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Geometrical optics and the diffraction phenomenon

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Contents

Abstract. This note outlines the principles of the geometrical optics of inhomogeneous waves whose description necessitates the use of complex values of the wave vector. Generalizing geometrical optics to inhomogeneous waves permits including in its scope the analysis of the diffraction phenomenon.

1. Introduction

It is widely believed that diffraction refers to a purely wave phenomenon and can therefore be treated only with the use of wave equations. Wave equations comprise partial differential equations of order no lower than the second (the order of the wave equations is normally increased if the spatial dispersion of the medium is taken into account). Recourse to these equations requires, as a rule, the use of a cumbersome mathematical apparatus. A simplified version of the wave equation appropriate for the analysis of short-wave envelope evolution was introduced by M A Leontovich in the investigation of radio wave diffraction on the terrestrial surface [1] (the parabolic Leontovich equation). However, analysis of the solutions of the parabolic equation in the general case of an inhomogeneous anisotropic medium turns out to be a rather intricate task.

An alternative and, in our view, simpler approach to the treatment of the diffraction phenomenon involves extending the notions of a wave vector $\mathbf{k}$ and an eikonal

$$\psi(\mathbf{r}) = \int^{\mathbf{r}} \mathbf{k}(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

to include complex values (inhomogeneous waves) [2].

The evolution of short homogeneous waves ($\mathrm{Im}\,\mathbf{k} = 0$) may be analyzed with the aid of geometrical optics (GO) equations. Unlike wave equations, these equations are ordinary differential equations (see, for instance, Ref. [3]). Augmenting GO to include inhomogeneous waves ($\mathrm{Im}\,\mathbf{k} \neq 0$) permits including in its scheme the analysis of diffraction phenomenon as well.

Interest in the problem of inhomogeneous wave evolution has recently grown in connection with the progress of the techniques of microwave plasma heating and current generation in thermonuclear systems. Microwave radiation is normally injected into such systems in the form of wave beams. In paper [4], the general scheme of the complex GO of Ref. [2] (see also Refs [5, 6]) was realized by the example of the simplest wave beam with a Gaussian intensity distribution over the beam cross section. It turned out that this system of ordinary differential equations coincided with the system derived in the work [7] employing the parabolic equation formalism. The results of Refs [2, 7] were applied in numerical codes written to analyze the microwave – plasma interaction [8 – 10].

The aim of the present note is to outline the foundations of 'complex' GO which permits considering in its context from a unified standpoint the entire collection of phenomena (refraction, focusing – defocusing, diffraction) that determine the evolution of short waves.

2. Geometrical (ray) optics

In the investigation of electromagnetic waves their field is quite often defined on some surface and it is necessary to extend the field to the entire domain of wave propagation. Analysis of this problem presents serious difficulties in general, and therefore different approximations are commonly employed in its solution. When the characteristic wavelength is short in comparison with the dimension of the system, it is advantageous to resort to the short-wave quasiclassical approximation, whereby the variable electric field is described by the spatial dependence

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(\mathbf{r}) \exp\left(\mathrm{i}\psi(\mathbf{r})\right),$$

where $\mathbf{E}' = \mathbf{e}E$, and $\mathbf{e}$ is the unit polarization vector. The typical scale L of the $\mathbf{E}'(\mathbf{r})$-field amplitude variation is assumed to be much longer than the wavelength. This

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work is concerned with precisely such waves; it is also anticipated that the time dependence of the electromagnetic field will be harmonic, $\propto \exp(-\mathrm{i}\omega t)$, and the medium will be stationary.

We substitute into the wave equation, resulting from the Maxwell equations, the quasiclassical representation for the electric field to obtain a hierarchy of equations in powers of the small parameter $1/(kL)$. The zero-order equations are then algebraic:

$$a_{ij}(\mathbf{r}, \mathbf{k}, \omega) E_j' = 0\,, \tag{1}$$

where $a_{ij} = k_i k_j - \delta_{ij} k^2 + (\omega/c)^2 \varepsilon_{ij}$, and ε_{ij} is the permittivity tensor. It is believed that this tensor is Hermitian, which is equivalent to the assumption about the absence of irreversible processes of wave – medium energy exchange (dissipation as well as collisionless resonance interaction between charged particles and the waves in the case of plasma).

System of equations (1) is solvable when the so-called dispersion relation

$$D(\mathbf{r}, \mathbf{k}, \omega) = \left\| a_{ij}(\mathbf{r}, \mathbf{k}, \omega) \right\| = 0 \tag{2}$$

holds true for the waves.

