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Bose – Einstein condensates in a laser radiation field

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Contents

Abstract. The properties of Bose – Einstein condensates in an external potential produced by a laser light field are discussed. Considered are the condensates embedded in periodic lattices produced by standing laser waves and the condensates confined within 'optical traps' near the focus of a traveling wave. Observations of the Mott transition in a periodic lattice are described, as are experiments on atoms near the Feshbach resonance and experiments on the condensates in a double potential well, which permit investigating the tunnel dynamics of condensate phases.

1. Introduction

In his previous review, the author set forth the fundamental tenets of the Bose – Einstein condensation (BEC) theory [1]. The period of time elapsed since the experimental discovery of this amazing phenomenon has seen the performance of many interesting experiments of fundamental importance for the understanding of quantum processes in many-particle systems. Consequently, the theory has also gained a strong impetus.

In the proposed paper we dwell on a recently investigated range of phenomena. The case in point is the behavior of Bose – Einstein condensates in an external field produced by the light fields of laser sources. The reader should bear in mind that the number of papers concerning this matter is very large, although they account for only a small fraction of recent papers on BEC. I endeavored to describe papers of a fundamental nature. Of course, the author's preferences also play a part.

First of all, we should bear in mind some propositions of the BEC theory. Bose – Einstein condensation signifies that a macroscopically large number of atoms are 'condensed' in one quantum state. We consider the atom-annihilation operator $\hat{\psi}(\mathbf{r}, t)$ in the secondary quantization representation and separate out its $\hat{\psi}_0(\mathbf{r}, t)$ part which destructs the state into which the condensation takes place. Owing to the presence of a large number of bosons in this state, $\hat{\psi}_0(\mathbf{r}, t)$ may be replaced with the classical function $\psi_0(\mathbf{r}, t)$, which is referred to as the wave function of the condensate. This change has a profound physical significance. It is similar to the passage from quantum electrodynamics to the classical theory of electromagnetic phenomena. This transition is justified if a large number of photons are in a common quantum state. In this case, the noncommutativity of electromagnetic field operators is insignificant and the field may be described by the classical functions $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, which obey the Maxwell equations. In our case, the presence of a large number of atoms in the Bose – Einstein condensate permits introducing the classical function $\psi_0(\mathbf{r}, t)$. (In the subsequent discussion we omit the subscript '0'.)

When the density of gas in the condensation state is low enough, it is described by the mean field approximation, which corresponds to the first approximation of the Bogolyubov theory of a nonideal uniform Bose gas [2]. In this approximation, for $T = 0$ all particles reside in the condensate, so that the gas density is

$$n(\mathbf{r}, t) = \left|\psi(\mathbf{r}, t)\right|^2 . \tag{1}$$

The $\psi(\mathbf{r}, t)$ function itself satisfies the nonlinear equation [3, 4]

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{r}, t) + g\left|\psi(\mathbf{r}, t)\right|^2 \right] \psi(\mathbf{r}, t) . \tag{2}$$

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The nonlinear term in this equation describes the interaction between the condensate atoms, the interaction constant g being defined by the scattering length a for atoms in the s state:

$$g = \frac{4\pi\hbar^2 a}{m} . \tag{3}$$

The energy of interaction of a given atom with the remaining ones is proportional to the atomic number density. It is as if the atom travels in the mean field produced by other atoms. That is why the theory based on Eqn (2) is conventionally referred to as the mean field theory.

The Gross – Pitaevskii equation (2) for the condensate wave function $\psi(\mathbf{r}, t)$ plays a similar role to the Maxwell equations in classical electrodynamics. It is valid to say that this function is the classical limit of the de Broglie wave, in which the corpuscular properties of matter play no part. Unlike the Maxwell equations, however, Eqn (2) contains the Planck constant $\hbar$. This difference is underlain by the difference between the relations binding the energy ϵ and the momentum p for photons and atoms, which leads to different relations between the frequency $\omega = \epsilon/\hbar$ and the wave vector $k = p/\hbar$ of the corresponding classical waves. For photons, the equation $\epsilon = cp$ gives the classical dispersion relation $\omega = ck$, which does not contain the quantum constant. For atoms, the equation $\epsilon = p^2/2m$ gives the dispersion relation $\omega = \hbar k^2/2m$ instead, which contains the quantum constant $\hbar$ explicitly. This signifies, in particular, that the interference effects for classical waves of matter depend on the magnitude of $\hbar$ [see, for instance, Eqn (54) below].

In a stationary state, the wave function depends on the time according to the Josephson equation

$$\psi(\mathbf{r}, t) = \psi(\mathbf{r}) \exp\left(-\frac{\mathrm{i}\mu t}{\hbar}\right) , \tag{4}$$

where the constant μ has the meaning of the chemical potential of a given state. In this case, Eqn (2) reduces to the form

$$\left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{r}, t) + g\left|\psi(\mathbf{r}, t)\right|^2 - \mu\right]\psi(\mathbf{r}, t) = 0 . \tag{5}$$

A mathematically rigorous derivation of Eqn (5) for a dilute gas was made by Lieb, Seiringer, and Ingvason [5].

2. An atom in a light field

Let us consider a condensate embedded in the field of a monochromatic light wave. The electric field of the wave can be written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) \exp(-\mathrm{i}\omega t) + \mathrm{c.c.} . \tag{6}$$

The time-average force acting on a gas atom is

$$\mathbf{f}(\mathbf{r}) = \alpha(\omega)\,\nabla\,\overline{\frac{E^2}{2}} = \alpha(\omega)\nabla|E_0|^2 , \tag{7}$$

where $\alpha(\omega)$ is the polarizability of the atom. This formula is valid when the frequency ω is far enough from the atomic absorption line ω_0, so that $\alpha(\omega)$ is real. On the other hand, it is expedient to operate sufficiently close to ω_0, where the polarizability is high and hence the force is strong. Therefore, the frequency should satisfy the conditions

$$\omega_0 \gg |\omega - \omega_0| \gg \Gamma , \tag{8}$$

where Γ is the absorption line width. It is noteworthy that the Bose – Einstein condensate is a convenient phenomenon in this respect, too, because atomic absorption lines in the condensate are extremely narrow.

Under condition (8), the polarizability can be approximately written as

$$\alpha(\omega) \approx \frac{A}{\omega_0 - \omega} , \tag{9}$$

with $A > 0$. This signifies that the atoms are drawn into the strong-field region for $\omega < \omega_0$ ('red detuning') and forced out of it for $\omega > \omega_0$ ('blue detuning').

To force (7) there corresponds the average potential energy of an atom embedded in a light field

$$U_{\mathrm{opt}}(\mathbf{r}) = -\alpha(\omega)|E_0|^2 . \tag{10}$$

A one-dimensional periodic potential may be produced by a standing light wave. In such a wave, $|E_0|^2 \sim \sin^2(qz)$, where q is the wave vector. In this case, the potential energy is commonly written as

$$U_{\mathrm{opt}}(z) = sE_r \sin^2(qz) , \tag{11}$$

where E_r is the so-called 'recoil energy' (m is the atomic mass)

$$E_r = \frac{\hbar^2 q^2}{2m} \tag{12}$$

and s is the dimensionless parameter proportional to the laser beam intensity. In the majority of real experiments, $s < 20$. The potential (11) has a period $d = \pi/q$. We point out that the period of the reciprocal lattice is $2q$. Three mutually orthogonal laser beams make up a three-dimensional lattice:

$$U_{\mathrm{opt}}(\mathbf{r}) = sE_r\left[\sin^2(qx) + \sin^2(qy) + \sin^2(qz)\right] . \tag{13}$$

3. Bose – Einstein condensate in a one-dimensional periodic potential. Ground state

In this section we mostly consider, for the sake of simplicity, one-dimensional condensate motion in the periodic potential (11). We thereby assume that the condensate wave function is of the form $\psi(z)\exp(-\mathrm{i}\mu t/\hbar)$. Naturally, in experiment there is also, apart from this potential, the confining potential of the 'trap'. However, the condensate dimension is, as a rule, large in comparison with the lattice period d. The confining potential changes little for this distance and in many problems it may be neglected to a first approximation. Then, Eqn (5) for $\psi(z)$ reduces to the equation

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + sE_r\sin^2(qz) + g\left|\psi(z)\right|^2\right]\psi(z) = \mu\psi(z) . \tag{14}$$

It is common knowledge that the quantum-mechanical description of particle motion in a periodic field was elaborated by F Bloch many years ago. Up to now, however, experiments have primarily dealt with electrons, i.e., fermions, in a periodic crystal lattice.

We will see that the behavior of a Bose – Einstein condensate in a periodic field exhibits many nontrivial features. Furthermore, artificial 'optical' lattices differ favorably from 'natural' ones in many respects. Their period is macroscopically long, which facilitates observations, and is

controllable, as is the intensity of the periodic potential. They are practically devoid of defects. Lastly, it is easy to produce one- and two-dimensional structures which are hard to realize in solids.

It therefore comes as no surprise that even the first experiments on condensates in optical lattices yielded very interesting results. Special mention should be made of the discovery of a new physical effect: the Mott transition between the superfluid phase of Bose atoms in a lattice and the dielectric phase, in which superfluid flow is impossible (see Section 8 of this review).

Let us consider the ground state of the system. The wave function of this state is the real periodic function $\psi_0(z) = \psi_0(z + d)$. It may be expanded into the Fourier series

$$\psi_0(z) = \sum_{l=-\infty}^{\infty} \psi^{(l)} \exp(\mathrm{i}l2qz), \qquad \psi^{(l)} = \psi^{(-l)*}. \tag{15}$$

In accordance with the basic principles of quantum mechanics, expression (15) is the expansion of the ground state in terms of states with momentum values equal to $p_l = \hbar 2ql$. This statement is well known in the theory of electrons in a crystal lattice, where this statement is hard to verify experimentally. In a condensate, it can be verified in a direct and elegant experiment. To do this it would suffice to rapidly turn off the confining potential and the periodic lattice potential at some point in time. Then, atoms with $l = 0$ would remain immobile, while those with momenta p_l would recede from them by the law [1]

$$z_l(t) = \frac{\hbar l 2q}{m}\, t. \tag{16}$$

Figure 1 shows the spatial condensate density distribution obtained in experiments by Pedri et al. [6]. One can clearly see the central group of atoms with $l = 0$ and the 'detached' condensates with $l = \pm 1$. The ratio P_1 between the number of atoms with $l = 1$ and the number of atoms in the central peak, according to expression (15), is $|\psi^{(1)}|^2/|\psi^{(0)}|^2$. This ratio depends on the amplitude s of the periodic potential and (owing to the existence of interatomic interaction) on the gas density. The P_1 values measured in Ref. [6] and those calculated by solving Eqn (14) are compared in Fig. 2.

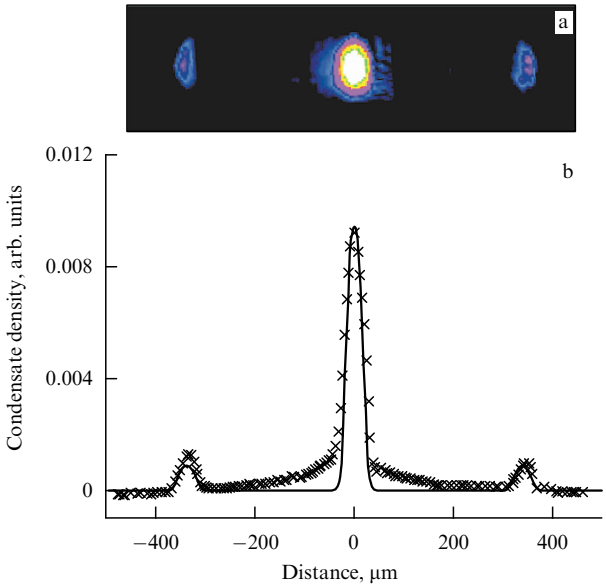

Figure 1. (a) Absorption image of the condensate obtained 10 s after its release from the optical lattice. (b) Spatial density distribution of the condensate. The ×'s represent the experimental data corresponding to the upper Figure. The solid line represents the theoretical predictions by Pedri et al. [6]. The parameter $s = 5$.

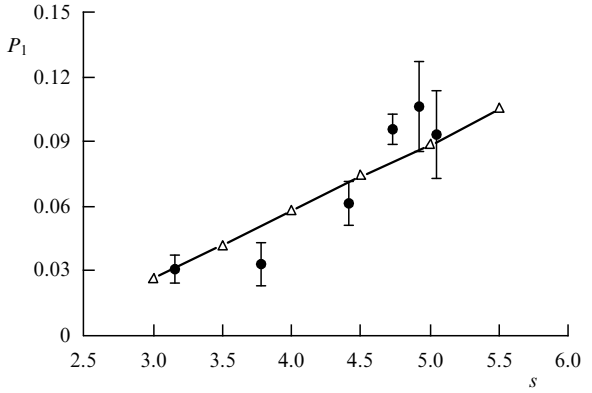

Figure 2. First additional-to-central peak population density ratio. Black circles represent the experimental data and triangles the theory of Ref. [6].

