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Theory of gauge-invariant response of superconductors to an external electromagnetic field

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Abstract. **A consistent method of calculating the linear response to an external magnetic field that allows obtaining the result in a manifestly gauge-invariant form is proposed. Within the diagram technique for nonequilibrium processes, the self-consistency equations for the order parameter in an arbitrary-gauge field allow deriving an equation that determines the phase of the order parameter as a function of the external field. Such a method automatically accounts for the existence of collective excitations in superconductors, which must be taken into consideration in accordance with the continuity equation. The possible types of collective excitations in pure superconductors at different temperatures are considered. The authors present a microscopic theory that explains the possibility of observing collective modes in superconducting tunnel junctions.**

1. Introduction

In 1957, Bardeen, Cooper, and Schrieffer for the first time formulated [1] a microscopic theory of superconductivity. The simple model proposed by the three authors (which became known as the BCS model) very successfully explained many properties of superconductors. However, at the beginning, several difficulties emerged in calculations of the linear response of a superconductor to an electromagnetic field in the BCS framework. Very soon it became clear [2 – 5] where the difficulties arise. It was found that the gauge invariance condition is satisfied in a superconductor in a nontrivial way. In the first works on the subject, only the current generated in the superconductor by the transverse field was calculated. Direct generalization of the results to the case of longitudinal fields led to violation of the gauge invariance condition for the response and to a contradiction with the continuity equation. To solve this problem, one must allow for excitation by an external field not only of Bogolyubov quasiparticles in the superconductor but also of collective modes specific to the superconducting state. The contribution to the current caused by collective modes eliminates the contradiction with the continuity equation and restores the gauge invariance of the theory. We note that even in the simplest case of stationary fields, correctly accounting for gauge invariance is extremely important. It is this requirement that modifies the well-known London equations such that it implies one of the most unusual properties of superconductors, the quantization of the magnetic flux trapped in a superconducting ring.

This review is an attempt to show how to consistently describe the linear response of superconductors to an electromagnetic field in the BCS model such that the gauge invariance condition is satisfied automatically. Sections 2 and 3 are devoted to the development of such a consistent microscopic theory. In Section 4, we discuss some simple corollaries of the general formulas that provide a clear relation between the general approach and the relations known from superconductivity theory. In Section 5, we briefly discuss the possible types of collective excitations in superconductors and their role in response functions. Finally, Section 6 is devoted to a topic closely related to the problem of collective modes in response functions, a theory that explains how collective modes may manifest themselves in measurements of the tunnel current flowing between two superconductors.

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2. Interaction of electrons and an electromagnetic field in the BCS model

We recall that the starting point in the BCS theory is a Hamiltonian of the form

$$\widehat{H} = \int \widehat{\psi}_\alpha^+(x) \left(\frac{\widehat{\mathbf{p}}^2}{2m} - \mu \right) \widehat{\psi}_\alpha(x) \, \mathrm{d}^3\mathbf{r} - g \int \widehat{\psi}_\downarrow^+(x) \, \widehat{\psi}_\uparrow^+(x) \, \widehat{\psi}_\uparrow(x) \, \widehat{\psi}_\downarrow(x) \, \mathrm{d}^3\mathbf{r} \,, \quad (1)$$

were $x = (t, \mathbf{r})$, ψ_α is the Heisenberg operator of annihilation of an electron with spin α ($\alpha = \uparrow, \downarrow$), and g is the electron–electron coupling constant. Here and in what follows, we use the notation proposed by Gor'kov [6] and assume summation over repeated spin indices. Superconductivity theory involves nonzero anomalous averages that determine the superconducting order parameter Δ:

$$\Delta(x) = g \left\langle \widehat{\psi}_\uparrow(x) \, \widehat{\psi}_\downarrow(x) \right\rangle . \quad (2)$$

In the mean-field approximation, the BCS Hamiltonian then assumes the simple form

$$\widehat{H} = \int \widehat{\psi}_\alpha^+(x) \left(\frac{\widehat{\mathbf{p}}^2}{2m} - \mu \right) \widehat{\psi}_\alpha(x) \, \mathrm{d}^3\mathbf{r} - \int \left[\Delta(x) \, \widehat{\psi}_\downarrow^+(x) \, \widehat{\psi}_\uparrow^+(x) + \text{h.c.} \right] \mathrm{d}^3\mathbf{r} \,. \quad (3)$$

This Hamiltonian can be diagonalized exactly via a Bogolyubov transformations [7], yielding the well-known spectrum of one-particle excitations of the gap type:

$$\widehat{H}' = \sum_p \sqrt{\left(\frac{p^2}{2m} - \mu \right)^2 + \Delta^2} \; a_p^+ a_p \,, \quad (4)$$

where a_p^+ is the quasiparticle creation operator.

We now describe the behavior of superconductors in an external electromagnetic field with potentials $\mathbf{A}(x)$ and $\varphi(x)$. With the electron–electron Coulomb interaction taken into account, the total Hamiltonian is

$$\widehat{H} = \int \widehat{\psi}_\alpha^+(x) \left\{ \frac{\left[\widehat{\mathbf{p}} - (e/c) \, \mathbf{A}(x) \right]^2}{2m} - \mu \right\} \widehat{\psi}_\alpha(x) \, \mathrm{d}^3\mathbf{r} + \int e\varphi(x) \, \delta\widehat{n}(x) \, \mathrm{d}^3\mathbf{r} - \int \left[\Delta(x) \, \widehat{\psi}_\downarrow^+(x) \, \widehat{\psi}_\uparrow^+(x) + \text{h.c.} \right] \mathrm{d}^3\mathbf{r} + \frac{1}{2} \int \delta\widehat{n}(x) \, V(\mathbf{r} - \mathbf{r}') \, \delta\widehat{n}(x') \, \mathrm{d}^3\mathbf{r} \,. \quad (5)$$

Hamiltonian (5) accounts for the presence of a uniformly distributed positive background, which ensures the electroneutrality of the system and gives rise to the density fluctuation operator

$$\delta\widehat{n}(x) \equiv \widehat{\psi}_\alpha^+(x) \, \widehat{\psi}_\alpha(x) - n \quad (6)$$

(n is the electron number density in a zero field) in the terms that contain the scalar potential and the electron–electron Coulomb interaction $V(\mathbf{r} - \mathbf{r}') = e^2/|\mathbf{r} - \mathbf{r}'|$.

The following remark is in order. If an external field is added to Hamiltonian (1), the Hamiltonian is unchanged under the gauge transformations

$$\mathbf{A}(x) \to \mathbf{A}'(x) = \mathbf{A}(x) + \nabla\chi(x) \,,$$
$$\varphi(x) \to \varphi'(x) = \varphi(x) - \frac{1}{c} \frac{\partial\chi(x)}{\partial t} \,, \quad (7)$$
$$\widehat{\psi}_\alpha(x) \to \widehat{\psi}_\alpha'(x) = \widehat{\psi}_\alpha(x) \exp\left[\mathrm{i} \frac{e}{c} \chi(x) \right] .$$

But the situation is different for Hamiltonian (5). If we consider Δ a fixed parameter, Hamiltonian (5) is no longer gauge invariant. Actually, the difficulties arise because of this fact. For the description of all effects associated with one-particle excitations in superconductors, Hamiltonian (3) or (4) is sufficient. But when an electromagnetic field comes into the picture, one must bear in mind that Δ must always be defined in a self-consistent manner by formula (2). If this fact is taken into account, then a completely gauge-invariant theory of the linear response of superconductors can be built such that only gauge-invariant combinations of the potentials forming the electromagnetic field enter the final expressions for the current and the electron number density.

The presence of an external field changes the states of the electrons in the superconductor or, in other words, the Heisenberg $\widehat{\psi}$-operators are dependent on the potentials $\mathbf{A}$ and φ. Therefore, order parameter (2) is, obviously, a function of the external field, and hence the total Hamiltonian of the excitation for a superconductor in an external field can be written as

$$\widehat{H}_{\text{int}} = -\frac{1}{c} \int \widehat{\mathbf{j}}^{\,0}(x) \, \mathbf{A}(x) \, \mathrm{d}^3\mathbf{r} + e \int \delta\widehat{n}(x) \left[\varphi(x) + \frac{1}{e} \int V(\mathbf{r} - \mathbf{r}') \left\langle \delta\widehat{n}(x') \right\rangle^{(1)} \mathrm{d}^3\mathbf{r}' \right] \mathrm{d}^3\mathbf{r} - \int \Delta^{(1)}(x) \, \widehat{\psi}_\downarrow^+(x) \, \widehat{\psi}_\uparrow^+(x) \, \mathrm{d}^3\mathbf{r} - \int \Delta^{(1)+}(x) \, \widehat{\psi}_\uparrow(x) \, \widehat{\psi}_\downarrow(x) \, \mathrm{d}^3\mathbf{r} \,, \quad (8)$$

were

$$\widehat{\mathbf{j}}^{\,0}(x) = \frac{\mathrm{i}e}{2m} \left[\left(\nabla\widehat{\psi}_\alpha^+(x) \right) \widehat{\psi}_\alpha(x) - \widehat{\psi}_\alpha^+(x) \, \nabla\widehat{\psi}_\alpha(x) \right] \quad (9)$$

is the 'paramagnetic' part of the current density operator, $\Delta^{(1)}$ is a correction to the order parameter caused by the field, and $\langle \delta\widehat{n}(x) \rangle^{(1)}$ is a correction to the electron number density. Because the Coulomb interaction between electrons leads to strong screening effects in real superconductors, the Coulomb interaction in the random-phase approximation (RPA) is also taken into account in Hamiltonian (8).

To determine the changes in the current and charge distributions due to the excitation Hamiltonian (8), we use the Keldysh diagram technique for nonequilibrium processes [8]. As the building blocks of the diagram technique for a superconductor, in addition to the normal Green's functions G, the anomalous Green's functions F are introduced (their form is given in Appendix 8.1). Here, it is most important to know the two functions $G_{\alpha\beta}^{-+}$ and $F_{\alpha\beta}^{-+}$ defined without the standard T-ordering symbol:

$$G_{\alpha\beta}^{-+}(x, x') = \mathrm{i} \left\langle \widehat{\psi}_\beta^+(x') \, \widehat{\psi}_\alpha(x) \right\rangle ,$$
$$F_{\alpha\beta}^{-+}(x, x') = \mathrm{i} \left\langle \widehat{\psi}_\beta(x') \, \widehat{\psi}_\alpha(x) \right\rangle . \quad (10)$$

We can now use (10) to write expressions for the fluctuations of density (6), current density, and order parameter (2) as

$$\delta n(x) = -2\mathrm{i}G_1^{-+}(x,x)\,, \tag{11}$$

$$\mathbf{j}(x) = -2\,\frac{\mathrm{i}e}{2m}\,(\nabla' - \nabla)\,G_1^{-+}(x,x')|_{x'=x} - \frac{e^2 n}{mc}\,\mathbf{A}(x)\,, \tag{12}$$

$$\Delta^{(1)}(x) = \mathrm{i}gF_1^{-+}(x,x)\,, \tag{13}$$

where the subscript 1 indicates a correction of the first order in perturbation operator (8) to the respective Green's function (we recall that we restrict ourselves to the theory of linear response). The factors 2 appear in (11) and (12) because of summation over spin (we neglect the interaction between the electron spin and the magnetic field, and therefore $G_{\alpha\beta} = \delta_{\alpha\beta}G$ and $F_{\alpha\beta} = \mathrm{i}\sigma^y_{\alpha\beta}F$).

The standard construction of the diagram technique in the perturbation theory with respect to the operator $\widehat{H}_{\mathrm{int}}$ in (8) leads to a first-order correction to the Green's function G^{-+} depicted by the diagrams in Fig. 1. We note that the quantity $\Delta^{(1)}$ on the right-hand side is determined by the anomalous function F^{-+} with coinciding arguments [see Eqn (13)]. For F^{-+}, there exists an expression that is very similar to that for G^{-+} and describes the variation of the anomalous function in the first order in $\widehat{H}_{\mathrm{int}}$. This leads to an equation for $\Delta^{(1)}$, shown in Fig. 2a, where the joint ends of an anomalous Green's function indicate that the arguments x and x' coincide. For the variation in the electron number density (11), we similarly arrive at the equation shown in Fig. 2b. For a normal metal, this equation describes the standard screening effect in the RPA framework. Thus, the equations represented in Fig. 2 are the self-consistency equations for the order parameter and the electron number density fluctuations.

Before we do specific calculations, we discuss some 'ideological' questions related to the self-consistency equation for the order parameter (Fig. 2a). To clarify matters, we temporarily ignore the effects associated with the Coulomb interaction. The self-consistency equations can then be formulated differently if we represent the right-hand side of the equation in Fig. 2a as the sum of all the terms in the series depicted in Fig. 3, where the vertices are due to the BCS coupling constant g. This series corresponds to the contribution of certain vertex corrections that modify the 'initial' vertex of the interaction between the electron and the electromagnetic field due to the electron–electron interaction that leads to superconducting pairing. It is known that the charge conservation law (the continuity equation) leads to a rigid relation between one-particle Green's functions and vertex functions, the Ward identity [9]. It turns out that taking the vertex corrections depicted in Fig. 3 into account is required for the Ward identity to be satisfied in the BCS model. We see in what follows that such vertex corrections contain a pole contribution corresponding to certain collective excitations specific of superconductors. There is a certain analogy with the appearance of a plasmon pole in normal metals when polarization corrections to the Coulomb interaction are taken into account. On the other hand, it can be shown that consistently implementing the requirement of gauge invariance in calculating the response functions automatically ensures that the induced currents and charges satisfy the continuity equation [9]. Thus, the concepts of gauge invariance, charge conservation, the Ward identity, and collective excitations used in different works mean the same thing in the given case. This was first noted by Bogolyubov [3] and Anderson [5].

The difficulties actually emerged at the first stage because Hamiltonian (4), which describes only one-particle excitations in superconductors, was used as the starting point in constructing the linear response. Now we can turn to technical details and show how to obtain manifestly gauge-invariant expressions for the linear response of superconductors.

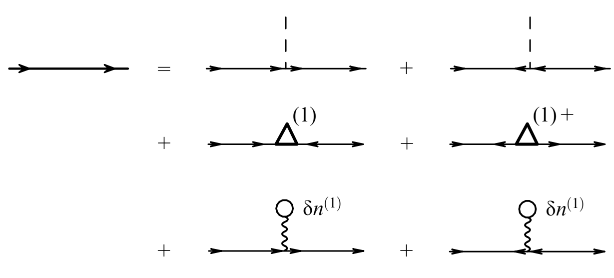

Figure 1. The diagram expression for the correction to the normal Green's function G_1^{-+} that is linear in the electromagnetic field. A dashed line corresponds to the 'initial' interaction of electrons with the potentials $\mathbf{A}$ and φ, a triangle represents the correction $\Delta^{(1)}$ to the order parameter, and a wavy line represents the electron–electron Coulomb interaction V. The open circles in the third line of the expression correspond to a change in the number density $\delta n^{(1)}$ due to the field.

