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LETTERS TO THE EDITORS

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On the relation between tensor and scalar perturbation modes in Friedmann cosmology

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Abstract. **An elementary derivation of the fundamental relation $T/S = 4\gamma$ between the tensor and scalar modes of cosmological perturbations in the early universe is given. Statements by L P Grishchuk on this problem are commented on.**

In a recent paper "Relic gravitational waves and cosmology" by L P Grishchuk [see *Phys. Usp.* **48** (12) 1235 (2005)], the author has reviewed his studies conducted over many years and devoted to quantum-gravity generation of gravitational waves (tensor mode T) and density perturbations (scalar mode S) in a homogeneous isotropic universe. The origin of primordial cosmological perturbations has been a key question of 20th century physics, initiated by the pioneering work by E M Lifshitz [2] and the first papers where quantization of T [3] and S [4] modes of perturbations in a flat Friedmann model has been done. The view by the author of Ref. [1] in its principal points contradicts the widely accepted result recognized as classical on the T-to-S mode ratio in the early universe, which is included in textbooks on cosmology.

Inasmuch as the central points of Ref. [1] and some earlier papers by Grishchuk, devoted to the relation between spectra of relic gravitational waves and density perturbations, are based on the statement that the "*final amplitudes of gravitational waves and density perturbations should be roughly equal to each other*" [see the discussion after formula [1] (33)*], in the present paper we shall consider only this key statement.

The cited statement runs counter to the generally accepted result that the *ratio of the squares of the amplitudes of the T- and S-modes of cosmological perturbations generated quantum-gravitationally in the early universe is proportional to* γ_i, where the parameter $\gamma \equiv -\dot{H}/H^2$ is taken at the initial time of the parametric amplification for perturbations of a given wavelength [2]. (We recall that at the inflationary stage $\gamma < 1$.)

Below, we shall present the elementary derivation of the classical relationship $T/S = 4\gamma$ without solving the equation for perturbations and then show what is erroneous in the statements by Grishchuk.

From the theoretical point of view, the problem of small linear cosmological perturbations is equivalent to the problem of the behavior of test fields in an unperturbed Friedmann model is reduced in turn to the problem of massless real fields in the Minkowski space – time, evolving under the influence of an external variable field:

$$S[q] = \int L \, d\eta \, d\mathbf{x}\,, \quad L = \frac{1}{2}\,\alpha^2 \eta^{\mu\nu} q_{,\mu} q_{,\nu}\,, \tag{1}$$

where S and L are the action and the Lagrangian density of the field q, respectively, and the comma in the subscript stands for the derivative over the Minkowskian coordinates $(\eta, \mathbf{x})$ with the metric tensor $\eta_{\mu\nu} = (1, -1, -1, -1)$.

The role of the external (parametric) field is played by the time-dependent function α^2. It is equal to $\alpha_T^2 = a^2/8\pi G$ for each of two polarizations of gravitational waves (in this case, q_T is the transverse-traceless component of the gravitational field) [3], and is $\alpha_S^2 = a^2\gamma/4\pi G$ for density perturbations (in that case, q_S is the gauge-invariant combination of the longitudinal gravitational potential and the potential of the 4-velocity of the medium multiplied by the Hubble parameter) [4].

In the Fourier representation, the field q is resolved into elementary time-dependent oscillators q_n with the Lagrangians (below we shall omit the subscript n of the Fourier modes)

$$L_n = \frac{1}{2n^3}\,\alpha^2 (q'^2 - n^2 q^2)\,. \tag{2}$$

The evolution of oscillator (2) depends on the function f determining its effective frequency:

$$\bar{q}'' + n^2 (1 - f)\,\bar{q} = 0\,,$$
$$\bar{q} = \frac{\alpha}{n}\,q\,, \quad f \equiv \frac{\alpha''}{\alpha n^2}\,. \tag{3}$$

When $|f| \ll 1$, the oscillator q stays in the free adiabatic oscillation regime and decays inversely proportional to α

[1] Hereinafter references to formulas from paper [1] are marked with an asterisk.