4. Quasimomentum and mass flux

The system of bosons in the Bose–Einstein condensation state is superfluid and can move without friction relative to the lattice. The solutions of Eqn (14), which describe the flow of condensate as a whole, are of the form of the Bloch functions

$$\psi_{kr}(z) = \exp(\mathrm{i}kz)\, u_{kr}(z). \tag{17}$$

Here, as usual, $\hbar k$ is the quasimomentum, r is the zone number, i.e., a discrete index that numbers the solutions for a given k, and $u_{kr}(z)$ is a complex periodic function of z with a period d. We emphasize that the very existence of solutions of the form of expression (17) to the nonlinear equation (14) is absolutely nontrivial and is intimately related to the gauge invariance of this equation. The chemical potential corresponding to solution (17) is also a function of k and r: $\mu = \mu_r(k)$. In the subsequent discussion we will usually omit the zone index r in all quantities. Integration of solution (17) yields the state energy. We will consider the energy per unit length of the lattice:

$$E(k) = \frac{1}{L}\int_0^L \left[\frac{\hbar^2}{2m}\left|\frac{\mathrm{d}\psi_k}{\mathrm{d}z}\right|^2 + sE_r \sin^2(qz)|\psi_k(z)|^2 + \frac{g}{2}|\psi_k(z)|^4\right]\mathrm{d}z, \tag{18}$$

with $\mu(k) = \partial E(k)/\partial n$, where $n = N/L$ is the number of atoms per unit length, L is the lattice length, and N is the

[1] This result, which we obtained by means of simplified reasoning, is in fact the special case of a general theorem of quantum mechanics. If the dimension of a system upon expansion is far greater than the initial one and the interatomic interaction during the expansion may be neglected, the spatial density distribution upon the expansion reproduces the initial momentum distribution: $n(\mathbf{r}, t) = (m/t)^3 n^{(p)}(\mathbf{p} = m\mathbf{r}/t, 0)$ for $t \to \infty$, where $n^{(p)}(\mathbf{p}, t)$ is the distribution function in the momentum space normalized by the condition $\int n^{(p)}(\mathbf{p}, t)\, \mathrm{d}^3 p = N$.

total number of atoms. In lieu of the energy $E(k)$, quite often it is more expedient to employ the energy of the grand canonical ensemble

$$E_\mu(k) = E(k) - \mu(k)n = E(k) - \mu(k)\int_0^L \frac{|\psi_k(z)|^2\,\mathrm{d}z}{L}\,, \qquad (19)$$

which coincides with the thermodynamic potential Ω for $T = 0$, so that $n = -\partial E_\mu(k)/\partial\mu$. It is significant for the subsequent discussion that Eqn (14) can be obtained as the condition for the minimum of the functional E in ψ for a constant n, or of the functional E_μ for a constant μ.

It is well known that the values of a wave vector k differing by the reciprocal lattice vector $2q$ are physically equivalent. It would therefore suffice to consider only the values of k lying in the first Brillouin zone $-q < k < q$. However, k is sometimes conveniently allowed to assume arbitrary values when $E(k)$ [and $j(k)$ below] are considered as periodic functions with a period $2q$.

Wave functions (17) describe the stationary states of the system. In these states, the condensate moves and there is a mass flux. It can be calculated if we resort to the general expression for the flux following from Eqn (2). It coincides with the conventional quantum-mechanical expression for the mass flux from the one-particle Schrödinger equation, and the nonlinear term plays no part in this case. In the one-dimensional case,

$$j = \frac{\mathrm{i}\hbar}{2}\left(\psi\,\frac{\mathrm{d}\psi^*}{\mathrm{d}z} - \psi^*\,\frac{\mathrm{d}\psi}{\mathrm{d}z}\right). \qquad (20)$$

In the stationary state, the flux j is constant and is independent of t and z. It can be calculated for the Bloch wave function (17) with the knowledge of $E(k)$ as a function of k.

To derive the corresponding formula, in Eqn (14) and accordingly in the energy functional (19) we replace the operator $\mathrm{d}/\mathrm{d}z$ with $[(\mathrm{d}/\mathrm{d}z) - \mathrm{i}A]$, where A is the constant 'vector potential'. Then, it is easily verified by direct differentiation that

$$-m\left[\frac{\partial E_\mu}{\partial\hbar A}\right]_{A=0,\,\mu} = \frac{\mathrm{i}\hbar}{2L}\int_0^L\left[\psi\,\frac{\mathrm{d}\psi^*}{\mathrm{d}z} - \psi^*\,\frac{\mathrm{d}\psi}{\mathrm{d}z}\right]\mathrm{d}z = j\,. \qquad (21)$$

In this case, it is significant that the wave function, which depends on A itself, should not be differentiated, because the variation of the functional in ψ is equal to zero by virtue of the equation for ψ. Introducing the constant A does not change the periodicity of the equation, and therefore the modified wave function $\psi(z, A)$ is of the Bloch form. However, the constant A can be eliminated from the equation by the gauge transformation

$$\psi = \exp(\mathrm{i}Az)\,\psi'\,. \qquad (22)$$

Therefore, the modified equation has the same dispersion law $E_\mu(k')$, where $\hbar k'$ is the quasimomentum corresponding to the function ψ'. Equality (22) signifies, however, that the function $\psi(z, A)$ possesses the quasimomentum $\hbar k = \hbar(k' + A)$, so that $E_\mu(k, A) = E_\mu(k - A)$. In view of this relation, formula (21) gives the desired expression for the flux:

$$j(k) = m\left(\frac{\partial E_\mu}{\hbar\,\partial k}\right)_\mu = m\left(\frac{\partial E}{\hbar\,\partial k}\right)_n. \qquad (23)$$

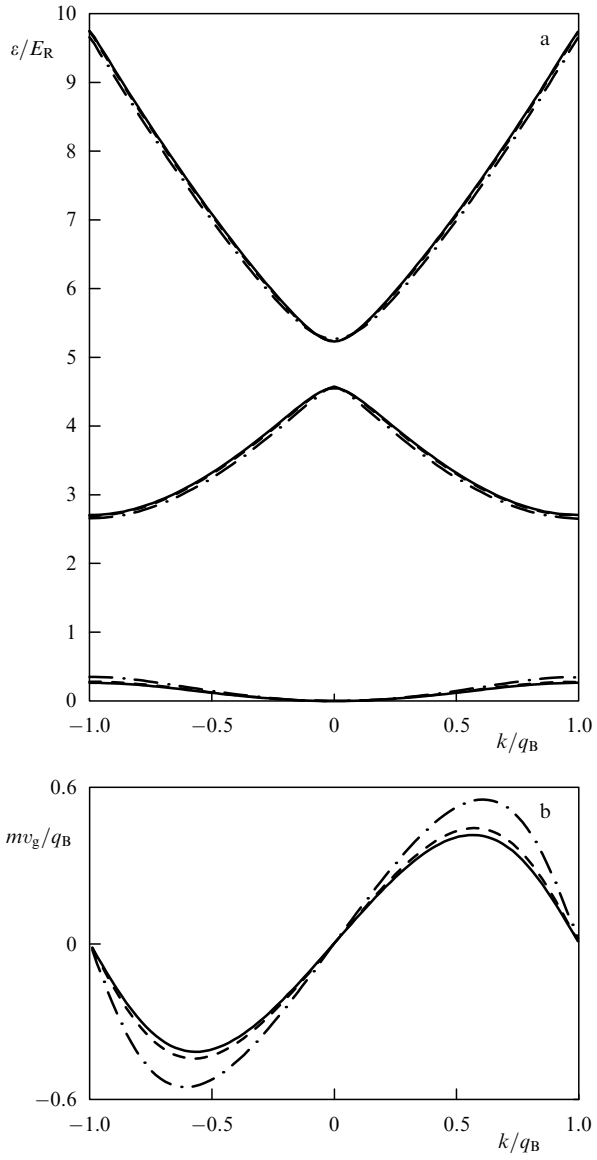

Figure 3. (a) Energy as a function of quasimomentum for the first three zones in a one-dimensional Bloch lattice with the potential (11), $s = 3$; (b) group velocity vs quasimomentum [7].

This equation plays an important part in the theory of metals. Its ordinary derivation, however, essentially relies on the linearity of the Schrödinger equation. Furthermore, in a linear theory it remains unclear whether the flux is defined by the functions $E(k)$ or $\mu(k)$, for in such a theory $E = n\mu$. That it why I adduced another derivation which invokes the gauge invariance of the Gross – Pitaevskii equation (2). This derivation is, of course, also suitable for the conventional Schrödinger equation. The last-given equality is expediently rewritten by introducing the 'group velocity':

$$v_\mathrm{g}(k) \equiv \frac{1}{\rho}\,j(k) = \frac{1}{n}\left(\frac{\partial E}{\partial\hbar k}\right)_n, \qquad (24)$$

where $\rho = mn$ is the linear density. The functions $E_r(k)$ for the first three zones and the group velocity $v_\mathrm{g}(k)$ in the first zone are plotted in Fig. 3 [7].

Equation (23) defines the mass flux in the lattice frame of reference. When the lattice moves with a velocity v_lat, the mass

flux in the laboratory system of coordinates is given by the general formula of the Galilean transformation:

$$j_{\rm lab} = j(k) + \rho v_{\rm lat} \,. \tag{25}$$

We now consider the condensate behavior in circumstances where the condensate experiences, apart from a periodic field, a weak uniform force F. Then, on the right-hand side of the temporal equation (2) one should add the term $-Fz\psi(z,t)$. Despite the smallness of F, this term significantly changes the solution, for it unrestrictedly increases with z. To compensate for it, we will seek the solution of Eqn (2) in the form of Bloch function (17) with a time-dependent quasimomentum $\hbar k$. Then, on the left-hand side of the equation there appears the term $-\hbar(\mathrm{d}k/\mathrm{d}t)z\psi(z,t)$, which is proportional to z. The terms proportional to z cancel out when

$$\frac{\mathrm{d}\hbar k}{\mathrm{d}t} = F. \tag{26}$$

We point out that the periodic part of Bloch function (17) should not be differentiated with respect to time. The corresponding terms do not contain z and are negligible for weak F. Equation (26) again is in the same form as in the conventional one-particle theory.

When the force F is constant, Eqn (26) has a simple solution:

$$\hbar k = Ft \,. \tag{27}$$

However, since the values of k that differ by a whole multiple of $2q$ are physically equivalent, all observable quantities will be periodic functions of time with the period

$$T_{\rm B} = \frac{2q}{\hbar F} \,. \tag{28}$$

Since the quantity q is known to a high degree of accuracy, observing these *Bloch oscillations* permits measuring the force F to a high degree of accuracy [8, 9]. Roati et al. [8] experimentally observed the periodic variation of the momentum distribution of atoms in the lattice. However, polarized fermions are better suited for this kind of measurement. The interatomic interaction leads to oscillation damping, while polarized cold Fermions barely interact with each other.

Employing the resulting equations, in experiment it is possible to directly determine the function $v_{\rm g}(k)$ [10]. In the experiment, the lattice was set in motion. To this end, a small frequency difference $\delta\omega$ was introduced between the laser beams that produced the standing wave. In this case, the lattice moves with a time-dependent velocity $v_{\rm lat}(t) = \delta\omega(t)/q$. In the lattice frame of reference, the condensate experiences the force of inertia $F = -m\,\mathrm{d}v_{\rm lat}/\mathrm{d}t$. According to Eqn (26), by the point in time t the condensate acquires the quasimomentum

$$\hbar k = -mv_{\rm lat}(t) \,. \tag{29}$$

At this point in time, the fields of the lattice and the trap are turned off and measurements are made of the average condensate motion velocity, which is, according to formula (25), equal to

$$v_{\rm m} \equiv \frac{j_{\rm lab}}{\rho} = v_{\rm lat} - v_{\rm g}\left(\frac{mv_{\rm lat}}{\hbar}\right). \tag{30}$$

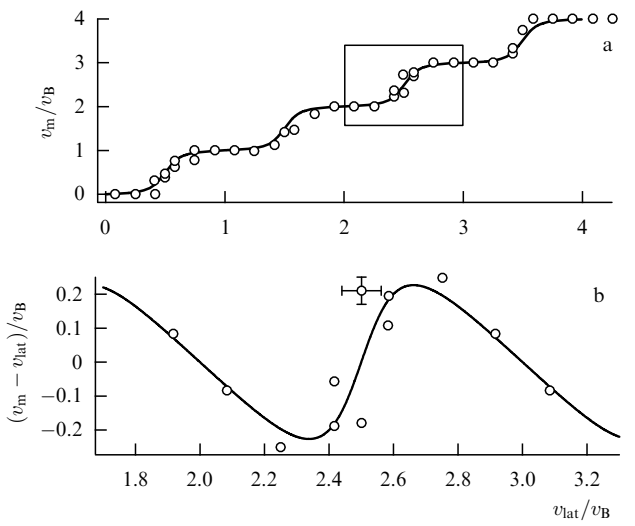

Figure 4. Measured dependences on the lattice velocity $v_{\rm lat}$ for (a) the average condensate velocity $v_{\rm m}$ and (b) the average velocity in the lattice frame of reference $v_{\rm m} - v_{\rm lat} = -v_{\rm g}$. In units of $v_{\rm B} = 2\pi\hbar/md$ [10].

By repeating this experiment for different values of the final lattice velocity $v_{\rm lat}$, it is possible to reconstruct the function $v_{\rm g}(k)$. The results of these measurements are plotted in Fig. 4.