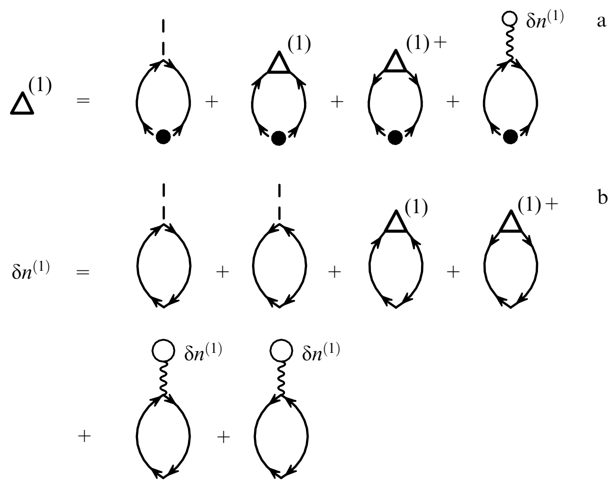

Figure 2. (a) Diagram representation of the self-consistency equation for the correction $\Delta^{(1)}$ to the order parameter that is linear in the external field. A black circle represents the BCS coupling constant g; (b) an equation for determining a change in the electron number density $\delta n^{(1)}$ in an approximation that is linear in the external field.

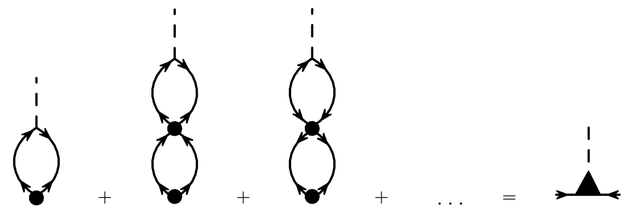

Figure 3. The diagram series corresponding to the renormalization of the vertex coupling the electrons to the electromagnetic field, a renormalization caused by the BCS interaction between electrons.

3. A self-consistent method of calculating the linear response

Instead of solving the problem by directly summing the required vertex diagrams, we can use the strategy first formulated in Ambegaokar and Kadanoff's paper [10]. We know that an external field changes the order parameter. We first assume that the variation $\Delta^{(1)}$ is a fixed but unknown function of the external fields. Then, knowing the first-order diagrams for G^{-+} (Fig. 1), we can easily determine $\mathbf{j}$ and δn as functions of the potentials $\mathbf{A}$ and φ and the quantity $\Delta^{(1)}$. Next, we require that the current and electron number densities satisfy the continuity equation $e(\partial\delta n/\partial t) + \operatorname{div}\mathbf{j} = 0$, which is solved for $\Delta^{(1)}$ as a function of the external fields $\mathbf{A}$ and φ. In accordance with what we said at the end of Section 2, such a procedure is equivalent to the correct summation of the vertex corrections in accordance with the Ward identity.

However, there is an approach that allows restoring gauge invariance at an earlier stage. In calculation of the linear response of a superconductor to an external longitudinal electric field, the change in the order parameter due to the external field can be represented in the form of corrections to the absolute value and the gradient of the phase of the order parameter, which are small quantities. Changes in the phase proper (hence, in the order parameter as a whole) may be of order 1. Such is the situation, e.g., with a superconducting ring placed in a magnetic field. The corresponding theory can be constructed as follows.

We explicitly separate the changes in the absolute value and phase of Δ in an external field:

$$\Delta(x) = (\Delta_0 + \Delta_1)\exp\left[\mathrm{i}\theta(\mathbf{A},\varphi)\right].$$

By applying a gauge transformation, we make the order parameter a real quantity. (We note that the phase θ is still an unknown function of $\mathbf{A}$ and φ.) Under such a transformation, the potentials acquire [in accordance with (7)] additional terms:

$$\mathbf{A}'(x) = \mathbf{A}(x) - \frac{c}{2e}\left(\nabla\theta(x)\right)^{(1)},$$
$$\varphi'(x) = \varphi(x) + \frac{1}{2e}\left(\frac{\partial\theta(x)}{\partial t}\right)^{(1)}. \tag{14}$$

To simplify matters, we begin by studying a superconductor without the Coulomb interaction. Substituting $\mathbf{A}'$ and φ' as external fields into the diagrams in Fig. 1, we readily arrive at the following expression for the current and charge densities (combined for the sake of compactness into a current 4-vector $j_\mu = (e\delta n, \mathbf{j})$):

$$j_\mu(q) = -\frac{e^2}{c}\left[Q^A_{\mu\nu}(q) + \frac{n}{m}\,\delta_{\mu\nu}(1-\delta_{\mu 0})\right]A'_\nu(q) - e\left[Q^\Delta_\mu(q) + Q^{\Delta *}_\mu(-q)\right]\Delta_1(q). \tag{15}$$

Here, for field potentials (14), we have introduced the 4-vector notation $A'(q) = (c\varphi'(q), \mathbf{A}'(q))$ and performed a Fourier transformation in the spatial and temporal variables, $q = (\omega, \mathbf{q})$, with the subscript $\mu = 0$ corresponding to the temporal component of the 4-vector and $\mu = 1, 2, 3$ to the spatial components. The explicit form of the kernels (polarization operators) Q, which are sums of convolutions of different pairs of Green's function, is given in Appendix 8.2 (so as not to clutter up the picture).

Equation (15) is still not the solution of the problem, because its right-hand side (a) contains the unknown variation Δ_1 of the absolute value of the order parameter, and (b) involves the potential A' that contains not only the potentials of the external field but also the phase $\theta(\mathbf{A},\varphi)$, which is to be determined. To find all these quantities, we use the self-consistency equation shown in Fig. 2a:

$$\Delta^{(1)}(q) = -\frac{e}{c}\,\Pi^A_\nu(q)\,A'_\nu(q) - \left[\Pi^\Delta(q) + \Pi^{\Delta^+}(q)\right]\Delta^{(1)}(q). \tag{16}$$

The polarization operators Π are calculated in the same way as the kernels Q and are given in Appendix 8.2. To avoid a misunderstanding, we note that Π^{Δ^+} is an independently defined kernel and not the complex conjugate of Π^Δ.

We introduced the phase θ such that the order parameter $\Delta^{(1)}$ becomes a real quantity. But because the functions Π are complex-valued, Eqn (16) actually amounts to a system of two equations, which makes it possible to simultaneously determine the real-valued correction Δ_1 and the relation between the phase θ and the external field. We define the real and imaginary parts of the functions Π and Q as

$$\Pi_1(q) = \frac{\Pi(q) + \Pi^*(-q)}{2}, \quad \Pi_2(q) = \frac{\Pi(q) - \Pi^*(-q)}{2\mathrm{i}},$$
$$Q_1(q) = \frac{Q(q) + Q^*(-q)}{2}, \quad Q_2(q) = \frac{Q(q) - Q^*(-q)}{2\mathrm{i}}. \tag{17}$$

Then, Eqn (16) with the Coulomb interaction taken into account is equivalent to the two conditions

$$\Delta_1(q) = -\frac{e}{c}\,\Pi^A_{1,l}(q)\left[A_l(q) - \mathrm{i}\,\frac{c}{2e}\,q_l\,\theta(q)\right] + e\,\Pi^A_{1,0}(q)\left[\varphi(q) - \mathrm{i}\,\frac{1}{2e}\,\omega\theta(q) + \frac{1}{e}\,V(q)\,\delta n(q)\right] - \left[\Pi^\Delta_1(q) + \Pi^{\Delta^+}_1(q)\right]\Delta_1(q), \tag{18}$$

$$0 = -\frac{e}{c}\,\Pi^A_{2,l}(q)\left[A_l(q) - \mathrm{i}\,\frac{c}{2e}\,q_l\theta(q)\right] + e\Pi^A_{2,0}(q)\left[\varphi(q) - \mathrm{i}\,\frac{1}{2e}\,\omega\theta(q) + \frac{1}{e}\,V(q)\,\delta n(q)\right] - \left[\Pi^\Delta_2(q) + \Pi^{\Delta^+}_2(q)\right]\Delta_1(q), \tag{19}$$

where (18) is the real part of Eqn (16) and (19) the imaginary part. Equation (19) is the condition for the order parameter to be real (after we explicitly isolated the phase of the order parameter). Because we have now returned to the complete problem with fluctuations of the electron number density and the Coulomb interaction taken into account, Eqns (18) and (19) must be supplemented with by a self-consistency equation for the fluctuations of the electron number density (see Fig. 2b):

$$\delta n(q) = -\frac{e}{c}\,Q^A_{0l}(q)\left[A_l(q) - \mathrm{i}\,\frac{c}{2e}\,q_l\theta(q)\right] + eQ^A_{00}(q)\left[\varphi(q) - \mathrm{i}\,\frac{1}{2e}\,\omega\theta(q) + \frac{1}{e}\,V(q)\,\delta n(q)\right] - 2Q^\Delta_{1,0}(q)\,\Delta_1(q), \tag{20}$$

where $V(q) = 4\pi e^2/q^2$ and the subscript l takes only the 'spatial' values 1, 2, and 3. Equations (18)–(20) form a closed system of three equations for the three unknown functions Δ_1, θ, and δn. By solving this system, we arrive at the following equation for the phase θ as a function of the external fields:

$$\frac{i}{2}\left\{\left[\overline{\Pi}^A_{2,l}(1 - V\overline{Q}^A_{00}) + \overline{\Pi}^A_{2,0}\, V\overline{Q}^A_{0l}\right]q_l - \overline{\Pi}^A_{2,0}\omega\right\}\theta$$

$$= \frac{e}{c}\left[\overline{\Pi}^A_{2,l}(1 - V\overline{Q}^A_{00}) + \overline{\Pi}^A_{2,0} V\overline{Q}^A_{0l}\right]A_l - e\overline{\Pi}^A_{2,0}\varphi\,. \qquad (21)$$

Equation (21) is at the center of the theory of a gauge-invariant response of superconductors. As long as the phase θ is an arbitrary function, Eqn (15) does not allow finding the current density and the fluctuations of the charge density in fixed external fields. Although Eqn (15) is not altered by gauge transformations, it depends on the choice of the initial gauge of the potentials. In solving the full system of three equations, renormalized functions $\overline{\Pi}$ and $\overline{Q}$ appear in (21):

$$\overline{Q}^A_{\mu\nu} = Q^A_{\mu\nu} - 2Q^\Delta_{1,\mu}\,\frac{\Pi^A_{1,\nu}}{1 + \Pi^\Delta_1 + \Pi^{\Delta^+}_1}\,, \qquad (22)$$

$$\overline{\Pi}^A_{2,\nu} = \Pi^A_{2,\nu} - (\Pi^\Delta_2 + \Pi^{\Delta^+}_2)\,\frac{\Pi^A_{1,\nu}}{1 + \Pi^\Delta_1 + \Pi^{\Delta^+}_1}\,. \qquad (23)$$

It can be verified that the additions to the initial functions Π and Q on the right-hand sides of Eqns (22) and (23) emerge when the changes in the absolute value of the order parameter due to the external field are taken into account (see also Ref. [11]). If we ignore these changes by excluding Eqn (18) and setting $\Delta_1 = 0$ in Eqns (19) and (20), the equation for the phase [Eqn (21)] retains its form, the only difference being that the initial operators Π and Q replace the polarization operators with the bar. The problem of how essential the corrections to the absolute value of the order parameter are can be resolved by analyzing the explicit expression for Δ_1 obtained from the same system of equations:

$$\Delta_1 = \frac{ie\left(\Pi^A_{2,l}\overline{\Pi}^A_{1,0} - \Pi^A_{2,0}\,\overline{\Pi}^A_{1,l}\right)E_l}{\left[\overline{\Pi}^A_{2,l'}(1 - V\overline{Q}^A_{00}) + \overline{\Pi}^A_{2,0}\; V\overline{Q}^A_{0l'}\right]q_{l'} - \overline{\Pi}^A_{2,0}\omega}\,, \qquad (24)$$

where

$$\overline{\Pi}^A_{1,\nu} = \frac{\Pi^A_{1,\nu}}{1 + \Pi^\Delta_1 + \Pi^{\Delta^+}_1}\,. \qquad (25)$$

In Appendix 8.3, we give estimates of the polarization operators, which suggest that for ordinary superconductors, there is always a small characteristic parameter Δ/ε_F with respect to which all corrections due to changes in the absolute value of the order parameter are small. In what follows, we therefore ignore the difference between polarization operators with and without a bar.

After we have used Eqn (21) to find the dependence of the phase θ on the potentials $\mathbf{A}$ and φ, it remains to substitute $\theta(\mathbf{A}, \varphi)$ in formulas (15) for the linear response. The final expressions determining the linear response of superconductors to an electromagnetic field are

$$\delta n = \frac{ie\left(\overline{\Pi}^A_{2,l}\,\overline{Q}^A_{00} - \overline{\Pi}^A_{2,0}\,\overline{Q}^A_{0l}\right)E_l}{\left[\overline{\Pi}^A_{2,l'}\,(1 - V\overline{Q}^A_{00}) + \overline{\Pi}^A_{2,0} V\overline{Q}^A_{0l'}\right]q_{l'} - \overline{\Pi}^A_{2,0}\omega}\,, \qquad (26)$$

$$j_k = -\frac{e^2}{c}\left(Q^A_{kl} + \frac{n}{m}\,\delta_{kl}\right)\left(A_l - \frac{q_l q_{l'}}{q^2}\,A_{l'}\right)$$

$$+\, ie^2\,\frac{\overline{Q}^A_{k0}\,\overline{\Pi}^A_{2,l} - \left[\overline{Q}^A_{kl} + (n/m)\,\delta_{kl}\right]\overline{\Pi}^A_{2,0}}{\left[\overline{\Pi}^A_{2,l'}(1 - V\overline{Q}^A_{00}) + \overline{\Pi}^A_{2,0} V\overline{Q}^A_{0l'}\right]q_{l'} - \overline{\Pi}^A_{2,0}\omega}\;\frac{q_l q_{l''}}{q^2}\,E_{l''}\,. \qquad (27)$$

Thus, we have succeeded in solving the problem, i.e., the final formulas contain the potentials $\mathbf{A}$ and φ only in gauge-invariant combinations: as the electric field $\mathbf{E} = -c^{-1}\partial\mathbf{A}/\partial t - \nabla\varphi$ and as the transverse part of the vector potential, $A_{\mathrm{tr},\,l} = \left[A_l - (q_l q_{l'}/q^2)\,A_{l'}\right]$. We note that by solving the equation for the phase of the order parameter, we actually required that the system be gauge invariant in arbitrary fields. Hence, it comes as no surprise that the self-consistency condition (15) in our approach is equivalent to the condition of continuity of the current used by Ambegaokar and Kadanoff [10].