The dispersion relation may be considered a constraint imposed on the wave-vector components for the waves of a given frequency. Owing to this, the modulus of the vector $\mathbf{k}$ becomes direction-dependent. In the short-wave quasiclassical approximation, the above-formulated problem of extending the solution of the wave equation reduces to selecting an analytical dependence $\mathbf{k}(\mathbf{r})$ that goes over into the given one on the 'initial' surface S. To this end, we equate to zero the total spatial derivative of the quantity D with respect to each of the coordinates:

$$\frac{\mathrm{d}D}{\mathrm{d}x_i} = \frac{\partial D}{\partial x_i} + \frac{\partial k_i}{\partial x_j}\frac{\partial D}{\partial k_j} = 0\,. \tag{3}$$

Here, use was made of the equality

$$\frac{\partial^2 \psi}{\partial x_i \, \partial x_j} = \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}\,.$$

We now consider the procedure for extending the dependence $\mathbf{k}(\mathbf{r})$ from the 'initial' surface S. Since the electromagnetic field is specified on S, it is possible to define the derivatives of the vectors $\mathbf{k}$ with respect to the directions tangent to S. After that we need to determine $(\mathbf{n}\nabla)\mathbf{k}$ from Eqn (3), where $\mathbf{n}$ is the vector normal to the surface S. The complete information about the quantities $\partial k_i/\partial x_j$ permits continuing the dependence $\mathbf{k}(\mathbf{r})$ through a distance $\mathrm{d}r$ along any direction. In this case, the surface S goes over into S', and so forth. It is noteworthy, however, that the projection of the quantity $\partial k_i/\partial x_j$ onto the direction of the vector $\partial D/\partial \mathbf{k}$ enters in Eqn (3), and it would therefore be natural to continue the dependence $\mathbf{k}(\mathbf{r})$ along precisely this direction. It coincides with the direction of the group velocity $\mathbf{V}_{\mathrm{gr}}$. Indeed, the dispersion relation (2) may also be treated as a condition which defines the frequency $\omega(\mathbf{r}, \mathbf{k})$ of natural (free) waves, with

$$\frac{\partial \omega}{\partial k_i} = -\frac{\partial D}{\partial k_i}\left(\frac{\partial D}{\partial \omega}\right)^{-1}. \tag{4}$$

In the continuation of the dependence $\mathbf{k}(\mathbf{r})$ along the direction of the vector $\partial D/\partial \mathbf{k}$, the following equalities hold true:

$$\begin{cases} \dfrac{\mathrm{d}k_i}{\mathrm{d}s} = -\dfrac{\partial D}{\partial x_i}\left|\dfrac{\partial D}{\partial \mathbf{k}}\right|^{-1}, \\[2ex] \dfrac{\mathrm{d}x_i}{\mathrm{d}s} = \dfrac{\partial D}{\partial k_i}\left|\dfrac{\partial D}{\partial \mathbf{k}}\right|^{-1}, \end{cases} \tag{5}$$

where s is the distance along the direction of the group velocity.

When the motion along the above direction takes place with the group velocity, Eqns (5) take on the form of conventional equations of ray (geometrical) optics:

$$\begin{cases} \dfrac{\mathrm{d}k_i}{\mathrm{d}t} = \dfrac{\partial D}{\partial x_i}\left(\dfrac{\partial D}{\partial \omega}\right)^{-1} = -\dfrac{\partial \omega}{\partial x_i}\,, \\[2ex] \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -\dfrac{\partial D}{\partial k_i}\left(\dfrac{\partial D}{\partial \omega}\right)^{-1} = \dfrac{\partial \omega}{\partial k_i}\,. \end{cases} \tag{6}$$

The zero-order equations in the parameter $1/(kL)$, which we are interested in, permit determination of the dependence $\mathbf{k}(\mathbf{r})$ and the electric field polarization characterized by the vector $\mathbf{e}$, but leave the amplitude E arbitrary. It is defined by the first-order equations. To derive them, we make the change $\mathbf{k} \to \mathbf{k} - \mathrm{i}\nabla$ in the quantities $a_{ij}(\mathbf{r}, \mathbf{k})$ in Eqn (1). The operator ∇ should be applied to the slowly varying quantities $\mathbf{k}$, $\mathbf{e}$, and E. In the first order in $1/(kL)$, we arrive at the system of inhomogeneous algebraic equations [11, 12]

$$a_{ij}\,\delta E_j' = \mathrm{i} f_i\,, \tag{7}$$

where $\delta E_i'$ are corrections to the solutions of the zero-order equations (1), and

$$f_i = \frac{\partial a_{ik}}{\partial k_j}\, e_k \frac{\partial E}{\partial x_j} + \left[\frac{\partial a_{ik}}{\partial k_j}\left(\frac{\partial e_k}{\partial x_j} + \frac{\partial e_k}{\partial k_l}\frac{\partial k_l}{\partial x_j}\right) + \frac{1}{2}\frac{\partial^2 a_{ik}}{\partial k_j\,\partial k_l}\frac{\partial k_l}{\partial x_j}\, e_k\right] E\,.$$