For small values of the quasimomentum, the function $E(k)$ may be expanded as

$$E(k) = E_0 + n\,\frac{\hbar^2 k^2}{2m^*} \,. \tag{31}$$

The quantity m^* means the effective mass of condensate atoms in the lattice. Accordingly,

$$v_{\rm g}(k) = \frac{\hbar}{m^*}\,k \,. \tag{32}$$

It is easily shown with the use of the quantum-mechanical perturbation theory that $m^* > m$ in the first Bloch zone. In Section 5 we will see that the effective mass defines the density of the superfluid gas fraction. A similar expansion of the chemical potential is of the form

$$\mu(k) = \mu_0 + \frac{\hbar^2 k^2}{2m_1} \,, \tag{33}$$

with

$$\mu_0 = \frac{\partial E_0}{\partial n} \,, \qquad \frac{1}{m_1} = \frac{\partial}{\partial n}\,\frac{n}{m^*} \,. \tag{34}$$

5. Density of the superfluid fraction and dipole vibrations

According to the fundamental tenets of the Landau theory of superfluidity [11], a homogeneous superfluid liquid for $T = 0$ is entirely superfluid, $\rho_{\rm s} = \rho$. This is not the case for a liquid in nonuniform ambient conditions, for instance, in the presence of impurities or, as in our case, of a periodic lattice field. In this case, $\rho_{\rm s} < \rho$ even for $T = 0$. However, the situation is different from that which occurs at a finite temperature. The normal liquid fraction, whose density is $\rho_{\rm n} = \rho - \rho_{\rm s}$, is at rest relative to the lattice and there is no point in introducing the velocity of the normal fraction.

The superfluid fraction in the Landau theory is 'that which cannot rotate'. This leads to the following definition of the superfluid density. Let us assume that the lattice is 'rolled up' in a ring of large radius R. Let the ring rotate slowly with an angular velocity Ω. Then, the angular momentum of the nonsuperfluid would be $M = \rho 2\pi R^2 \Omega$. If the liquid is superfluid, only its normal fraction rotates and the angular momentum is

$$M = \rho_n 2\pi R^2 \Omega = (\rho - \rho_s) 2\pi R^2 \Omega \,. \tag{35}$$

This is precisely the way the superfluid density was measured in the pioneering experiments of Andronikashvili. Bishop and Reppy [12] employed this method to measure ρ_s in spatially inhomogeneous systems: superfluid ^{4}He films on a Vycor substrate. For a condensate in an optical lattice, this experiment should be considered as an imaginary experiment.

Our derivation calls for the Galilean transformation of the wave function. Let

$$\psi(z, t) = \psi_k(z) \exp\left(-\frac{i\mu t}{\hbar}\right)$$

be the wave function in the lattice frame of reference. Then, the wave function in the laboratory system of coordinates, in which the lattice moves with a velocity v_{lat}, is

$$\psi_{lab}(z, t) = \exp\left(\frac{i m v_{lat} z}{\hbar}\right) \psi(z - v_{lat} t, t) \exp\left(-\frac{i m v_{lat}^2 t}{2\hbar}\right). \tag{36}$$

[See Ref. [13], the problem to § 17. It is also easy to directly verify that function (36) satisfies Eqn (2) when $\psi_k(z)$ is the solution of Eqn (14).]

We employ expression (17) for ψ_k to find that

$$\psi_{lab}(z, t) = \exp\left[i\left(k + \frac{m v_{lat}}{\hbar}\right) z\right] u_k(z - v_{lat} t) \times \exp\left\{-\frac{i\left[\mu(k) + \hbar k v_{lat} + m v_{lat}^2/2\right] t}{\hbar}\right\}. \tag{37}$$

Now let the lattice be rolled up in a ring. When the ring radius R is sufficiently large, we may take advantage of this equation, where z is taken to mean the coordinate along the ring and the change $v_{lat} = \Omega R$ is made. The unambiguity of the wave function now requires that it be periodic in z:

$$\psi_{lab}(z, t) = \psi_{lab}(z + 2\pi R, t) \,. \tag{38}$$

According to Eqn (36) this signifies that

$$\left(k + \frac{m v_{lat}}{\hbar}\right) 2\pi R = 2\pi l, \qquad l = 0, \pm 1, \ldots \,. \tag{39}$$

In the presence of rotation, $l = 0$. This condition should therefore be fulfilled for a slow rotation as well. This gives $\hbar k = -m v_{lat}$ again. We calculate the mass flux according to expression (25) to find for a low rotation velocity that

$$j_{lab} = m n v_{lat} + \frac{m}{\hbar}\left[\frac{\partial E(k)}{\partial k}\right]_{k = -m v_{lat}/\hbar} \approx m n v_{lat}\left(1 - \frac{m}{m^*}\right), \tag{40}$$

so that the angular momentum is

$$M = j_{lab} 2\pi R = m n \Omega 2\pi R^2 \left(1 - \frac{m}{m^*}\right). \tag{41}$$

By way of comparison with expression (35) we find the superfluid density:

$$\rho_s = m n \frac{m}{m^*} = \rho \frac{m}{m^*} \,. \tag{42}$$

We return to the lattice frame of reference. Knowing ρ_s lets us determine the superfluid velocity according to $j = \rho_s v_s$. Therefore,

$$v_s = \frac{\hbar}{m} k \,. \tag{43}$$

Expression (42) for the superfluid density was obtained in a different way in the papers by Eggington [14] and Ambegaokar et al. [15] concerned with the properties of superfluid ^{4}He films. However, recently there have appeared papers dedicated to the Bose gas in a one-dimensional lattice in the presence of disorder whose authors adhere to the opinion that in the absence of interaction $\rho_s = \rho$. This is evidently at variance with expression (42), because $m^* \neq m$ even in the absence of interaction. In particular, Rapsch et al. [16] believe that $\rho_s/\rho = 1$ in the absence of interaction and disorder. Similarly, Roth and Burnett [17] state that the superfluid fraction is equal to unity for a noninteracting system. This variance impelled me to make a more rigorous, in my view, derivation of the expression for the superfluid fraction density. My result coincides, as noted above, with that of Refs [14, 15]. As regards the disagreement with Refs [16, 17], it is likely to arise merely from misunderstanding. The authors of these papers, it seems to me, identified the effective mass of the Hamiltonian in the Bose–Hubbard model with the physical mass of a free particle.

The first experimental confirmation of formula (42) for the superfluid fraction density was obtained by Cataliotti et al. [18]. The authors observed the vibrations of the condensate embedded in a periodic optical potential and the harmonic field of a magnetic trap. To describe this effect requires some development of the theory outlined.

The wave vector k in solution (17) is constant. However, one may consider approximate solutions of the form

$$\psi(z, t) = \exp\left[i\phi(z, t)\right] u_{k = \mathrm{d}\phi/\mathrm{d}z}(z) \,, \tag{44}$$

where the phase $\phi(z, t)$ slowly varies in distances on the order of the lattice period d. The superfluid velocity will then be coordinate- and time-dependent and formula (43) will assume the form

$$v_s = \frac{\hbar}{m} \frac{\partial}{\partial z} \phi \,. \tag{45}$$

The equations for the density ρ and the superfluid velocity can be derived similarly to the Landau equations [11] for the fluid dynamics of a superfluid liquid when it is considered that the entropy and velocity of the normal fraction are equal to zero in our case.

The continuity equation is of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho_s v_s)}{\partial z} = 0 \,. \tag{46}$$

When writing the equation of motion we assume that there is also, apart from the periodic potential, a slowly varying one

$U_{\rm ext}(z)$. Then,

$$\frac{\partial v_{\rm s}}{\partial t}+\frac{\partial}{\partial z}\left[\frac{m}{m_1}\,\frac{v_{\rm s}^2}{2}+\mu_0(n)+U_{\rm ext}(z)\right]=0\,, \tag{47}$$

where we neglected terms of higher order than $v_{\rm s}^2$.

With the aid of these equations we consider the condensate vibrations in a periodic lattice in the presence of the harmonic potential $U_{\rm ext}(z)=m\omega_z^2 z^2/2$, which were investigated by Cataliotti et al. [18]. We assume that the effective mass m^* is independent of the gas density and hence of the coordinates and the time. (Under experimental conditions, this dependence is quite insignificant.) Then, as is easily verified, the linearized system of equations (46), (47) has a solution in the form $\rho(z,t)=\rho_0[z-z_0(t)]$, which describes the vibrations of the condensate as a whole with a velocity $\mathrm{d}z_0/\mathrm{d}t$. Here, $\rho_0(z)$ is the equilibrium condensate density, which satisfies the condition

$$\frac{\partial}{\partial z}\left(\mu_0\left[\rho_0(z)\right]+U_{\rm ext}(z)\right)=0\,.$$

The equations then yield

$$\frac{\mathrm{d}z_0}{\mathrm{d}t}=\frac{m}{m^*}\,v_{\rm s}\,,\qquad \frac{\mathrm{d}v_{\rm s}}{\mathrm{d}t}=-m\omega_z^2 z_0\,. \tag{48}$$

The former of these equations shows that the condensate travels with a group velocity $(m/m^*)v_{\rm s}=v_{\rm g}$ rather than with a superfluid velocity $v_{\rm s}$. According to Eqn (48), the frequency of these 'dipole' vibrations is [18]

$$\omega_{\rm d}=\sqrt{\frac{m}{m^*}}\,\omega_z\,. \tag{49}$$

In experiments, at some point in time the lattice was swiftly shifted from its initial position and thereby excited the condensate vibrations. After some 'time delay', the lattice and the confining potential were removed, the gas expanded, and the density distribution was recorded. Subsequently, the experiment was repeated with a different delay in the lattice removal relative to the onset of vibrations.

The dependence of the dipole vibrations frequency on the amplitude s of the periodic potential measured in Ref. [18] is depicted in Fig. 5. The experimental data are compared with the theoretical formula (49). The values of the effective mass were calculated by Krämer et al. [19] by way of a numerical solution of Eqn (14). In this case, owing to the diluteness of gas, the interaction between the atoms and therefore the nonlinear term in this equation turned out to be practically insignificant.

It is noteworthy that the m/m^* ratio corresponding to the maximum attained value $s=9$ turns out to be approximately equal to 6.3. Under these conditions, the atoms can move only by way of tunneling across the potential barriers separating the minima of the periodic potential. Therefore, the quantum tunneling of a macroscopic body was observed in the experiment. Here, the superfluid nature of condensate flow is of significance. A thermal atomic cloud cannot vibrate under these conditions (Fig. 6).

Wave function (17) can be expanded in terms of the states with momentum values $p(k,l)=\hbar(k+2ql)$, $l=0,\pm1,\ldots$. Similarly to expansion (15), we have:

$$\psi_{rk}(z)=\sum_{l=-\infty}^{\infty}\psi_{rk}^{(l)}\exp\left[\mathrm{i}(k+l2q)z\right]. \tag{50}$$

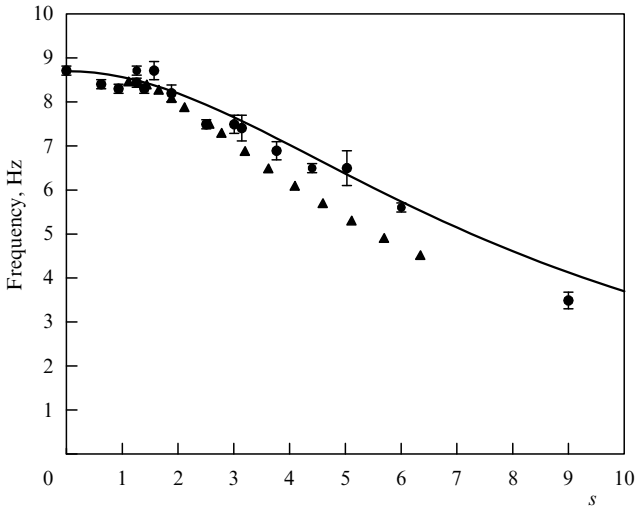

Figure 5. The frequency of dipole vibrations as a function of the amplitude s of the periodic lattice potential. The circles and triangles represent, respectively, experimental and calculated data [18]. The solid line represents a more exact theory [19].

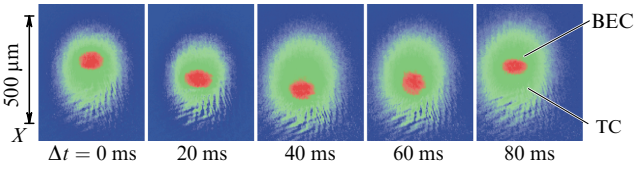

Figure 6. Gas density distribution during dipole vibrations at different points in time. One can see that only the condensate (BEC) vibrates, while the lower-density thermal component (TC) stays put. (The picture was rendered by M Inguscio.)