Using the relations between polarization operators (equivalent to the Ward identity; see Appendix 8.2), we can write Eqns (26) and (27) in a more convenient form, which contains the kernels Q^A_{kl}, Q^A_{k0}, and Q^A_{00} standard for the electromagnetic response:

$$\delta n = ieq_k\left[\left(\overline{Q}^A_{kl} + \frac{n}{m}\,\delta_{kl}\right)\overline{Q}^A_{00} - \overline{Q}^A_{k0}\,\overline{Q}^A_{0l}\right]E_l$$

$$\times\left\{q_{k'}\left(Q^A_{k'l'} + \frac{n}{m}\,\delta_{k'l'}\right)q_{l'} - 2\omega\,Q^A_{0l'}\,q_{l'} + \omega^2 Q^A_{00}\right.$$

$$\left. -\, Vq_{k'}q_{l'}\left[\left(\overline{Q}^A_{k'l'} + \frac{n}{m}\,\delta_{k'l'}\right)\overline{Q}^A_{00} - \overline{Q}^A_{k'0}\,\overline{Q}^A_{0l'}\right]\right\}^{-1}, \qquad (28)$$

$$j_k = -\frac{e^2}{c}\left(Q^A_{kl} + \frac{n}{m}\,\delta_{kl}\right)\left(A_l - \frac{q_l q_{l'}}{q^2}\,A_{l'}\right)$$

$$+\, ie^2\left[\left(\overline{Q}^A_{kl} + \frac{n}{m}\,\delta_{kl}\right)\overline{Q}^A_{00} - \overline{Q}^A_{k0}\,\overline{Q}^A_{0l}\right]\omega$$

$$\times\left\{q_{k'}\left(\overline{Q}^A_{k'l'} + \frac{n}{m}\,\delta_{k'l'}\right)q_{l'} - 2\omega\,Q^A_{0l'}\,q_{l'} + \omega^2 Q^A_{00}\right.$$

$$\left. -\, Vq_{k'}q_{l'}\left[\left(\overline{Q}\,A_{k'l'} + \frac{n}{m}\,\delta_{kl}\right)\overline{Q}^A_{00} - \overline{Q}^A_{k'0}\,\overline{Q}^A_{0l'}\right]\right\}^{-1}\frac{q_l q_{l''}}{q^2}\,E_{l''}\,. \qquad (29)$$

Linear response (27) contains a term related to the longitudinal electric field. For certain frequencies ω and wave vectors $\mathbf{q}$, this term may be of a resonant nature. Comparing (27) with (21), we see that a resonance in the response corresponds to a nontrivial solution of the homogeneous equation for the phase, i.e., a solution of Eqn (21) with zero external fields $\mathbf{A}$ and φ. The existence of such solutions means that there are specific collective excitations in the superconductor spectrum for which the order parameter phase is one of the collective variables of the system. In an infinite system, solving the homogeneous equation corresponding to (21) allows finding the dispersion law for such modes, $\omega(q)$. Actually, because the complete expressions for the response necessarily contain a resonant term related to collective excitations, we can say that to restore the gauge invariance, we must, in addition to taking one-particle excitations into account, allow for collective modes (these are discussed later). We next examine several simple limit cases for the general expressions with the aim to relate cumbersome formulas (28) and (29) to the well-known, simpler expressions.

4. Simple limit cases

We begin with the case of static fields, $\omega = 0$. In (29), we then have the contribution to the current density from only the first term containing the transverse part of the vector potential:

$$j_k = -\frac{e^2}{c}\left[Q^A_{kl}(\mathbf{q},0) + \frac{n}{m}\,\delta_{kl}\right]\left(A_l - \frac{q_l q_{l'}}{q^2}\,A_{l'}\right). \tag{30}$$

This is a standard equation of the London type, which, however, fully accounts for spatial dispersion effects. At absolute zero and as $\mathbf{q} \to 0$, we easily establish (see Section 2) that $Q^A_{kl} \to 0$. Then, Eqn (30) becomes the London equation, derived in 1935 via phenomenological reasoning and given in many textbooks on superconductivity (e.g., see Ref. [12]):

$$\mathbf{j} = -\frac{ne^2}{mc}\,\mathbf{A}_{\rm tr}\,. \tag{31}$$

As the temperature is raised to $T_{\rm c}$, with $T_{\rm c}$ the critical, or transition, temperature, and the order parameter vanishes, the kernel $Q^A_{kl} \to -(n/m)\,\delta_{kl}$, and hence there is no London response in a normal metal. We note that for a uniform superconductor, it automatically occurs that $\mathbf{j}$ is proportional precisely to the transverse part of the vector potential, $\mathbf{A}_{\rm tr}$, for any initial gauge of the fields. This is achieved in our approach by determining the phase of the order parameter in an arbitrary external field through Eqn (21). We examine this equation more closely. In the static limit and for finite $\mathbf{q}$, this equation has a very simple form,

$$q^2\theta = -\frac{2{\rm i}e}{c}\,q_l A_l\,, \tag{32}$$

or in the coordinate space,

$$\nabla^2\theta = \frac{2e}{c}\,{\rm div}\,\mathbf{A}\,. \tag{33}$$

In the limit in question, Eqn (15) for the current density becomes

$$\mathbf{j} = -\frac{ne^2}{mc}\left(\mathbf{A} - \frac{c}{2e}\,\nabla\theta\right). \tag{34}$$

We see that with the phase θ, the solution of inhomogeneous equation (33) in infinite space, substituted in (34), the longitudinal part of the vector potential is eliminated from the response. If we select the transverse gauge with ${\rm div}\,\mathbf{A} = 0$ from the start, the only possible solution of Eqn (33) is $\theta = {\rm const}$. But in a finite superconductor, e.g., in a superconducting ring, there can be nonzero solutions of Eqn (33) even in the gauge ${\rm div}\,\mathbf{A} = 0$. In this case, we must return to Eqn (34) with the $\mathbf{A}$-independent additional 'field' $\nabla\theta_0$, which is the solution of the homogeneous equation corresponding to (33). substituted into it. Instead of (31), we then have

$$\mathbf{j} = -\frac{ne^2}{mc}\left(\mathbf{A}_{\rm tr} - \frac{c}{2e}\,\nabla\theta_0\right). \tag{35}$$

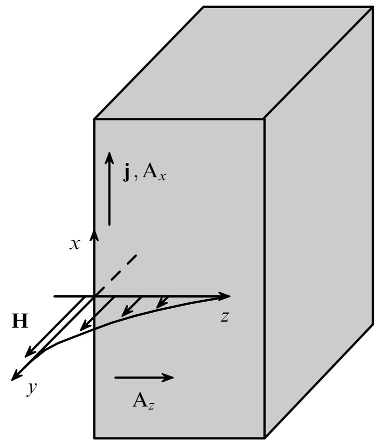

Figure 4. A semi-infinite superconductor in a magnetic field **H** applied parallel to the superconductor surface.

It turns out that in a superconductor with boundaries, only the total solution of the Maxwell equations allows finding $\nabla\theta_0$ in (35), which is an independent quantity from the standpoint of the theory of linear response. Because this aspect is discussed in the literature very scantily, we examine a simple example showing to what extent θ is determined by an external field $\mathbf{A}$ at the superconductor boundary.

We suppose that a magnetic field is applied parallel to the flat surface of a semi-infinite superconductor, as shown in Fig. 4. Equation (31) [or (34)] and the Maxwell equations immediately imply an equation for the magnetic field strength leading to the Meissner effect, i.e., magnetic field damping inside a superconductor. We do not repeat the derivation of this equation. Instead, we simply mention the well-known result that the magnetic field decays in accordance with the law

$$H_y(z) = H_0 \exp\left(-\frac{z}{\lambda}\right), \tag{36}$$

where $\lambda = (4\pi ne^2/mc)^{1/2}$ is the decay length. Such a magnetic field can be described by a vector potential in the transverse gauge as well as in the longitudinal gauge. If we select the transverse gauge with $A_x = -\lambda H_0 \exp\{-z/\lambda\}$, then ${\rm div}\,\mathbf{A} = 0$. The only solution of the homogeneous equation for the phase, $\Delta\theta = 0$, that agrees with the Maxwell equations is $\theta = {\rm const}$. London equations (31) have a distinct meaning: the current is directed along the vector potential $\mathbf{A}$. But there is nothing to prevent us from describing *the same* magnetic field H in terms of the vector potential $A_z = xH_0\exp(-z/\lambda)$, which has a longitudinal part, i.e., for which ${\rm div}\,\mathbf{A} = -A_z/\lambda \neq 0$. The equation for the phase then becomes

$$\Delta\theta = -\frac{2e}{c}\,\frac{A_z}{\lambda}\,, \tag{37}$$

and is solved by

$$\theta(x,z) = -\frac{2e}{c}\,\lambda x H_0 \exp\left(-\frac{z}{\lambda}\right). \tag{38}$$

Equation (34) yields

$$j_z = -\frac{ne^2}{mc}\left[A_z - \frac{c}{2e}\,\frac{\partial\theta}{\partial z}\right] = 0$$

and

$$j_x = \frac{ne}{2m} \frac{\partial \theta}{\partial x} \, .$$

If we substitute (38) in the expression for j_x, we, of course, arrive at the same result as would be obtained with the 'natural' transverse gauge. However, if we select the 'non-standard' gauge for the vector potential, the current density **j** is no longer directed along **A**. Only by solving the equation for θ and substituting the result in (34) we correctly restore the gauge-invariant current density.

We now suppose that we have a massive ring of superconducting material with a cylindrical hole in the middle. We can assume that Fig. 4 shows a small segment of the inner cylindrical surface and the x axis as a whole is 'folded' into a circle. Nonzero solutions of the equation for the phase, $\Delta\theta = 0$, such that $\partial\theta/\partial x = \text{const}$ can then also exist in the transverse gauge $A_x \neq 0$. The physical restriction for such a solution is that for all quantities to be uniquely defined, the total change in the phase due to going around a closed loop must be a multiple of 2π. It is precisely this condition that leads to the magnetic flux quantization in a hole in the superconductor.

The last problem that we wish to consider for static fields is to determine how strongly the screening of a charge in superconductors differs from that in normal metals. Formula (28) simplifies considerably at $\omega = 0$:

$$\delta n = \frac{\mathrm{i} e Q_{00}^A(q)}{q^2 - 4\pi e^2 Q_{00}^A(q)} \, q_l E_l \, . \qquad (39)$$

For small values of q (see Appendix 8.3), we have $Q_{00}^A = -2N_0$, where N_0 is the density of states at the Fermi level. Superconductivity yields corrections of the order of $(\Delta/\varepsilon_{\mathrm{F}})^2$ to this value. Instead of (39), we then have

$$e\delta n = -\frac{1}{4\pi} \frac{\varkappa^2}{q^2 + \varkappa^2} \, \mathbf{q}\mathbf{E}_{\text{ext}} \, , \qquad (40)$$

where $\varkappa$ is the reciprocal Debye (or Thomas – Fermi) screening radius, with $\varkappa^2 = 8\pi N_0 e^2$. Equation (40) clearly shows that static screening in superconductors is described, to within corrections of the order $(\Delta/\varepsilon_{\mathrm{F}})^2$, by the ordinary dielectric constant $\varepsilon(q) = 1 + \varkappa^2/q^2$, as could be expected. The presence of superconducting pairing, which is interpreted as a kind of effective attraction, in no way alters the static dielectric constant and leads to no real attraction of electrons in the coordinate space.

The calculation of the response of superconductors at an arbitrary frequency is a much more difficult problem than in the static case. Here, the difficulties are not only of an 'arithmetic' but also of a 'conceptual' nature. The thing is that only in infinite space can we conveniently separate fields into longitudinal and transverse [as we did explicitly in Eqns (28) and (29)]. Even with a static magnetic field in superconductors with boundaries (of, a finite size), there appear nontrivial solutions of boundary value problems, as we have seen above. One of the main problems confronting the theory is the calculation of the reflection of electromagnetic waves from a superconducting surface. Thus, generally speaking, to correctly interpret the results of experiments, we must know how to solve boundary value problems in a certain geometry. One of the few attempts in this area of research was that of Lozovik and Apenko [30], who described nontrivial modes of a superconducting surface.

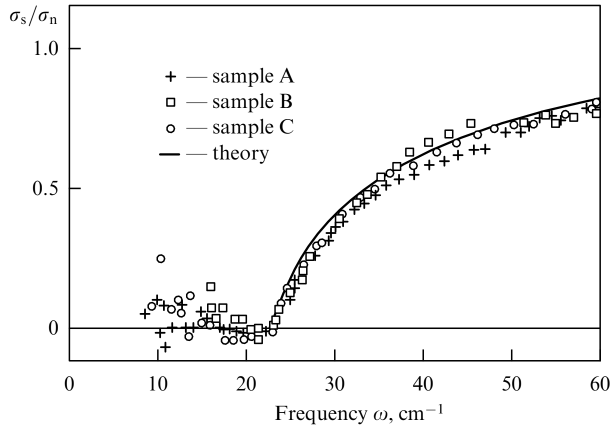

Figure 5. Comparison of the experimentally measured real part of the optical conductivity with the results of the Mattis – Bardeen theory (taken from Ref. [14]).

Unfortunately, up to now, the common approach to solving the problem of reflection of electromagnetic waves has been to study only strictly normal incidence of an electromagnetic wave on a flat, infinite surface of the superconductor. In this case, indeed, only the transverse 'London' part survives in the current density [the first term on the right-hand side of Eqn (29)]. The only quantity that fully determines the response is the kernel Q_{kl}^A, which is calculated for an arbitrary frequency ω but at $\mathbf{q} = 0$ ($Q_{kl}^A = \delta_{kl} Q^A$). It is more convenient to describe the superconductor in this limit not by the function Q_{kl}^A itself but by the optical conductivity σ, which is related to Q_{kl}^A as $\sigma(\omega) = -e^2[Q^A(\omega) + n/m]/\mathrm{i}\omega$. The current density is then related to the electric field in the standard way, as $j = \sigma(\omega)\, E$. The first to calculate the transverse conductivity were Mattis and Bardeen [13] for a superconductor with impurities. They arrived at the following result (at $T = 0$):

$$\frac{\operatorname{Re}\sigma_{\mathrm{s}}}{\operatorname{Re}\sigma_{\mathrm{n}}} = \left[\left(1 + \frac{2\Delta}{\omega}\right) E(\Omega) - \frac{4\Delta}{\omega} K(\Omega)\right] \theta(\omega - 2\Delta) \, . \qquad (41)$$

Here, E and K are elliptic integrals with the argument $\Omega = (\omega - 2\Delta)/(\omega + 2\Delta)$, and the conductivity of the superconductor is normalized to the conductivity in the normal state. Figure 5 shows the result (taken from Ref. [14]) of comparing formula (41) with the experimental data. The real part of the conductivity determines the absorption in the system, and therefore, due to the gap in the spectrum of one-particle states, $\operatorname{Re}\sigma_{\mathrm{s}} = 0$ in the BCS model up to the frequencies $\omega = 2\Delta$. Above this threshold, quasiparticles and quasiholes occur in pairs, and absorption gradually increases. In the framework of a more detailed approach based on the Eliashberg equations, it is also possible to calculate the transverse conductivity of superconductors. However, the behavior of $\operatorname{Re}\sigma_{\mathrm{s}}(\omega)$ is more complicated in this case. The real part of the conductivity is then nonzero even at frequencies below Δ. The details of calculations in the strong-coupling theory can be found in Maksimov's review article [15] and in the literature cited therein.