System of equations (7) is solvable when the condition

$$e_i^{\mathrm{T}} f_i = 0 \tag{8}$$

is fulfilled, where e_i^{T} is the solution of the system transposed to Eqn (1):

$$e_i^{\mathrm{T}} a_{ij} = 0\,.$$

It is pertinent to note that the tensor $a_{ij}(\mathbf{r}, \mathbf{k})$ is Hermitian in the absence of irreversible processes: $a_{ij}(\mathbf{r}, \mathbf{k}) = a_{ji}^*(\mathbf{r}, \mathbf{k})$. Then, $e_i^{\mathrm{T}} = e_i^*$.

Simple calculations show that the vector

$$e_j^{\mathrm{T}}\,\frac{\partial a_{jk}}{\partial k_i}\, e_k$$

is parallel to the vector $\partial D/\partial k_i$, and hence to the group velocity [see expression (4)]. We continue the dependence $E(\mathbf{r})$ over the direction of ray propagation to arrive at the equation

$$\frac{\mathrm{d}E}{\mathrm{d}s} = -e_i^{\mathrm{T}} \left[\frac{\partial a_{ik}}{\partial k_j} \left(\frac{\partial e_k}{\partial x_j} + \frac{\partial e_k}{\partial k_l} \frac{\partial k_l}{\partial x_j} \right) + \frac{1}{2} \frac{\partial^2 a_{ik}}{\partial k_j \partial k_l} \frac{\partial k_l}{\partial x_j} e_k \right] \times \left| e_i^{\mathrm{T}} \frac{\partial a_{ik}}{\partial k_j} e_k \right|^{-1} E \,, \tag{9}$$

which should supplement the system of equations (5).

The passage from the spatial derivative to the time derivative is effected, as for systems (5) and (6), by way of introducing the factor

$$-\left| \frac{\partial D}{\partial \mathbf{k}} \right| \left(\frac{\partial D}{\partial \omega} \right)^{-1} .$$

3. 'Complex' geometrical optics

The propagation of electromagnetic waves through a nonuniform anisotropic medium is accompanied by changes in their polarization, wave vector (refraction), and diffraction. The first two phenomena can be analyzed in the framework of conventional geometrical optics (see the previous section). Choudhary and Felsen [2] showed that the diffraction phenomenon may be included in the extended scheme of geometrical optics which operates on complex values of the wave vector and describes the evolution of inhomogeneous waves.

Following paper [2] we illustrate this approach by the example of two-dimensional waves propagating through an isotropic medium with a permittivity ε. Such waves are described by the Helmholtz equation

$$\Delta\Phi + \left(\frac{\omega}{c} \right)^2 \varepsilon \Phi = 0 \,. \tag{10}$$

Here, $\Phi(x, y)$ is the only nonzero component of the Hertz vector (the z-component).

The wave vector and the eikonal will be considered to be complex: $\mathbf{k} = \mathrm{Re}\,\mathbf{k} + \mathrm{i}\,\mathrm{Im}\,\mathbf{k}$. When $\mathrm{Im}\,\mathbf{k} \neq 0$, an imaginary part appears in the dispersion relation, even in the absence of irreversible processes of wave–medium energy exchange:

$$D = \mathrm{Re}\,D + \mathrm{i}\,\mathrm{Im}\,D = -(\mathrm{Re}\,\mathbf{k})^2 + (\mathrm{Im}\,\mathbf{k})^2 + \left(\frac{\omega}{c} \right)^2 \varepsilon - 2\mathrm{i}\,\mathrm{Re}\,\mathbf{k}\,\mathrm{Im}\,\mathbf{k} = 0 \,. \tag{11}$$

To continue the dependence $\mathbf{k}(\mathbf{r})$, as in the previous section, we need to equate the gradient (11) to zero. The real part of the equation thus obtained is of the form

$$\nabla(\mathrm{Re}\,D) = -2(\mathrm{Re}\,\mathbf{k}\nabla)\,\mathrm{Re}\,\mathbf{k} + \left(\frac{\omega}{c} \right)^2 \nabla\varepsilon + 2(\mathrm{Im}\,\mathbf{k}\nabla)\,\mathrm{Im}\,\mathbf{k} = 0 \,. \tag{12}$$

This equation shows that the real part of the wave vector changes not only due to medium nonuniformity (the second term) which manifests itself, in particular, in wave refraction, but also due to the inhomogeneity of the waves themselves (the third term). The latter effect should be considered the manifestation of diffraction.

