tag{2.6}$$

Here, r is the distance to the center of the sphere. Similarly, for a dielectric sphere rotating in a static magnetic field of strength $\mathbf{H}_0$ directed along the axis of rotation, the electric potential Φ outside the sphere has the form

$$\Phi = \frac{4}{15}\,\frac{\varepsilon\mu - 1}{\varepsilon c}\,H_0\Omega\,\frac{a^5}{r^3}\left(1 - 3\,\frac{z^2}{r^2}\right). \tag{2.7}$$

In can be shown that the same result is obtainable by the traditional approach based on matching solutions at the interface between the rotating sphere and the stationary medium. To this effect, it is necessary to generalize the solution of the problem posed in Ref. [11] where the authors considered the rotation of a dielectric sphere in a vacuum. (The problem of the surrounding medium with an arbitrary permittivity cannot be reduced to this case because the velocity-dependent terms in the Minkowski constitutive equations (1.2) disappear for vacuum). The result obtained confirms the validity of the integral approach. At the same time, the integral approach and the general relationship (2.5) make it possible to solve a broader circle of problems, e.g., for a rotating finite cylinder [15].

3. Geometrical optics of moving media

3.1 Basic relationships and an eikonal equation

Let us consider the distribution of monochromatic radiation with frequency ω in a medium possessing a smooth (compared to the radiation wavelength) nonuniformity of the velocity of motion [20]. This motion is assumed to be stationary, and its velocity is independent of time at each point of the space. Then, as is easy to see from the general relationships (1.1) and (1.2), the radiation remains mono-

chromatic even after it passes through the moving medium (no Doppler frequency shifts are apparent). We are interested in the trajectories of light rays and the changes in polarization of light associated with the motion of the medium.

In the case of small velocities $\mathbf{v}$ of medium motion and with the utilization of complex field representation [with factor $\exp(-\mathrm{i}\omega t)$ being omitted], the Maxwell equations (1.1) are substituted by

$$\operatorname{rot}\mathbf{E} = \mathrm{i}\,\frac{\omega}{c}\,\mathbf{B}\,, \qquad \operatorname{rot}\mathbf{H} = -\mathrm{i}\,\frac{\omega}{c}\,\mathbf{D}\,. \tag{3.1}$$

Whence, the following wave equations for a nonuniformly moving medium can be deduced:

$$\begin{aligned}
&\nabla^2\mathbf{E} + k_0^2\varepsilon\mu\mathbf{E} + [\operatorname{grad}\ln\mu\times\operatorname{rot}\mathbf{E}] + \operatorname{grad}(\operatorname{grad}\ln\varepsilon\cdot\mathbf{E})\\
&\quad = -\frac{\varepsilon\mu-1}{c}\Big\{k_0^2\mu[\mathbf{vH}] + \mathrm{i}k_0\operatorname{rot}[\mathbf{Ev}]\\
&\quad + \frac{1}{\varepsilon}\operatorname{grad}\operatorname{div}[\mathbf{vH}] - \mathrm{i}k_0\big[\operatorname{grad}\ln\mu\times[\mathbf{Ev}]\big]\Big\}\\
&\quad - \mathrm{i}k_0\frac{2n}{c}\big[\operatorname{grad}n\times[\mathbf{Ev}]\big] - \operatorname{div}[\mathbf{vH}]\operatorname{grad}\frac{\varepsilon\mu-1}{c\varepsilon}\\
&\quad - \frac{2}{c}\operatorname{grad}\left(\frac{n}{\varepsilon}\operatorname{grad}n\,[\mathbf{vH}]\right),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\nabla^2\mathbf{H} + k_0^2\varepsilon\mu\mathbf{H} + [\operatorname{grad}\ln\varepsilon\times\operatorname{rot}\mathbf{H}] + \operatorname{grad}(\operatorname{grad}\ln\mu\cdot\mathbf{H})\\
&\quad = -\frac{\varepsilon\mu-1}{c}\Big\{k_0^2\varepsilon[\mathbf{Ev}] - \mathrm{i}k_0\operatorname{rot}[\mathbf{vH}]\\
&\quad + \frac{1}{\mu}\operatorname{grad}\operatorname{div}[\mathbf{Ev}] + \mathrm{i}k_0\big[\operatorname{grad}\ln\varepsilon\times[\mathbf{Ev}]\big]\Big\}\\
&\quad - \mathrm{i}k_0\frac{2n}{c}\big[\operatorname{grad}n\times[\mathbf{vH}]\big] - \operatorname{rot}[\mathbf{Ev}]\operatorname{grad}\frac{\varepsilon\mu-1}{c\mu}\\
&\quad - \frac{2}{c}\operatorname{grad}\left(\frac{n}{\mu}\operatorname{grad}n\cdot[\mathbf{Ev}]\right),
\end{aligned} \tag{3.3}$$

where $k_0 = \omega/c$ is the wave number in a vacuum, and $n = (\varepsilon\mu)^{1/2}$ is the medium refractive index. It is worth noting that the right-hand sides of these equations contain a small multiplier v/c.

In the geometrical optics approximation, the local field structure is the same as for a plane wave propagating in a homogeneous medium. It is important to note that even in a medium moving with a constant velocity $\mathbf{v}$, the directions of the wave vector $\mathbf{k}$ and the time-averaged energy flux (Poynting vector)

$$\mathbf{S} = \frac{c}{8\pi}\operatorname{Re}[\mathbf{EH}^*] \tag{3.4}$$

are different (as in an anisotropic medium, because the motion of the medium distinguishes a certain direction in the space [9]).

Indeed, in the case under consideration, it follows from Eqns (3.1) and (1.2) for a plane wave of the form

$$\mathbf{E} = \mathbf{e}_0\exp(\mathrm{i}\mathbf{kr})\,, \qquad \mathbf{H} = \mathbf{h}_0\exp(\mathrm{i}\mathbf{kr})$$

that

$$[\mathbf{qe}_0] = \frac{\omega}{c}\,\mu\mathbf{h}_0\,, \qquad [\mathbf{qh}_0] = -\frac{\omega}{c}\,\varepsilon\mathbf{e}_0\,, \tag{3.5}$$

where

$$\mathbf{q} = \mathbf{k} + \frac{\omega}{c}\,\frac{\varepsilon\mu-1}{c}\,\mathbf{v}\,. \tag{3.6}$$

It is readily seen that after the substitution of $\mathbf{q}$ by $\mathbf{k}$ equations (3.5) coincide with the equations for a plane wave propagating in a stationary medium. Therefore, the Poynting vector in a moving medium is directed along vector $\mathbf{q}$ (nonnormalized ray vector).

Let us consider now a medium with a smooth spatial change in the motion velocity $\mathbf{v} = \mathbf{v}(\mathbf{r})$. The solution is sought after in the form [11, 21]

$$\begin{aligned}
\mathbf{E} &= \mathbf{e}(\mathbf{r})\exp\big(\mathrm{i}k_0\,L(\mathbf{r})\big)\,,\\
\mathbf{H} &= \mathbf{h}(\mathbf{r})\exp\big(\mathrm{i}k_0\,L(\mathbf{r})\big)\,,
\end{aligned} \tag{3.7}$$

where the wave number k_0 is regarded as a large parameter of the asymptotic theory, while vectors $\mathbf{e}(\mathbf{r})$ and $\mathbf{h}(\mathbf{r})$ are complex in the general case.

By substituting expressions (3.7) into Eqns (3.1) and (1.2) we arrive, in the first approximation in v/c and in the main order with respect to the small parameter k_0^{-1}, at the following:

$$\begin{aligned}
\mathbf{e} &= -\frac{1}{\varepsilon}\left([\operatorname{grad}L\,\mathbf{h}] + \frac{\varepsilon\mu-1}{c}\,[\mathbf{vh}]\right),\\
\mathbf{h} &= -\frac{1}{\mu}\left([\mathbf{e}\operatorname{grad}L] + \frac{\varepsilon\mu-1}{c}\,[\mathbf{ev}]\right),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathbf{e}\operatorname{grad}L &= -\frac{\varepsilon\mu-1}{c}\,\mathbf{ev}\,,\\
\mathbf{h}\operatorname{grad}L &= -\frac{\varepsilon\mu-1}{c}\,\mathbf{hv}\,.
\end{aligned} \tag{3.9}$$

It follows from Eqn (3.8) that

$$n^2 - (\operatorname{grad}L)^2 - 2\,\frac{\varepsilon\mu-1}{c}\,\mathbf{v}\operatorname{grad}L = 0\,. \tag{3.10}$$

This is the eikonal equation for the case of a slowly moving medium. (At $\mathbf{v} = 0$, it transforms to the standard eikonal equation). It can also be written in the form

$$\left(\operatorname{grad}L + \frac{\varepsilon\mu-1}{c}\,\mathbf{v}\right)^2 = n^2\,. \tag{3.11}$$

The arbitrary distribution of the velocity $\mathbf{v}$ can be decomposed into irrotational (index 1) and solenoidal (without sources, index 2) to obtain

$$\begin{aligned}
&\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2\,, \qquad \operatorname{rot}\mathbf{v}_1 = 0\,, \qquad \mathbf{v}_1 = \operatorname{grad}U\,,\\
&\operatorname{div}\mathbf{v}_2 = 0\,, \qquad \mathbf{v}_2 = \operatorname{rot}\mathbf{W}\,,\\
&U = -\frac{1}{4\pi}\int\frac{\operatorname{div}\mathbf{v}}{r}\,\mathrm{d}V\,, \qquad \mathbf{W} = \frac{1}{4\pi}\int\frac{\operatorname{rot}\mathbf{v}}{r}\,\mathrm{d}V\,,
\end{aligned} \tag{3.12}$$

or

$$\mathbf{v}_1 = -\frac{1}{4\pi}\operatorname{grad}\int\frac{\operatorname{div}\mathbf{v}}{r}\,\mathrm{d}V\,, \qquad \mathbf{v}_2 = -\frac{1}{4\pi}\operatorname{rot}\int\frac{\operatorname{rot}\mathbf{v}}{r}\,\mathrm{d}V\,, \tag{3.13}$$

where r is the distance from the integration volume element $\mathrm{d}V$ to the point of observation.

For the motion with velocity $\mathbf{v}_1$ (potential flow, vorticity $\operatorname{rot}\mathbf{v}_1 = 0$), the eikonal equation acquires the form

$$(\operatorname{grad} M)^2 = n^2\,, \qquad M = L + \frac{\varepsilon\mu - 1}{c}\,U\,. \tag{3.14}$$

The eikonal equation for M is applicable to a medium at rest. In the case of a homogeneous medium, equation (3.14) has the solution in the form of a flat wave front. Knowing M, it is possible to find L from formula (3.14), but wave fronts in a nonuniformly moving medium will be bent.

The second case (movement with velocity $\mathbf{v}_2$) is characteristic of an incompressible fluid. The eikonal equation acquires the form

$$\left(\operatorname{grad} M + \frac{\varepsilon\mu - 1}{c}\,\mathbf{v}_2\right)^2 = n^2\,, \qquad \operatorname{div}\mathbf{v}_2 = 0\,. \tag{3.15}$$

3.2 Energy relations and ray trajectories

It follows from Eqns (3.7)–(3.9) that the time-averaged Poynting vector (3.4) has the form

$$\mathbf{S} = \frac{c}{8\pi\mu}\,\mathbf{e}\mathbf{e}^*\left(\operatorname{grad} L + \frac{\varepsilon\mu - 1}{c}\,\mathbf{v}\right). \tag{3.16}$$

Let us introduce time-averaged electric and magnetic energy densities, as well as the total energy density [9, 21]:

$$\begin{aligned} \langle w_{\mathrm{e}}\rangle &= \frac{1}{16\pi}\,\mathbf{e}\mathbf{d}^*\,, \\ \langle w_{\mathrm{m}}\rangle &= \frac{1}{16\pi}\,\mathbf{h}\mathbf{b}^*\,, \\ \langle w\rangle &= \langle w_{\mathrm{e}}\rangle + \langle w_{\mathrm{m}}\rangle\,, \end{aligned} \tag{3.17}$$

where $\mathbf{d}$ and $\mathbf{b}$ are the amplitudes of electric and magnetic induction, respectively. It can be shown that

$$\begin{aligned} \langle w_{\mathrm{e}}\rangle &= -\frac{1}{16\pi}\,\mathbf{e}\,[\operatorname{grad} L\,\mathbf{h}^*] = \frac{1}{16\pi}\,[\mathbf{e}\mathbf{h}^*]\operatorname{grad} L\,, \\ \langle w_{\mathrm{m}}\rangle &= \frac{1}{16\pi}\,\mathbf{h}^*\,[\operatorname{grad} L\,\mathbf{e}] = \frac{1}{16\pi}\,[\mathbf{e}\mathbf{h}^*]\operatorname{grad} L\,, \end{aligned} \tag{3.18}$$

$$\langle\mathbf{S}\rangle = \mathbf{s}c_{\mathrm{p}}\langle w\rangle\left(\mathbf{s}\,\frac{\operatorname{grad} L}{n}\right)^{-1}. \tag{3.19}$$