Accordingly, the expansion of a plane wave in terms of the Bloch functions is of the form

$$\exp\left(\mathrm{i}kz\right)=\frac{1}{2\pi}\sum_{r=0}^{\infty}\psi_{rk}^{(0)*}\psi_{rk}(z)\,. \tag{51}$$

The coefficients $\psi_{rk}^{(l)}$ are easy to calculate theoretically. The results of calculations were employed to interpret experimental data [21]. At the beginning of the experiment, there was no periodic potential and the condensate moved as a whole with a wave vector k. At the point in time $t=0$, the periodic potential was quickly ('diabatically') turned on. The wave function has no time to change under this engagement and is given by expression (51). In the experiment, the gas was quite dilute and interatomic interaction was negligible, so that Eqn (14) reduces to the ordinary linear Schrödinger equation. Then, the expansion in the form of (51) holds true in what follows as well, and the temporal dependence reduces to the appearance of factors of the form $\exp\left[-\mathrm{i}\mu_r(k)t/\hbar\right]$ in every term:

$$\psi(z,t)=\frac{1}{2\pi}\sum_{r=0}^{\infty}\psi_{rk}^{(0)*}\,\psi_{rk}(z)\exp\left[-\frac{\mathrm{i}\mu_r(k)t}{\hbar}\right], \tag{52}$$

where $\mu_r(k)$ are the eigenvalues of the linearized equation (14), i.e., the single-particle energy levels. At the point in time $t=\tau$, the confining and periodic potentials were disengaged and the condensate was allowed to expand freely. As in the

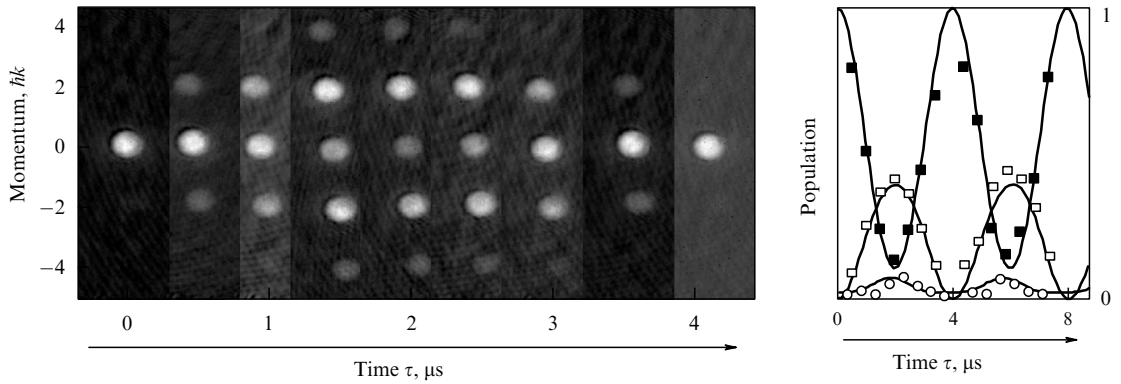

Figure 7. Shown at the left of the figure is the condensate density distribution at different points in time τ upon release from the lattice. In the experiment, $s = 14$ and the lattice was immobile ($k = 0$). On the right is the τ-dependence of the populations for the momentum values $p = 0, +2\hbar q$, and $+4\hbar q$, which are represented, respectively, by black squares, white squares, and circles. The data calculated in Ref. [21] are represented by solid lines.

experiment of Ref. [6], this allowed measuring the coefficients of the wave function expansion (52) in terms of the states with specific values of momentum. Expanding the Bloch functions on the right-hand side of expression (52) according to expansion (15), for the expansion coefficients we find

$$\psi_{rk}^{(l)}(\tau) = \frac{1}{2\pi}\sum_{r=0}^{\infty} \psi_{rk}^{(0)*}\,\psi_{rk}^{(l)} \exp\left[-\frac{\mathrm{i}\mu_r(k)\tau}{\hbar}\right]. \qquad (53)$$

Of course, the 'populations' $|\psi_{rk}^{(l)}(\tau)|^2$ rather than the coefficients $\psi_{rk}^{(l)}$ themselves are experimentally measured. The results of the measurements are presented in Fig. 7.

The situation was radically different when the optical potential was slowly imposed adiabatically. Here, the condensate, which had initially been in the stationary state of a free particle, transited to a Bloch state, which was also stationary. In this case, when the initial condensate momentum p (in the lattice frame of reference) was smaller than $\hbar q$ in the modulus, the condensate found itself in the first Bloch zone with a quasimomentum $\hbar k = p$. When the initial pulse was, for instance, in the limits $\hbar q < p < 2\hbar q$, this state, as is easy to see, passed into the state with the quasimomentum $\hbar k = p - 2\hbar q$ in the second Bloch zone, so that the adiabatic engagement of the lattice enabled transferring the condensate to a predetermined Bloch state.[2]

The nontrivial character of the quantum dynamics of the condensate in a lattice is clearly seen in an elegant experiment described in the same Ref. [21]. A lattice with a condensate in the ground state was adiabatically accelerated to a relatively high velocity v_{lat}, which was much higher than the maximum value of the group velocity in the first zone. Then, the first term in Eqn (25) is small, and the condensate momentum is simply equal to ρv_{lat} and increases as the lattice is accelerated. The wave function in the laboratory system of coordinates is defined by Eqn (37). According to Eqn (29), however, during acceleration $\hbar k + m v_{\mathrm{lat}} = 0$ and the wave function remains periodic, so that all atoms are in states with moments that are whole multiples of $2\hbar q$. The momentum distribution was measured by observing the free expansion of the condensate, as described above. The measured data are shown in Fig. 8. One can see that the total condensate momentum increases steadily due to the increase in the number of atoms with higher values of the 'multiplicity' l.

It is pertinent to note that experiments in which the condensate travels with a considerable velocity relative to the lattice are possible only when the interatomic interaction is sufficiently weak, i.e., the gas is diluted enough. The point is that the condensate becomes unstable at such velocities [22, 23]. However, the characteristic buildup times for these instabilities turn out to be long for a weak interaction.

The energy $E_r(k)$ introduced above, which describes the motion of a condensate as a whole, should not be confused with elementary condensate excitation energies. For a uniform periodic lattice, these energies $\epsilon_r(k)$ are also dependent on the zone number r and the quasimomentum of an

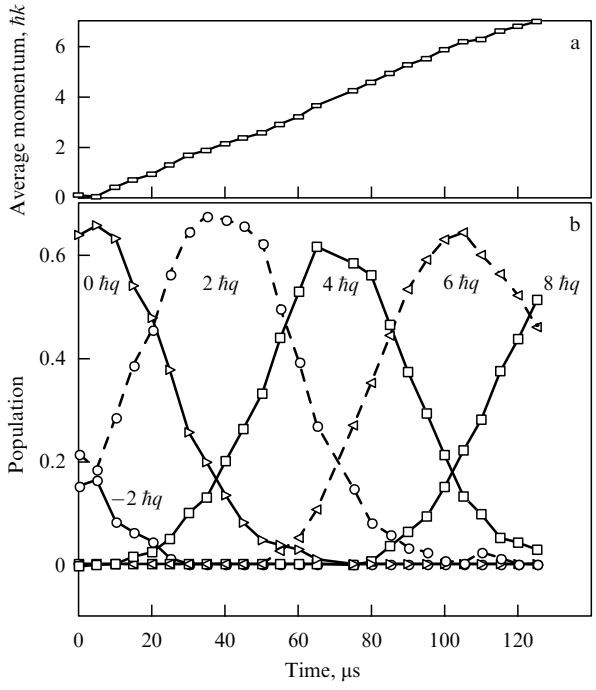

Figure 8. Time dependence of the average condensate momentum in the laboratory system of coordinates for a constant lattice acceleration is plotted in the upper part of the drawing (a). Shown at the bottom (b) is the time dependence of the populations for different values of momentum [21].

[2] Recall that the engagement of an arbitrarily weak potential changes the classification of states without changing the particle energies. For instance, states in the momentum interval $q < |p| < 2q$ should be considered as states in the second Bloch zone with the dispersion law $(|\hbar k| - 2q)^2/2m$ (see Ref. [20], problem 2 to § 55).

elementary excitation k. In the framework of the Bogolyubov approximation, $\epsilon_r(k) = \hbar\omega_r(k)$, where $\omega_r(k)$ are the frequencies of small condensate vibrations. To find $\omega_r(k)$, one should represent the wave function of the condensate in a lattice as $\psi(z,t) = [\psi(z) + \psi'(z,t)]\exp(-i\mu t/\hbar)$, where $\psi(z)$ is the wave function of the ground state and $\psi'(z,t)$ is a small addition, substitute $\psi(z,t)$ into the temporal equation (2), and linearize this equation in ψ'. The eigenfrequencies of the system of linear equations for ψ' and ψ'^* determine $\omega_r(k)$. I shall not discuss this issue in detail (see Refs [7, 24]).

To summarize this section, I mention several new lines of investigation related to optical lattices, which I cannot discuss in greater detail.

Two mutually orthogonal standing light lattices produce a two-dimensional optical lattice. When the field intensity is high enough, a system of virtually independent one-dimensional (more precisely, needle-shaped) condensates results [25]. In this system it has been possible to verify for the first time several predictions of the theory of one-dimensional Bose gases [26, 27]. This theory, which predicts quite unusual properties of the gases, was constructed many years ago but has heretofore had no experimental applications.

An appropriately selected set of laser beams makes it possible to produce a 'Kagome lattice' [28]. The atoms in such a lattice possess, for a specific number of atoms per site, special magnetic properties.

Stöferle et al. [25] investigated gas vibrations excited by the periodic modulation of the lattice depth. Krämer et al. [29] constructed the theory of this excitation. The damping of these oscillations at absolute zero constitutes an intricate theoretical problem. It was solved by Yu M Kagan and L A Maksimov [30].

6. Optical traps

For several years early on, experiments on the BEC were carried out on atoms confined near the minimum of a quadrupole magnetic field. Of course, in this way it is possible to confine only those atoms that possess a magnetic moment and are in a state in which the magnetic moment is opposed to the field. Comparatively recently, however, it has been possible to make purely optical traps, where atoms are confined near the intensity peak of a light beam [31]. Such a peak is formed by focusing the laser beam with a lens. The light intensity near the peak rapidly decreases with distance to the axis and decreases much more slowly along the axis. That is why the condensate in this trap possesses a strongly elongated 'cigar-like' shape.

This method of condensate retention opens up entirely new possibilities. Optical traps are much smaller than magnetic ones, and their parameters are easy to change in a very short time. In particular, it has been possible to form a double potential well, which permits a quantitative investigation of condensate phase dynamics. To this end, Shin et al. [31] allowed the laser beam to pass through a special device — an *acousto-optical modulator* (AOM). It is a transparent dielectric, in which two standing sound waves with close frequencies are excited. The laser beam experiences Bragg reflection from these waves and, owing to the difference in their wavelengths, splits into two almost parallel beams with closely located foci, where the condensates are located. In this case, the mutual arrangement of the foci can be adjusted by changing the frequency difference of the sound waves.

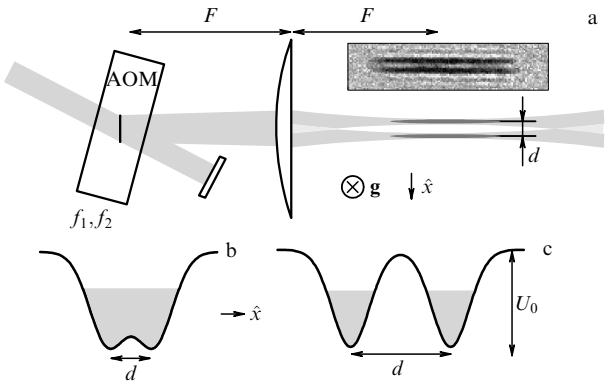

Figure 9. (a) Schematic representation of the experimental setup for measuring the phase difference between condensates. An acousto-optical modulator AOM is placed in the focal plane of the lens, which concentrates the split laser beam in the other focus to produce two traps separated by a distance d. Shown at the upper right of the figure is the absorption image of the condensates in the trap. The field of view measures 70 by 300 μm. (b) Potential energy as a function of the x-coordinate for a nonsplit trap for $d = 6$ μm. (c) Potential energy in a double potential well for $d = 13$ μm. Trap parameters: energy $U_0 = h \times 5$ kHz in both cases (b) and (c), the potential barrier in the case (c) is $h \times 4.7$ kHz, the interaction energy of atoms is $\sim h \times 3$ kHz [31].

The setup is schematically represented in Fig. 9. It is significant that it permits measuring the phase difference between the condensates. To do this requires removing the laser field and observing the pattern of interference between the condensates, which overlap in the course of expansion, i.e., measuring the spatial density distribution. When the condensates are sufficiently widely spaced, the interaction between the atoms is insignificant and the wave function is simply equal to the sum of the wave functions of the condensates: $\psi = \psi_1 + \psi_2$. To determine the phases of the wave functions we note that the point with a radius vector $\mathbf{r} - \mathbf{r}_1$ at a point in time t is reached by the atoms of condensate 1 with a velocity $\mathbf{v} = (\mathbf{r} - \mathbf{r}_1)/t$, where $\mathbf{r}_1$ is the condensate center (it is assumed that the distance $|\mathbf{r}|$ is much longer than the initial condensate dimension). Accordingly, the wave function ψ_1 acquires an additional phase $\Phi_1 = m\int \mathbf{v}\,d\mathbf{r}/\hbar = m(\mathbf{r} - \mathbf{r}_1)^2/2t\hbar$. Then, as is easily shown, the density distribution in the interference of condensates spaced at a distance d along the x-axis is

$$n(\mathbf{r},t) = \left[n_1 + n_2 + 2\sqrt{n_1 n_2}\,\cos\left(\frac{md}{\hbar t}\,x + \Phi\right)\right], \qquad (54)$$

where $\Phi = \Phi_1 - \Phi_2$ is the initial phase difference, which can thus be determined from the location of interference fringes. The interference pattern is exemplified in Fig. 10.