[2] In the exact theory, the parameter γ/β^2 in this formula and ensuing ones should be substituted for γ, where β is the speed of sound in the medium in units of the speed of light [4]. However, in most applications considered $\beta \sim 1$, so in what follows we omit the parameter β. We also use units where $c = 1$ and $l_{\rm Pl} = (G\hbar)^{1/2}$, the scale factor is $a \equiv (1+z)^{-1}$, $\mathbf{x}$ are the spatial (comoving) coordinates in the flat Friedmann model, $\eta = \int dt/a$ and t are the conformal and physical time, respectively, $H = \dot{a}/a = a'/a^2$ is the Hubble parameter, and a dot or prime over a function means its derivative with respect to the physical or conformal time, respectively. (We shall mark with a subscript T or S any variable while considering T and S perturbation modes, respectively. — *Author's amendment to English proofs*.)

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($q \propto \exp(-\mathrm{i}n\eta)/\alpha$). When $f \geqslant 1$, a parametric amplification occurs and the field q 'freezes out' ($q \propto \mathrm{const}$). In variables $(\bar{q}, \bar{p})$, the Lagrangian takes the standard canonical form

$$L_n = \frac{n}{2}\left(\bar{p}^2 - \bar{q}^2\right),$$
$$\bar{p} = \frac{\partial L_n}{\partial \bar{q}'} = \frac{\alpha q'}{n^2} = \frac{\bar{q}'}{n} - s\bar{q}, \quad s \equiv \frac{\alpha'}{\alpha n}, \tag{4}$$

where $\bar{p}$ is the field momentum conjugate to $\bar{q}$.

The key to understanding the T/S ratio lies in choosing the initial conditions for elementary q-oscillators. It is convenient to determine them in the adiabatic zone as states with minimum energy for all T- and S-oscillators (vacuum). The quantization of systems (1) and (2) is a standard procedure that does not require explanation. The question is in an unambiguous choice of the initial vacuum state for the q-oscillator. We are reminded that a free (noninteracting) oscillator possesses a unique ground state.

In most inflationary scenarios, the adiabatic condition is realized in a microscopic region ($\eta < \eta_i$), where the period of oscillations of the q-oscillator is smaller than the characteristic variability time of the parameter α:

$$|s| < 1, \quad |f| < 1, \tag{5}$$

and the Hamiltonian of system (2) is positively determined. When both conditions (5) are satisfied, quantum-mechanical operators $\bar{q}$ and $\bar{p}$ describe in the leading order a free oscillator of external action [see Eqns (3) and (4)]. This allows us to use the standard procedure for the frequency decomposition (into positive and negative sets) and for the determination of the ground state [the absence of particles at stage (5)]:

$$\langle \bar{p}^2 \rangle = \langle \bar{q}^2 \rangle = \frac{\hbar}{2}, \quad \eta < \eta_i, \tag{6}$$

where brackets $\langle \ldots \rangle$ signify averaging over the given (vacuum) state.

Equivalently, the ground state at stage (5) can be constructed using the 'normalized' variables q:

$$\tilde{q} = \frac{\alpha_0}{n}\, q,$$
$$L_n = \frac{1}{2n}\,\bar{\alpha}^2(\tilde{q}'^2 - n^2\tilde{q}^2) = \frac{n}{2}\left(\frac{\tilde{p}^2}{\bar{\alpha}^2} - \bar{\alpha}^2\tilde{q}^2\right),$$
$$\bar{\alpha} \equiv \frac{\alpha}{\alpha_0}, \tag{7}$$

which canonize the Lagrangian within some period of time at any instant η_0 (with $\eta_0 < \eta_i$), within which the value of α can be considered constant ($\bar{\alpha} \simeq 1$). Therefore, canonical pairs $\tilde{q} \simeq \bar{q}$ and $\tilde{p} \simeq \bar{p}$ and vacuum conditions (6), (8) turn out to be identical:

$$\langle \tilde{p}^2 \rangle = \langle \tilde{q}^2 \rangle = \frac{\hbar}{2}, \quad \forall \quad \eta_0 < \eta_i. \tag{8}$$

(One can say that variables $\tilde{q}$ form a tangent space to the function $\bar{q}$).