Here, $c_{\mathrm{p}} = c/n$ is the phase velocity of light in a stationary medium, and $\mathbf{s}$ is the unit ray vector directed along the Poynting vector:

$$\mathbf{s} = \frac{1}{n}\left(\operatorname{grad} L + \frac{\varepsilon\mu - 1}{c}\,\mathbf{v}\right). \tag{3.20}$$

It is easily seen that a ray vector in a medium moving with the uniform velocity is directed parallel to vector $\mathbf{q}$. According to Ref. [9], the component of the velocity of light directed along vector $\mathbf{s}$ is given by

$$\mathbf{c}_{\mathrm{s}} = \frac{\langle\mathbf{S}\rangle}{\langle w\rangle} = \mathbf{s}c_{\mathrm{p}}\left(\mathbf{s}\,\frac{\operatorname{grad} L}{n}\right)^{-1}. \tag{3.21}$$

The velocity component normal to the wave front equals phase velocity c_{p} [9, 21]. Indeed, projecting $\mathbf{c}_{\mathrm{s}}$ onto this direction and using formula (3.21) yield

$$c_{\mathrm{p}} = \mathbf{c}_{\mathrm{s}}\,\frac{\operatorname{grad} L}{n}\,. \tag{3.22}$$

The ray may be defined as a trajectory, the tangent to which at each point is directed along the ray vector $\mathbf{s}$. If the radius vector $\mathbf{r}(s)$ of a point on the ray is regarded as a function of the ray length s, the equation for the ray in a moving medium takes the form

$$n\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\operatorname{grad} L + \frac{\varepsilon\mu - 1}{c}\,\mathbf{v}\right) \tag{3.23}$$

or

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left(n\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) &= \operatorname{grad} n + \frac{2}{c}\left(\mathbf{v}(\mathbf{s}\operatorname{grad} n) - \operatorname{grad} n\,(\mathbf{s}\mathbf{v})\right) \\ &\quad + \frac{n^2 - 1}{c}\,[\mathbf{s}\operatorname{rot}\mathbf{v}]\,. \end{aligned} \tag{3.24}$$

Equations (3.23) and (3.24) may also be used to derive an expression for the intensity ratio at two arbitrary points of the ray:

$$\begin{aligned} \frac{I_2}{I_1} &= \frac{n_2}{n_1}\exp\left\{-\int_{s_1}^{s_2}\frac{\nabla^2 L}{n}\,\mathrm{d}s\right\} \\ &\quad\times\left\{1 - \int_{s_1}^{s_2}\frac{1}{n}\left(\frac{2n}{c}\operatorname{grad} n\,\mathbf{v} + \frac{n^2 - 1}{c}\operatorname{div}\mathbf{v}\right)\mathrm{d}s\right\}. \end{aligned} \tag{3.25}$$

Here, integration is performed along the ray, and the light intensity I is defined as the absolute value of vector $\langle\mathbf{S}\rangle$.

The above results are analogous to those in traditional geometrical optics, with the exception of the term proportional to the velocity in the eikonal (3.10), (3.11) and in the ray vector (3.20). The presence of this term results in the distortion of the geometric wave front and effective anisotropy of the moving medium. For a medium with a spatially uniform refractive index, it follows from Eqn (3.24) that

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \frac{n^2 - 1}{cn}\,[\mathbf{s}\operatorname{rot}\mathbf{v}]\,. \tag{3.26}$$

Evidently, the rays remain straight in an irrotationally moving medium (i.e., when vorticity $\operatorname{rot}\mathbf{v} = 0$) with a uniform refractive index, and bent in the presence of vortices. A change in light intensity in a uniform incompressible medium ($\operatorname{grad} n = 0$, $\operatorname{div}\mathbf{v} = 0$) due to wave front curvature is analogous to a change in intensity in a vacuum, for example, during propagation of a spherical wave [20].

3.3 Polarization effects

We now go over to analyzing the changes in the amplitudes of vectors $\mathbf{e}$ and $\mathbf{h}$ during propagation of radiation. In the general case of an inhomogeneous medium, the equations for amplitudes are cumbersome; therefore, we confine the consideration to a medium with a uniform refractive index (but a nonuniform velocity distribution). Then, the wave equations acquire the form

$$\begin{aligned} \nabla^2\mathbf{E} + k_0^2\varepsilon\mu\mathbf{E} &= -\frac{\varepsilon\mu - 1}{c} \\ &\quad\times\left(k_0^2\mu\,[\mathbf{v}\mathbf{H}] + \mathrm{i}k_0\operatorname{rot}[\mathbf{E}\mathbf{v}] + \frac{1}{\varepsilon}\operatorname{grad}\operatorname{div}[\mathbf{v}\mathbf{H}]\right), \end{aligned} \tag{3.27}$$

$$\begin{aligned} \nabla^2\mathbf{H} + k_0^2\varepsilon\mu\mathbf{H} &= -\frac{\varepsilon\mu - 1}{c} \\ &\quad\times\left(k_0^2\varepsilon\,[\mathbf{E}\mathbf{v}] - \mathrm{i}k_0\operatorname{rot}[\mathbf{v}\mathbf{H}] + \frac{1}{\mu}\operatorname{grad}\operatorname{div}[\mathbf{E}\mathbf{v}]\right). \end{aligned} \tag{3.28}$$

Substituting expressions (3.7) into Eqns (3.27) and (3.28) and equating the terms with equal powers of k_0 yield not only the eikonal equation (3.10) (terms with k_0^2) but also the equations for the amplitudes of **e** and **h** (terms involving $\mathrm{i}k_0$):

$$(\mathrm{grad}\, L\ \mathrm{grad})\mathbf{e} + \frac{1}{2}\,\mathbf{e}\,\nabla^2 L + \frac{\varepsilon\mu - 1}{c}\left\{[\mathbf{e}\,\mathrm{rot}\,\mathbf{v}] + (\mathbf{v}\,\mathrm{grad})\mathbf{e} + \frac{1}{2}\,\mathbf{e}\,\mathrm{div}\,\mathbf{v} - \frac{1}{2\varepsilon\mu}\,(\mathrm{rot}\,\mathbf{v}\,\mathrm{grad}\,L)\,[\mathbf{e}\,\mathrm{grad}\,L]\right\} = 0\,, \tag{3.29}$$

$$(\mathrm{grad}\, L\ \mathrm{grad})\mathbf{h} + \frac{1}{2}\,\mathbf{h}\,\nabla^2 L + \frac{\varepsilon\mu - 1}{c}\left\{[\mathbf{h}\,\mathrm{rot}\,\mathbf{v}] + (\mathbf{v}\,\mathrm{grad})\mathbf{h} - \frac{1}{2}\,\mathbf{h}\,\mathrm{div}\,\mathbf{v} - \frac{1}{2\varepsilon\mu}\,(\mathrm{rot}\,\mathbf{v}\,\mathrm{grad}\,L)\,[\mathbf{h}\,\mathrm{grad}\,L]\right\} = 0\,. \tag{3.30}$$

These are the vector transport equations being sought that describe the changes in **e** and **h** along the ray. Let us introduce the complex vectors with a unit modulus:

$$\mathbf{u} = \frac{\mathbf{e}}{(\mathbf{e}\mathbf{e}^*)^{1/2}}\,, \qquad \mathbf{w} = \frac{\mathbf{h}}{(\mathbf{h}\mathbf{h}^*)^{1/2}}\,, \tag{3.31}$$

and the parameter τ characterizing the position along the ray, with

$$\frac{\partial}{\partial\tau} = \left(\left(\mathrm{grad}L + \frac{\varepsilon\mu - 1}{c}\,\mathbf{v}\right)\mathrm{grad}\right). \tag{3.32}$$

Then, it follows from Eqns (3.29) – (3.32) that

$$\frac{\partial}{\partial\tau}(\mathbf{e}\mathbf{e}^*) + \mathbf{e}\mathbf{e}^*\left(\nabla^2 L + \frac{\varepsilon\mu - 1}{c}\,\mathrm{div}\,\mathbf{v}\right) = 0\,, \tag{3.33}$$

$$\frac{\partial}{\partial\tau}(\mathbf{h}\mathbf{h}^*) + \mathbf{h}\mathbf{h}^*\left(\nabla^2 L - \frac{\varepsilon\mu - 1}{c}\,\mathrm{div}\,\mathbf{v}\right) = 0\,. \tag{3.34}$$

For the amplitudes themselves, taking into consideration the identity $\mathrm{d}/\mathrm{d}\tau = n\,\mathrm{d}/\mathrm{d}s$, one obtains

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = -\frac{\varepsilon\mu - 1}{cn}\left\{[\mathbf{u}\,\mathrm{rot}\,\mathbf{v}] + \frac{1}{2}\,\mathbf{w}(\mathbf{s}\,\mathrm{rot}\,\mathbf{v})\right\}, \tag{3.35}$$

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}s} = -\frac{\varepsilon\mu - 1}{cn}\left\{[\mathbf{w}\,\mathrm{rot}\,\mathbf{v}] - \frac{1}{2}\,\mathbf{u}(\mathbf{s}\,\mathrm{rot}\,\mathbf{v})\right\}. \tag{3.36}$$

The last two relationships may be used to find changes in polarization of radiation resulting from the motion of the medium. The right-hand sides of Eqns (3.35), (3.36) are proportional to the small parameter v/c; therefore, **u** and **w** may be substituted by their values in a medium at rest. This accounts for the uncoupling of these equations.

The solution is convenient to present in the matrix form

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix}\begin{pmatrix} u_{x0} \\ u_{y0} \end{pmatrix} \tag{3.37}$$

(zero indices correspond to a stationary medium). Equation (3.37) contains the Jones matrix [22, 23] describing a turn of the polarization vector through the angle

$$\theta = \int A\,\mathrm{d}s\,, \tag{3.38}$$

where

$$A = \frac{\varepsilon\mu - 1}{cn}\left\{(\mathrm{rot}\,\mathbf{v})_z - \frac{1}{2}\,\mathbf{s}\,\mathrm{rot}\,\mathbf{v}\right\}. \tag{3.39}$$

The expression for the turn of the polarization plane (3.38) is consistent with the Fermi prediction [12] (see also Ref. [11]) that should be regarded more likely as an evaluating one because when the initial plane wave passes through a layer of the rotating medium it may be considered plane only locally, in the geometrical optics approximation. In a more accurate description (see Section 5), the spatial nonuniformity of the medium velocity leads to the scattering (diffraction) of incident radiation. Neglect of diffraction effects is justified for light trajectories much shorter than the characteristic diffraction length $L_0 \sim kl^2$, where l is the characteristic nonuniformity size.

4. Nonreciprocal waveguides and lenses

In this section, we shall demonstrate that the Fresnel–Fizeau effect of partial light entrainment by a moving fluid in the case of transverse nonuniformity of the medium velocity may lead to the waveguide propagation of radiation and its focusing. It turns out to be a nonreciprocal effect (different for opposite directions of radiation propagation) [24].

Let us consider first a plane monochromatic wave propagating towards the x-axis and having frequency ω, wave number k, and electric and magnetic field strengths (written in the complex form)

$$\mathbf{E} = \mathbf{E}_0 \exp(\mathrm{i}kx - \mathrm{i}\omega t)\,, \qquad \mathbf{H} = \mathbf{H}_0 \exp(\mathrm{i}kx - \mathrm{i}\omega t)\,. \tag{4.1}$$

Let the medium in which the plane wave propagates also move parallel to the x-axis with the velocity v ($v > 0$ if the directions of motion of the medium and the light coincide, and $v < 0$ if they move in opposite directions). Then, for the phase velocity $v_{\mathrm{ph}} = \omega/k$ of the wave in the first approximation with respect to parameter $v/c \ll 1$ [10, 11], one finds

$$v_{\mathrm{ph}} = c_{\mathrm{p}} + v\left(1 - \frac{1}{n^2}\right). \tag{4.2}$$

Here, n is the refractive index of a stationary medium at frequency ω (the radiation frequency in the medium is meant that differs by virtue of the Doppler effect from the frequency of radiation incident from the outside on the moving medium). This relation can be rewritten in the form

$$v_{\mathrm{ph}} = \frac{c}{n_{\mathrm{eff}}}\,, \tag{4.3}$$

where an effective refractive index was introduced:

$$n_{\mathrm{eff}} = n - \frac{v}{c}(n^2 - 1)\,. \tag{4.4}$$

For gases at moderate pressures, the refractive index is close to unity and the light entrainment effect (the dependence of the effective refractive index on the velocity of medium motion) is insignificant. As the estimates imply, this effect for

gases is small even in the anomalous dispersion region. However, the same effect is not so small for liquids. For ordinary liquids with a refractive index higher than unity, the effective refractive index for side flow (counter flow) decreases (increases) with the velocity of motion. Here, the above conditions at which it is possible to neglect any additional effects caused by the medium motion (e.g., density changes) are considered to be fulfilled.