5. Collective excitations in superconductors

We now return to the discussion of the possibility of resonances appearing in the response due to the excitation of collective modes. We recall Eqn (21) for the phase in the absence of fields. We write it in the same way as Eqns (28) and (29), but in terms of the polarization operators Q:

$$\left\{ q_k \left(Q_{kl}^A + \frac{n}{m}\,\delta_{kl} \right) q_l - 2\omega Q_{0l}^A q_l + \omega^2 Q_{00}^A - V(q)\, q_k q_l \left[\left(Q_{kl}^A + \frac{n}{m}\,\delta_{kl} \right) Q_{00}^A - Q_{k0}^A Q_{0l}^A \right] \right\} \theta = 0\,. \quad (42)$$

The first (and the simplest) type of collective excitation is generated in an uncharged Fermi gas with attraction according to the BCS model. The term 'uncharged Fermi gas' means that we have excluded the Coulomb interaction between particles and have set $V(q) = 0$ in (42). At absolute zero in the limit of small q and ω, Eqn (42) acquires the simple form

$$(q^2 v^2 - \omega^2)\, \theta(q, \omega) = 0\,. \quad (43)$$

In deriving (43), we used the limit expressions of polarization operators for small q and ω given in Appendix 8.3. Bogolyubov et al. [3] and Anderson [5] were the first to discuss the possibility of the existence of collective oscillations with an acoustic spectrum [3, 5]. They found that with a quadratic dispersion law for Fermi particles, the velocity of 'sound' is given by the simple formula $v = v_{\mathrm{F}}/\sqrt{3}$ (where v_{F} is the Fermi velocity). These modes are characteristic features of superconductors and are related to long-wave oscillations of the order parameter phase, and the oscillation propagation speed is always of the order of v_{F}.

Ambegaokar and Kadanoff [10] calculated the current density in a superconductor with the Bogolyubov mode taken into account and arrived at an expression that is the limit of the general formula (29) at $V(q) = 0$ and $T = 0$:

$$\mathbf{j} = -\frac{ne^2}{mc} \left[\mathbf{A}_{\mathrm{tr}} + \frac{\mathrm{i}c\omega \mathbf{E}_{\|}}{(q^2 v^2 - \omega^2)} \right], \quad (44)$$

where $\mathbf{E}_{\|}$ is the longitudinal part of the electric field. We note that at $\omega = 0$ and finite $\mathbf{q}$, this expression reduces to London formula (31).

It is unclear whether uncharged Fermi systems (e.g., neutron stars) with such a Bogolyubov mode can exist. It is also unclear whether formula (44) has a clear-cut meaning, because it was derived for Fermi particles interacting with an electromagnetic field but not experiencing mutual Coulomb repulsion. There is no way in which the Coulomb interaction between electrons in ordinary superconductors can be neglected. In the long-wave limit as $\mathbf{q} \to 0$, the contribution of the Coulomb interaction $V(q) = 4\pi e^2/q^2$ dramatically changes the collective-mode spectrum. While the Bogolyubov mode was determined by the first row in Eqn (42), now, due to the divergence of $V(q)$ at small q in (42), the leading terms are those proportional to Q_{00}^A, because $Q_{kl}^A \to 0$ and $Q_{k0}^A \to 0$ at low temperatures (see Appendix 8.3):

$$\left[\omega^2 Q_{00}^A - \frac{4\pi e^2}{q^2}\, q_k q_l\, \frac{n}{m}\, \delta_{kl} Q_{00}^A \right] \theta = 0\,, \quad (45)$$

$$\left[\omega^2 - \omega_{\mathrm{p}}^2 \right] \theta = 0\,. \quad (46)$$

Equation (46) describes plasma oscillations with the frequency determined by the standard formula $\omega_{\mathrm{p}}^2 = 4\pi ne^2/m$. We have arrived at the fact, known from Anderson's paper [5], that at small q, the Coulomb interaction 'pushes,' as it is sometimes said, the frequency of acoustic oscillations to the plasma frequency ω_{p}. Here is a qualitative explanation of this effect. As noted earlier, the Bogolyubov mode is related to the appearance of oscillations in the order parameter phase, necessarily causing oscillations of the current [see Eqn (15)], which in turn give rise to oscillations in the electron number density. Any changes in the charge density in metals, including superconductors, generate strong longitudinal electric fields, which result in oscillations at the plasma frequency.

A somewhat unexpected fact is that despite what we have said about the role of the strong Coulomb interaction, there may still be long-wave oscillations of the acoustic type in superconductors. The first traces of such modes of the acoustic type were observed in 1973 – 1975 in the experiments of Carlson and Goldman [16, 17], and these excitations therefore became known as the Carlson – Goldman modes. In Section 6, we show how to build a theory that describes these experiments. Here, we try to explain in what case solutions of the acoustic type may appear in Eqn (42). We take the long-wave limit, $\omega \to 0$ and $\mathbf{q} \to 0$, and assume that the ratio $\omega/q = v_0$ is finite and equal to the velocity of the acoustic oscillations. In this case, the leading part in Eqn (42) is the one containing the Coulomb potential $V(q)$. In the limit, all the polarization operators are functions of only v_0 and the temperature T, and we arrive at the equation

$$\frac{q_k q_l}{q^2} \left\{ \left[Q_{kl}^A(v_0, T) + \frac{n}{m}\,\delta_{kl} \right] Q_{00}^A(v_0, T) - Q_{k0}^A(v_0, T)\, Q_{0l}^A(v_0, T) \right\} \theta = 0\,. \quad (47)$$

Finding a simple analytic solution of Eqn (42) in the general case is impossible. The existence of a solution of this equation depends on the temperature and on how strong the scattering on impurities in a given superconductor is. The conditions needed for the existence of the Carlson – Goldman modes are different for pure and impure superconductors. In any case, such modes can exist only at high temperatures close to the transition temperature. By studying the properties of polarization kernels, we can easily show that at temperatures close to absolute zero in the long-wave limit, the only oscillations that are possible are those at the plasma frequency and acoustic modes cannot appear. Here is a qualitative explanation of what the Carlson – Goldman modes are. As noted earlier, the low-frequency Bogolyubov sound cannot exist in real superconductors because oscillations of the phase generate charge density oscillations. But at temperatures close to T_{c}, in addition to the 'superconducting' current, there is the current of quasiparticles, or the 'normal' component of the current. If oscillations of the order parameter phase generate oscillations of the superconducting current, the 'normal' current can oscillate in antiphase, to prevent the charge density from changing. This opens the possibility of soft oscillations of the acoustic type existing in a superconductor. As the temperature decreases, the number of quasiparticles decreases, and at some moment the 'normal' current is not strong enough to compensate the 'superconducting' contribution. At this temperature, the soft modes disappear and the collective-mode frequency is 'pushed' to the plasma frequency.

This qualitative explanation already suggests that the Carlson – Goldman modes are damped modes, in contrast to the Bogolyubov modes. First, in the presence of impurities, the normal component of the current is always dissipative. Second, it turns out that the damping associated with the

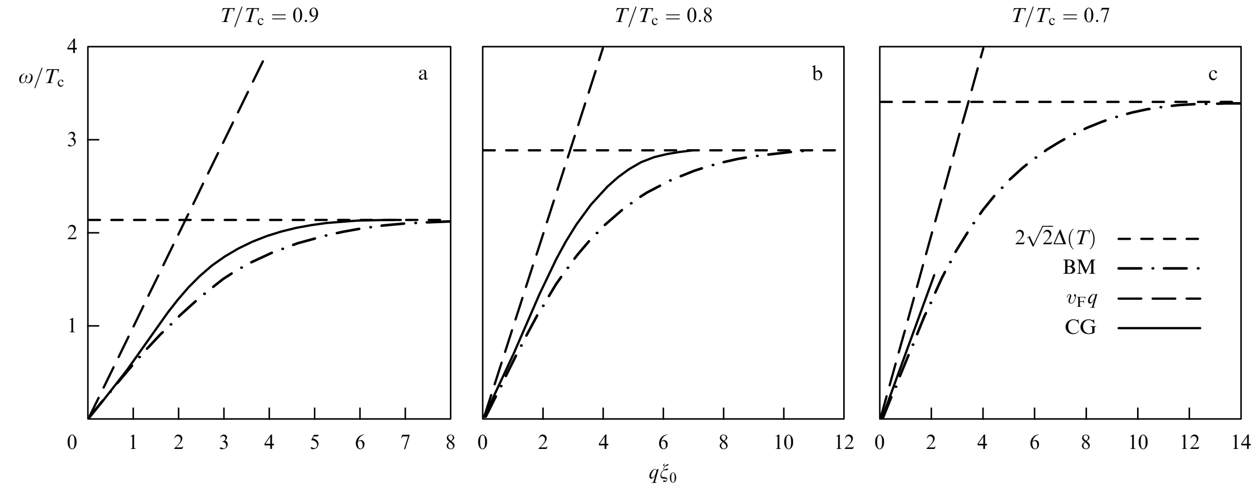

Figure 6. Dispersion of collective excitations for a pure superconductor with a d-wave order parameter, at different temperatures: (a) $T_c = 0.9$, (b) $T/T_c = 0.8$, and (c) $T/T_c = 0.7$. The solid line corresponds to a Carlson – Goldman (CG) mode and the dotted-dashed line to a Bogolyubov mode (BM). The figure is taken from Ref. [18].

quasiparticle – quasihole pair production in pure superconductors has an even more destructive effect on acoustic modes. This effect is so strong that for pure s-type superconductors in the nondissipative regime, there can be no Carlson – Goldman modes. But in pure d-type superconductors, the situation is different because many normal excitations remain down to low temperatures. Figure 6, taken from paper [18], schematically shows how the region where the Carlson – Goldman modes exist in a d-wave superconductor changes as the temperature decreases from T_c. It can be seen from the figure that the speed of the soft mode in this case is close to v_F (see also Ref. [19]). Artemenko and Volkov [20], Ovchinnikov [21], and Schön [22] studied s-type superconductors with impurities. Similar investigations based on the Green's function formalism within the temperature technique have been performed by many authors (see Refs [23 – 27] and Refs [28, 29]). Despite the differences in the approaches, the soft-mode speed at temperatures close to the transition temperature is of the same order in most works: $v_0^2 \sim (\Delta(T)/T)v_F^2$.

We note that the phrase 'Carlson – Goldman modes exist in such and such conditions' should be interpreted with caution. The point is that Eqn (47) has a real part and an imaginary part. In many works, the condition needed for the existence of a Carlson – Goldman mode is assumed to be the requirement that the real part of Eqn (47) vanish. But the magnitude and role of the imaginary part requires a separate investigation. There is also another problem that is rarely discussed. We return to expression (29) for the current density. We see that condition (47) corresponds not to a resonance in the response, as would be the case when the frequency of the external field coincides with the frequency of an eigenmode of the system, but to the condition that the current density must vanish. This agrees with the general qualitative understanding that low-frequency modes exist because the superconducting current and the normal current balance each other. However, the total equation for eigenoscillations of the phase, Eqn (42), also contains terms proportional to q^2 and ω^2, which were neglected in Eqn (47). And if exact solutions of the equation for the phase existed, then instead of the current vanishing, a resonance would appear in the current.

The common approach in works devoted to the Carlson – Goldman mode is to study not the response function but only an equation for the phase similar to Eqn (42) but obtained within one or another simplified model of the problem. At the same time, we see that such an approach may have no immediate physical meaning, because the response functions may have no resonances at these frequencies due to damping or because the zeros in the numerators and denominators in (28) and (29) are close to each other. But then how do these modes manifest themselves in physically measurable quantities, as resonances or as zeros of response functions? One answer we already know: in tunnel measurements involving superconductors, which we examine in the next section.

The following remark is in order in concluding this section. The question of whether collective eigenmodes exist in superconductors is closely related to the problem discussed in Section 3 concerning the homogeneous solution of the equation determining the order parameter phase as a function of external fields. For an infinite superconductor, solving such equations is reduced to solving an algebraic equation in the Fourier space. The possibility of nontrivial solutions emerging in the case of a finite superconductor was briefly discussed above only in the static case. Actually, a problem that has been left unstudied is how the collective modes in finite superconductors, e.g., films and small crystals in resonators, change. The difficulty lies in the fact that Eqn (42) becomes an integro-differential equation. The problem of the imposed boundary conditions is also essential here. The only work in this area of research, we believe, is Lozovik and Apenko's paper [30] written in 1981. And yet the problem is highly important for interpreting the results of many experiments in measuring the conductivity of high-T_c superconductors.

6. Excitation of collective modes in tunnel experiments

Soft modes in superconductors were first observed in experiments in measuring the tunneling conductivity described by Carlson and Goldman [16, 17]. A schematic of a tunnel junction that was used in the experiments is shown in Fig. 7. A magnetic field directed parallel to the junction plane is applied in a flat gap between the two superconductors of the

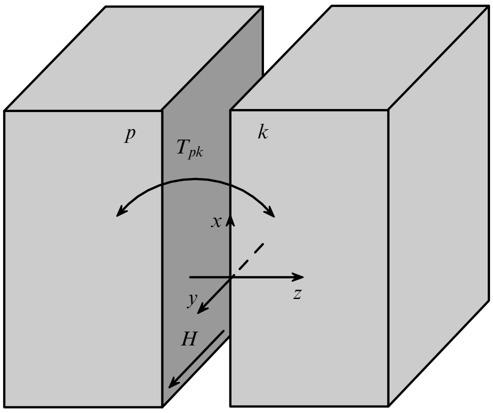

Figure 7. Schematic of a tunnel junction used to observe a collective Carlson – Goldman mode.

tunnel junction. As we see later, it was the introduction of a magnetic field into the experiment that made it possible to determine the dispersion law of collective modes, $\omega(q)$, within a certain region.

The tunneling of electrons between two superconductors is described by the Hamiltonian

$$H_t = \sum_{k,p} (T_{kp} \hat{c}_k^+ \hat{c}_p + \text{h.c.}) \,, \tag{48}$$

where $\hat{c}_k^+$ and $\hat{c}_p^+$ are the electron creation operators in the left and right superconductors and T_{kp} is the matrix element describing the transition from the state k of one superconductor to the state p of the other superconductor. Here, we have omitted the spin indices because we assume that tunneling occurs without spin flip. The tunnel current is determined by the off-diagonal Green's functions G_{kp}^{-+} [31], which become nonzero as soon as electron transitions described by Hamiltonian (48) occur:

$$J_{\text{tun}} = 2e \sum_{k,p} (T_{pk}\, G_{kp}^{-+} - T_{kp}\, G_{pk}^{-+}) \,. \tag{49}$$

Just as we sought the perturbation of the system due to an external field in linear-response theory, we here have a similar problem of building a perturbation theory with respect to tunnel Hamiltonian (48). We can say that we are calculating the 'response' of a system consisting of two superconductors to the transition amplitude T_{kp}, while the applied potential and the magnetic field are taken into account exactly in the zeroth-order Green's functions.