This method was employed to demonstrate the validity of Eqn (4), which defines the time evolution of the condensate phase. From this equation it follows that the condensates with different chemical potentials acquire in a time τ the phase difference

$$\Phi = -\frac{i\tau(\mu_1 - \mu_2)}{\hbar}\,. \qquad (55)$$

In the experiment by Shin et al. [31], the laser beam which produced one of the potential wells was turned off for a short time τ_p. This produced a difference in chemical potentials equal to the condensate-averaged potential energy proportional to the depth of the potential well. The condensate did

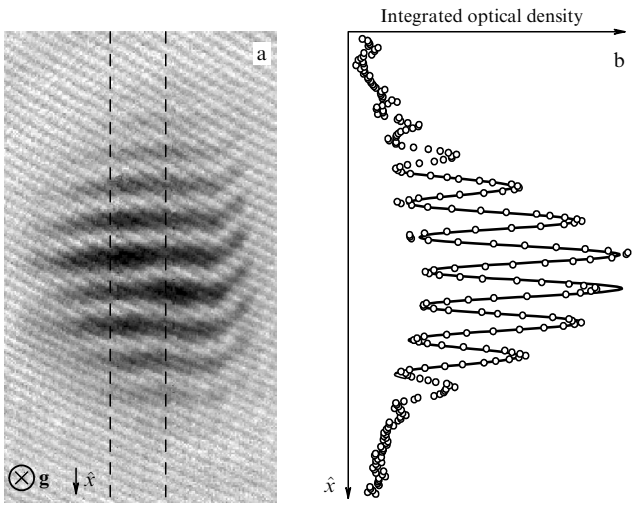

Figure 10. (a) Absorption image of the condensates interference pattern upon expansion from the trap shown in Fig. 9c. (b) Density distribution integrated over the condensate section [31].

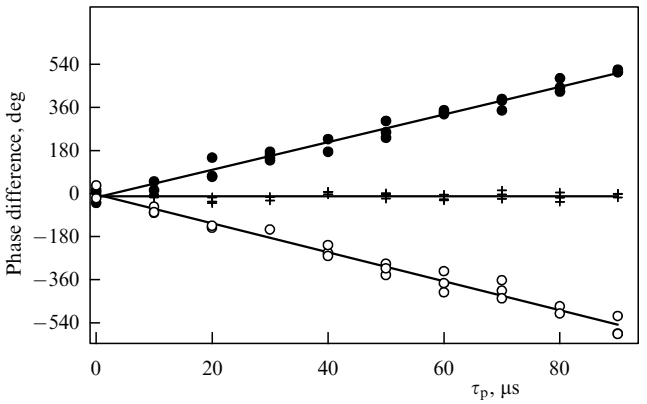

Figure 11. Difference in condensate phases as a function of the turn-off time τ_p in the cases when one trap or the other (black and white circles) or both traps simultaneously (crosses) were turned off.

not manage to noticeably expand in the time τ_p, but the condensates acquired a phase difference according to expression (55). After that, the second trap was also turned off and the phase difference was measured from the interference pattern. The measuring data are given in Fig. 11.

The above method of measuring the phase difference has a significant drawback. It is 'destructive'. The condensates are lost upon every phase measurement, and an individual measurement had to be performed to obtain each point in Fig. 11.

However, it is also possible to continuously measure the phase difference. This method was proposed by Stringari and the author [32] and is based on the phenomenon termed the *interference in the momentum space*. The momentum distribution of two condensates considered as a single quantum-mechanical system depends on their relative phase. For instance, for two identical quantum wells,

$$n(\mathbf{p}) = \left|\psi_1(\mathbf{p}) + \psi_2(\mathbf{p})\right|^2 = 2\left[1 + \cos\left(\frac{p_z d}{\hbar} + \Phi\right)\right] n_1(\mathbf{p})\,. \tag{56}$$

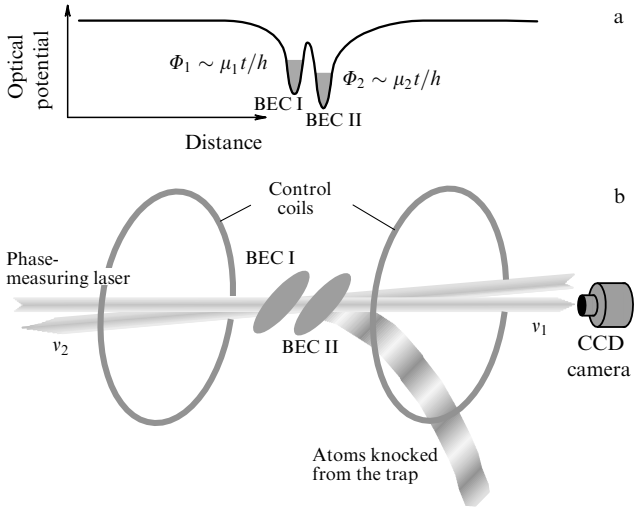

Figure 12. Continuous measurement of the difference between condensate phases. (a) Potential energy of the traps as a function of the coordinate. Trap parameters: radial atomic vibration frequencies are $\simeq 325$ kHz and the axial ones $\simeq 10$ kHz, the trap depths are $\simeq h \times 5$ kHz. (b) Schematic of the experimental setup. Two laser beams v_1 and v_2 illuminate the condensate and knock atoms out of the traps. The probability of the process is measured by recording the number of atoms 'knocked out' of the traps or recording with a CCD camera the increase in the number of photons in one of the beams. The control coils produce a magnetic field gradient and enable changing the difference in the chemical potentials [34].

Expression (56) predicts characteristic oscillations in the p_z-variable, the phase of which depends on the relative phase of the condensates.

It is significant that the momentum distribution can be measured and, what is more, measured in principle nondestructively. The measurement is performed by way of observation of *two-photon Bragg light scattering*. The condensate is illuminated by two laser beams with a small frequency difference $\Omega = \omega_1 - \omega_2$ and a wave-vector difference $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$ (use is usually made of oppositely directed beams, so that $\mathbf{k}_1 \approx -\mathbf{k}_2$). The scattering may be considered as the atomic absorption of a photon of the first beam and the stimulated emission of a photon of the second beam. As this takes place, the atom gains an energy $\hbar\Omega$ and a momentum $\hbar\mathbf{q}$. When the frequency Ω is high enough, the interatomic interaction has no effect on the scattering and its probability, i.e., the number of photons absorbed from one beam, is proportional to the quantity

$$S(\mathbf{q}, \Omega) = \frac{m}{\hbar q}\, n_z(Y)\,, \tag{57}$$

where $n_z(p_z) = \int n(\mathbf{p})\, \mathrm{d}p_x\, \mathrm{d}p_y$ and the Y variable is

$$Y = \frac{m(\Omega - \hbar q^2/2m)}{q}\,. \tag{58}$$

By changing the frequency difference Ω and measuring S, it is possible to determine the momentum distribution and, according to expression (56), the difference between the condensate phases. This method was first applied by Stenger et al. [33] to measure the momentum distribution of condensate atoms related to the nonuniformity of the condensate in a magnetic trap.

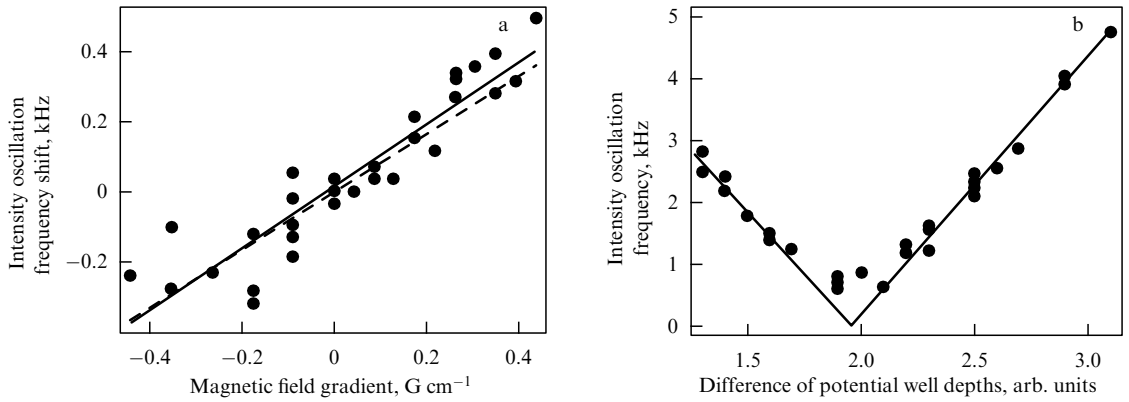

Figure 13. Continuous measurement of the difference in the condensate phases. (a) Scattering intensity oscillation frequency as a function of magnetic field gradient. The difference in the chemical potentials in the absence of this gradient was $\simeq h \times 0.53$ kHz. The solid line is the linear interpolation of the experimental data and the dashed line gives the theoretical dependence $\delta\omega = \mu_B B' d/\hbar$, where B' is the magnetic field gradient, d is coil separation, and μ_B is the Bohr magneton. (b) Dependence of the oscillation frequency on the difference in the chemical potentials in the case when this difference was formed due to inequality of the intensities of the trap-producing light beams [34].

Recently, Saba et al. [34] employed two-photon Bragg scattering for the continuous measurement of the difference between condensate phases. The setup is schematically represented in Fig. 12.

As in the preceding experiment of Shin et al. [31], the condensates were confined in the optical traps in the vicinity of the foci of two parallel laser beams. As explained above, the scattering of atoms by two parallel laser beams ν_1 and ν_2 with close frequencies propagating in opposite directions is observed. The number of scattering events was measured by recording the number of scattered atoms and therefore the atoms 'knocked out' of the traps, as well as by recording the variation of the difference in the number of photons in the measuring light beams. Additional control coils produced the magnetic field to enable changing the difference of the chemical potentials of the condensates and, according to Eqn (4), the time dependence of the phase difference. According to Eqns (56) and (57), this results in scattering intensity oscillations with a frequency $\omega_J = (\mu_2 - \mu_1)/\hbar$.

Figure 13 shows the results of measurements of the oscillation frequency as a function of the difference of the chemical potentials of the condensates. We emphasize that this experiment opens up the possibility of measuring weak forces experienced by atoms with the use of BECs.

The authors of Ref. [34] also provided the answer to the long-standing question: "Is interference possible between initially independent condensates?" The question is not quite trivial. Indeed, let as assume that two widely separated condensates were independently produced and have not interacted. Then, as is easy to see, the ground state of the system corresponds to the given numbers of atoms in each of the condensates and the value of their phase difference as a quantum-mechanical variable is not defined, i.e., the state of the system is not an eigenfunction of the phase difference operator. Now let the condensates be allowed to overlap. Will the interference pattern be visible? The correct answer is "Yes, it will" [35]. The observation of an interference pattern signifies the quantum-mechanical measurement of the phase difference. As a result of this measurement, the system transits to a state with a specific phase difference. The fact that this difference was indeterminate in the initial state manifests itself in the measurements of similarly prepared systems each time yielding different phase values.[3]

The interference of independent condensates was investigated in Ref. [34]. At the beginning of the experiment, the optical traps were spaced far apart (at about 14 μm). Under these conditions, the condensates produced in the traps were independent and did not possess a specific phase difference. At some point in time, the laser beams ν_1 and ν_2 were engaged and measurements were made of the relative phase, which yielded a value Φ_1. After the measurement, the phase difference changed with time according to Eqn (55). In this case, the difference in chemical potentials was defined by the difference in the depth of the optical traps and by the gradient of the magnetic field produced by control coils. After a time, the phase difference measurement was repeated and yielded a value Φ_2. Measurements of similarly prepared condensates were repeated several times. As would be expected, the values of the phase differences Φ_1 and Φ_2 were randomly distributed for different repetitions of the experiment. However, the measured values $\Phi_1 - \Phi_2$ corresponded to Eqn (55).

In the above experiments, the tunneling of condensate atoms between the traps was negligible. The condensate dynamics in the presence of such a tunneling was experimentally investigated recently by Albiez et al. [37]. However,

[3] Here, I have outlined a viewpoint on this interesting problem which, I believe, is shared by the majority of experts. In my view, however, the classical nature of the condensate wave function ψ allows one to arrive at the same correct conclusion without invoking the quantum theory of measurements. From this viewpoint it may be assumed that the wave function ψ, being a classical quantity, possesses a certain phase even prior to the measurement, which merely reveals the existing value of the phase difference. That is why stating that the measured value of the relative phase is produced in the course of measurement makes little more sense than saying that a parked-car position is produced by the observer who looks out of the window. It is not merely a matter of words. The statement that the wave function phase possessed a value prior to the measurement does not, owing to its classical nature, lead to any contradictions in this case. By contrast, the assumption that such a 'true quantum' quantity as the spin projection of a 1/2-spin particle possessed the measured value even prior to the measurement is well known to lead to contradictions. From this assumption there follow *Bell inequalities*, which are at contradiction with quantum mechanics and experiment (see Ref. [36], Ch. 6). Delicate experiments, which would reveal the 'true quantum properties' of the phase are discussed in Ref. [36a].

before describing these experiments, I will set forth some theoretical considerations.