Let us consider now a medium (liquid) moving with a spatially nonuniform velocity. More precisely, let us assume that velocity v depends on the transverse (relative to the x-axis) coordinates. If the velocity varies smoothly, the radiation is locally close to the plane wave and then the effective refractive index can be described by expression (4.4). In that case, the effective refractive index changes in the transverse direction. The medium possesses either a waveguide or antiwaveguide character depending on the sign of velocity v. This suggests the possibility of creating a nonreciprocal waveguide (with different properties for waves propagating in opposite directions).

It is worth noting that the nonuniform velocity of a fluid motion is characteristic of the majority of variants of fluid flow in pipes because the velocity of stationary motion is usually highest on the tube axis, whereas $\mathbf{v} = 0$ on the walls.

Let us consider two variants of a stationary flow of a viscous incompressible fluid [18]. In the first variant, the fluid flows between two parallel planes separated by distance h (along the y-axis orthogonal to the planes), i.e., coordinates of the planes are $y = \pm h/2$.

The rate of the laminar flow shows the quadratic dependence on the coordinate y:

$$v(y) = v_0\left(1 - 4\,\frac{y^2}{h^2}\right), \tag{4.5}$$

and is maximum on the symmetry axis:

$$v_0 = v(0) = \frac{\Delta p}{2\eta l}\,\frac{h^2}{4}\,. \tag{4.6}$$

Here, η is the dynamic fluid viscosity, p is the pressure, and the pressure gradient $\Delta p/l = \text{const}$ (where Δp is the pressure difference at the pipe ends, and l is the pipe length). In this case, relationship (4.4) takes the form

$$n_{\text{eff}}(y) = n_{0y} - \frac{1}{2}\,n_{1y}y^2\,, \tag{4.7}$$

where

$$n_{0y} = n - \frac{v_0}{c}(n^2 - 1)\,, \qquad n_{1y} = -\frac{8}{h^2}\,\frac{v_0}{c}(n^2 - 1)\,. \tag{4.8}$$

The second variant corresponds to a pipe of circular section of radius R. In this case, there are analogous relations for the rate of flow (r is the radial coordinate):

$$v(r) = \frac{\Delta p}{4\eta l}(R^2 - r^2)\,, \qquad v_0 = v(0) = \frac{\Delta p}{4\eta l}\,R^2\,, \tag{4.9}$$

and for the radial profile of the effective refractive index:

$$n_{\text{eff}}(r) = n_{0r} - \frac{1}{2}\,n_{1r}\,r^2\,, \tag{4.10}$$

$$n_{0r} = n - \frac{v_0}{c}(n^2 - 1)\,, \qquad n_{1r} = -\frac{2}{R^2}\,\frac{v_0}{c}(n^2 - 1)\,. \tag{4.11}$$

It can be seen from Eqns (4.7) and (4.10) that the effective refractive index in either variant is quadratically dependent on the transverse coordinate. The radiation propagation in such a quadratic media has been described at length in the literature (see, for instance, Ref. [23]). The waveguide is realized for $n_1 > 0$, i.e., as $v_0 < 0$. The medium may be regarded as unbounded in the transverse direction if the lateral dimensions of the pipe are much larger than the width of the mode. The half-width of the lowest (fundamental) mode w is given by the relationship

$$w^2 = \frac{2}{k_0(nn_1)^{1/2}}\,. \tag{4.12}$$

Taking into account the smallness of the parameter v/c in formula (4.1), it is possible to disregard the difference between n_0 and n, and determine the wave number k_0 as in a homogeneous medium at rest. For the two above variants of the flow, one obtains

$$w_y^2 = \frac{h}{k_0}\left(\frac{c}{2v_0\,n(n^2-1)}\right)^{1/2}, \qquad w_r^2 = \frac{d}{k_0}\left(\frac{c}{2v_0\,n(n^2-1)}\right)^{1/2}, \tag{4.13}$$

respectively. The latter formula is derived from the former by substituting h with the pipe diameter $d = 2R$.

The waveguide effect is realized in a pipe whose length exceeds the characteristic diffraction length of a light beam with a half-width w:

$$l \gg L_{\text{d}} = k_0 w^2\,. \tag{4.14}$$

In the opposite case of a short pipe, it is equivalent to a lens with the focal distance

$$f = \frac{1}{n_1 L}\,. \tag{4.15}$$

Although the effect in question heightens with increasing rate of the fluid flow, the possibility of such an increase is limited due to the constraints imposed by the laminarity of the flow. The flow can transform into a turbulent one if the Reynolds number

$$\text{Re} = \frac{vD}{\nu_m} \tag{4.16}$$

exceeds a certain critical value Re_{cr} [6]. Here, ν_m is the kinematic viscosity, and D is the characteristic lateral dimension (in a pipe — its diameter). Generally speaking, the critical value Re_{cr} is not universal.

For a pipe of circular section, the motion is stable with respect to infinitesimally small perturbations at any Reynolds number. By eliminating the perturbations at the entrance to the pipe in thoroughly performed experiments, the laminar flow is maintained up to $\text{Re} \approx 10^5$ [18]. Also, fluid viscosity determining the Reynolds number is strongly dependent on temperature and increases as the temperature decreases [18, 25].

Let us evaluate the effect under conditions close to those realized in the classical Fizeau experiment [2]. In this experiment, water travelled with velocity $v = 7$ m s^{-1} in a pipe of length $L = 150$ cm and diameter $d = 5.3$ mm. For radiation with wavelength $\lambda = 0.5$ μm, formula (4.13) gives $w_r = 1.38$ mm, which is much smaller than the pipe radius; therefore, the medium may be regarded as unbounded in the transverse direction.

However, the diffraction length under these conditions is $L_{\mathrm{d}} = 22.8$ m; hence, the length of the pipe must be increased if the waveguide effect is to be observed. When the pipe length is large, it is necessary to take into account that the radiation becomes weaker due to its absorption and scattering by the water (extinction). This attenuation is characterized by the total extinction coefficient or the damping decrement of radiation flux density during its propagation in the absorbing and scattering medium [11]. For distilled water and a light wavelength range from 440 to 480 nm, this coefficient equals 0.012 m^{-1}, with the contribution from the light scattering being less than 30% [26]. For this reason, both attenuation of the light beam and scattering that causes its broadening are small within the diffraction length.

Substituting parameters of the Fizeau experimental setup and viscosity $\nu_m = 0.01\ \mathrm{cm}^2\ \mathrm{s}^{-1}$ into formula (4.16) gives the Reynolds number $\mathrm{Re} \approx 3.7 \times 10^4$ at a water temperature of 20 °C. At 0 °C, the Reynolds number $\mathrm{Re} \approx 2 \times 10^4$. This presumably accounts for the turbulent water motion in the Fizeau experiments, although it can be made laminar provided the experiment is accurately performed.

It should be borne in mind that the question of turbulence in pipes of such a small diameter is not as simple as it may seem. The point is that the characteristic minimal size of vortices (Kolmogorov scale) ranges 1 – 10 mm, in which case they are no longer splitted but simply disperse due to viscosity. Therefore, it may be supposed that turbulence is absent altogether in the pipes with such a diameter. Note also that relationship (4.15) gives the focal distance $f = 129$ m of an equivalent lens for the parameters of the Fizeau setup, which is quite possible to register in experiment.

Thus, the passage of a light beam through water flowing in a pipe in the opposite direction results in its waveguide propagation; when the directions of the water flow and the light beam coincide, the latter broadens. Both the beam (mode) radius and the focal distance can be decreased by using a transparent fluid with a higher refractive index than that of water.

To conclude the present section, it should be noted that the rotation of a dielectric waveguide (optical fiber) affects the structure of its modes [27]. An example is a fiber whose refractive index undergoes axisymmetric smooth or jumplike transverse variations. Its rotation about the axis makes unequal the clockwise and counterclockwise directions of the axis bypass; this results in the removal of degeneracy of the spectrum of modes with different signs of the azimuthal index.

This situation is especially well-apparent in media with a constant (spatially uniform) refractive index, where a single nonuniformity corresponds to the transverse (with respect to the rotation axis) dependence of the angular rotation velocity. Naturally, a homogeneous medium exhibits no waveguide properties in the absence of its rotation.

Interestingly, the exact solutions of the Maxwell equations for the quadratic transverse dependence of the rotational velocity are presented by the Gaussian beams (Gauss – Hermite or Gauss – Laguerre modes), whereas for ordinary media with quadratic transverse variations in the refractive index, the Gaussian beams appear only in the quasioptical (paraxial) approximation on neglecting polarization effects [23]. To be more precise, the Gaussian profile is intrinsic to longitudinal components of electric and magnetic fields E_z and H_z, whereas their transverse components (defined by the application of differential operators to them) contain an admixture of higher-order Gaussian modes. Moreover, degeneracy of mode frequencies in the sign of the azimuthal index is removed in a rotating waveguide. Due to this, such a waveguide may serve as a rotation sensor [27].

5. Light scattering by velocity nonuniformities of a moving medium

5.1 General relationships

Let us consider the problem of diffraction of electromagnetic radiation by nonuniformities of the medium motion velocity [15]. To this end, it is convenient to employ the wave equations for **D** and **B** derived from Eqns (1.1) and (1.3):

$$\Box\,\mathbf{D} = \mathbf{f}_D\,, \qquad \Box\,\mathbf{B} = \mathbf{f}_B\,,$$
$$\Box = \Delta - \frac{\varepsilon_0\mu_0}{c^2}\frac{\partial^2}{\partial t^2}\,, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,, \tag{5.1}$$

where

$$\mathbf{f}_D = -\frac{\varepsilon_0\mu_0 - 1}{c}\left\{\mathrm{rot\,rot}\,[\mathbf{vH}] - \frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\,\mathrm{rot}\,[\mathbf{Ev}]\right\} - \mathrm{rot\,rot}\,\delta\mathbf{D} + \frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\,\mathrm{rot}\,\delta\mathbf{B}\,,$$
$$\mathbf{f}_B = -\frac{\varepsilon_0\mu_0 - 1}{c}\left\{\mathrm{rot\,rot}\,[\mathbf{Ev}] + \frac{\mu_0}{c}\frac{\partial}{\partial t}\,\mathrm{rot}\,[\mathbf{vH}]\right\} - \mathrm{rot\,rot}\,\delta\mathbf{B} - \frac{\mu_0}{c}\frac{\partial}{\partial t}\,\mathrm{rot}\,\delta\mathbf{D}\,. \tag{5.2}$$

Because the velocity of medium motion is small, Eqns (5.1), (5.2) can be solved in the framework of the perturbation theory. In this case, the scattered radiation is found from the first-order perturbation theory by solving Eqn (5.1) in the form of retarded potentials [28] and considering the right-hand sides of these equations as given [expressed through zero-order solutions: see Eqns (5.3) – (5.5)]. The terms in braces in Eqn (5.2) represent relativistic effects, and the terms following them (outside the brackets) nonrelativistic ones. In what follows we take into consideration only relativistic effects, i.e., we neglect $\delta\mathbf{D}$ and $\delta\mathbf{B}$ assuming that the velocity of the moving medium satisfies the condition $v \ll v_0$.