We assume for definiteness that the potential difference ϕ across the tunnel junction is described as a change in the potential of only the left side of the junction. Then, in the space – time representation, the Green's functions of the superconductors involves oscillating factors of the form [32, 33]

$$\begin{aligned} G_k(r_1, t_1; r, t) &= \exp\left(-\frac{\mathrm{i}}{2}\, q_H^k x_1\right) G_k(r_1 - r, t_1 - t) \\ &\quad \times \exp\left(\frac{\mathrm{i}}{2}\, q_H^k x\right), \\ F_k(r_1, t_1; r, t) &= \exp\left(-\frac{\mathrm{i}}{2}\, q_H^k x\right) F_k(r_1 - r, t_1 - t) \\ &\quad \times \exp\left(-\frac{\mathrm{i}}{2}\, q_H^k x_1\right), \\ G_p(r_1, t_1; r, t) &= \exp\left(\frac{\mathrm{i}}{2}\, q_H^p x_1 - \mathrm{i}e\phi t_1\right) G_p(r_1 - r, t_1 - t) \\ &\quad \times \exp\left(-\frac{\mathrm{i}}{2}\, q_H^p x + \mathrm{i}e\phi t\right), \\ F_p(r_1, t_1; r, t) &= \exp\left(\frac{\mathrm{i}}{2}\, q_H^p x_1 - \mathrm{i}e\phi t_1\right) F_p(r_1 - r, t_1 - t) \\ &\quad \times \exp\left(\frac{\mathrm{i}}{2}\, q_H^p x - \mathrm{i}e\phi t\right). \end{aligned} \tag{50}$$

Hence, the order parameters become oscillating functions of the form

$$\begin{aligned} \Delta_p(x, t) &= \Delta_p \exp(\mathrm{i}q_H^p x - \mathrm{i}2e\phi t), \\ \Delta_k(x, t) &= \Delta_k \exp(-\mathrm{i}q_H^k x) \,, \end{aligned} \tag{51}$$

where $q_H^p = 2e\lambda_p/\hbar cH$ and $q_H^k = 2e\lambda_k/\hbar cH$. It is important that the magnetic field controls the magnitude of the wave vector and that the voltage applied to the junction controls the oscillation frequency.

The perturbation theory with respect to Hamiltonian (48) can be built in exactly the same way as in the case of a response to an external field. The first-order correction to the Green's function G_{kp}^{-+} is

$$G_{kp}^{(1)-+} = (G_k T_{kp} G_p)^{-+} + (F_k T_{kp} F_p^+)^{-+} \,. \tag{52}$$

This approximation is sufficient if we want to derive the ordinary expression for the current (including the Josephson current) in a tunnel junction of two superconductors. The resonance in which we are interested (related to collective modes) may manifest itself in the DC tunnel characteristics only in the third order in T_{kp}, as we see shortly. However, in calculating higher-order corrections, we must again take into account that turning on the tunneling between superconductors leads to corrections in the order parameter and the electron number density fluctuations. For simplicity, we assume that the transition temperatures of the superconductors differ substantially and that the temperature is closer to T_c of the 'weaker' superconductor (which agrees with the conditions of a real experiment). Then the variation of the order parameter is stronger for the superconductor with the lower value of T_c, and we can ignore the variation of the order parameter in the other superconductor. We assume that the junction side labeled by k is just such a superconductor. As a result of calculations of G_{kp}^{-+} in the third order in T_{kp}, there appear corrections, shown in Fig. 8, in which the variation of the off-diagonal Green's function is determined by the action of the tunnel matrix element T_{kp} both 'explicitly' and 'implicitly' through the variation of the order parameter Δ_k and the electron number density δn_k:

$$\begin{aligned} G_{kp}^{(3)-+} &= \left(G_k \Delta_k^{(1)} G_{-k} T_{kp} F_p^+\right)^{-+} + \left(F_k \Delta_k^{(1)+} F_k T_{kp} F_p^+\right)^{-+} \\ &\quad + \left(G_k (V\delta n_k)\, F_k T_{kp} F_p^+\right)^{-+} \\ &\quad + \left(F_k (V\delta n_k)\, G_{-k} T_{kp} F_p^+\right)^{-+} \ldots \,. \end{aligned} \tag{53}$$

According to the general approach, the corrections to the order parameter, $\Delta_k^{(1)}$, and the electron number density, δn_k, must satisfy a self-consistency equation similar to Eqns (16) and (20) (Fig. 9a). What happens next depends very strongly on the type of tunnel junction that we consider, because different types of junctions have different dependences of the matrix element T_{kp} on the momenta k and p. For instance, for

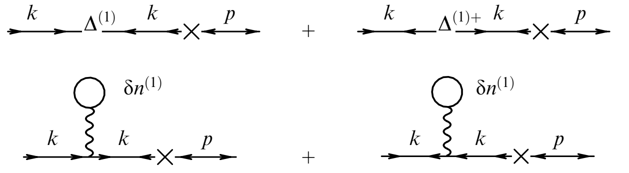

Figure 8. The part of the diagrams of the third order in the tunnel matrix element T_{kp} (a $\times$ in the figure) for the function G^{-+}_{kp} containing corrections to the order parameter, $\Delta^{(1)}$, and fluctuations of the electron number density, $\delta n^{(1)}$, of superconductor k.

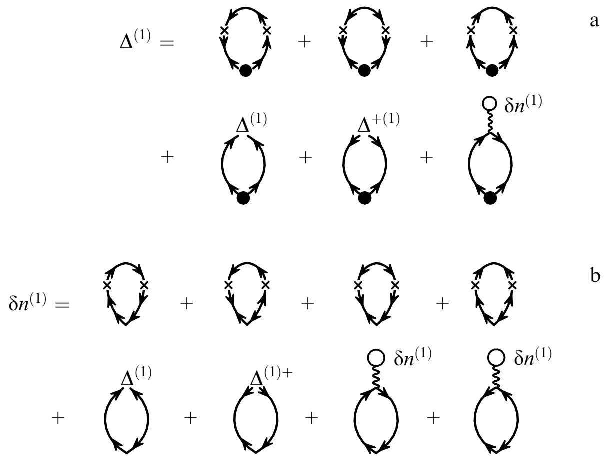

Figure 9. Self-consistency equations for (a) the correction to the order parameter of superconductor k that is quadratic in T_{kp}, and (b) the fluctuation of the electron number density of superconductor k in the second order in the tunnel matrix element T_{kp}.

a point junction, a good approximation is the substitution of a momentum-independent constant for T_{kp}. Summation over the momenta of the left and right sides of the junction is then performed independently, which simplifies calculations considerably. Such an approximation is often used to derive the classical expression for the tunnel current in the first order in T^2_{kp} [33]. But the replacement of T_{kp} with a constant in higher-order terms is too rough an approximation in our case. For the real junction of the type considered here, due to the roughness of the its surfaces and defects on them and to other technological reasons that lead to the junction not being ideal, we must assume that the tunnel matrix element is a random function of the coordinates x and y of both sides of the junction. Then, assuming that this random function is characterized by a Gaussian distribution and that there is no correlation between the random inhomogeneities of the surfaces of the two superconductors, we conclude that the following averages appear in the theory:

$$\langle T_{kp} T_{p_1 k_1} \rangle = T^2_{k-k_1, p-p_1} \quad \text{or} \quad \langle T_{\mathbf{r}_1 \mathbf{r}'_2} \, T_{\mathbf{r}'_3 \mathbf{r}_4} \rangle = T^2_{\mathbf{r}_1 - \mathbf{r}'_2} \, \delta(\mathbf{r}_1 - \mathbf{r}_4) \, \delta(\mathbf{r}'_2 - \mathbf{r}'_3) \,, \tag{54}$$

where $\mathbf{r} = (x, y)$ is a point on the surface of one superconductor, $\mathbf{r}' = (x', y')$ is a point on the surface of the other superconductor, and only the dependence on the momentum components parallel to the junction plane is written explicitly. The use of such averages leads to the situation where second-order corrections to the order parameter and to the electron number density are distinctly separated into three space–time 'harmonics': one constant contribution and two proportional to $\exp\left[\mathrm{i}(q^p_H + q^k_H)\,x - \mathrm{i}2e\phi t\right]$ and $\exp\left[-\mathrm{i}(q^p_H + q^k_H)\,x + \mathrm{i}2e\phi t\right]$. To explain all this, we use the expression corresponding to the third diagram in Fig. 9a. We let $D_3(\mathbf{r}, t)$ denote the contribution of this diagram:

$$D_3(\mathbf{r},t) = \mathrm{i}g \int \left[G_k(\mathbf{r}, t; \mathbf{r}_1, t_1) \, T_{\mathbf{r}_1 \mathbf{r}'_2} \, F_p(\mathbf{r}'_2, t_1; \mathbf{r}'_3, t_2) \times T_{\mathbf{r}'_3 \mathbf{r}_4} G_k(\mathbf{r}, t; \mathbf{r}_4, t_2) \right]^{-+} \mathrm{d}\mathbf{r}_1 \, \mathrm{d}\mathbf{r}_2 \, \mathrm{d}\mathbf{r}_3 \, \mathrm{d}\mathbf{r}_4 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \,. \tag{55}$$

Using the averaging in (54), we obtain

$$D_3(\mathbf{r}, t) = \mathrm{i}g \int \left[G_k(\mathbf{r}, t; \mathbf{r}_1, t_1) T^2_{\mathbf{r}_1 - \mathbf{r}'_2} F_p(\mathbf{r}'_2, t_1; \mathbf{r}'_2, t_2) \times G_k(\mathbf{r}, t; \mathbf{r}_1, t_2) \right]^{-+} \mathrm{d}\mathbf{r}_1 \, \mathrm{d}\mathbf{r}_2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \,. \tag{56}$$

If we separate the explicit dependence of the Green's functions on the phase oscillations [Eqn (50)], we can write the expression for $D_3(\mathbf{r}, t)$ in terms of the superconductor Green's functions that depend solely on the differences of the $\mathbf{r}$ and t variables. The final contribution of $D_3(\mathbf{r}, t)$ can be written as

$$\begin{aligned} D_3(\mathbf{r}, t) &= D_3 \exp(\mathrm{i} q^p_H x - \mathrm{i}2e\phi t) \,, \\ D_3 &= \mathrm{i}g \widetilde{T^2} \int \mathrm{d}\varepsilon \int \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{p}_1 \left[G_k(\mathbf{p} + \mathbf{q}_H, \varepsilon + e\phi) \times F_p(\mathbf{p}_1, \varepsilon) \, G_k(-\mathbf{p}, -\varepsilon + e\phi) \right]^{-+} , \end{aligned} \tag{57}$$

where

$$\mathbf{q}_H = (q^p_H + q^k_H) \, \hat{\mathbf{x}} \,, \quad \widetilde{T^2} = \int \exp\left[-\mathrm{i} q^p_H (x_1 - x) \right] T^2_{\mathbf{r}_1 - \mathbf{r}} \, \mathrm{d}\mathbf{r}_1 \,, \tag{58}$$

and $\hat{\mathbf{x}}$ is the unit vector along the x axis. Reasoning along the same lines, we can easily show that the second and third diagrams on the right-hand side of Fig. 9a make the respective contributions proportional to $\exp[-\mathrm{i}(q^p_H x + 2q^k_H x - 2e\phi t)]$ and $\exp(-\mathrm{i} q^k_H x)$. The same separation into three harmonics can be done in calculations of the corrections to the electron number density, whose diagram series is shown in Fig. 9b. Hence, it is convenient to write δn_k and $\Delta^{(1)}_k$ as a sum of contributions of three different harmonics:

$$\begin{aligned} \Delta^{(1)}_k(x, t) &= \Big\{ \Delta_1 + \Delta_2 \exp\left[-\mathrm{i}\,(q^p_H x + q^k_H x - 2e\phi t) \right] + \Delta_3 \exp\left[\mathrm{i}\,(q^p_H x + q^k_H x - 2e\phi t) \right] \Big\} \exp(-\mathrm{i} q^k_H x) \,, \\ \delta n_k(x, t) &= \delta n_1 + \delta n_2 \exp\left[-\mathrm{i}\,(q^p_H x + q^k_H x - 2e\phi t) \right] + \delta n_3 \exp\left[i\,(q^p_H x + q^k_H x - 2e\phi t) \right] . \end{aligned} \tag{59}$$

The superconductor with the lower value of T_c used in the Carlson–Goldman experiments was a thin film whose thickness was much smaller than the penetration depth and the coherence length. In these conditions, both δn and Δ are practically constant across the film, and Eqn (58) describes the variation of these quantities (averaged over the film thickness) as depending on the time and the coordinate x along the film. Strictly speaking, in this case, the definition of q^k_H involves the film thickness d rather than the penetration depth λ [see Eqn (51)], but this does not significantly alter the result. In formula (57) for D_3, we intentionally indicated only the dependence on the coordinates x and y in the junction plane and did not explicitly write the integral over the transverse coordinate z. The point is that the specific value of the integral with respect to z is, generally speaking, model-dependent and differs for different types of junctions and different geometries. This, however, is unimportant in our further discussion.

If in (53) we isolate the oscillating factors in the Green's functions (50), we immediately see that the DC contribution is obtained if we substitute Δ_3 in the first term on the right-hand side of Eqn (53), Δ_2^+ in the second term, and δn_3 in the third and fourth terms. The remaining possible combinations yield corrections to the oscillating part of the tunnel current (the Josephson current and its harmonics) and trivial corrections to the DC amplitude, which do not contain the resonance contribution from the collective modes. Hence, the problem amounts to solving only the system of equations for the constants Δ_2^+, Δ_3, and δn_3, a system that follows from the self-consistency equations (see Fig. 9) after the expansion in harmonics (59) have been substituted in them and the coefficients at each harmonic have been equated. The system of equations has the form

$$\Delta_2^+\left[1+\Pi^{\Delta*}(-q)\right]+\Pi^{\Delta^+*}(-q)\,\Delta_3-\Pi_0^{A*}(-q)\,V(q)\delta n_2^*=D_2^+(q),$$
$$\Delta_3\left[1+\Pi^{\Delta}(q)\right]+\Pi^{\Delta^+}(q)\,\Delta_2^+-\Pi_0^A(q)\,V(q)\,\delta n_3=D_3(q)\,,\tag{60}$$

$$\delta n_2^*\left[1-V(q)\,Q_{00}^{A*}(-q)\right]+Q_0^{\Delta}(q)\,\Delta_3+Q_0^{\Delta*}(-q)\,\Delta_2^+=\delta N_2^*(q)\,,$$
$$\delta n_3\left[1-V(q)\,Q_{00}^A(q)\right]+Q_0^{\Delta}(q)\,\Delta_3+Q_0^{\Delta*}(-q)\,\Delta_2^+=\delta N_3(q)\,.\tag{61}$$

For brevity, we have combined the components of the Fourier transform in t and x into one quantity: $q\equiv(\omega=2e\phi, q_x=q_H^p+q_H^k)$. We note that the polarization kernels involved here are the same as in the linear-response theory, but instead of the frequency and wave vector of the applied electromagnetic field, they now involve the Josephson frequency and the wave vector specified in the given junction geometry by the constant magnetic field.