7. Tunnel condensate dynamics in a double potential well

We consider two condensates separated by a high, but not infinite, potential barrier. The wave function inside the barrier is small but nevertheless finite, and atoms can tunnel between the condensates. When the tunneling probability is low enough, the condensates inside each trap will be in equilibrium and the system will be described by the assignment of only three variables: the numbers N_1 and N_2 of atoms in the traps and the phase difference Φ of the condensate wave functions.

In the presence of tunneling, the difference between the condensate phases signifies that atoms 'go over' from one condensate to the other. First of all we note that the wave function of the system may be written, despite the nonlinearity of Eqn (2), as $\psi = \psi_1 + \psi_2$, where ψ_1 and ψ_2 are the solutions of Eqn (2) localized respectively in the first and second traps. In the barrier area, where ψ_1 and ψ_2 overlap, the wave functions are small and the nonlinearity of the equation is insignificant. Therefore, the wave function can be written in the form

$$\psi(x,t) = \sqrt{n_1(x,t)}\,\exp\left[\mathrm{i}\Phi_1(t)\right] + \sqrt{n_2(x,t)}\,\exp\left[\mathrm{i}\Phi_2(t)\right]. \tag{59}$$

With the aid of Eqn (20) we now find the current density at the point $x = 0$ in the middle of the barrier. Simple calculation gives

$$j = -I_\mathrm{J}\sin\Phi\,, \tag{60}$$

where

$$I_\mathrm{J} = \frac{\hbar}{m}\left[\sqrt{n_1}\,\partial_x(\sqrt{n_2}\,) - \sqrt{n_2}\,\partial_x(\sqrt{n_1}\,)\right]_{x=0}, \qquad \Phi = \Phi_1 - \Phi_2\,.$$

By definition of the current, Eqn (60) defines the temporal dependence of the numbers of particles in the condensate: $j = \partial N_1/\partial t = -\partial N_2/\partial t$.

The phase equation can be derived by generalizing the Josephson equation (4), which was written under the assumption that the chemical potentials were time-independent. When they are sufficiently slow functions of the time, in lieu of Eqn (4) the differential equation

$$\frac{\partial\Phi}{\partial t} = -\frac{1}{\hbar}(\mu_1 - \mu_2) \tag{61}$$

holds. In the subsequent discussion it will be assumed for simplicity that the numbers of atoms in the condensates differ little: $|N_1 - N_2| \ll N_1, N_2$. In this case, one may treat the coefficient I_J in expression (60) as a constant and expand the right-hand side of Eqn (61) in terms of $N_1 - N_2$, with the result that this equation takes on the form

$$\frac{\partial\Phi}{\partial t} = -\frac{E_\mathrm{C}}{\hbar}\,k\,, \tag{62}$$

where $E_\mathrm{C} = [2\mathrm{d}\mu_1/\mathrm{d}N_1]_{N_1=N_2}$ and a convenient variable

$$k = \frac{N_1 - N_2}{2}\,, \tag{63}$$

which characterizes the difference of the potential well populations was introduced. Similarly, Eqn (60) can be rewritten as

$$\frac{\partial k}{\partial t} = I_\mathrm{J}\sin\Phi\,. \tag{64}$$

Equations (62) and (64) make up the system of equations of the *tunnel* or *Josephson* dynamics of a condensate in a double potential well. When Φ and k are small enough, Eqn (64) may be linearized by putting $\sin\Phi \approx \Phi$. Then, the system describes harmonic oscillations with a frequency

$$\omega_\mathrm{J} = \frac{\sqrt{E_\mathrm{C}E_\mathrm{J}}}{\hbar}\,, \tag{65}$$

where the energy dimension quantity $E_\mathrm{J} = \hbar I_\mathrm{J}$ is introduced. We note that the theory under consideration is valid only when the tunneling probability is sufficiently low. This condition can be qualitatively formulated in the form of the inequality $\hbar\omega_J \ll \varepsilon_\mathrm{ex}$, where ε_ex is the energy difference between the ground and first excited atomic states in a potential well.

It is significant that the system (62), (64) can be rewritten in the form of the Hamilton equations for the canonically conjugate quantities Φ and $\hbar k$:

$$\frac{\partial\Phi}{\partial t} = -\frac{\partial H_\mathrm{J}}{\partial(\hbar k)} \tag{66}$$

and

$$\frac{\partial(\hbar k)}{\partial t} = -\frac{\partial H_\mathrm{J}}{\partial\Phi} \tag{67}$$

with the Hamiltonian

$$H_\mathrm{J} = \frac{E_\mathrm{C}k^2}{2} - E_\mathrm{J}(\cos\Phi - 1) \tag{68}$$

(see, for instance, Ref. [38]). The solutions of these equations are qualitatively different in character for different initial conditions. Let us assume that the condensates initially possess equal phases, $\Phi = 0$, and that $k = k_0 > 0$. Then, when $k_0 < k_\mathrm{c} \equiv \sqrt{2E_\mathrm{J}/E_\mathrm{C}}$, the variation in both the phase and the quantity k is oscillatory in character. On the strength of the temporal constancy of the Hamiltonian H_J, the quantity k oscillates in the limits $-k_0 < k < k_0$ and $\cos\Phi$ between unity and $1 - E_\mathrm{C}k_0^2/2E_\mathrm{J}$.

By contrast, when the initial value $k_0 > k_\mathrm{c}$, the phase unrestrictedly grows with time and k oscillates about the k_0 value without changing sign:

$$\sqrt{k_0^2 - \frac{4E_\mathrm{J}}{E_\mathrm{C}}} < k < k_0\,.$$

Therefore, the populations of the potential wells do not become equal in this case, despite the tunneling. The so-called '*self-trapping*' effect occurs [39, 40].

This effect has a simple mechanical analogue. Hamiltonian (68) is equivalent to that of a physical pendulum, with Φ playing the part of the angular displacement and k of the angular velocity. When the initial angular velocity is high enough, the pendulum rotates about the support point rather than oscillating.

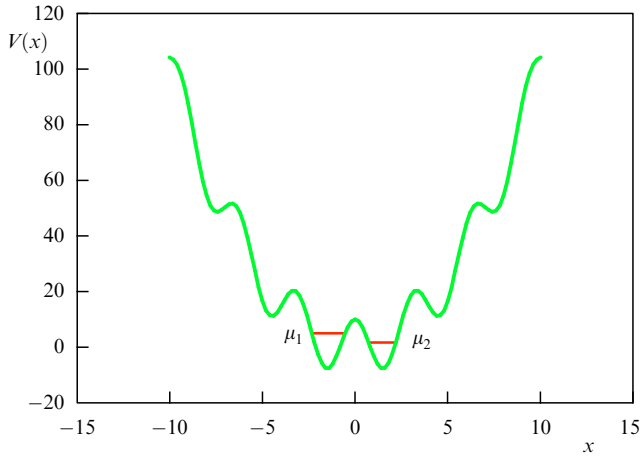

Figure 14. Schematic plot of the dependence of the potential energy on the coordinate along the beam for an optical trap [37].

These interesting effects were investigated in Ref. [37]. The authors employed an optical trap formed by a focused laser beam. Unlike the preceding experimental work, however, the double trap was produced by imposing a standing light wave directed along the focused laser beam. This produced the potential shown schematically in Fig. 14. It has several minima. The condensate is trapped in the two deepest. The barrier height can be adjusted by changing the amplitude of the standing wave. To produce the initial potential-well population difference, the standing wave was somewhat shifted relative to the focal point. In this case, one well became deeper than the other and more atoms found themselves in this trap upon establishment of equilibrium. After that, the wells were rapidly shifted to the symmetric position and the tunneling motion of the condensate began. At some point in time the laser field was turned off and the condensate was observed to expand. As this took place, the researchers managed to measure the number of atoms in the traps and, from the interference pattern, the phase difference. The measurement data are presented in Fig. 15, which exemplifies the oscillatory regime and the self-trapping regime.

Once again, I draw the reader's attention to the fact that in the photographs given we can see the quantum tunneling of a macroscopic body virtually with the unaided eye.

The above-outlined experimental device holds great promise. For instance, it can underlie quantitative investigations into the quantum and classical fluctuations of the condensate wave function (see Section 8). It is possible to verify the unexpected prediction made by Kagan, Kovrizhin, and Maksimov that the phonons excited in the condensate do not experience reflection from the potential barrier [41].

8. The breaking of coherence by quantum fluctuations and the Mott transition

The effects discussed thus far are, of course, of a quantum nature. However, they can be adequately described within the approximation of classical de Broglie waves, which reduces to the Gross–Pitaevskii theory. It is noteworthy, in particular, that the parameter I_J, which describes tunneling, can be calculated by solving Eqn (2). However, when the tunneling probability is low enough, investigations of the systems described above are of special interest

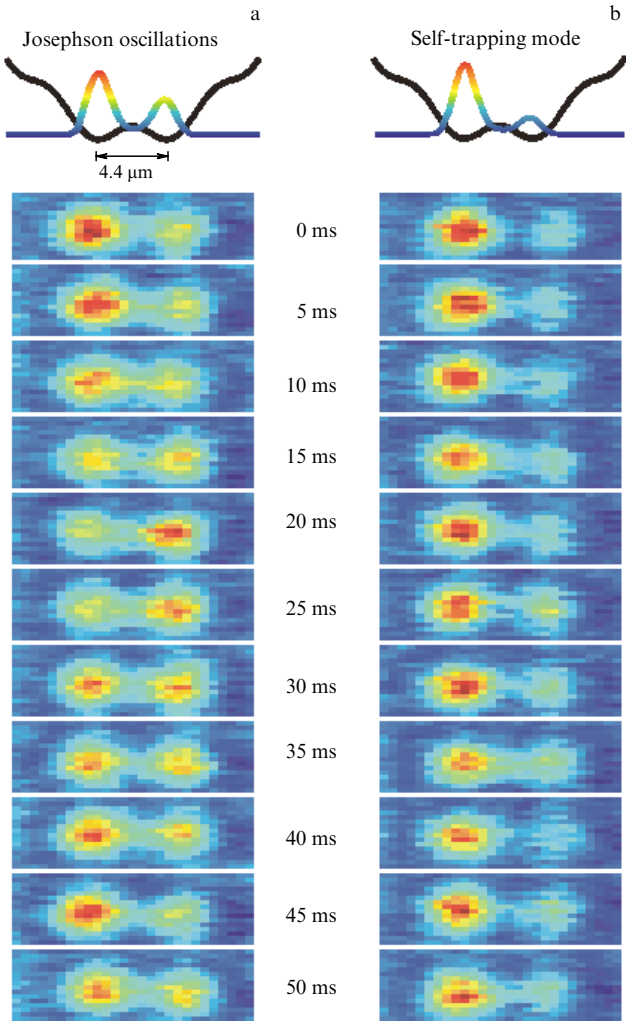

Figure 15. Tunnel dynamics of the condensate in a symmetric double well. The temporal dependence of the condensate density is immediately evident from the absorption images. The field of view is 19.4 by 9.2 mm. Figure 15a shows the condensate evolution in the Josephson oscillation mode, when $k < k_c$. Figure 15b corresponds to the self-trapping mode $k > k_c$. When the images were obtained, the condensates were spaced at 6.7 μm [37].

because they permit revealing the departures from the mean field theory.

In a homogeneous gas, suchlike departures are described by the correction terms of the Bogolyubov theory, which may be considered as a manifestation of the quantum fluctuations of the wave function ψ. These corrections, generally speaking, are of the relative order $(na^3)^{1/2}$, where n is the number of atoms per unit gas volume. This formula also applies to a condensate in a trap by the order of magnitude, if n is considered to mean the gas density at its center. Assuming the trap potential to be harmonic, it is easy to verify that the corrections to the mean field theory are of the relative order of magnitude

$$\frac{1}{N}\left[\frac{Na}{a_H}\right]^{6/5},$$

where

$$a_H = \left(\frac{\hbar}{m\omega_H}\right)^{1/2}$$

and ω_H is the atomic oscillation frequency in the trap. The dimensionless parameter Na/a_H is of great importance in BEC physics. When this parameter is large, as is usually the case, advantage can be taken of the so-called Thomas – Fermi approximation (see Ref. [1]) in theory.

Exhibiting the most promise is the possibility of measuring the departures of the condensate oscillation eigenfrequencies in a trap from the results of mean field theory. The corresponding corrections, which increase the oscillation frequency, were calculated by Pitaevskii and Stringari [42] and by Braaten and Pearson [43]. Despite the large magnitude of the Thomas – Fermi parameter, owing to the small factor $1/N$ the corrections turn out to be small under typical experimental conditions and have so far escaped reliable measurement. It is likely that Chevy at el. [44] observed them in the shift of the oscillation frequency of a strongly elongated condensate.

In this respect, the situation near the so-called *Feshbach resonances*, where the scattering length depends on the magnetic field and may assume very large values, is of considerable promise [54]. This effect is discussed in Section 9.

Departures from the mean field theory result, however, in appreciable observable effects for weakly coupled condensates, i.e., those separated by high potential barriers. Here, the case in point is the condensates both in optical lattices and in double potential wells. The interatomic interaction leads to quantum fluctuations of condensate phases, which can be measured in sensitive interference experiments.