In order to solve equations (5.1), assume that $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \ldots$, $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1 + \ldots$, etc. In the zero approximation in the parameter v/c (a continuous uniform medium at rest), the unperturbed solution $\mathbf{E}_0$, $\mathbf{H}_0$ is regarded as known. In the first approximation of the perturbation theory, Eqns (5.1) give

$$\Box\,\mathbf{E}_1 = \mathbf{f}_E\,, \qquad \Box\,\mathbf{H}_1 = \mathbf{f}_H\,, \tag{5.3}$$

$$\mathbf{f}_E = -\frac{\varepsilon_0\mu_0 - 1}{c\varepsilon_0}\left\{\mathrm{grad\,div}\,[\mathbf{vH}_0] - \frac{\varepsilon_0\mu_0}{c^2}\frac{\partial^2}{\partial t^2}\,[\mathbf{vH}_0] - \frac{\varepsilon_0}{c}\frac{\partial}{\partial t}\,\mathrm{rot}\,[\mathbf{E}_0\mathbf{v}]\right\},$$
$$\mathbf{f}_H = -\frac{\varepsilon_0\mu_0 - 1}{c\mu_0}\left\{\mathrm{grad\,div}\,[\mathbf{E}_0\mathbf{v}] - \frac{\varepsilon_0\mu_0}{c^2}\frac{\partial^2}{\partial t^2}\,[\mathbf{E}_0\mathbf{v}] + \frac{\mu_0}{c}\frac{\partial}{\partial t}\,\mathrm{rot}\,[\mathbf{vH}_0]\right\}. \tag{5.4}$$

The right-hand sides of Eqns (5.3) being known, their solutions are presented as retarded potentials [28]:

$$\mathbf{E}_1 = -\frac{1}{4\pi}\int \frac{1}{R}\,\mathbf{f}_E\left(t-\frac{R}{c}\right)\mathrm{d}V\,,\qquad \mathbf{H}_1 = -\frac{1}{4\pi}\int \frac{1}{R}\,\mathbf{f}_H\left(t-\frac{R}{c}\right)\mathrm{d}V\,. \tag{5.5}$$

Here, integration is performed over the bulk of the moving medium, and R is the distance between the elementary volume $\mathrm{d}V$ and the point at which the fields are sought. Naturally, the amplitudes of a scattered field (in the first approximation) are proportional, like $\mathbf{f}$, to the parameter v/c. Let us assume that velocity nonuniformities are concentrated within a certain bounded region of the medium, and place the origin of the coordinates at an 'average' point of this region.

Following Ref. [11], we denote the radius vector connecting the origin of coordinates and the point of observation as $\mathbf{R}_0$, the unit vector in the same direction as $\mathbf{n}$, the radius vector of the integration volume as $\mathbf{r}$, and the radius vector drawn from the integration volume to the point of observation as $\mathbf{R}$, so that $\mathbf{R}_0 = \mathbf{R} + \mathbf{r}$. In the far field (at a sufficiently large distance from the velocity nonuniformity region), R in the denominator of the integrands in formulas (5.5) may be substituted by R_0, and in the arguments of the functions in the numerators by a more exact expression $R = R_0 - \mathbf{nr}$. Then, Eqns (5.5) acquire the form

$$\mathbf{E}_1 = -\frac{1}{4\pi R_0}\int \mathbf{f}_E\left(t-\frac{R_0}{c}+\frac{\mathbf{nr}}{c}\right)\mathrm{d}V\,,\qquad \mathbf{H}_1 = -\frac{1}{4\pi R_0}\int \mathbf{f}_H\left(t-\frac{R_0}{c}+\frac{\mathbf{nr}}{c}\right)\mathrm{d}V\,. \tag{5.6}$$

Expressions (5.5) and (5.6) give the general solution to the problem of radiation diffraction (scattering) by arbitrary velocity nonuniformities of a moving medium. Incident radiation also has an arbitrary form. Moreover, it is sufficient to solve only one equation, (5.3) or (5.4), and the other can be obtained by the simple substitution of the initial parameters. Indeed, the right-hand sides of these equations transform into each other upon the substitution of $\varepsilon \leftrightarrow -\mu$ and of $\mathbf{E}_0 \leftrightarrow \mathbf{H}_0$ [29].

Another important case is represented by the stationary motion of a medium when its velocity at each point is time-independent. In this case, the radiation spectrum remains unaltered — that is, the scattered radiation frequency coincides with the incident radiation frequency (with the Doppler frequency shift being absent). This makes it possible to reduce the linear problem of electromagnetic radiation pulse scattering to the consideration of scattering of individual plane monochromatic waves.

Below, we shall apply this result to the case of incident radiation exemplified by a plane monochromatic wave with frequency ω and wave number $k = n\omega/c$:

$$\mathbf{E}_0 = \exp\left(-\mathrm{i}\omega t + \mathrm{i}k\mathbf{rm}\right)(b_{\mathrm{I}}, b_{\mathrm{II}}, 0)\,,\qquad \mathbf{H}_0 = \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\exp\left(-\mathrm{i}\omega t + \mathrm{i}k\mathbf{rm}\right)(b_{\mathrm{I}}, -b_{\mathrm{II}}, 0)\,, \tag{5.7}$$

where $\mathbf{m}$ is the unit vector of wave incidence direction, and $b_{\mathrm{I,II}} = |b_{\mathrm{I,II}}|\exp\left(\mathrm{i}\Delta_{\mathrm{I,II}}\right)$ are the complex amplitudes of the incident radiation with elliptic polarization; the result is also applicable to concrete variants of the velocity distribution.

Practically speaking, it is also possible to reduce to this problem (owing to the high speed of light) a case of quasi-stationary motion of the medium, when its velocity is not substantially altered as the light propagates through the region of a nonuniform velocity distribution. We believe that the scattering object (nonuniformity) is a body of revolution with a linear rotational velocity

$$\mathbf{v} = \begin{cases} [\mathbf{\Omega}\,\mathbf{r}]\,, & r \in V\,,\\ 0\,, & r \notin V\,,\end{cases} \tag{5.8}$$

where V is the volume of the rotating body, and $\mathbf{\Omega}$ is the angular rotational velocity.

This section deals with the scattering of electromagnetic waves in situations where the solution of a problem is represented in an analytical form. One such problem concerns light scattering by rotational velocity nonuniformities, the sizes of which are small compared with the wavelength λ, and which have an arbitrary axisymmetric shape. Another problem is that of the light scattering by a velocity nonuniformity having a cylindrical form.

Let us separate polarizations by representing the electric and magnetic fields of an incident wave as

$$\mathbf{E}_0 = b_{\mathrm{I}}\mathbf{E}_{0\mathrm{I}} + b_{\mathrm{II}}\mathbf{E}_{0\mathrm{II}}\,,\qquad \mathbf{H}_0 = b_{\mathrm{I}}\mathbf{H}_{0\mathrm{I}} + b_{\mathrm{II}}\mathbf{H}_{0\mathrm{II}}\,, \tag{5.9}$$

$$\begin{aligned} \mathbf{E}_{0\mathrm{I}} &= \tilde{\mathbf{E}}_0\exp\left(-\mathrm{i}\omega t + \mathrm{i}k\mathbf{rm}\right),\\ \mathbf{H}_{0\mathrm{I}} &= -\left(\frac{\varepsilon}{\mu}\right)^{1/2}\tilde{\mathbf{H}}_0\exp\left(-\mathrm{i}\omega t + \mathrm{i}k\mathbf{rm}\right),\\ \mathbf{E}_{0\mathrm{II}} &= \tilde{\mathbf{H}}_0\exp\left(-\mathrm{i}\omega t + \mathrm{i}k\mathbf{rm}\right),\\ \mathbf{H}_{0\mathrm{II}} &= \left(\frac{\varepsilon}{\mu}\right)^{1/2}\tilde{\mathbf{E}}_0\exp\left(-\mathrm{i}\omega t + \mathrm{i}k\mathbf{rm}\right), \end{aligned} \tag{5.10}$$

where $\tilde{\mathbf{E}}_0$, $\tilde{\mathbf{H}}_0$ are the constant vectors. Because the problem is linear, the general solution in the case of the incidence of such a wave is entirely similar to formulas (5.9):

$$\mathbf{E}_1 = b_{\mathrm{I}}\mathbf{E}_{\mathrm{I}} + b_{\mathrm{II}}\mathbf{E}_{\mathrm{II}}\,,\qquad \mathbf{H}_1 = b_{\mathrm{I}}\mathbf{H}_{\mathrm{I}} + b_{\mathrm{II}}\mathbf{H}_{\mathrm{II}}\,. \tag{5.11}$$

Due to the aforementioned symmetry of the problem, the fields $\mathbf{E}_{\mathrm{II}}$ and $\mathbf{H}_{\mathrm{II}}$ are related to $\mathbf{E}_{\mathrm{I}}$ and $\mathbf{H}_{\mathrm{I}}$ by the simple relationships

$$\mathbf{E}_{\mathrm{II}} = -\left(\frac{\mu}{\varepsilon}\right)^{1/2}\mathbf{H}_{\mathrm{I}}\,,\qquad \mathbf{H}_{\mathrm{II}} = \left(\frac{\varepsilon}{\mu}\right)^{1/2}\mathbf{E}_{\mathrm{I}}\,. \tag{5.12}$$

Note that the vectors in triples $\mathbf{E}_{\mathrm{I}}$, $\mathbf{H}_{\mathrm{I}}$, $\mathbf{n}$ and $\mathbf{E}_{\mathrm{II}}$, $\mathbf{H}_{\mathrm{II}}$, $\mathbf{n}$ are mutually orthogonal.

By introducing the dimensionless variable $\mathbf{q} = k\mathbf{r}$, the expressions for the scattered field can be presented, based on Eqns (5.3), (5.4), and (5.8)–(5.10), in the form

$$\begin{aligned} \mathbf{E}_1 = S_E\iiint \mathrm{d}\mathbf{q}\,\big\{&\mathbf{\Omega}_\perp(\tilde{\mathbf{H}}_0\mathbf{q}) - \mathbf{q}_\perp(\tilde{\mathbf{H}}_0\mathbf{\Omega})\\ &+ [\mathbf{nq}]\,(\tilde{\mathbf{E}}_0\mathbf{\Omega}) - [\mathbf{n\Omega}]\,(\tilde{\mathbf{E}}_0\mathbf{q})\big\}\,\Phi(\mathbf{q})\exp\left\{-\mathrm{i}\mathbf{q}(\mathbf{n}-\mathbf{m})\right\}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \mathbf{H}_1 = S_H\iiint \mathrm{d}\mathbf{q}\,\big\{&-\mathbf{\Omega}_\perp(\tilde{\mathbf{E}}_0\mathbf{q}) + \mathbf{q}_\perp(\tilde{\mathbf{E}}_0\mathbf{\Omega})\\ &+ [\mathbf{nq}]\,(\tilde{\mathbf{H}}_0\mathbf{\Omega}) - [\mathbf{n\Omega}]\,(\tilde{\mathbf{H}}_0\mathbf{q})\big\}\,\Phi(\mathbf{q})\exp\left\{-\mathrm{i}\mathbf{q}(\mathbf{n}-\mathbf{m})\right\}. \end{aligned} \tag{5.14}$$

Here, $\mathbf{\Omega}_\perp$, $\mathbf{q}_\perp$ are the two-dimensional vectors perpendicular to the direction of the incident wave $\mathbf{m}$, and the coefficients of the integrals have the form

$$S_E = \frac{n^2-1}{4\pi c n k^2}\,\frac{\exp\left(-\mathrm{i}\omega t + \mathrm{i}kR_0\right)}{R_0}\,, \qquad S_H = \left(\frac{\varepsilon}{\mu}\right)^{1/2} S_E\,, \tag{5.15}$$

while

$$\Phi(\mathbf{q}) = \begin{cases} 1\,, & r \in V\,, \\ 0\,, & r \notin V\,. \end{cases}$$

In the case of direct registering of scattered radiation unmixed with the reference wave, the energy characteristics are quadratic in angular velocity (and in parameter v/c) and defined by the differential scattering cross section $P(\theta,\phi,\alpha)$, where α is the angle between the rotation axis and the incident direction of the electromagnetic wave. The cross section is introduced by the relationship

$$P(\theta,\phi,\alpha) = \frac{w}{w_0}\,R_0^2\,. \tag{5.16}$$

Here, w_0 is the energy density in the incident wave, and w is the density of the scattered electromagnetic energy given by the expression

$$w = \frac{\varepsilon_0}{4\pi}\left(\mathrm{Re}\,\mathbf{E}_1\right)^2 = \frac{\mu_0}{4\pi}\left(\mathrm{Re}\,\mathbf{H}_1\right)^2. \tag{5.17}$$

Integration of the differential cross section $P(\theta,\phi,\alpha)$ over the solid angle yields the total scattering cross section σ.

Before passing to concrete results, it is worthwhile to specify the coordinate systems being used. The xyz-system shown in Fig. 1 is associated with the rotating medium region, and the y-axis coincides with the nonuniformity rotation axis. System $x'y'z'$ is associated with the incident plane wave. The direction of wave propagation coincides with the z'-axis and forms an angle α with the y-axis of rotation. Axes x and x' coincide; therefore, z'- and y'-axes lie in the plane zy. The radius vector OR_0 connecting the center of the coordinate system and the point of observation forms an angle θ with the z'-axis, while its projection R_0' onto the plane $x'y'$ forms an angle ϕ with the x'-axis.