We stress that the adiabatic conditions (5) provide the unique choice of the state (6), (8) as the initial state of elementary oscillators (2) that corresponds to the minimal initial level of their excitations (vacuum of the field q). In a later evolution, q-oscillators enter the zone of the parametric amplification, equalities (6) and (8) are violated, and their state becomes multiparticle (the generation of cosmological perturbations).

Assuming the existence of the adiabatic stage (5) in the early universe, which at the instant η_i ($f \sim s = 1$) changed to the parametric amplification stage, we obtain from condition (6) the amplitude of the q-oscillator in the 'freezing-out' zone ($\eta > \eta_i$):

$$\langle q^2 \rangle \simeq \langle q_i^2 \rangle = \frac{n^2}{\alpha_i^2}\,\langle \bar{q}_i^2 \rangle \approx \frac{\hbar n^2}{2\alpha_i^2}. \tag{9}$$

Hence follows the validity of the generally recognized statement for the ratio of the perturbation modes of a given wavelength [3]:

$$\frac{T}{S} \equiv 2\,\frac{\langle q_T^2 \rangle}{\langle q_S^2 \rangle}\bigg|_{\eta > \eta_i} \simeq 2\left(\frac{\alpha_S}{\alpha_T}\right)_i^2 = 4\gamma_i \tag{10}$$

(both polarizations of gravitational waves were taken into account).

Now let us consider Grishchuk's error.

The dimensional amplitudes of elementary oscillators correspond to the following notations from Ref. [1]:

$$q_T \equiv h, \quad q_S \equiv \frac{\zeta}{2}$$

[see formulas (11)*, (20)*]. When determining the state of the T-oscillator, the author follows equations (7) and (8) ($\tilde{q}_T \equiv \bar{h}$; see Eqns (12)–(17)*). However, when moving to the S-mode, instead of normalized variable $\tilde{q}_S$ he introduces an asymmetric (with respect to $\alpha_S \propto a\sqrt{\gamma}$) variable $\bar{\zeta}$ [see Eqn (21)*]:

$$\bar{\zeta} \equiv q_{\mathrm{LPG}} = \frac{\tilde{q}_S}{\sqrt{\gamma_0}},$$
$$L_n = \frac{1}{2n}\,\bar{\alpha}_S^2\gamma_0(\bar{\zeta}'^2 - n^2\bar{\zeta}^2) = \frac{n}{2}\left(\frac{p_{\mathrm{LPG}}^2}{\bar{\alpha}_S^2\gamma_0} - \bar{\alpha}_S^2\gamma_0\, q_{\mathrm{LPG}}^2\right), \tag{11}$$
$$\bar{\alpha}_S \equiv \frac{a\sqrt{\gamma}}{a_0\sqrt{\gamma_0}},$$

for which the Lagrangian explicitly depends on γ_0. Then equations (8), rewritten for the pair

$$q_{\mathrm{LPG}} \equiv \bar{\zeta}, \quad p_{\mathrm{LPG}} = \frac{\partial L_n}{\partial \bar{\zeta}'} = \tilde{p}\sqrt{\gamma_0}$$

[$\mathbf{q}, p$ in notations of Eqns (24)*, (25)*], also acquire the explicit dependence on the parameter γ:

$$\langle q_{\mathrm{LPG}}^2 \rangle = \frac{\hbar}{2\gamma_0}, \quad \langle p_{\mathrm{LPG}}^2 \rangle = \frac{\hbar}{2}\,\gamma_0, \quad \forall \quad \eta_0 < \eta_i. \tag{12}$$