The simplest way of describing quantum fluctuations of the condensate phase difference in a double potential well involves the quantization of the classical tunnel dynamics equations (66), (67). To do this requires replacing the canonically conjugate variables Φ and k by the operators with the commutation relations

$$[\hat{\Phi}, \hat{k}] = \mathrm{i}\,. \tag{69}$$

In this case, it is expedient to operate in the 'Φ representation', whereby the operator

$$\hat{k} = -\mathrm{i}\,\frac{\partial}{\partial \Phi} \tag{70}$$

acts on the wave function, which should be a periodic function of Φ with the period 2π. Accordingly, the Hamilton operator is the quantum generalization of Hamiltonian (68):

$$\hat{H}_J = \frac{E_C}{2}\,\frac{\partial^2}{\partial \Phi^2} - E_J(\cos\Phi - 1)\,. \tag{71}$$

Hamiltonian (71) is sufficiently simple, and with its aid it is possible to provide a complete description of phase fluctuations. First of all, we consider the case when these fluctuations are small. Then, we can approximately put $1 - \cos\Phi \approx \Phi^2/2$, with the result that Hamiltonian (71) reduces to the Hamiltonian of a harmonic oscillator. By taking advantage of the well-known results, we find that the ground state of the system involved exhibits the following fluctuations:

$$\langle \Phi^2 \rangle = \frac{1}{2}\,\sqrt{\frac{E_C}{E_J}}\,. \tag{72}$$

The above assumption about the smallness of fluctuations therefore necessitates the fulfillment of the inequality

$$\frac{E_C}{E_J} \ll 1\,, \tag{73}$$

which signifies that the interaction is sufficiently weak (E_C is low) and the tunneling coefficient E_J is large enough. In the absence of interaction, when $E_C = 0$, the phase fluctuations are absent, as they must be. All atoms are condensed in one state with a certain phase [4].

The quantity E_J is, generally speaking, proportional to the number of condensate atoms N. However, since it decreases exponentially with the barrier height, inequality (73) may be violated even for high N.

Phase fluctuations have the effect that the interference pattern shown in Fig. 10 turns out to be blurred *upon averaging over a set of many measurements*. Indeed, every measurement would yield a density distribution in accordance with expression (54), but with different phase values in different realizations of the experiment. Averaging the resultant density distributions over different measurements would then yield (in the case of similar condensates with $n_2 = n_1$)

$$\begin{aligned}\langle n(\mathbf{r}, t)\rangle &= 2n_1\left[1 + \left\langle \cos\left(\frac{md}{\hbar t}\,x + \Phi\right)\right\rangle\right] \\ &= 2n_1\left[1 + \cos\left(\frac{md}{\hbar t}\,x\right)\langle\cos\Phi\rangle\right].\end{aligned} \tag{74}$$

Therefore, the sharpness of the interference pattern is determined by the *coherence factor*

$$\alpha \equiv \langle\cos\Phi\rangle \approx 1 - \frac{\langle\Phi^2\rangle}{2}\,. \tag{75}$$

In particular, the minimal-to-maximal intensity ratio turns out to be equal to

$$\frac{1-\alpha}{1+\alpha} \approx \frac{\langle\Phi^2\rangle}{4}\,.$$

We emphasize once again that the interference pattern blurring occurs due to averaging over different realizations of the experiment. Each individual experiment yields an interference pattern with a certain phase Φ.

In the opposite limiting case of weak tunneling ($E_J/E_C \ll 1$), the fluctuations of the population difference $\langle(\Delta k)^2\rangle = 2(E_J/E_C)^2$ are found to be small. In this case, the condensates are in the states with fixed numbers of particles (they are referred to as the *Fock states*). The sharpness of the averaged interference pattern is quite poor in this case: $\alpha = \langle\cos\Phi\rangle = 2E_J/E_C$.

Up to the present time, the phase fluctuations of a condensate in a double potential well have not been investigated (recall that the tunneling itself in this system has been observed quite recently [37]). Optical lattices open up greater opportunities to observe these fluctuations. Here, the

[4] The transition to the case of noninteracting particles is in fact not quite trivial. This is evident even from the following fact: in the same approximation as that of formula (72), when $E_C \to 0$, the fluctuation of the population difference $\langle(\Delta k)^2\rangle = \sqrt{(E_J/E_C)} \to \infty$, which is impossible. This issue is discussed in greater detail in Ref. [45], § 15.7.

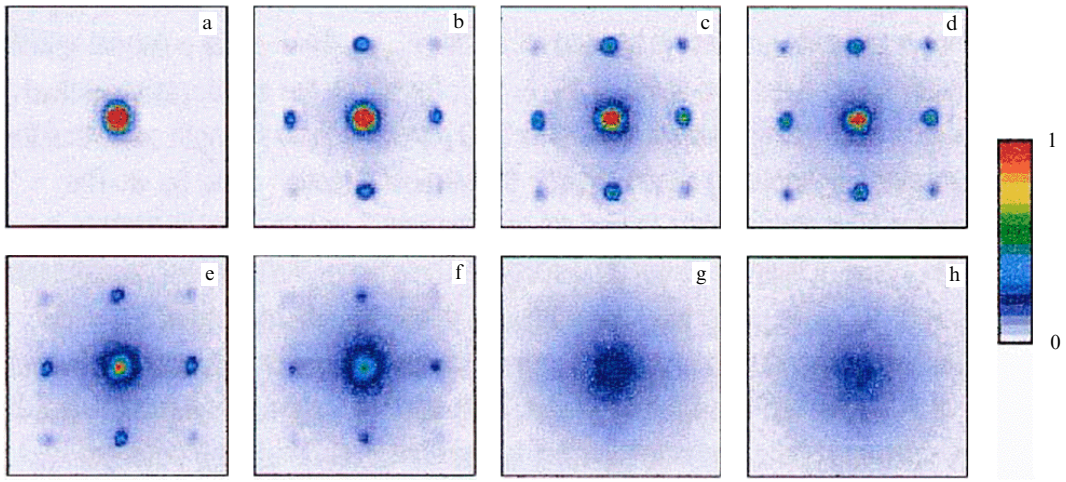

Figure 16. Interference pattern upon Bose-gas expansion from a three-dimensional lattice for different values of the parameter s: (a) $s = 0$, (b) 3, (c) 7, (d) 10, (e) 13, (f) 14, (g) 16, (h) 20. The disappearance of diffraction spots for $s > 13$ signifies the Mott transition to the dielectric phase [48].

presence of fluctuations signifies that the condensates located at different lattice sites possess different phases. This is responsible for the blurring of the interference pattern even in every individual measurement in an experiment like that shown in Fig. 1.

The first experiments of this kind were performed by Anderson and Kasevich [46] and by Orzel et al. [47]. They observed the Bloch oscillations of the condensate in a one-dimensional optical lattice under the effect of gravity. As in other experiments, at some point in time the lattice was turned off and the interference pattern was recorded. In Ref. [47], the interference pattern, which was sharp in a low-intensity lattice, was discovered to vanish with an increase in barrier heights. However, it should be borne in mind that the interpretation of these results is not quite unequivocal. It is likely that the aforementioned instability of the condensate moving in the lattice also played a part.

Greiner et al. [48] obtained important results in the investigation of condensates in a three-dimensional lattice with a potential of the form (13). Unlike the preceding experimental papers discussed above, the number of atoms in each lattice site was small: on the order of 1 – 3. Because of this, the departures from the mean field theory turned out to be large. The authors observed the interference pattern upon turning off the external field. The results are given in Fig. 16 for different values of the parameter s. Small s parameter values signify that the lattice potential is low. In this case, the condensate is almost uniform and the side peaks of the interference pattern are weak, as is evident from Fig. 16a. As s increases, these peaks rise in intensity. However, there simultaneously appears an incoherent background related to phase fluctuations. For $s > 13$, the discrete interference peaks are 'absorbed' by the background and disappear completely. At this s value, the system undergoes a phase transition to a new state. Unlike the superfluid condensate phase for small s values, this phase is termed dielectric, because the gas can no longer flow through the lattice. The absence of interference peaks signifies the absence of non-diagonal long range ordering, i.e., the absence of BEC.

The critical value of s is in agreement with the theoretical predictions by Fisher et al. [49] (see also Jaksch et al. [50]). The dielectric phase is characterized by the existence of a gap in the energy spectrum. The gap has a simple physical significance. This energy is required to transfer an atom from one lattice site to the neighboring one. Owing to the repulsion between the atoms, this process requires energy. The gap width was measured by imposing a nonuniform external potential. When the difference in potential between the neighboring sites was as large as the gap, the atoms were set in motion, which resulted in an irreversible heating of the system. When the s parameter was decreased after that again, the system did not return to the superfluid state, which showed up in the absence of the interference pattern.

Mott predicted a transition of this type for electrons in a metal, and it is commonly referred to as the *Mott transition*.

Recently, Xu et al. [51] performed a quantitative investigation of departures from the mean field theory in conditions when they are not too small. The aim of the work was to measure the number of 'supracondensate' atoms. In the mean field approximation, for $T = 0$ all atoms are in the condensate. The fluctuations of the wave function give rise to out-of-condensate atoms. According to predictions by the Bogolyubov theory, for a uniform condensate at $T = 0$ the ratio η between the out-of-condensate number of atoms and the total number (the quantum depletion) is $\eta = 1.5(na^3)^{1/2}$. Direct generalization of the Bogolyubov theory for condensates in a lattice yields for this ratio the expression $\eta = \int v_{\mathbf{k}}^2 \, \mathrm{d}\mathbf{k}/n$, where $\mathbf{k}$ is the quasimomentum and $v_{\mathbf{k}}^2$ is the parameter similar to the parameter v of the Bogolyubov transformation expressed in terms of the elementary spectrum $\epsilon(\mathbf{k})$ of excitations in the lattice, according to van Oosten et al. [52], as

$$v_{\mathbf{k}}^2 = \frac{\epsilon(\mathbf{k}) + gn_0 - \left[2\epsilon(\mathbf{k})gn_0 + \epsilon^2(\mathbf{k})\right]^{1/2}}{2\left[2\epsilon(\mathbf{k})gn_0 + \epsilon^2(\mathbf{k})\right]^{1/2}} . \tag{76}$$

It is easily shown that increasing the lattice potential amplitude s increases the η ratio, so that the supracondensate atoms become observable.

The experimental setup was in principle similar to that of Ref. [48], which was discussed above. The condensate, containing up to 5×10^5 atoms of ^{23}Na, was placed in an optical trap produced in the vicinity of the intersecting foci of

two infrared laser beams with a wavelength of 1064 nm (for a more detailed description of the experimental facility, see Xu et al. [53]). The atomic eigenfrequencies in the trap were $\omega_{x,y,z} = 2\pi \times 60, 60, 85$ Hz and the condensate radius was equal to about 12 μm for 1.7×10^5 atoms. Three mutually perpendicular standing laser waves were imposed on the condensate to produce a three-dimensional optical lattice. It was also possible to produce one- or two-dimensional lattices by using one or two of these waves. There were approximately 7 atoms per lattice site, i.e., appreciably more than in Ref. [48]. The experiment was performed up to parameter values $s = 22$.

To measure the number of supracondensate atoms, the trap and lattice potentials were rapidly (in a time shorter than 1 μs) turned off and the condensate expanded freely for 10 ms. After that, the absorption condensate image was obtained, i.e., measurements were made of the spatial density distribution. As discussed above (see footnote 1), this distribution reproduces the initial momentum distribution. In this experiment, the condensate atoms produced sharp interference peaks and the supracondensate atoms produced the diffuse background. In the experiment, the peaks were shaded and the number of background-producing atoms was counted, thereby giving the number of the supracondensate atoms. We do well to bear in mind that many pitfalls were encountered when processing the data of this extremely difficult experiment. For values $s > 16$, part of the condensate at the center of the trap underwent the Mott transition to the dielectric phase. The nonuniformity of the system obstructed the application of the theory. Nevertheless, the measuring data turned out to be in reasonable agreement with the theory for three-, two-, and one-dimensional lattices alike.