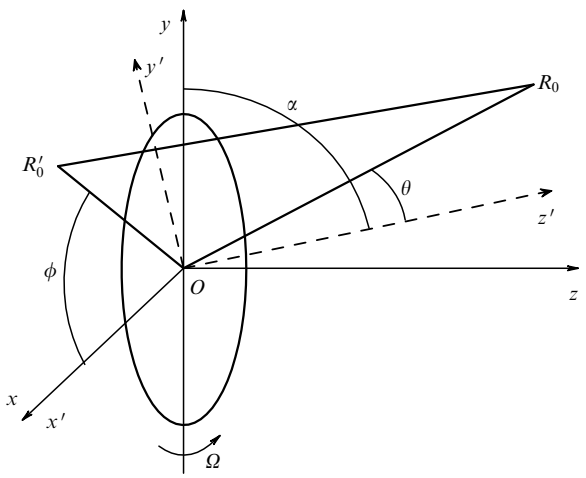

Figure 1. The frames of reference (see the text).

5.2 Small region of a rotating medium

Let us consider the case of small scattering nonuniformities with $|\mathbf{q}| < 1$ in Eqns (5.13), (5.14). It is well known [11, 30] that in the case of small-sized nonuniformities a scattering particle resides in an effectively uniform field and is characterized by certain electric and magnetic moments $\mathbf{P}$ and $\mathbf{M}$, respectively. Their time dependence is given by the factor $\exp\left(-\mathrm{i}\omega t\right)$. When the exponent under the integrands in Eqns (5.13), (5.14) is expanded in a Taylor series, the first nonvanishing term turns out quadratic in $\mathbf{q}$. Regarding the rotation-axis direction as the y-axis and taking into account the symmetry of the solution in x and z for small scattering nonuniformities, it is easy to reduce relationships (5.13), (5.14) to the following [29]:

$$\mathbf{E}_1 = K_E\,\mathbf{\Lambda}(\mathbf{m},\mathbf{n},\mathbf{\Omega})\,, \qquad \mathbf{H}_1 = K_H\,\mathbf{M}(\mathbf{m},\mathbf{n},\mathbf{\Omega})\,, \tag{5.18}$$

$$K_{E,H} = -\mathrm{i}f S_{E,H}\,, \qquad f = \int \mathrm{d}\mathbf{q}\,q_x^2 = \frac{\pi}{4}\,k^5 \int \mathrm{d}y\,\rho^4(y)\,. \tag{5.19}$$

The vector functions on the right-hand sides of Eqn (5.18) depend only on the angular characteristics of the problem, while the dependence on size and form contains the factor f.

As follows from Eqn (5.18), the angular dependence of the scattered field is identical for small rotating nonuniformities of an arbitrary form. The fields are different only in form factor f determined by the nonuniformity size. Given below is the expression for $\mathbf{\Lambda}$ because $\mathbf{M}$ is straightforwardly derived from it in accordance with formulas (5.11) and (5.12):

$$\mathbf{\Lambda}(\mathbf{m},\mathbf{n},\mathbf{\Omega}) = 4\Omega\big(A(\mathbf{m},\mathbf{n})\,\mathbf{e}_x + B(\mathbf{m},\mathbf{n})\,\mathbf{e}_y + C(\mathbf{m},\mathbf{n})\,\mathbf{e}_z\big)\,, \tag{5.20}$$

$$A = b_{\mathrm{I}}A_1 + b_{\mathrm{II}}A_2\,, \quad B = b_{\mathrm{I}}B_1 + b_{\mathrm{II}}B_2\,, \quad C = b_{\mathrm{I}}C_1 + b_{\mathrm{II}}C_2\,.$$

Here, the following notation was used:

$$\begin{aligned} A_1 &= 2\gamma^2 n_{x'}\sin\alpha + \left(1+\sin^2\alpha - n_{y'}\cos\alpha\right)2n_{x'}(n_{z'}-\sin\alpha)\,, \\ A_2 &= \gamma^2(n_{y'}+\cos\alpha)\left(1 + 2\sin\alpha\,\frac{n_{z'}-\sin\alpha}{\gamma^2}\right) \\ &\quad + (n_{y'}+\cos\alpha)\big(n_{x'}^2 - (n_{z'}-\sin\alpha)^2\big)\,, \end{aligned} \tag{5.21}$$

$$\begin{aligned} B_1 &= 2(n_{z'}-\sin\alpha)(n_{y'}+\cos\alpha)(1-n_{y'}\cos\alpha) + 2n_{y'}\gamma^2\sin\alpha\,, \\ B_2 &= 2n_{x'}(n_{y'}^2 - \cos^2\alpha)\,, \end{aligned} \tag{5.22}$$

$$\begin{aligned} C_1 &= 2\gamma^2 n_{z'}\sin\alpha \\ &\quad - \cos\alpha\,(n_{y'}+\cos\alpha)\big(\gamma^2 + 2(n_{z'}-\sin\alpha)\sin\alpha\big) \\ &\quad - (1+\sin^2\alpha - n_{y'}\cos\alpha)\big(n_{x'}^2 - (n_{z'}-\sin\alpha)^2\big)\,, \\ C_2 &= 2n_{x'}(n_{z'}-\sin\alpha)(n_{y'}-\cos\alpha)\,. \end{aligned} \tag{5.23}$$

Introduced into formulas (5.21)–(5.23) are the projections of the unit vector specifying the scattering direction:

$$\begin{aligned} n_{x'} &= \sin\theta\cos\phi\,, \\ n_{y'} &= \sin\theta\sin\phi\sin\alpha - \cos\theta\cos\alpha\,, \\ n_{z'} &= \sin\theta\sin\phi\cos\alpha + \cos\theta\sin\alpha\,, \end{aligned} \tag{5.24}$$

and the parameter γ is related to them by the formula

$$\gamma = \big(n_{x'}^2 + (n_{z'}-\sin\alpha)^2\big)^{1/2}\,.$$

When an incident wave is perpendicular to the rotation axis, the differential cross sections of light scattering by small

nonuniformities have the form [31]

$$P_s\left(\theta, \phi, \frac{\pi}{2}\right) = |S|^2 f^2 (1 - \cos\theta)^2 \sin^2\phi \sin^2\theta \,. \tag{5.25}$$

Their maxima in angle ϕ occur at $\phi = \pi/2, 3\pi/2$, and in angle θ at $\theta = 2\pi/3$. Similarly, when an incident wave is parallel to the rotation axis ($\alpha = 0$) [32], one has

$$P_s(\theta, \phi, 0) = |S|^2 f^2 \sin^4\theta \,, \tag{5.26}$$

in other words, maximum scattering occurs through the angle $\theta = \pi/2$. Moreover, the cross section in this case does not depend on the polar angle ϕ; therefore, the scattering of polarized and natural light differs only in the numerical coefficient.

Here are expressions for the total scattering cross sections

$$\sigma_s\left(\frac{\pi}{2}\right) = \frac{8\pi}{5}\,|S|^2 f^2 \,, \qquad \sigma_s(0) = \frac{32\pi}{15}\,|S|^2 f^2 \,. \tag{5.27}$$

The total cross sections in formulas (5.27) differ only in the close-to-unity numerical multiplier. Figure 2 depicts (in two forms) normalized differential cross sections $p(\theta, \phi) = P(\theta, \phi)/\sigma$ for a small nonuniformity. Unlike the total cross section σ, these functions are independent of the angular rotational velocity.

Calculations indicate that, similar to the Mie theory, the small-particle approximation may be used up to dimensions l defined by the relation $kl < 0.5$. A substantial difference between the light scattering by small-sized velocity nonuniformities and that by small particles is due to the multipole character of the former [15, 21]. This inference ensues from the shape of the dependence of scattering cross sections on the incident radiation wavelength (λ^{-6} for a velocity nonuniformity, instead of λ^{-4} for a particle) and on the size of the scatterer (l^{10} in the former case, and l^6 in the latter).

5.3 Finite region of a rotating medium (analytics)

Let us consider now the problem of the scattering of a plane wave by a dielectric nonuniformity in the form of a circular cylinder (see Fig. 1) with a base radius ρ_0 and height $2h$ (rotation axis y), rotating with a constant angular velocity Ω in a medium having the same refractive index. We shall assume that the incident direction of the plane wave is perpendicular to the rotation axis. In this case, integration in formulas (5.6) gives

$$\mathbf{E}_1 = b_{\mathrm{I}}\mathbf{E}_{\mathrm{I}} + b_{\mathrm{II}}\mathbf{E}_{\mathrm{II}} \,, \tag{5.28}$$

$$\begin{aligned}\mathbf{E}_{\mathrm{I}} &= K_E\big\{-\mathbf{e}_x \cos\phi \sin\phi \sin^2\theta \\ &\quad + \mathbf{e}_y(1 - \cos\theta - \sin^2\phi \sin^2\theta) \\ &\quad + \mathbf{e}_z \sin\phi \sin\theta(1 - \cos\theta)\big\} \,, \\ \mathbf{E}_{\mathrm{II}} &= K_E\big\{-\mathbf{e}_x(1 - \cos\theta - \cos^2\phi \sin^2\theta) \\ &\quad + \mathbf{e}_y \cos\phi \sin\phi \sin^2\theta - \mathbf{e}_z \cos\phi \sin\theta(1 - \cos\theta)\big\} \,,\end{aligned} \tag{5.29}$$

$$\begin{aligned}K_E &= -\mathrm{i}\,\frac{n^2 - 1}{cn}\,\frac{\Omega}{R_0}\exp(-\mathrm{i}\omega t + \mathrm{i}kR_0) \\ &\quad \times \frac{\rho_0^2 J_2(k\rho_0\gamma)}{\gamma^2}\sin(kh\sin\phi\sin\theta) \,, \\ \gamma &= \left((1 - \cos\theta)^2 + \cos^2\theta \sin^2\phi\right)^{1/2} \,,\end{aligned} \tag{5.30}$$

where J_2 is the second-order Bessel function.

The magnetic field is obtained from Eqns (5.28)–(5.30) in accordance with formulas (5.12). Generalization of these results to the case of oblique incidence of a plane wave on a cylinder was reported in Ref. [15]. Here is the value of the coefficient K_E alone:

$$\begin{aligned}K_E &= -\mathrm{i}\,\frac{n^2 - 1}{2cn}\,\frac{\Omega}{R_0}\exp(-\mathrm{i}\omega t + \mathrm{i}kR_0) \\ &\quad \times \frac{\rho_0^2 J_2(k\rho_0\gamma)}{\gamma^2}\,\frac{\sin\big(kh(n_y' - \cos\alpha)\big)}{n_y' - \cos\alpha} \,.\end{aligned} \tag{5.31}$$

The angular dependence for this case is described exactly as for small particles, i.e., by the functions $\mathbf{\Lambda}(\mathbf{m}, \mathbf{n}, \mathbf{\Omega})$ and $\mathbf{M}(\mathbf{m}, \mathbf{n}, \mathbf{\Omega})$ [see Eqns (5.18), (5.20)].

The differential cross section of light scattering by a finite circular cylinder, when the incident radiation is perpendicular to the rotation axis, has the form [15]

$$P\left(\theta, \phi, \frac{\pi}{2}\right) = \left|K_E\left(\frac{\pi}{2}\right)\right|^2 \sin^2(kh\sin\phi\sin\theta)\,(1 - \cos\theta)^2 \,. \tag{5.32}$$

When the wave vector of the incident wave is parallel to the rotation angle, one obtains

$$P(\theta, \phi, 0) = \left|K_E(0)\right|^2 \frac{\sin^2\big(kh(1 - \cos\theta)\big)}{(1 - \cos\theta)^2} \,. \tag{5.33}$$

Let us analyze the angular dependences of differential cross sections (5.32), (5.33) in the limiting cases of a small-sized cylinder and a long thin cylinder. For the small-sized cylinder ($k\rho_0,\ kh \ll 1$), the differential scattering cross section $P(\theta, \phi, \pi/2)$ is obtained by substituting

$$f = \frac{\pi}{2}\,(k\rho_0)^4 kh \tag{5.34}$$

into Eqns (5.25)–(5.27). It should be noted that for a small sphere ($kr_0 \ll 1$) of radius r_0, one has

$$f = \frac{4\pi}{15}\,(kr_0)^5 \,. \tag{5.35}$$

For the elongated cylinder ($kh \gg 1$), the square of the sine in formula (5.32) accounts for high-frequency oscillations. Considering their envelope, when substituting the sine squared by $1/2$, reveals a weak dependence on the angle ϕ (only via the parameter γ). It totally disappears when the diameter of the cylinder is small. The dependence on angle θ has the form

$$P_{\mathrm{t}}\left(\theta, \phi, \frac{\pi}{2}\right) = \frac{(n^2 - 1)^2}{128n^2}\,\frac{\Omega^2}{c^2}\,k^4\rho_0^8(1 - \cos\theta)^2 \,. \tag{5.36}$$

The maximum value is reached at $\theta = \pi$. It is important to note that expression (5.36) holds beyond a small region of angles, where the product $\sin\phi\sin\theta$ is close to zero. At small angles θ, the differential scattering cross section is proportional to θ^6, whereas as $\theta \to \pi$ it is only proportional to $(\theta - \pi)^2$.