Bearing in mind our remarks concerning the possibility of integrating the Green's functions over the coordinate z in different ways, we find that the functions on the right-hand sides of Eqns (60) and (61) are determined by formulas similar to (57):

$$D_2^+=-\mathrm{i}g\widetilde{T^2}\int\mathrm{d}\varepsilon\int\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{p}_1\left[F_k^+(\mathbf{p}+\mathbf{q}_H,\varepsilon+e\phi)\right.$$
$$\left.\times F_p(\mathbf{p}_1,\varepsilon)\,F_k^+(\mathbf{p},\varepsilon-e\phi)\right]^{-+},$$
$$\delta N_3=-2\mathrm{i}\widetilde{T^2}\int\mathrm{d}\varepsilon\int\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{p}_1\left[G_k(\mathbf{p}+\mathbf{q}_H,\varepsilon+e\phi)\right.$$
$$\left.\times F_p(\mathbf{p}_1,\varepsilon)\,F_k^+(\mathbf{p},\varepsilon-e\phi)\right]^{-+},$$
$$\delta N_2^*=\delta N_3\,.\tag{62}$$

Clearly, the condition $\delta n_2^*=\delta n_3$ is always satisfied, which follows directly from the fact that electron number density fluctuation (59) is a real quantity. As a result, we arrive at a simpler system of three equations for the amplitudes Δ_2^+, Δ_3, and δn_3 of the harmonics. The system of equations becomes even simpler if, as discussed in Section 3, we neglect contributions of the order of $\Delta/\varepsilon_{\mathrm{F}}$ in the polarization kernels that are present in Eqns (60) and (61). In this approximation (see Appendix 8.3), $\Pi_0^A(q)=-\Pi_0^{A*}(-q)$, $\Pi^{\Delta}(q)=\Pi^{\Delta*}(-q)$, $\Pi^{\Delta^+}(q)=\Pi^{\Delta^+_*}(-q)$, and $Q_0^{\Delta}(q)=-Q_0^{\Delta*}(-q)$. Then, the sought expressions for Δ_2^+, Δ_3, and δn_3 are given by

$$\Delta_2^+(q)=\frac{1}{2}\,\frac{D_2^+(q)+D_3(q)}{1+\Pi^{\Delta}(q)+\Pi^{\Delta^+}(q)}$$
$$+\frac{1}{2}\,\frac{[D_2^+(q)-D_3(q)][1-V(q)Q_{00}^A(q)]-2\delta N_3(q)V(q)\Pi_0^A(q)}{[1+\Pi^{\Delta}(q)-\Pi^{\Delta^+}(q)][1-V(q)\,Q_{00}^A(q)]+2\Pi_0^A(q)V(q)\,Q_0^{\Delta}(q)}\,,\tag{63}$$

$$\Delta_3(q)=\frac{1}{2}\,\frac{D_2^+(q)+D_3(q)}{1+\Pi^{\Delta}(q)+\Pi^{\Delta^+}(q)}$$
$$-\frac{1}{2}\,\frac{\left[D_2^+(q)-D_3(q)\right]\left[1-V(q)\,Q_{00}^A(q)\right]-2\delta N_3(q)V(q)\,\Pi_0^A(q)}{\left[1+\Pi^{\Delta}(q)-\Pi^{\Delta^+}(q)\right]\left[1-V(q)\,Q_{00}^A(q)\right]+2\Pi_0^A(q)V(q)\,Q_0^{\Delta}(q)}\,,\tag{64}$$

$$\delta n_3(q)=\frac{\left[D_2^+(q)-D_3(q)\right]Q_0^{\Delta}(q)+\delta N_3(q)\left[1+\Pi^{\Delta}(q)-\Pi^{\Delta^+}(q)\right]}{\left[1+\Pi^{\Delta}(q)-\Pi^{\Delta^+}(q)\right]\left[1-V(q)\,Q_{00}^A(q)\right]+2\Pi_0^A(q)\,V(q)\,Q_0^{\Delta}(q)}\,.\tag{65}$$

Using the relations between the polarization kernels (see Appendix 8.2), we can easily show that the expression in the denominator of (65) and the expressions in the denominators in the second terms on the right-hand side of Eqns (63) and (64) coincide with the expressions in the denominators of the linear-response formulas (26)–(29). Hence, the condition that the denominators in (63) and (65) vanish coincides with the condition that collective modes (42) with $\omega=2e\phi$ and $\mathbf{q}=\mathbf{q}_H$ exist in the superconductor we labeled k. We see that the quantity $(\Delta_2^+-\Delta_3)$ behaves in the manner of a resonance, while the sum $(\Delta_2^++\Delta_3)$ contains no such resonance contribution. (There is a certain analogy with the fact that the collective modes in a linear response are related mainly to the oscillations of the order parameter phase, while variations in the absolute value of the order parameter are unessential.) Keeping only terms that may contain resonances corresponding to the excitation of collective modes in (53), we can write the contribution to DC current (49) in the compact form

$$J_{\mathrm{tun}}^{(4)}(H,\phi)$$
$$=2e\,\mathrm{Re}\left[(\widetilde{D}_2-\widetilde{D}_3^+)(\Delta_2^+-\Delta_3)+2(\widetilde{\delta N}_2+\widetilde{\delta N}_3^+)\,V(q_H)\,\delta n_3\right],\tag{66}$$

where we use the notation similar to that in (57) and (62):

$$\widetilde{D}_2=\widetilde{T^2}\int\mathrm{d}\varepsilon\int\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{p}_1\left[F_k(\mathbf{p},\varepsilon+e\phi)\right.$$
$$\left.\times F_k(\mathbf{p}-\mathbf{q}_H,\varepsilon-e\phi)\,F_p^+(\mathbf{p}_1,\varepsilon)\right]^{-+},$$
$$\widetilde{D}_3^+=-\widetilde{T^2}\int\mathrm{d}\varepsilon\int\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{p}_1\left[G_k(\mathbf{p},\varepsilon+e\phi)\right.$$
$$\left.\times G_k(-\mathbf{p}+\mathbf{q}_H,-\varepsilon+e\phi)\,F_p^+(\mathbf{p}_1,\varepsilon)\right]^{-+},$$
$$\widetilde{\delta N}_2=\widetilde{T^2}\int\mathrm{d}\varepsilon\int\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{p}_1\left[G_k(\mathbf{p},\varepsilon+e\phi)\right.$$
$$\left.\times F_k(\mathbf{p}-\mathbf{q}_H,\varepsilon-e\phi)\,F_p^+(\mathbf{p}_1,\varepsilon)\right]^{-+},$$
$$\widetilde{\delta N}_3^+=-\widetilde{T^2}\int\mathrm{d}\varepsilon\int\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{p}_1\left[F_k(\mathbf{p},\varepsilon+e\phi)\right.$$
$$\left.\times G_k(-\mathbf{p}+\mathbf{q}_H,-\varepsilon+e\phi)\,F_p^+(\mathbf{p}_1,\varepsilon)\right]^{-+}.\tag{67}$$

Although the contribution (66) to the current is of the order of T_{kp}^4 (while the ordinary current is of the order $\sim T_{kp}^2$), it may be significant. If the denominator in (63)–(65) vanishes (or becomes small) at certain applied voltages that are smaller than the total gap $\Delta_p + \Delta_k$, a resonance peak appears in the range of voltages where the ordinary quasiparticle current is suppressed. As noted above, the condition for this resonance to appear coincides with the condition for the collective modes with $\omega = 2e\phi$ and $q = q_H$ to exist. Varying the magnetic field strength and determining the voltage at which the current–voltage characteristic exhibits a peak, we can determine the dispersion law for these modes. We note than the tunnel characteristics may also exhibit resonances related to other collective excitations. For instance, as a result of their experiments, Ponomarev et al. [34] discovered peaks in the current–voltage characteristics of tunnel junctions that corresponded to frequencies of certain optical phonons. A theoretical explanation of this effect was given by Maksimov et al. [35], who used an approach similar to the theory discussed in the present section to describe tunnel experiments with excitation or absorption of optical phonons. Of course, it is not accidental that the resonances in the tunnel characteristics in Carlson and Goldman's experiments coincided with the resonances in the response to a longitudinal electric field. A possible qualitative explanation of this fact is that any tunneling process is accompanied by a transfer of charge (electrons) from one superconductor to another, giving rise to a response to this introduced charge, which is the source of the longitudinal field. At the phenomenological level, the contribution of fluctuation corrections to the tunnel characteristics had been examined by Scalapino [36] even before Carlson and Goldman's research. Later, such a phenomenological theory was described in greater detail by Shenoy and Lee [37] and Kadin and Goldman [38]. A microscopic theory within the temperature diagram technique similar to the approach discussed in the present section was first developed by Dinter [27] in 1978.

7. Conclusion

Our primary goal in this review was to show the general principles on which the theory of response for superconductors is based. In order not to make the review too long, we did not touch on the problem of superconductors with impurities. Most of the formulas were derived for pure superconductors (or in the collisionless limit, where there is scattering on impurities). But our approach can easily be extended to incorporate the case of dirty superconductors with impurities. All the formulas for the current and charge (26)–(29) will retain their form, as well as Eqn (42) for the collective modes. The difference is that the polarization operators in these formulas must be calculated with the scattering on impurities taken into account. How such calculations should be done is shown very well in Refs [18, 28, 29, 32]. The form of the polarization kernels with the scattering by impurities taken into account can be found, e.g., in Ref. [18].

In conclusion, we once more discuss the question of what we consider the key problem for the future. The problem of how collective modes are modified due to the presence of the boundaries of a superconductor and whether surface oscillations of a specific sort can emerge has been poorly studied. This problem is very important when one has to deal with small superconductors. To correctly interpret the numerous experiments involving conductivity (impedance) measurements for superconductors in a broad range of frequencies, one needs to know how to correctly estimate effects associated with charges at the boundary, i.e., with the longitudinal electric field. The difficulty of a theoretical description lies in the fact that one must solve equations for phase variations [Eqns (21) and (42)] not in infinite space but within a limited volume. In real space, these equations are very complicated integral equations and can be solved, probably, only in certain limit cases.

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8. Appendices

8.1 Normal and anomalous Green's functions in the Keldysh diagram technique

To find the linear response of superconductors to an external perturbation via the Keldysh diagram technique [8], we must know the following Green's functions in addition to (10):

$$
\begin{aligned}
&G_{\alpha\beta}^{+-}(x,x') = -\mathrm{i}\left\langle \hat\psi_\alpha(x)\,\hat\psi_\beta^+(x')\right\rangle ,\\
&F_{\alpha\beta}^{+-}(x,x') = -\mathrm{i}\left\langle \hat\psi_\alpha(x)\,\hat\psi_\beta(x')\right\rangle ,\\
&G_{\alpha\beta}^{R}(x,x') = -\mathrm{i}\left\langle \left\{\hat\psi_\alpha(x)\,\hat\psi_\beta^+(x')\right\}\right\rangle \theta(t-t') ,\\
&F_{\alpha\beta}^{R}(x,x') = -\mathrm{i}\left\langle \left\{\hat\psi_\alpha(x)\,\hat\psi_\beta(x')\right\}\right\rangle \theta(t-t') ,\\
&G_{\alpha\beta}^{A}(x,x') = \mathrm{i}\left\langle \left\{\hat\psi_\alpha(x)\,\hat\psi_\beta^+(x')\right\}\right\rangle \theta(t'-t) ,\\
&F_{\alpha\beta}^{A}(x,x') = \mathrm{i}\left\langle \left\{\hat\psi_\alpha(x)\,\hat\psi_\beta(x')\right\}\right\rangle \theta(t'-t) .
\end{aligned}
\tag{68}
$$

Here, the braces denote the anticommutator of the two operators, and $x = (t, \mathbf{r})$. We limit ourselves to functions (68) in the absence of external fields, because these functions enter the expressions for the polarization operators Π and Q (Appendix 8.2). They depend only on the difference of coordinates and times, $x - x'$, and their spin dependence is $G_{\alpha\beta} = \delta_{\alpha\beta} G$ and $F_{\alpha\beta} = \mathrm{i}\sigma_{\alpha\beta}^y F$.

The Fourier transforms of functions (68), introduced according to the formulas

$$
\begin{aligned}
&G(p) = \iint G(x-x')\exp\left[-\mathrm{i}p(x-x')\right]\mathrm{d}^4(x-x') ,\\
&F(p) = \iint F(x-x')\exp\left[-\mathrm{i}p(x-x')\right]\mathrm{d}^4(x-x') ,\quad p=(\varepsilon,\mathbf{p}),
\end{aligned}
$$

satisfy Gor'kov's equations, which can be conveniently written in matrix form as

$$
\begin{aligned}
&\widehat{G_0}^{-1}(p)\,\hat G^{R,A}(p) = \hat 1 ,\\
&\widehat{G_0}^{-1}(p)\,\hat G^{-+}(p) = 0 ,
\end{aligned}
\tag{69}
$$

where we introduce the notation

$$
\begin{aligned}
&\widehat{G_0}^{-1}(p) = \begin{pmatrix} \varepsilon-\xi_{\mathbf{p}} & -\Delta_0 \\ -\Delta_0^+ & \varepsilon+\xi_{\mathbf{p}} \end{pmatrix} ,\\
&\hat G^{R,A}(p) = \begin{pmatrix} G_0^{R,A}(p) & F_0^{R,A}(p) \\ -F_0^{+R,A}(p) & -G_0^{A,R}(-p) \end{pmatrix} ,\\
&\hat G^{-+}(p) = \begin{pmatrix} G_0^{-+}(p) & F_0^{-+}(p) \\ -(F^+)_0^{-+}(p) & -G_0^{+-}(-p) \end{pmatrix} ,\\
&\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu .
\end{aligned}
$$

Solving Eqns (69), we obtain

$$G_0^{-+}(p) = 2\pi i \left[u_p^2 n_\mathbf{p} \delta(\varepsilon - \varepsilon_\mathbf{p}) + v_p^2 (1 - n_\mathbf{p})\, \delta(\varepsilon + \varepsilon_\mathbf{p})\right],$$

$$F_0^{-+}(p) = 2\pi i u_p v_p \left[n_\mathbf{p} \delta(\varepsilon - \varepsilon_\mathbf{p}) - (1 - n_\mathbf{p})\, \delta(\varepsilon + \varepsilon_\mathbf{p})\right],$$

$$G_0^{+-}(p) = -2\pi i \left[u_p^2 (1 - n_\mathbf{p})\, \delta(\varepsilon - \varepsilon_\mathbf{p}) + v_p^2 n_\mathbf{p} \delta(\varepsilon + \varepsilon_\mathbf{p})\right],$$

$$G_0^R(p) = \frac{u_p^2}{\varepsilon - \varepsilon_\mathbf{p} + i0} + \frac{v_p^2}{\varepsilon + \varepsilon_\mathbf{p} + i0},$$

$$F_0^R(p) = u_p v_p \left(\frac{1}{\varepsilon - \varepsilon_\mathbf{p} + i0} - \frac{1}{\varepsilon + \varepsilon_\mathbf{p} + i0}\right)$$

were $n_\mathbf{p} = (\exp(\varepsilon_\mathbf{p}/T) + 1)^{-1}$ is the Fermi distribution for quasiparticles, and $u_\mathbf{p}^2 = (1 + \xi_\mathbf{p}/\varepsilon_\mathbf{p})/2$ and $v_\mathbf{p}^2 = (1 - \xi_\mathbf{p}/\varepsilon_\mathbf{p})/2$ are the coherence factors. The expressions for the other Green's functions can easily be obtained using relations that follow from definitions (68): $G_0^A(p) = G_0^{R*}(p)$, $F_0^A(p) = F_0^R(-p)$, and $F_0^{+-}(p) = F_0^{-+}(-p)$.