9. Feshbach resonance

As mentioned above, broad experimental possibilities are opened up by the existence of *Feshbach resonances*, in the vicinity of which the scattering length is long and depends on the magnetic field. Before describing several experiments on ultracold gases near the Feshbach resonances, we are reminded of the central tenets of the scattering theory for slow particles under conditions when a system of two atoms possesses a bound state with a negative energy ϵ close to zero (see Ref. [13], § 133). The term 'close to zero' implies in this context that the characteristic distance $\hbar/\sqrt{m|\epsilon|}$ between the bound atoms is large in comparison with the interatomic potential range r_0:

$$\frac{\hbar}{\sqrt{m|\epsilon|}} \gg r_0 \,. \tag{77}$$

In this case, a quantum-mechanical description of the interatomic interaction does not necessitate knowledge of the specific properties of the potential. It would suffice to consider the free atomic motion, but impose the following boundary condition for $r \to 0$ on the wave function of the s state:

$$\left[\frac{\mathrm{d}(r\psi)}{\mathrm{d}r}\right]_{r=0} = -\frac{1}{a}(r\psi)\,, \tag{78}$$

where a is a constant subject to the condition $a \gg r_0$. When $a > 0$, the Schrödinger equation for the relative motion of two atoms possesses a solution of the form

$$\psi = \mathrm{const} \cdot \frac{\exp(-r/a)}{r}\,,$$

which describes the bound state with an energy

$$\epsilon = -\frac{\hbar^2}{ma^2}\,. \tag{79}$$

When a is negative, the bound state is nonexistent. In this case, the two atoms are said to have a 'virtual level' as opposed to the 'real' level (79). However, irrespective of the sign of a, the solution with a positive energy E is of the form $\psi = \mathrm{const} \cdot \sin(kr + \delta_0)/r$, where $k = p/\hbar = \sqrt{mE}/\hbar$. The s-scattering phase is determined with the aid of Eqn (78), which leads to the formula $\tan\delta_0 = -ka$. The scattering amplitude is

$$f = \frac{1}{k(\cot\delta_0 - \mathrm{i})} = -\frac{1}{1/a + \mathrm{i}k}\,. \tag{80}$$

Therefore, the constant $a = -(f)_{k\to 0}$ is nothing but the scattering length. If $\epsilon \to 0$, the length $a \to \infty$. Then, the scattering amplitude tends to its universal 'unitarity limit':

$$f \to \frac{\mathrm{i}}{k}\,. \tag{81}$$

We note that the a-positivity condition, which ensures the condensate stability, signifies that the system of two atoms possesses a real level with a low negative energy for large values of a. It may appear that the occurrence of this level is an unlikely fortuity. However, alkali-earth atoms possess levels whose position depends strongly on the magnetic field. We consider the system of two atoms as a diatomic molecule. The electronic terms, i.e., the potential energy curves of atomic interaction, are determined by their electronic states. In this case, it is significant that different electronic states possess different magnetic moments. That is why the relative positions of the terms depend on the external magnetic field. We consider two terms: the upper term and the lower one, which corresponds to the ground state. By changing the magnetic field, it is possible to make the energy of one of the bound states of the upper term approach zero. This is just the Feshbach resonance [54]. In the absence of transitions between the terms, the existence of this level has no effect on the scattering in the ground state. However, the small terms in the Hamiltonian which describe the nuclear – electronic motion interaction give rise to these transitions. And so the existence of the upper term level manifests itself in atomic scattering according to formulas (79), (80). As for the weakness of the interaction between the terms, it shows up in these formulas being valid only for very low energies, and the lower the energies, the weaker the interaction.

The constant $1/a$, which enters into boundary condition (78), is a regular function of magnetic induction B, and it can be expanded to about the value B_0, whereby the level reaches zero: $1/a \approx (B - B_0)/b$. When the constant b is sufficiently small, it is appropriate to include the next term of the expansion as well. Eventually the scattering length in the neighborhood of the Feshbach resonance can be written as

$$a = a_\mathrm{g} + \frac{b}{B - B_0}\,. \tag{82}$$

The employment of the Feshbach resonance opens up radically new experimental possibilities, for it permits controlling the interaction intensity. As B_0 is approached, the scattering amplitude increases, so that attempts to obtain rather large gas parameter values for a low gas density meet with success. On the other hand, the interaction becomes quite weak when the magnetic field is close to the value $B \approx B^* = B_0 - b/a_g$, making it possible to investigate the properties of a virtually perfect Bose gas. Interestingly, under these conditions there is good reason to include in Eqn (2) the fifth-order term as well, i.e., to describe the gas by an equation in the form

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}, t) + v(B - B^*) \left|\psi(\mathbf{r}, t)\right|^2 + g_3 \left|\psi(\mathbf{r}, t)\right|^4 \right] \psi(\mathbf{r}, t) . \quad (83)$$

The constant g_3 accounts for the three-particle interatomic interaction.

There emerges a wide range of interesting effects when the magnetic field changes rapidly in time. As mentioned above, the condensate is stable in the vicinity of the resonance only when the system of two atoms possesses a bound state with a low binding energy. In stationary conditions, such molecules are produced by three-body recombination, which has a low probability in a dilute gas. In an alternative magnetic field, however, atoms combine to form molecules for a two-particle interaction as well. As a result, along with atomic condensate there also emerges a condensate of molecules [55]. The molecular condensate is described by its own wave function. The interference between the two condensates manifests itself in characteristic beats — periodic variations in atomic condensate density with a frequency $\hbar|\epsilon|$, where ϵ is the molecular binding energy (79).

The possibility of controlling the magnitude of the scattering length makes it possible to radically change the interaction properties of two bosons placed in a cylindrical harmonic trap. When the frequency $\omega_\perp$ of transverse vibrations in the trap is sufficiently high, the atomic motion may be treated as one-dimensional. The interaction of two slow atoms may then be described employing a one-dimensional effective potential of the form

$$U_{1D}(z - z') = g_{1D}\, \delta(z - z') . \quad (84)$$

It is possible to express the interaction constant g_{1D} in terms of the 'three-dimensional' atomic scattering length a in free space [56]:

$$g_{1D} = \frac{4\hbar^2}{m a_\perp} \frac{a}{1 - C(a/a_\perp)} , \quad (85)$$

where $a_\perp = (2\hbar/m\omega_\perp)$ and $C = 0.56$. One can see from formula (85) that by changing the magnetic field and hence the scattering length a it is possible to greatly increase the one-dimensional interaction constant and change its sign. It is noteworthy that the above formulas also apply to the interaction of slow fermions with antiparallel spins.

Experimental BEC research in the immediate vicinity of the Feshbach resonance is hindered by the short condensate lifetime. Increasing the scattering length increases the probability of three-body recombination. However, there exist systems that are free from this limitation. The case in point is diatomic molecules made up of fermions with opposite spins. (Slow fermions with parallel spins interact only slightly because the s-scattering length of fermions with parallel spins is equal to zero.) Such molecules evidently obey the Bose–Einstein statistics and have the capability to form a condensate. However, their properties near the Feshbach resonance are completely different from the properties of atomic bosons. These properties were elucidated in an important paper by Petrov, Salomon, and Shlyapnikov [57], whose findings are discussed below.

First of all, it is possible to express the molecule–molecule scattering length a_m in terms of the scattering length for atoms with opposite spins. The problem allows an elegant solution, which yields the result

$$a_m = 0.6a , \quad (86)$$

which is valid near the resonance, i.e., in the limit $a \to \infty$. It is noteworthy that the Fermi statistics of the atoms are of significance. For molecules made up of two bosons, it is impossible to express the molecular scattering length in terms of the atomic one. The result in Ref. [57] was obtained by solving the Schrödinger equation in the coordinate space. Brodsky et al. [58] developed a diagram approach to the problem. Also investigated in Ref. [57] were the molecular transitions to deep molecular levels occurring in the collisions of two molecules. It turned out that the probability α_{rel} of this relaxation process, which is responsible for the loss of molecules from the condensate, lowers with an increase in scattering length:

$$\alpha_{rel} \propto a^{-2.55} . \quad (87)$$

This signifies that the lifetime of the condensate of molecules consisting of Fermi atoms lengthens rapidly as the Feshbach resonance is approached. (Regal et al. [59] experimentally observed a power dependence for the relaxation rate with an exponent -2.3, which is close to expression (87).) I emphasize that this effect is related to precisely the Fermi nature of the atoms. For molecules made up of bosons, the molecular relaxation probability near the resonance increases proportionally to a^4. At present, Fermi gases near the Feshbach resonance are the concern of numerous experimental and theoretical papers and their discussion is beyond the scope of the present review. Here, I will restrict myself to only brief remarks.

It is significant that the system turns out to be stable for all values of the scattering length a, both positive and negative. For positive a, we are dealing with molecular bosons with a positive, according to expression (86), scattering length. In this case, the system is described by Eqn (2) with replacement of the mass by $2m$ and of the scattering length by $0.6a$. A negative a signifies the absence of a bound state, and we are dealing with a Fermi gas whose stability is provided by the statistical properties of fermions.

Of special interest is the situation that takes place in the immediate vicinity of the Feshbach resonance, when the formally calculated scattering length a turns out to be much longer than the average interatomic distance:

$$a \gg n^{-1/3} . \quad (88)$$

It is significant that in this case the gas remains dilute in the sense that the atomic dimension is much smaller than the

interatomic distances

$$r_0 \ll n^{-1/3} . \tag{89}$$

The properties of the system should then be determined by the parameters of the two-particle scattering amplitude. In a gas with a density n, the majority of atoms possess momenta on the order of $\bar{p} \sim \hbar n^{-1/3} \gg \hbar/a$. The scattering amplitude for such momenta reaches the unitary limit (81) and is independent of a. In this situation, there remain no parameters at our disposal, with the exception of the density, characterizing the system.

Simple dimension considerations show that the equation of state is of the same form as the equation of state of a perfect Fermi gas. For instance, for $T = 0$ the energy E_0/N accounted for by one atom should be of the form

$$\frac{E_0}{N} = \frac{\xi p_F^2}{2m} = \xi \frac{\hbar^2 (3\pi^2 n)^{2/3}}{2m} , \tag{90}$$

with a universal coefficient ξ, i.e., one that is independent of the kind of gas. Naturally, there is no way of calculating the ξ coefficient analytically because the theory does not contain a small parameter in the unitary domain. However, numerical calculations by the Monte Carlo technique yield a reasonably reliable value $\xi = 0.44$ [60, 61], which is in good agreement with experimental data [62, 63]. It is pertinent to note that the chemical potential in the unitary domain for $T = 0$ and the energy are related by the same equation as in the ideal gas: $\mu = (5/3)E_0/N$. At finite temperatures, the curve $E(T)$ should also be universal.

There is good reason to believe that a Fermi gas close to the resonance is, for $T = 0$, superfluid for all values of the scattering length. Indeed, for relatively small positive values of a we are dealing with a dilute gas of Bose molecules (the BEC domain). For small negative a values we have a Fermi gas with a weak attraction between the atoms. According to the Bardeen – Cooper – Schrieffer – Bogolyubov theory, Cooper pairs form in the gas in this case and the state of the gas turns out to be similar to the state of electrons in a superconducting metal. The calculation of the gap in the spectrum and of the transition temperature is, however, an arduous task, which was carried out by Gor'kov and Melik-Barkhudarov [64]. I mention that the transition between the two specified regimes is continuous, not involving a phase transition.

In the unitary near-resonance regime, the superconducting gap for $T = 0$ should be on the order of $p_F^2/2m$, i.e., $\Delta(0) = \gamma p_F^2/2m$ with a universal coefficient γ. Calculations of Ref. [60] yield a value $\gamma \approx 0.88$. The transition temperature should also be on the order of $p_F^2/2m$. Recent calculations employing the diagram Monte Carlo technique [67] yield the value [68]

$$T_c = (0.152 \pm 0.003) \frac{p_F^2}{2m} . \tag{91}$$

A direct confirmation of superfluidity is the existence of quantized vortices, which were observed in experiments [65]. I should mention that these vortices in a Fermi gas possess a velocity circulation $\pi\hbar/m$ — two times lower than in a Bose superfluid liquid. The circulation can be determined by calculating the number of vortices per unit area for a given rotation velocity.

As discussed above, for a sufficiently small a the system is a molecular Bose gas. To a first approximation, the equation of state of this gas may be written as $\mu = (2\pi\hbar^2/m)(0.6an)$. Of fundamental importance is the question of whether, with the aid of the Bogolyubov – Lee – Yang – Huang theory, it is possible to calculate the next correction as well, which is of the relative order of magnitude $(na^3)^{1/2}$ in this theory (see the beginning of the previous section). This is not evident, because the Fermi properties of the atoms that make up a molecule are of significance, but the orders of magnitude of statistics-related corrections are hard to estimate. However, I am convinced that the correction $(na^3)^{1/2}$ is the principal one. This signifies that the frequency of collective gas oscillations in the trap should rise with increasing a in the BEC domain. At the same time, the oscillation frequency in the unitary limit should be lower than in the BEC domain. Indeed, for a 'polytropic' equation of state of the form $\mu = n^s$, the frequencies lower with decreasing s, with $s = 1$ in the BEC domain and $s = 2/3$ in the unitary domain. Therefore, the oscillation frequencies as functions of a should have a maximum [66]. Experimental evidence on this matter is discrepant, but recent experiments by Grimm [69] seem to bear out the existence of this maximum.

10. Conclusion

Investigations involving optical traps and optical lattices play an increasingly important part in BEC physics. It is just this technique that is actually employed in the majority of new experiments. However, the technology of magnetic traps is not standing still, either. Rapid strides are being made by 'microtraps', in which the magnetic field is produced by the currents flowing through thin electrodes deposited on a dielectric substrate, near which the atoms are confined. (See, for instance, Treutlein et al. [70] and references therein).

To summarize, I note that the work on BECs and ultracold gases generally has entered a new phase. While initial studies were concerned with the amazing properties of the condensates themselves, at present the condensates are used primarily for investigating different physical problems that defy investigation by other methods. Among them is the study of quantum phase transitions, which are exemplified by the Mott transition discussed in the foregoing. Also worth mentioning is the measurement of the force of molecular attraction between an atom and a solid surface in the measurement of near-surface condensate oscillations [71]. However, a truly vast range of applications of these systems opens up in connection with their possible use for quantum information processing and development of quantum computers. Ultracold gases in optical traps and microtraps show the greatest promise as candidates for the elements of such computers, even though the idea of a gaseous, at least partly, computer seems strange. There have already been many achievements on this path. Although this problem is extremely difficult, it is my belief that the present generation of the Earth's population will witness the advent of quantum computers. This topic undoubtedly deserves to be comprehensively reviewed on the pages of *Physics – Uspekhi*.

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