If the total scattering cross sections for the small and long cylinders are denoted as σ_{s} and σ_{t}, respectively, then

$$\begin{aligned}\sigma_{\mathrm{s}}\left(\frac{\pi}{2}\right) &= \frac{(n^2 - 1)^2}{40n^2}\,\frac{\Omega^2}{c^2}\,\pi k^6\rho_0^8 h^2 \,, \\ \sigma_{\mathrm{t}}\left(\frac{\pi}{2}\right) &= \frac{(n^2 - 1)^2}{96n^2}\,\frac{\Omega^2}{c^2}\,\pi k^4\rho_0^8 \,.\end{aligned} \tag{5.37}$$

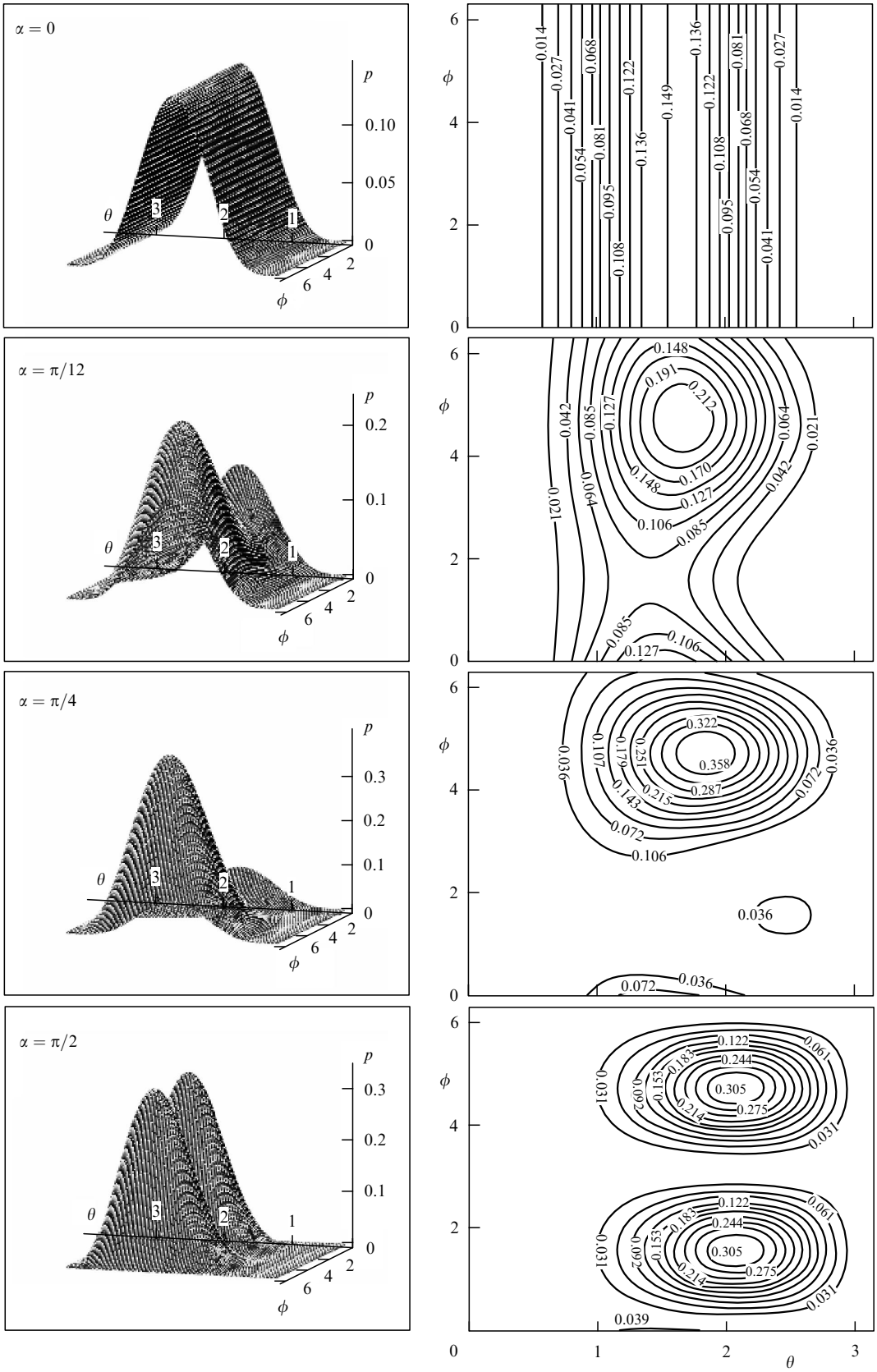

Figure 2. Normalized differential cross sections of light scattering by a small nonuniformity at different tilt angles α: left — spatial graphs, and right — flat images (top view).

The dependence of differential and total scattering cross sections on the light wavelength λ is of special interest. It follows from formulas (5.37) that $\sigma_s \sim \lambda^{-6}$ for the small cylinder, while $\sigma_t \sim \lambda^{-4}$ for the long thin cylinder (as in the case of Rayleigh scattering).

In the case described by Eqn (5.33), one finds for small sizes of the cylinder:

$$P_{\mathrm{s}}(\theta,\phi,0)=\frac{(n^2-1)^2}{64n^2}\,\frac{\Omega^2}{c^2}\,k^6\rho_0^8h^2\sin^4\theta\,, \tag{5.38}$$

i.e., the maximum scattering by small nonuniformities is directed toward the angle $\theta=\pi/2$. For the elongated cylinder, one obtains

$$P_{\mathrm{t}}(\theta,\phi,0)=\frac{(n^2-1)^2}{128n^2}\,\frac{\Omega^2}{c^2}\,k^4\rho_0^8\sin^4\theta\,. \tag{5.39}$$

The total cross sections have the form

$$\begin{aligned}\sigma_{\mathrm{s}}(0)&=\frac{(n^2-1)^2}{30n^2}\,\frac{\Omega^2}{c^2}\,\pi k^6\rho_0^8h^2\,,\\ \sigma_{\mathrm{t}}(0)&=\frac{(n^2-1)^2}{60n^2}\,\frac{\Omega^2}{c^2}\,\pi k^4\rho_0^8\,.\end{aligned} \tag{5.40}$$

Naturally, the scattered radiation amplitude found in the first-order perturbation theory is linear in the angular rotational velocity of the body. Such a linear effect is possible to register if the scattered radiation is mixed with the reference (unscattered) coherent radiation of the same frequency.

Let the mixing (interference) take place in the plane $z=Z_0$, and the reference radiation be a plane wave

$$\mathbf{E}_s=\exp\left(\mathrm{i}kL+\mathrm{i}k\mathbf{q}\boldsymbol{\alpha}-\mathrm{i}\omega t\right)\left(\mathbf{e}_xa_{\mathrm{I}}+\mathbf{e}_ya_{\mathrm{II}}\right). \tag{5.41}$$

Here, the first term in the exponent stands for the constant phase incursion, and the second term takes into account the turn of the wave front through a small two-dimensional angle $\boldsymbol{\alpha}$ with components α_x and α_y in the Cartesian coordinate system, and $\mathbf{q}=(x,y)$ is the two-dimensional vector of the transverse coordinates. Quantities a_{I} and a_{II} characterize the state of polarization of radiation and are represented in the form

$$a_j=|a_j|\exp\left(\mathrm{i}\delta_j\right),\qquad j=\mathrm{I},\mathrm{II}\,.$$

Then, the interference term has the form

$$I_{12}=2\,\mathrm{Re}\,\langle\mathbf{E}_s\mathbf{E}_1^*\rangle\,, \tag{5.42}$$

where the angle brackets denote averaging over time for a period significantly longer than the light oscillation period.

Taking into account formulas (5.28)–(5.31) and (5.41) yields

$$\begin{aligned}I_{12}=&\frac{\Omega\rho_0}{c}\,\frac{\rho_0}{R_0}\,\frac{n^2-1}{n}\,\frac{J_2(k\rho_0\gamma)}{\gamma^2}\\ &\times\sin\left(kh\sin\phi\sin\theta\right)\big\{\cos\phi\sin\phi\sin^2\theta\\ &\times\left[|a_{\mathrm{I}}|\,|b_{\mathrm{I}}|\sin\left(S_{\mathrm{I}}-\Lambda_{\mathrm{I}}\right)-|a_{\mathrm{II}}|\,|b_{\mathrm{II}}|\sin\left(S_{\mathrm{II}}-\Lambda_{\mathrm{II}}\right)\right]\\ &+\left(1-\cos\theta-\sin^2\phi\sin^2\theta\right)\\ &\times\left[|a_{\mathrm{I}}|\,|b_{\mathrm{II}}|\sin\left(S_{\mathrm{I}}-\Lambda_{\mathrm{II}}\right)-|a_{\mathrm{II}}|\,|b_{\mathrm{I}}|\sin\left(S_{\mathrm{II}}-\Lambda_{\mathrm{I}}\right)\right]\big\}\,.\end{aligned} \tag{5.43}$$

$$S_j=kL+k\,\mathbf{q}\boldsymbol{\alpha}+\delta_j\,,\qquad \Lambda_j=kR_0+\Delta_j\,,\qquad j=\mathrm{I},\,\mathrm{II}\,. \tag{5.44}$$

The first multiplier in expression (5.43) coincides with the ratio v_{c}/c of the linear rotational velocity at the lateral face of the cylinder to the speed of light; the second one is the ratio of the cylinder radius to the distance from the receipt point, and the third is proportional to the Fresnel entrainment coefficient.

In the region not far from the axis, i.e., at small angles θ, one has

$$\begin{aligned}I_{12}=&\frac{\Omega\rho_0}{c}\,\frac{\rho_0}{R_0}\,\frac{n^2-1}{2n}\,(k\rho_0)^2\theta^2\sin\left(kh\theta\sin\phi\right)\\ &\times\big\{\sin2\phi\left[|a_{\mathrm{I}}|\,|b_{\mathrm{I}}|\sin\left(S_{\mathrm{I}}-\Lambda_{\mathrm{I}}\right)-|a_{\mathrm{II}}|\,|b_{\mathrm{II}}|\sin\left(S_{\mathrm{II}}-\Lambda_{\mathrm{II}}\right)\right]\\ &+\cos2\phi\left[|a_{\mathrm{I}}|\,|b_{\mathrm{II}}|\sin\left(S_{\mathrm{I}}-\Lambda_{\mathrm{II}}\right)-|a_{\mathrm{II}}|\,|b_{\mathrm{I}}|\sin\left(S_{\mathrm{II}}-\Lambda_{\mathrm{I}}\right)\right]\big\}\,.\end{aligned} \tag{5.45}$$

Here, it was assumed that $kh\theta>1$. In the Fresnel diffraction zone $R_0\approx k\rho_0^2$, and the interference term is estimated as

$$I_{12}\sim k\rho_0\theta^2\,\frac{v_{\mathrm{c}}}{c}\,,$$

i.e., it is essentially determined by the velocity ratio. The relationships (5.43) and (5.45) acquire the simplest form when the electric vector in the wave incident on the cylinder and in the reference wave is directed along the x-axis: $a_{\mathrm{II}}=b_{\mathrm{II}}=0$, or the y-axis: $a_{\mathrm{I}}=b_{\mathrm{I}}=0$.

5.4 Finite region of a rotating medium (numerical simulation)

It appears from the above expressions for fields and scattering cross sections by a cylinder that the scattering by finite velocity nonuniformities differs from the scattering by small particles in the form of both the angular dependence and the dependence on the radiation wavelength λ. However, no explicit solution has been obtained to the problem of light scattering by nonuniformities of other types (e.g., a finite sphere). In this case, a numerical approach may be used to study differential scattering cross section [15, 31, 32].

Figure 3 depicts (in two forms) normalized differential cross sections $p(\theta,\phi)=P(\theta,\phi)/\sigma$ for a rotating cylinder of finite dimensions. In all calculations of σ, the angular velocity Ω was assumed to differ, and the linear rotational velocity at the boundary ($v_{\mathrm{c}}=1$ cm s$^{-1}\ll v_0$) to be constant, which allowed dynamo-optical effects in water to be disregarded. In the glass medium, the rotational velocity may be as high as 10 m s^{-1} [15], which accounts for the significant enlargement of the scattering cross section.

The results of calculations presented in Fig. 3 were obtained for a cylinder with the dimensions $k\rho_0=5$, $kh=5$. The total scattering cross section reached $\sigma(0)=4.653\times10^{-30}$ cm^2. A 1000-fold rise in the rotational velocity resulted in an increase in the cross section by six orders of magnitude. It was shown in Ref. [15] how strongly distorted the scattering field practically on a 'filament' ($\rho_0\ll h$) is, and what an important role is played by diffraction in this case.