8.2 Polarization operators Q and Π. The Ward identity

The polarization operators Q and Π play an important role in expressions (26) – (29) for the linear response. Their explicit form can be obtained by using the diagrams in Fig. 1 (with close free ends of the solid lines) and Fig. 2b:

$$Q_{kl}^A(q) = -2i \frac{1}{m^2} \int \left(p_k + \frac{q_k}{2}\right)\left(p_l + \frac{q_l}{2}\right) \left[G_0^R(p+q)\, G_0^{-+}(p) + G_0^{-+}(p+q)\, G_0^A(p) - F_0^R(p+q)(F^+)_0^{-+}(p) - F_0^{-+}(p+q)(F^+)_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (70)$$

$$Q_{k0}^A(q) = Q_{0k}^A(q) = -2i \frac{1}{m} \int \left(p_k + \frac{q_k}{2}\right) \left[G_0^R(p+q)\, G_0^{-+}(p) + G_0^{-+}(p+q)\, G_0^A(p) + F_0^R(p+q)(F^+)_0^{-+}(p) + F_0^{-+}(p+q)(F^+)_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (71)$$

$$Q_{00}^A(q) = -2i \int \left[G_0^R(p+q)\, G_0^{-+}(p) + G_0^{-+}(p+q)\, G_0^A(p) - F_0^R(p+q)(F^+)_0^{-+}(p) - F_0^{-+}(p+q)(F^+)_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (72)$$

$$Q_k^\Delta(q) = -2i \frac{1}{m} \int \left(p_k + \frac{q_k}{2}\right) \left[G_0^R(p+q)(F^+)_0^{-+}(p) + G_0^{-+}(p+q)(F^+)_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (73)$$

$$Q_0^\Delta(q) = -2i \int \left[G_0^R(p+q)(F^+)_0^{-+}(p) + G_0^{-+}(p+q)(F^+)_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (74)$$

and also Fig. 2a:

$$\Pi_l^A(q) = 2ig \frac{1}{m} \int \left(p_l + \frac{q_l}{2}\right) \left[G_0^R(p+q)\, F_0^{-+}(p) + G_0^{-+}(p+q)\, F_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (75)$$

$$\Pi_0^A(q) = 2ig \int \left[G_0^R(p+q)\, F_0^{-+}(p) + G_0^{-+}(p+q)\, F_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (76)$$

$$\Pi^\Delta(q) = ig \int \left[G_0^R(p+q)\, G_0^{+-}(-p) + G_0^{-+}(p+q)\, G_0^R(-p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (77)$$

$$\Pi^{\Delta^+}(q) = -ig \int \left[F_0^R(p+q)\, F_0^{-+}(p) + F_0^{-+}(p+q)\, F_0^A(p)\right] \frac{d^4 p}{(2\pi)^4}, \quad (78)$$

where k and l are the 'spatial' indices, and integration is performed with respect to the 4-momentum $p \equiv (\varepsilon, \mathbf{p})$. In the absence of external fields, the order parameter Δ_0 is chosen real. In this case, the polarization operators Π^A and Q^Δ are equal up to the coefficients: $Q_k^\Delta = \Pi_k^A/g$ and $Q_0^\Delta = \Pi_0^A/g$, and in addition $\Pi^{\Delta^+}(q) = \Pi^{\Delta^+ *}(-q)$. The requirement that the linear response be gauge invariant (or equivalently, that the current be continuous) means, in terms of the Π- and Q-kernels, that the following identities hold:

$$\left[Q_{kl}^A(q) + \frac{n}{m}\,\delta_{kl}\right] q_l - Q_{k0}^A(q)\omega = -4i\Delta_0 Q_{2k}^\Delta(q), \quad (79)$$

$$Q_{0l}^A(q)\, q_l - Q_{00}^A(q)\omega = -4i\Delta_0 Q_{20}^\Delta(q), \quad (80)$$

$$\Pi_{1l}^A(q)\, q_l - \Pi_{10}^A(q)\, \omega = -2i\Delta_0 \left[\Pi_2^\Delta(q) - \Pi_2^{\Delta^+}(q)\right], \quad (81)$$

$$\Pi_{2l}^A(q)\, q_l - \Pi_{20}^A(q)\, \omega = 2i\Delta_0 \left[1 + \Pi_1^\Delta(q) - \Pi_1^{\Delta^+}(q)\right]. \quad (82)$$

Here, the first subscripts 1 and 2 denote the real and imaginary parts of the respective polarization operators [Eqns (17)]. Similar identities were obtained earlier in the limit of small $\mathbf{q}$ and ω in Ref. [29].

It can be directly shown that Eqns (79) – (82) are related to the Ward identity for a superconductor, which in the standard form can be written as [9]

$$\sum_{\nu=0}^{3} q_\nu \widehat{\Gamma}_\nu(p+q, p) = \widehat{\tau}_3 \widehat{G}_0^{-1}(p) - \widehat{G}_0^{-1}(p+q)\,\widehat{\tau}_3, \quad (83)$$

were

$$\widehat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the definition of the matrix $\widehat{G}_0^{-1}(p)$ is given in Appendix 8.1. The vertex function $\widehat{\Gamma}_\nu$ in (83) enters the expression for the correction to the one-particle Green's function that is linear in the potentials $A = (c\varphi, \mathbf{A})$ as

$$\widehat{G}^{-+(1)}(p+q, p) = -\frac{e}{c}\, \widehat{G}_0^R(p+q) \sum_\nu \widehat{\Gamma}_\nu(p+q, p) \times \widehat{G}_0^{-+}(p)\, A_\nu(q) - \frac{e}{c}\, \widehat{G}_0^{-+}(p+q) \sum_\nu \widehat{\Gamma}_\nu(p+q, p) \widehat{G}_0^A(p)\, A_\nu(q). \quad (84)$$

For the interaction in the BCS model, this function does not contain Keldysh's indices (−,+) and is determined by the formula $\widehat{\Gamma}_v = \widehat{\Gamma}_v^{--,-} + \widehat{\Gamma}_v^{--,+} = -\widehat{\Gamma}_v^{++,-} - \widehat{\Gamma}_v^{++,+}$.

We first consider a superconductor without the electron–electron Coulomb interaction. In this case, the vertex function is given by

$$\widehat{\Gamma}_v(p+q,p) = \begin{pmatrix} \gamma_v(p+q,p) & \Pi_v(q) \\ \Pi_v^*(-q) & -\gamma_v(-p,-p-q) \end{pmatrix},$$

where $\gamma(p+q,p) = [1, m^{-1}(\mathbf{p}+\mathbf{q}/2)]$ is the 'free' vertex function (the vertex function in the normal state) and Π_v is the kernel whereby the correction to the order parameter, $\Delta^{(1)}$, is related to the external potential A_v, $\Delta^{(1)}(q) = \Pi_v(q)\,A_v(q)$ (see Fig. 3). To find the function Π_v, we must solve self-consistency equation (16) (at $V=0$). We note that Π_v ensures the renormalization of the 'free' vertex by the attractive interaction between electrons that leads to superconductivity ($\Pi_v \propto g$). Thus, in the case under consideration, Eqn (16) (at $V=0$) is precisely the equation that determines the vertex matrix. Solving this equation, we express the kernel Π_v in the absence of the Coulomb interaction as

$$\Pi_v(q) = \frac{\Pi_v^A(q)\left[1+\Pi^{\Delta*}(-q)\right] - \Pi_v^{A*}(-q)\,\Pi^{\Delta^+}(q)}{\left[1+\Pi^{\Delta}(q)\right]\left[1+\Pi^{\Delta*}(-q)\right] - \Pi^{\Delta^+}(q)\,\Pi^{\Delta^+*}(-q)}\,. \tag{85}$$

To derive the Ward identity in form (8.3), we must evaluate the combination $\sum_v q_v \Pi_v(q)$. Using relations (81) and (82) between the Π-kernels, we obtain

$$\begin{aligned}\sum_v q_v \Pi_v(q) &= \frac{\sum_v q_v \Pi_v^A(q)\left[1+\Pi^{\Delta*}(-q)\right] - \sum_v q_v \Pi_v^{A*}(-q)\,\Pi^{\Delta^+}(q)}{\left[1+\Pi^{\Delta}(q)\right]\left[1+\Pi^{\Delta*}(-q)\right] - \Pi^{\Delta^+}(q)\,\Pi^{\Delta^+*}(-q)} \\ &= -2\Delta_0\left\{\frac{\left[1+\Pi^{\Delta}(q) - \Pi^{\Delta^+}(q)\right]\left[1+\Pi^{\Delta*}(-q)\right]}{\left[1+\Pi^{\Delta}(q)\right]\left[1+\Pi^{\Delta*}(-q)\right] - \Pi^{\Delta^+}(q)\,\Pi^{\Delta^+*}(-q)}\right. \\ &\quad \left. + \frac{\left[1+\Pi^{\Delta*}(-q) - \Pi^{\Delta^+*}(-q)\right]\Pi^{\Delta^+}(q)}{\left[1+\Pi^{\Delta}(q)\right]\left[1+\Pi^{\Delta*}(-q)\right] - \Pi^{\Delta^+}(q)\,\Pi^{\Delta^+*}(-q)}\right\} = -2\Delta_0\,. \end{aligned} \tag{86}$$

Similar combinations for the diagonal elements of the vertex matrix $\widehat{\Gamma}_v$ are given by

$$\begin{aligned} \sum_v q_v\,\gamma_v(p+q,p) &= \xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} - \omega\,, \\ \sum_v q_v\,\gamma_v(-p,-p-q) &= \xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} + \omega\,. \end{aligned} \tag{87}$$

As a result, the left-hand side of Eqn (83) becomes

$$\sum_v q_v \widehat{\Gamma}_v(p+q,p) = \begin{pmatrix} \xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} - \omega & -2\Delta_0 \\ 2\Delta_0 & \xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} + \omega \end{pmatrix}. \tag{88}$$

Clearly, the matrix on the right-hand side of (88) coincides with $\widehat{\tau}_3 \widehat{G}_0^{-1}(p) - \widehat{G}_0^{-1}(p+q)\,\widehat{\tau}_3$. Thus, we have shown that the vertex matrix $\widehat{\Gamma}_v$ satisfies Ward identity (82) if identities (81) and (82) hold. At the end of this section, we show one of the ways of proving (79)–(82).

In the RPA, the Coulomb interaction is taken into account via a certain modification of the field potentials, by replacing A_v with $A_v - \delta_{v0} c e^{-1} V \delta n$, where the electron number density fluctuations are determined in a self-consistent manner. In formula (84), such a substitution leads to the following renormalization of the vertex matrix:

$$\begin{aligned} \widehat{\Gamma}_v(p+q,p) \to \widehat{\Gamma}'_v(p+q,p) &= \widehat{\Gamma}_v(p+q,p) \\ &\quad + \widehat{\Gamma}_0(p+q,p)\,V(q)\,Q_{0v}(q) \\ &= \begin{pmatrix} \gamma_v(p+q,p) + V(q)\,Q_{0v}(q) & \Pi_v(q) + \Pi_0(q)V(q)\,Q_{0v}(q) \\ \Pi_v^*(-q) + \Pi_0^*(-q)V(q)\,Q_{0v}(q) & -\gamma_v(-p,-p-q) - V(q)\,Q_{0v}(q) \end{pmatrix}. \end{aligned} \tag{89}$$

Here, Q_{0v} is the kernel relating δn to the potential A_v,

$$\delta n(q) = -\frac{e}{c}\,Q_{0v}(q)\,A_v(q)\,, \tag{90}$$

and Π_v is defined in (85). The explicit form of Q_{0v} can easily be established by comparing (90) and (26).