Clearly, the vacuum state is in no way related to the choice of one pair of canonical variables or another. Equations (8) [and identical to them Eqns (12)] bears a transparent invariant sense: the equality of quantities $\langle \tilde{q}^2 \rangle = \langle \tilde{p}^2 \rangle$ (or $\gamma_0 \langle q_{\mathrm{LPG}}^2 \rangle = \langle p_{\mathrm{LPG}}^2 \rangle / \gamma_0$) means the equality of the mean kinetic and potential energies of the elementary oscillator (2), while the equality of each of these quantities to $\hbar/2$ means choosing

[3] Accounting for $\alpha_i H_i \approx n$, from formulas (9) and (10) we obtain the well-known expressions for the spectra of perturbations and their slopes:

$$\langle q_T^2 \rangle^{1/2} \approx l_{\mathrm{Pl}} H_i\,, n_T \equiv \frac{\mathrm{d}\ln\langle q_T^2 \rangle}{\mathrm{d}\ln n} \simeq -2\gamma_i \simeq -0.5\,\frac{T}{S},$$

$$\langle q_S^2 \rangle^{1/2} \approx \frac{l_{\mathrm{Pl}} H_i}{(2\gamma_i)^{1/2}}, \quad n_S \equiv \frac{\mathrm{d}\ln\langle q_S^2 \rangle}{\mathrm{d}\ln n} \simeq -\left(2\gamma + \frac{\dot{\gamma}}{\gamma H}\right)_i.$$

the minimum possible energy level of the oscillator (i.e., the vacuum state) at the adiabatic stage (5).

Nevertheless, the author of Ref. [1] erroneously interprets the state (12) for S-oscillators as a squeezed one [multiparticle; see formulas after Eqns (26)*, (27)*], ignoring the fact that the asymmetry of equations (12) has nothing to do with the choice of the state over which the averaging was performed, but to the choice of the variable [4] that is explicitly dependent on γ. This leads him to introduce another initial state (we shall mark this state by the subscript LPG) which he calls "*the genuine vacuum state for the variable* ζ" [see formulas after Eqn (32)*]:

$$\langle q_{\rm LPG}^2\rangle_{\rm LPG} = \langle p_{\rm LPG}^2\rangle_{\rm LPG} = \frac{\hbar}{2}\,, \quad \forall \quad \eta_0 < \eta_i\,, \tag{13}$$

and, as a consequence, to the statement that $T/S \sim 1$ [see Eqn (33)*], since for $\eta > \eta_i$ one finds

$$\langle q_S^2\rangle_{\rm LPG} \simeq \gamma_i \langle q_S^2\rangle \approx \left(\frac{l_{\rm Pl}\, n}{a_i}\right)^2 \approx \langle q_T^2\rangle\,. \tag{14}$$

Grishchuk's error consists in setting an incorrect initial vacuum state for the S-oscillator, while his choice of the initial state for the T-oscillator is correct. It should be noted that the vacuum state of the elementary oscillator (2) is unique at stage (5) and is determined exclusively by methods of quantum mechanics [5], i.e., knowledge of physics of T- and S-modes is not required here. In this sense, all oscillators are formally similar: their relation to the external field is determined solely by the function $\alpha(t)$ irrespective of its physical content (whether it be the scale factor a for the T-oscillator or $a\sqrt{\gamma}$ for the S-oscillator). In particular, this means that the amplitude of excitation of the S-oscillator [under the action of the field $\alpha(t)$] from the minimum-energy state can only depend on the product $a\sqrt{\gamma}$, and not separately on γ or a, as was found in paper [1] [cf. Eqns (9) and (14)].