Figure 4 presents the dependences of total scattering cross sections on the angle of incidence for small nonuniformities and cylinders of different sizes [32]. Interestingly, the total scattering cross sections for both the small scatterer and the 'filament' exhibit only a weak dependence on the angle of rotation axis tilt toward the direction of incidence. In the latter case, however, weak oscillations of the cross section are apparent (curves *1* and *2*). Simultaneously, the total cross section changes considerably (a threefold variation, see

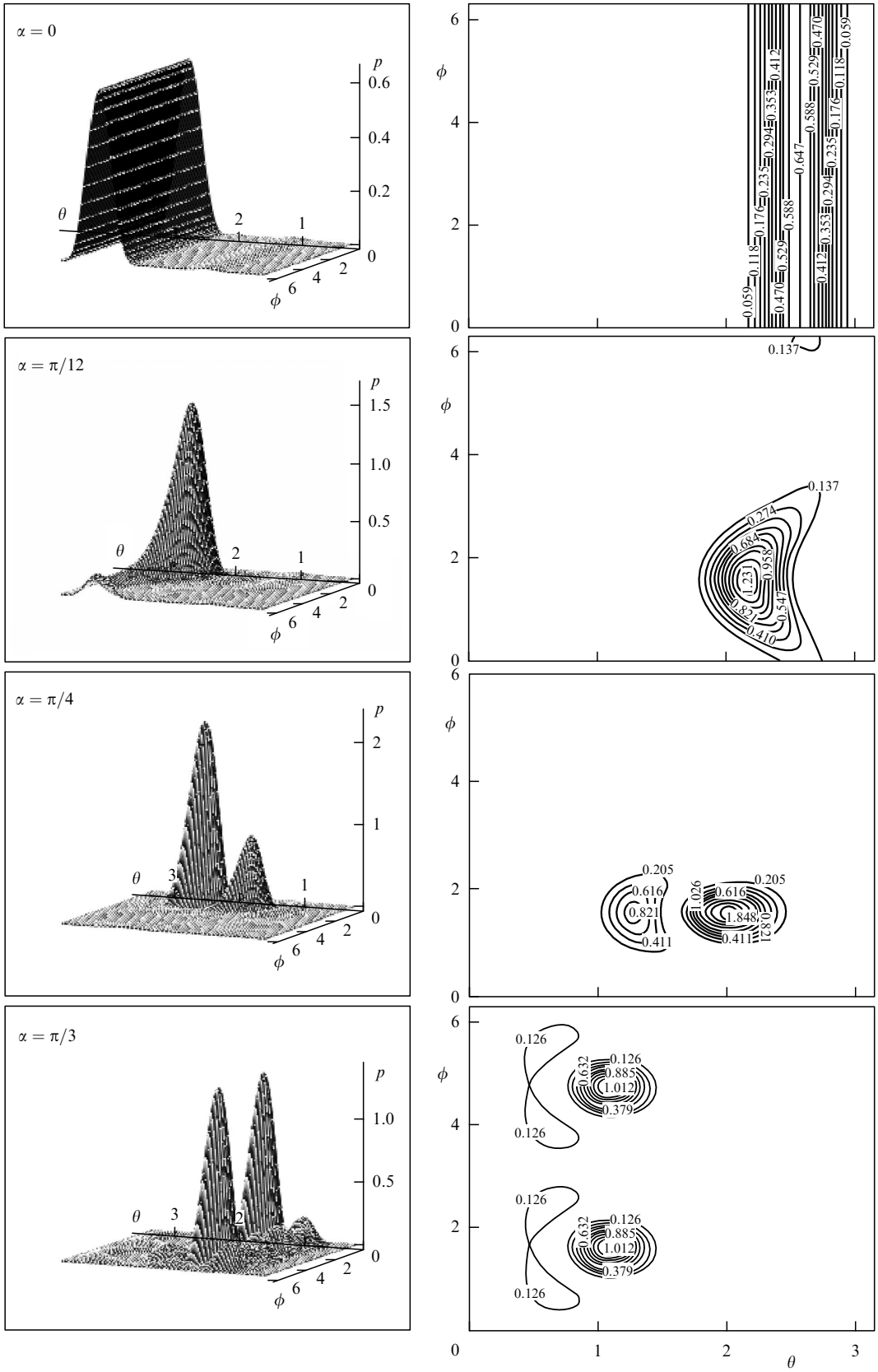

Figure 3. Normalized differential cross sections of light scattering by a finite cylinder at different tilt angles α: left — spatial graphs, and right — flat images (top view).

curve *3*). An unexpected result (curve *4*) was obtained when the base diameter ($2\rho_0$) of the cylinder was equal to its height ($2h$). As α was altered from 0 to $\pi/2$, the cross section changed by almost one order of magnitude. This means that in the case of scattering from a large number of scatterers the averaging over the tilt angles may acquire significance.

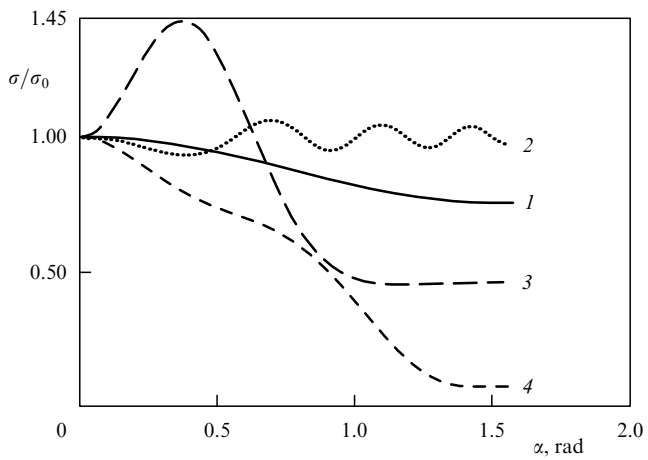

Figure 4. Tilt-angle dependence of the total scattering cross section for a small scatterer (curve *1*) and cylinders of different sizes (curve *2* — $k\rho_0 = 0.02$, $kh = 10$; curve *3* — $k\rho_0 = 5$, $kh = 0.05$, and curve *4* — $k\rho_0 = 5$, $kh = 5$); $\sigma_0 = \sigma(\alpha = 0)$.

For spherical nonuniformities, there is a single geometric parameter — the sphere radius r_0. Differential cross sections for a sphere with $kr_0 = 5$ are shown in Fig. 5 (without normalization to the total cross section). It can be seen that scattering into the back hemisphere at $\alpha = 0$ is replaced by that into the front hemisphere as α grows, with all the maxima being symmetrically localized along the line $\phi = \pi$. Simultaneously, the scattering cross section decreases two-fold, from $\sigma(0) = 4.11 \times 10^{-28}$ cm^2 to $\sigma(\pi/2) = 2.45 \times 10^{-28}$ cm^2. It is noteworthy that the scattering cross section from a sphere is more by two orders of magnitude than that from a cylinder of approximately the same size.

6. X-ray radiation in moving media

In the preceding sections, we considered light scattering by velocity nonuniformities of moving continuous media, based on the macroscopic Maxwell equations and the Minkowski constitutive relationships. However, the averaging of microscopic equations over an infinitesimally small volume is inapplicable to short-wave (X-ray) radiation with a wavelength on the order of interatomic distances. The present section is focused on the propagation and diffraction of short-wavelength (X-ray) radiation in a (nonuniformly) moving crystal examined by a generally accepted method [11].

The following microscopic Maxwell equations

$$\operatorname{div} \mathbf{H}_{\mathrm{m}} = 0\,, \qquad \operatorname{rot} \mathbf{E}_{\mathrm{m}} = -\frac{1}{c}\frac{\partial \mathbf{H}_{\mathrm{m}}}{\partial t}\,,$$
$$\operatorname{div} \mathbf{E}_{\mathrm{m}} = 0\,, \qquad \operatorname{rot} \mathbf{H}_{\mathrm{m}} = \frac{1}{c}\frac{\partial \mathbf{E}_{\mathrm{m}}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j}_{\mathrm{m}}\,, \tag{6.1}$$

are needed to describe the strengths of the electric and magnetic microfields $\mathbf{E}_{\mathrm{m}}$ and $\mathbf{H}_{\mathrm{m}}$, and the current density $\mathbf{j}_{\mathrm{m}}$ created by microfield-induced electron motions (in what follows, subscripts on field strengths and currents are omitted). Let a monochromatic plane wave be incident on the crystal, so that the field strengths in a laboratory system of coordinates have the form

$$\mathbf{E} = \mathbf{E}_0 \exp\left(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr}\right),$$
$$\mathbf{H} = \mathbf{H}_0 \exp\left(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr}\right), \tag{6.2}$$

where the vectors $\mathbf{E}_0$ and $\mathbf{H}_0$ are coordinate- and time-independent. The local (and instantaneous) crystal velocity in the laboratory frame of reference is a slowly varying function of the coordinates and time: $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$. The crystal is regarded as an absolutely solid body, which is justified by its rather small sizes. Quantities in the frame of reference rigidly aligned with the crystal are labelled by primes.

The equation of electron motion, neglecting the terms quadratic in the velocity of motion and in the field strengths, has the form

$$m\,\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t'} = e\mathbf{E}'\,, \tag{6.3}$$

where m is the mass, e is the charge, and $\mathbf{v}'$ is the electron velocity. In the crystal-aligned system of coordinates, an electron is under the action of the field

$$\mathbf{E}' = \left(\mathbf{E}_0 + \frac{1}{c}\,[\mathbf{u}\mathbf{H}_0]\right) \exp\left(-\mathrm{i}\omega' t' + \mathrm{i}\mathbf{k}'\mathbf{r}'\right), \tag{6.4}$$

where

$$\omega' = \omega - \mathbf{uk}\,, \qquad \mathbf{k}' = \mathbf{k} - \frac{\omega}{c}\,\frac{\mathbf{u}}{c}\,.$$

The steady solution of Eqn (6.3) has the form

$$\mathbf{v}' = \mathrm{i}\,\frac{e}{m\omega'^2}\left(\mathbf{E}_0 + \frac{1}{c}\,[\mathbf{u}\mathbf{H}_0]\right) \exp\left(-\mathrm{i}\omega' t' + \mathrm{i}\mathbf{k}'\mathbf{r}'\right). \tag{6.5}$$

It is supposed that the field-induced electron displacements are small; therefore, the velocity $\mathbf{u}$ may be regarded as practically constant. The expression for the current density $\mathbf{j}$ created by electrons with the number density n takes the form

$$\mathbf{j}' = e\,n(\mathbf{r}')\,\mathbf{v}'\,. \tag{6.6}$$

In order to solve the system of equations (6.1), it is necessary to go over to a laboratory (stationary) system of coordinates. According to Ref. [9], in such a system, with the quadratic-in-velocity terms being neglected, the expression for the current density remains unaltered:

$$\mathbf{j} = \mathrm{i}\,\frac{e\,n(\mathbf{r}, t)}{m\omega^2}\left(\mathbf{E}_0 + \frac{1}{c}\,[\mathbf{u}\mathbf{H}_0]\right) \exp\left(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr}\right). \tag{6.7}$$

We specially singled out the time- and coordinate-dependences of electron number density (without giving them an explicit form) because all other quantities, including $\mathbf{u}$, acquire their former values in the lab system.

Using the standard method [11, 33], it is not difficult to derive from Eqn (6.1) the equation defining the scattered field:

$$\Delta\mathbf{E}_1 - \frac{1}{c^2}\,\frac{\partial^2 \mathbf{E}_1}{\partial t^2} = \frac{4\pi}{c}\,\frac{\partial \mathbf{j}}{\partial t}\,. \tag{6.8}$$

Here, both the fields and the coordinates refer to the laboratory (stationary) frame of reference.