Taking the Coulomb interaction into account in the RPA has no effect on the Green's functions $\widehat{G}_0$, because the electron number density fluctuations are zero in the absence of an external field. Hence, the right-hand side of Ward identity (83) remains unchanged. Thus, the vertex matrix defined by (89) satisfies the Ward identity if

$$\sum_v Q_{0v}(q)\,q_v = 0\,. \tag{91}$$

We note that (91) is the condition that guarantees the gauge invariance of expression (90) for the electron number density fluctuations $\delta n(q)$. It is satisfied automatically with the present approach used to find the linear response of a superconductor, because formula (26) is manifestly gauge invariant. As a result, for the vertex matrix $\widehat{\Gamma}'_v$, we have the same identity as for $\widehat{\Gamma}_v$:

$$\begin{aligned} \sum_v q_v \widehat{\Gamma}'_v(p+q,p) &= \sum_v q_v\,\widehat{\Gamma}_v(p+q,p) \\ &= \widehat{\tau}_3 \widehat{G}_0^{-1}(p) - \widehat{G}_0^{-1}(p+q)\,\widehat{\tau}_3\,. \end{aligned} \tag{92}$$

We now return to identities (79)–(82). Equations (70)–(78) show that the polarization operators are determined by expressions of the same type, sums of convolutions of 'free' normal and anomalous Green's functions in different combinations. The functions G_0 and F_0 are related through Gor'kov's equations (69). One of the ways of proving the validity of (79)–(82) is based on these equations, and we demonstrate this by proving (79). We multiply (70) by q_l and then subtract (71) multiplied by ω from the product:

$$\begin{aligned} Q_{kl}^A(q)\,q_l - Q_{k0}^A(q)\,\omega &= -2\mathrm{i}\,\frac{1}{m}\int\left(p_k + \frac{q_k}{2}\right)\left[\frac{(p_l + q_l/2)\,q_l}{2m} - \omega\right] \\ &\quad \times \left[G_0^R(p+q)\,G_0^{-+}(p) + G_0^{-+}(p+q)\,G_0^A(p)\right]\frac{\mathrm{d}^4 p}{(2\pi)^4} \\ &\quad - 2\mathrm{i}\,\frac{1}{m}\int\left(p_k + \frac{q_k}{2}\right)\left[-\frac{(p_l + q_l/2)\,q_l}{2m} - \omega\right] \\ &\quad \times \left[F_0^R(p+q)(F^+)_0^{-+}(p) - F_0^{-+}(p+q)(F^+)_0^A(p)\right]\frac{\mathrm{d}^4 p}{(2\pi)^4}\,. \end{aligned} \tag{93}$$

We write the expression in the square brackets as

$$\frac{(p_l+q_l/2)\,q_l}{2m}-\omega=\xi_{\mathbf{p+q}}-\xi_{\mathbf{p}}-\omega$$
$$=(\xi_{\mathbf{p+q}}-\varepsilon-\omega)-(\xi_{\mathbf{p}}-\varepsilon)$$

and, similarly,

$$-\frac{(p_l+q_l/2)\,q_l}{2m}-\omega=-(\xi_{\mathbf{p+q}}+\varepsilon+\omega)+(\xi_{\mathbf{p}}+\varepsilon)\,.$$

Clearly, the combinations in parentheses are the diagonal elements of the matrices $\widehat{G_0}^{-1}(p+q)$ and $\widehat{G_0}^{-1}(p)$ involved in Gor'kov's equations (see Appendix 8.1). This allows using Eqns (69) for further simplification:

$$\begin{aligned}
&Q^A_{kl}(q)\,q_l-Q^A_{k0}(q)\omega\\
&=2\mathrm{i}\,\frac{1}{m}\int\left(p_k+\frac{q_k}{2}\right)\left[G_0^{-+}(p)-G_0^{-+}(p+q)\right]\frac{\mathrm{d}^4p}{(2\pi)^4}\\
&+4\mathrm{i}\Delta_0\,\frac{1}{m}\int\left(p_k+\frac{q_k}{2}\right)\left[G_0^R(p+q)(F^+)_0^{-+}(p)\right.\\
&\left.+G_0^{-+}(p+q)(F^+)_0^A(p)\right]\frac{\mathrm{d}^4p}{(2\pi)^4}\\
&+4\mathrm{i}\Delta_0\,\frac{1}{m}\int\left(p_k+\frac{q_k}{2}\right)\left[F_0^R(p+q)\,G_0^{-+}(p)\right.\\
&\left.+F_0^{-+}(p+q)\,G_0^A(p)\right]\frac{\mathrm{d}^4p}{(2\pi)^4}\,. \qquad (94)
\end{aligned}$$

After simple transformations, we find that the first integral on the right-hand side of Eqn (94) becomes

$$2\mathrm{i}\,\frac{1}{m}\,q_k\int G_0^{-+}(p)\,\frac{\mathrm{d}^4p}{(2\pi)^4}=-q_k\,\frac{n}{m}\,. \qquad (95)$$

Noting that, according to definition (73), the second integral on the right-hand side of (94) is equal (with the coefficient included) to $-2\Delta_0 Q^\Delta_k(q)$, and the third integral is equal to $2\Delta_0 Q^{\Delta *}_k(-q)$, we arrive at identity (79). The proof of the other identities, Eqns (80)–(82), may be carried out similarly.

8.3 Behavior of the polarization operators Q and Π in the limit of small q and ω

Finding the explicit form of the polarization operators as functions of $\mathbf{q}$ and ω is quite a complicated problem in general. Even in the 'pure' case considered here, only the integral over the frequency ε can be calculated exactly in (70)–(78). However, in most cases, it suffices to know the expressions for the Π- and Q-kernels in the limit of small $\mathbf{q}$ ($q\ll p_{\mathrm{F}}$) and ω ($\omega\ll\Delta_0$). After integration with respect to ε has been done, the polarization operators Q^A_{kl}, Q^A_{k0}, Q^A_{00}, Q^Δ_{1k}, and Q^Δ_{10} become

$$\begin{aligned}
Q^A_{kl}(q)&=\frac{1}{m^2}\int p_k\,p_l\left\{\left(1+\frac{\xi_+\xi_-+\Delta_0^2}{\varepsilon_+\,\varepsilon_-}\right)(n_--n_+)\right.\\
&\times\frac{\varepsilon_+-\varepsilon_-}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_+-\varepsilon_-)^2}+\left(1-\frac{\xi_+\xi_-+\Delta_0^2}{\varepsilon_+\varepsilon_-}\right)\\
&\left.\times(1-n_--n_+)\frac{\varepsilon_++\varepsilon_-}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_++\varepsilon_-)^2}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3}\,, \qquad (96)
\end{aligned}$$

$$\begin{aligned}
Q^A_{k0}(q)&=\frac{1}{m}\int p_k\,\frac{1}{\varepsilon_-\varepsilon_+}\left\{\Big(\xi_-\varepsilon_++\xi_+\varepsilon_-\Big)(n_--n_+)\right.\\
&\times\frac{\omega+\mathrm{i}\delta}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_+-\varepsilon_-)^2}+(\xi_+\varepsilon_--\xi_-\varepsilon_+)\\
&\left.\times(1-n_--n_+)\frac{\omega+\mathrm{i}\delta}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_++\varepsilon_-)^2}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3}\,, \qquad (97)
\end{aligned}$$

$$\begin{aligned}
Q^A_{00}(q)&=\int\left\{\left(1+\frac{\xi_+\,\xi_--\Delta_0^2}{\varepsilon_+\varepsilon_-}\right)(n_--n_+)\right.\\
&\times\frac{\varepsilon_+-\varepsilon_-}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_+-\varepsilon_-)^2}+\left(1-\frac{\xi_+\xi_--\Delta_0^2}{\varepsilon_+\varepsilon_-}\right)\\
&\left.\times(1-n_--n_+)\frac{\varepsilon_++\varepsilon_-}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_++\varepsilon_-)^2}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3}\,, \qquad (98)
\end{aligned}$$

$$\begin{aligned}
Q^\Delta_{1k}(q)&=-\frac{1}{m}\int p_k\,\frac{\Delta_0}{2\varepsilon_-\varepsilon_+}\left\{(n_--n_+)\frac{(\omega+\mathrm{i}\delta)(\varepsilon_++\varepsilon_-)}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_+-\varepsilon_-)^2}\right.\\
&\left.-(1-n_--n_+)\frac{(\omega+\mathrm{i}\delta)(\varepsilon_+-\varepsilon_-)}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_++\varepsilon_-)^2}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3}\,, \qquad (99)
\end{aligned}$$

$$\begin{aligned}
Q^\Delta_{10}(q)&=-\int\frac{\Delta_0}{2\varepsilon_-\varepsilon_+}\left\{(n_--n_+)\frac{(\xi_++\xi_-)(\varepsilon_+-\varepsilon_-)}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_+-\varepsilon_-)^2}\right.\\
&\left.-(1-n_--n_+)\frac{(\xi_++\xi_-)(\varepsilon_++\varepsilon_-)}{(\omega+\mathrm{i}\delta)^2-(\varepsilon_++\varepsilon_-)^2}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3}\,, \qquad (100)
\end{aligned}$$

where $\xi_\pm=\xi_{\mathbf{p}\pm\mathbf{q}/2}$, $\varepsilon_\pm=\varepsilon_{\mathbf{p}\pm\mathbf{q}/2}$, $n_\pm=n_{\mathbf{p}\pm\mathbf{q}/2}$, and $\delta=+0$.

We now analyze the explicit form of the kernels in specific limit cases. We begin with the static case ($\omega=0$).

To find the first London equation, we must know the kernel Q^A_{kl} in the limit as $\mathbf{q}\to 0$. At a temperature $T\leqslant T_{\mathrm{c}}$, this kernel is given by

$$\begin{aligned}
Q^A_{kl}(\mathbf{q}\to 0,0)&=\delta_{kl}\,\frac{2}{3m^2}\int p^2\left(-\frac{\partial n_{\mathbf{p}}}{\partial\varepsilon_{\mathbf{p}}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}\\
&=\delta_{kl}\,\frac{n_{\mathrm{s}}(T)-n}{m}\,. \qquad (101)
\end{aligned}$$

This formula is the definition of the superfluid electron number density n_{s}. Clearly, at $T=0$, the quasiparticle distribution function $n(\varepsilon_{\mathbf{p}})=0$, and hence $Q^A_{kl}\to 0$ in the given limit. If the temperature is equal to the transition temperature, $T=T_{\mathrm{c}}$, the order parameter vanishes and the integral in (101) is given by

$$\begin{aligned}
Q^A_{kl}(\mathbf{q}\to 0,0)&=\delta_{kl}\,\frac{2}{3m^2}\int p^2\left(-\frac{\partial n(\xi_{\mathbf{p}})}{\partial\xi_{\mathbf{p}}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}\\
&=-\delta_{kl}\,\frac{n}{m}\,. \qquad (102)
\end{aligned}$$

This result could be expected for the normal state and is in full agreement with identity (79). We note that in this case, $n_{\mathrm{s}}=0$.

The polarization operator Q^A_{00} found for the same limit $\mathbf{q}\to 0$ contains all the necessary information about the static screening in the superconductor. We calculate it at $T = 0$ as

$$Q_{00}^{A}(\mathbf{q} \to 0, 0) = -\int \left(1 - \frac{\xi_{\mathbf{p}}^2 - \Delta_0^2}{\varepsilon_{\mathbf{p}}^2}\right) \frac{1}{2\varepsilon_{\mathbf{p}}} \frac{\mathrm{d}^3 p}{(2\pi)^3}$$

$$= -\int \frac{\Delta_0^2}{(\xi_{\mathbf{p}}^2 + \Delta_0^2)^{3/2}} \frac{\mathrm{d}^3 p}{(2\pi)^3}$$

$$\approx -N_0 \int_{-\infty}^{+\infty} \frac{\mathrm{d}(\xi/\Delta_0)}{\left(1 + (\xi/\Delta_0)^2\right)^{3/2}} = -2N_0 \,, \qquad (103)$$

where N_0 is the density of states at the Fermi level.

For the kernels Q_{kl}^{A} and Q_{k0}^{A} at absolute zero, finite frequencies $\omega \neq 0$, and $\mathbf{q} = 0$, the equality $Q_{kl}^{A}(0, \omega) = Q_{k0}^{A}(0, \omega) = 0$ holds.

We now see whether the correction to the absolute value of the order parameter, Δ_1, can be neglected in the expression for the linear response in the limit of small but finite ω and $\mathbf{q}$. Allowing for the correction, Δ_1 in the final expressions for the electron number density functions δn, Eqns (26) and (28), and the current density j, Eqns (27) and (29), results in replacing the initial polarization operators by the polarization operators with a bar. From (22) and (23), it follows that the role played by such a substitution is determined mainly by the magnitude of the kernels Q_1^{Δ} (we recall that the Π_1^{A} are equal to them up to a coefficient) and Π_2^{Δ} (we have noted that $\Pi_2^{\Delta^+} = 0$). We find the expression for the kernel Q_{10}^{Δ} (for simplicity, we assume that $T = 0$):

$$Q_{10}^{\Delta}(\mathbf{q}, \omega) \approx -\int \frac{\Delta_0 \xi_{\mathbf{p}}}{2\varepsilon_{\mathbf{p}}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}$$

$$\approx -\frac{1}{2} \int_{-\varepsilon_{\mathrm{F}}}^{+\infty} \frac{\xi}{(\xi^2 + \Delta_0^2)^{3/2}} N(\xi + \varepsilon_{\mathrm{F}}) \, \mathrm{d}\xi \propto -N_0 \frac{\Delta_0}{\varepsilon_{\mathrm{F}}} \,. \qquad (104)$$

It follows from (104) that a small characteristic parameter $\Delta_0/\varepsilon_{\mathrm{F}}$ emerges as a result of the calculation. The reason is that in the zeroth order in $\Delta_0/\varepsilon_{\mathrm{F}}$, the kernel Q_{10}^{Δ} vanishes, because the integrand is an odd function of ξ in this approximation, and therefore the integration is performed within symmetric limits. As a result, to within $\Delta_0/\varepsilon_{\mathrm{F}}$, the quantity Q_0^{Δ} proves to be purely imaginary, i.e., $Q_0^{\Delta}(q) = -Q_0^{\Delta *}(-q) = \mathrm{i} Q_{20}^{\Delta}(q)$. Reasoning along the same lines, we can easily show that with the same approximation, we have $\Pi_0^{A}(q) = -\Pi_0^{A*}(-q) = \Pi_{20}^{A}(q)$, $\Pi^{\Delta}(q) = \Pi^{\Delta *}(-q) = \Pi_1^{\Delta}(q)$, and $\Pi^{\Delta^+}(q) = \Pi^{\Delta^+ *}(-q) = \Pi_1^{\Delta^+}(q)$. We now calculate the relative contribution to the kernel $\overline{Q}_{00}^{A}$ from the second term in (22). Substituting (104) in (22), we obtain

$$\frac{Q_{00}^{A} - \overline{Q}_{00}^{A}}{Q_{00}^{A}} = -\frac{2 Q_{10}^{\Delta} g Q_{10}^{\Delta}}{Q_{00}^{A}(1 + \Pi_1^{\Delta} + \Pi_1^{\Delta^+})}$$

$$\approx \frac{g\left[(N_0/2)(\Delta_0/\varepsilon_{\mathrm{F}})\right]^2}{g N_0^2} \propto \left(\frac{\Delta_0}{\varepsilon_{\mathrm{F}}}\right)^2 \ll 1 \,. \qquad (105)$$

Thus, in the limit of small $\mathbf{q}$ and ω, the correction to Q_{00}^{A} due to the second term in (22) is unessential. The same approach can be used to show that the difference between the polarization operators Π_2^{A} and the operator $\overline{\Pi}_2^{A}$ given by (23) is small in the parameter $(\Delta_0/\varepsilon_{\mathrm{F}})^2$, which emerges, as in the previous case, because the integrand in the expression for Π_2^{Δ} is an odd function (in the zeroth approximation) of ξ.

The above estimates show that not only the contribution of the correction to the absolute value of the order parameter is small in the expressions for the electron number density and current density fluctuations, but the correction itself is also a small quantity [see Eqn (24)].

We note that this result is not universal and to a great extent emerges because for a quadratic dispersion law for the free electrons, the density of states near the Fermi surface changes slowly and can be assumed constant (N_0). However, the electron spectrum may be such that the density of states has a singularity near the Fermi level, and therefore the integrands in the expressions for the kernels Q_{1k}^{Δ} and Π_2^{A} can no longer be approximately considered odd functions of ξ (in the above sense). In this case, the correction to the absolute value of the order parameter and its contribution to j and δn may become significant.

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