In the paper [1], the Lagrangian of the S-mode (22)* was derived from the Lagrangian of the T-mode (13)* with $\tilde{a} \equiv a\sqrt{\gamma}$ substituted for a and $\bar{\zeta}$ for $\bar{h}$. However, the correct transition from T to S, as seen from formula (2), occurs when substituting α_S for α_T and q_S for q_T. Here, to within a numerical factor of order unity, $a_0\bar{h} \propto h$ goes over into $\tilde{a}_0\zeta \propto \sqrt{\gamma_0}\,\bar{\zeta}$ and not into $\bar{\zeta}$, as Grishchuk believes. As a result, the correct Lagrangian (11) is obtained by multiplying Eqn (22)* by the factor γ_0.

The Lagrangian (22)* is inconsistent with other formulas from paper [1]. For example, in the high-frequency limit gravitational effects are insignificant, and the Lagrangian for the S-mode should turn into the Lagrangian for sound waves in a medium:

$$L_{n \gg aH} = \frac{1}{2n^3}\, a^2(\varphi_1'^2 - n^2\varphi_1^2)\,,$$

where

$$\varphi_1 \simeq \left(\frac{\gamma}{16\pi G}\right)^{1/2}\zeta = \frac{\alpha_S}{a}\, q_S$$

is the potential of the matter field for $n \gg aH$; see Eqns (19)*–(20)*. Clearly, the Lagrangian (22)* does not satisfy this limit.

Another inconsistency: the canonical variables $q_{\rm LPG}$ and $p_{\rm LPG}$, considered [6] in Eqns (24)* and (25)*, are canonical with regard to Lagrangian (11), but not (22)*, which is easily verified by directly inserting expressions (11) and (22)* into equation (25)*. In addition, by rewriting Lagrangian (22)* in terms of the initial field variable ζ, we see that it turns to be dependent on the arbitrary instant of time η_0, which is inadmissible. Removal of the inconsistency in formula (22)* would eliminate these contradictions.

Summarizing, we can state that Lagrangian (22)* does not follow from the field Lagrangian for a scalar field minimally coupled with gravity [see the formula preceding Eqn (19)*], and then further discussion is senseless. If one considers Eqn (22)* as a result of a technical inaccuracy and uses the correct Lagrangian, then the statement on the 'false character' of the standard inflationary result (see, for example, the title of Section 4 in Ref. [1]) is due to the incorrect choice of initial conditions.

Our second remark deals with the measurements of the value of T/S. It is not negligibly small, as the author of paper [1] repeatedly states [see, for example, his commentary to formula (6)*].

The estimate $T/S \simeq 4\gamma_i$ is confirmed by exact calculations for a broad range of inflationary models (see, for example, the quantity r in Ref. [5]). In particular, all models of the chaotic inflation with $p > 1$ ($V(\varphi) \propto \varphi^{2p}$, p is a natural number) contradict observations since they predict a significant value of the T/S ratio and a deviation from the Zel'dovich spectrum:

$$\frac{T}{S} \simeq \frac{2p}{N} \simeq (1 - n_S)\,\frac{2p}{1+p} \approx 0.04p\,, \tag{15}$$

where $N = 2\pi G\varphi^2/p \approx 50$ on a scale on the order of 10^3 Mpc.

The case of a massive scalar field ($p = 1$) is exceptional in providing a gravitational-wave mode amplitude only five times smaller than the scalar one ($\sqrt{0.04} = 1/5$), which does not contradict observational constraints at the 95% confidence level (see, for example, Ref. [6]).

It should be emphasized that all values $T/S > 0.2$ are excluded by the modern observations, since in that case the amplitude of the S-mode is insufficient to produce the observed large-scale structure of the universe (we should remember that the sum $T + S$ is fixed by the data on the anisotropy of cosmic microwave background).

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[4] We recall that S-oscillators are coupled to the product $a\sqrt{\gamma}$ and not to a or γ separately.

[5] In essence, it is mathematics (of Lagrangian systems), or "*the art of calling different things by the same names*", in the definition by Henry Poincaré.

[6] Note that the factor γ_0 in Eqns (24)* and (25)* should be substituted for γ, since ζ decreases inversely proportional to $\tilde{a} = a\sqrt{\gamma}$ in the adiabatic limit [see Eqn (20)*].

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