The solution of Eqn (6.8) has the form

$$\mathbf{E}_1 = -\int \frac{1}{R}\,\mathbf{f}_E\left(t - \frac{R}{c}\right) \mathrm{d}V\,, \tag{6.9}$$

where

$$\mathbf{f}_E(t) = \frac{1}{c}\,\frac{\partial \mathbf{j}}{\partial t}\,,$$

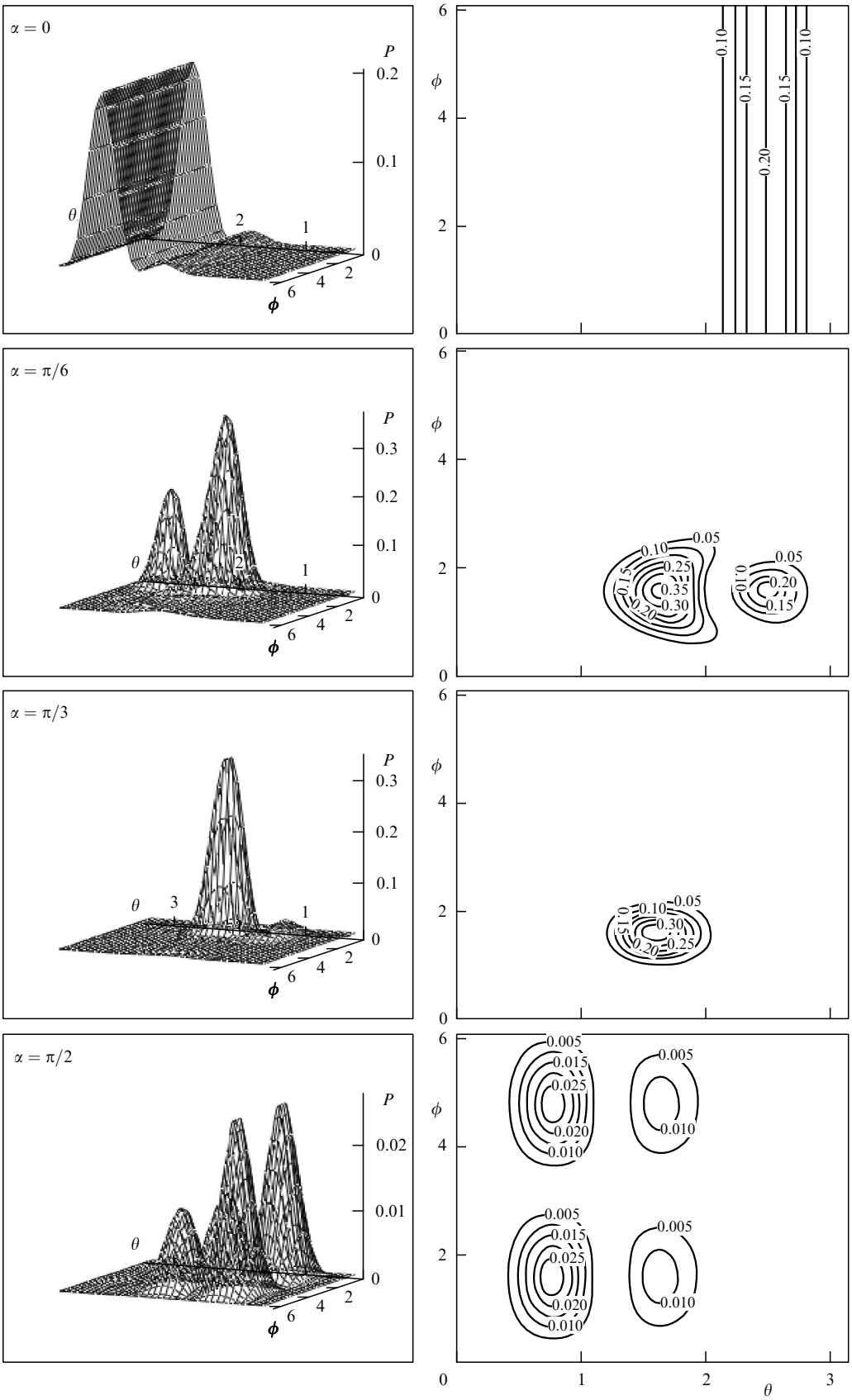

Figure 5. Differential cross section of light scattering by a sphere at different tilt angles α. Dimension of the sphere: $kr_0 = 5$; $\sigma(0) = 4.11 \times 10^{-28}$ cm^2, and $\sigma(\pi/2) = 2.45 \times 10^{-28}$ cm^2.

integration is carried out over the crystal volume, and R is the distance between the elementary volume $\mathrm{d}V$ and the point at which the fields are sought (see Section 5).

In the far zone, Eqn (6.9) can be rewritten as

$$\mathbf{E}_1 = -\frac{1}{R_0}\int \mathbf{f}_E\left(t - \frac{R_0}{c} + \frac{\mathbf{nr}}{c}\right) \mathrm{d}V. \tag{6.10}$$

The frequency of changes in density and velocity of motion being much smaller than ω, it may be assumed that

$$\frac{1}{c}\frac{\partial \mathbf{j}}{\partial t} \cong -\mathrm{i}\,\frac{\omega}{c}\,\mathbf{j}\,.$$

In this model, both the electron number density and the current in the crystal-aligned frame of reference are time-independent and the solution can be represented in the following form

$$\begin{aligned}\mathbf{E}_1 = {} & \frac{e^2}{m\omega^2}\,\frac{1}{R_0}\exp\left(-\mathrm{i}\omega t + \mathrm{i}kR_0\right) \\ & \times \int \mathrm{d}\mathbf{r}\, n(\mathbf{r}, t)\exp\left\{\mathrm{i}\mathbf{r}(\mathbf{k} - \mathbf{k}_s)\right\} \\ & \times \left\{\left[\mathbf{k}_s\left[\mathbf{k}_s\left(\mathbf{E}_0 + \frac{1}{c}\,[\mathbf{uH}_0]\right)\right]\right]\right\}. \end{aligned} \tag{6.11}$$

Here, $\mathbf{k}_s$ is the wave number of the scattered radiation, and

$$\mathbf{H}_0 = -\left[\frac{\mathbf{k}}{k}\,\mathbf{E}_0\right] = -[\mathbf{mE}_0]\,,$$

where $\mathbf{m}$ is the unit vector of the direction of the incident wave.

By introducing the unit vector in the scattering direction $\mathbf{n}$ [see formula (5.7)], Eqn (6.11) may be rewritten in the form

$$\begin{aligned}\mathbf{E}_1 = {} & \frac{e^2 k_s^2}{m\omega^2}\,\frac{1}{R_0}\exp\left(-\mathrm{i}\omega t + \mathrm{i}kR_0\right) \\ & \times \int \mathrm{d}\mathbf{r}\, n(\mathbf{r}, t)\exp\left\{-\mathrm{i}\mathbf{r}(\mathbf{k}_s - \mathbf{k})\right\} \\ & \times \left\{\left(1 - \frac{\mathbf{um}}{c}\right)\left[\mathbf{n}\,[\mathbf{nE}_0]\right] + \frac{\mathbf{uE}_0}{c}\,\left[\mathbf{n}\,[\mathbf{nm}]\right]\right\}. \end{aligned} \tag{6.12}$$

It is worth noting that the scattered radiation is no longer monochromatic. It can be accounted for by the spatial nonuniformity of the electron number density in the crystal-aligned lab frame, and by the corresponding nonstationary distribution in the fixed coordinate system.

Indeed, the crystal motion (including turns) is accompanied by a change in the mutual orientation of the incident radiation wave vector and the crystallographic axes. Specifically, rotation of a crystal with the angular velocity Ω must give rise to scattered radiation with the frequencies $\omega \pm \Omega$. Disregarding fluctuations, the electron number density in the crystal may be expanded in a Fourier series over all vectors of the reciprocal lattice $\mathbf{b}'$ [11, 34, 35], the direction of which is given by its translational periods [35]:

$$n(\mathbf{r}') = \sum_{\mathbf{b}'} n_{\mathbf{b}'}\exp\left(\mathrm{i}2\pi\,\mathbf{r}'\mathbf{b}'\right). \tag{6.13}$$

For further calculations, the form of the crystal motion needs to be given (a rule for coordinate transformations during passage from the system co-moving with the crystal to the lab system). In the quasistatic approximation, time plays the role of the parameter determining the angle of wave incidence on an effectively fixed crystal. It follows from Ref. [11] that the location of angular maxima of diffracted radiation is defined by the expression (the Laue equation)

$$\mathbf{k}_s - \mathbf{k} = 2\pi\,\mathbf{b}'(t)\,. \tag{6.14}$$

Naturally, their positions alter with time. This case corresponds to the Bragg method employed in the X-ray structural analysis to determine lattice constants or to the 'swing' and rotating sample method. In the general case, the relationship (6.2) describes additional relativistic corrections.

7. Conclusions

Let us consider the magnitudes of the above effects starting with the assessment of geometro-optical effects arising when radiation propagates through a moving medium. The first one is a deflection of the ray trajectory from the rectilinear direction as the radiation propagates across a moving medium layer of thickness z_0. By the order of magnitude, the deflection angle modulus equals

$$\alpha = \frac{n^2 - 1}{n}\,\frac{|\mathbf{v}|}{c}\,, \tag{7.1}$$

and the shift in the ray's transverse coordinates $\delta \sim \alpha z_0$.

Section 1 of this paper and Ref. [15] present estimates of medium motion velocities at which relativistic effects prevail over dynamo-optical ones. The critical velocity v_0 for water is on the order of a few centimeters per second, while for glass the value of v_0 may reach 10 m s^{-1}. Then, the angle of deflection for water (having refractive index $n = 1.34$) is $\alpha \approx 10^{-10}$, and for glass ($n = 1.5$) $\alpha \approx 3 \times 10^{-8}$. Ray shifts become apparent for sufficiently long trajectories.

The second effect concerns a change in polarization characteristics described by relationships (3.37)–(3.39). Ray bending being insignificant, integration in formula (3.38) may be replaced by integration along the z-axis, i.e., $\mathrm{d}s \cong \mathrm{d}z$ (paraxial approximation [23]). If the medium being considered rotates about the light propagation direction with an angular velocity Ω, the angle of rotation equals

$$\theta = \frac{n^2 - 1}{n}\,\frac{\Omega z_0}{c} = \alpha\left(\frac{z_0}{r_0}\right), \tag{7.2}$$

where z_0 is the thickness of the medium layer, and r_0 is the distance of the ray from the rotation axis.

As can be seen from Eqn (7.2), the angle of rotation of the polarization vector may be considerably larger than the ray deflection angle α. If the thickness z_0 of the layer is 100 times larger than r_0, the rotation angle in water $\theta \approx 10^{-8}$, while in glass the angle θ may be as large as 10^{-6}. Such rotations are readily observable due to the high accuracy of polarization measurements. They may have to be taken into account in long-haul optical communication schemes. There is a real opportunity to use these effects for diagnostics of the velocity distribution in moving liquids and, probably, gases.

It was shown in Section 4 that the focal distance of an equivalent lens resulting from the distinction of the flow velocity profile from a rectilinear one is large enough for the parameters of the Fizeau setup to be examined in experiment. In this way, allowing a light beam to pass through a pipe with water flowing in the opposite direction, it is possible to obtain its waveguide propagation; when the light and the water are moving in the same direction, the light beam broadens. The achievement of the waveguide propagation and focusing due to the Fresnel–Fizeau effect in the optical range is facilitated by the use of a transparent fluid with a larger refractive index than in water. Liquid xenon may serve as such transparent fluid at a wavelength of 10 μm and a pressure of 40 atm.

These considerations suggest the possibility of realizing nonreciprocal elements that in the previous laser technique

were largely built up around the Faraday effect in magneto-optic materials [24]. The application of such nonreciprocal elements can probably ensure the reliable unilateral generation in ring lasers. The relativistic effects discussed above may also be essential when applied to gas-dynamic lasers.

In light scattering by velocity nonuniformities, the scattering cross sections are small. However, if the nonuniformities are as large as a few radiation wavelengths, they are commensurate with the Raman scattering cross sections. It is also worth noting that disturbances in a laminar fluid flow are responsible for the appearance of an instability (fluctuation) domain [18, 36] in which light scattering is observable, even if weak. When it exceeds the light scattering by rotating nonuniformities, the changes introduced by these uniformities into the shape of the angular dependence of cross sections may be significant. The development of laser-based precision techniques [16, 17] gives hope that it will be possible to reliably observe relativistic effects. For example, it is not very difficult to detect a nonreciprocal lens in a ring laser, because its presence makes it even possible to ensure stable generation in a single propagation direction.

A few words about acoustic analogs of the above effects are in order. It was shown in Ref. [37] that waveguide propagation of acoustic oscillations may occur in a homogeneous moving medium in the case of a nonuniform velocity of motion. An important sign of the contribution from this mechanism (compared to the standard mechanism of waveguide propagation) is the difference in the conditions of sound propagation in the opposite directions. This nonreciprocity could be used to protect powerful acoustic radiators from the action of reflected radiation. Another analog is exemplified by acoustic wave scattering from velocity nonuniformities in a moving medium, described in Ref. [38]. Estimates reported in this work suggest the possibility of acoustic diagnostics of hydrodynamic velocity nonuniformities.

To summarize, the consideration of velocity nonuniformities in moving media leads to a wide circle of first-order electrodynamic relativistic effects. Their theory is fairly well described by the macroscopic Maxwell and Minkowski equations. For all that, further experiments are needed both because of the lack of observed data in support of moving media electrodynamics and because these effects have important technical implications, for example, for the development of non-Doppler methods with which to measure the velocity of drowned flows.

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