Let us compare these considerations with the results of analysis of the same process performed with the aid of a parabolic equation. To derive the parabolic equation, in Eqn (10) we put

$$\Phi(\mathbf{r}) = \exp(\mathrm{i}ky)\, F(x, y) \,,$$

where $k = (\omega/c)\varepsilon$, and $F(x, y)$ is the slowly varying 'envelope'. In this case, for the function F we obtain the (parabolic) equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{i}}{2k} \frac{\partial^2 F}{\partial x^2} = 0 \,.$$

The automodel solution of this equation, which describes the evolution of a wave beam with the Gaussian amplitude distribution

$$F(x, y) = \frac{1}{(y - \mathrm{i}a)^{1/2}} \exp\left(\frac{\mathrm{i}kx^2}{2(y - \mathrm{i}a)} \right) \tag{13}$$

on the 'initial' surface $y = 0$, is well known.

The wave vector corresponding to this solution is given by the expression

$$\mathbf{k} = k \left\{ \frac{x(y + \mathrm{i}a)}{y^2 + a^2} \,,\ 0,\ 1 - \frac{x^2(y^2 - a^2 + 2\mathrm{i}ay)}{2(y^2 + a^2)^2} \right\} . \tag{14}$$

It shows that in the $y = 0$ plane the vector $\mathrm{Re}\,\mathbf{k}$ is parallel to the $0Y$-axis throughout the entire beam cross section. However, since $\partial(\mathrm{Im}\,\mathbf{k})^2/\partial x > 0$ for the Gaussian distribution, in accordance with expressions (13) and (14) it forms a vector fan directed away from the axis, thus leading to beam expansion. For $y \gg a$, in the sector $x/y \leqslant 1/\sqrt{ak}$, in which the beam is mainly concentrated, it asymptotically turns into a radially diverging beam.

One can conceive a non-Gaussian wave beam, in a certain domain of which the condition $\partial(\mathrm{Im}\,\mathbf{k})^2/\partial x < 0$ is satisfied. In this domain, the beam will converge rather than diverge. If the amplitude decreases by a simple exponential law $\mathrm{Im}\,\mathbf{k} = \mathrm{const}$ in the transverse direction, this law will remain unchanged. Therefore, the evolution of inhomogeneous waves may be rather complicated and not reduced merely to the smoothing of the amplitude distribution.

Generally, for inhomogeneous waves ($\mathrm{Im}\,\mathbf{k} \neq 0$) in an arbitrary medium, the problem of electromagnetic field extension from the 'initial' surface S to the entire domain accessible to propagation is a natural generalization of the problem considered in the previous section. We put $\mathrm{Im}\,\mathbf{k} \neq 0$ in expression (3) and separate the real and imaginary parts to obtain

$$\begin{cases} \dfrac{\partial}{\partial x_i} \mathrm{Re}\,D + \dfrac{\partial}{\partial x_j} \mathrm{Re}\,k_i \dfrac{\partial}{\partial k_j} \mathrm{Re}\,D - \dfrac{\partial}{\partial x_j} \mathrm{Im}\,k_i \dfrac{\partial}{\partial k_j} \mathrm{Im}\,D = 0 \,, \\ \dfrac{\partial}{\partial x_i} \mathrm{Im}\,D + \dfrac{\partial}{\partial x_j} \mathrm{Im}\,k_i \dfrac{\partial}{\partial k_j} \mathrm{Re}\,D + \dfrac{\partial}{\partial x_j} \mathrm{Re}\,k_i \dfrac{\partial}{\partial k_j} \mathrm{Im}\,D = 0 \,. \end{cases} \tag{15}$$

Broadly speaking, the directions $\nabla\,\mathrm{Re}\,D$ and $\nabla\,\mathrm{Im}\,D$ do not coincide. That is why to continue the dependence $\mathbf{k}(\mathbf{r})$ from the 'initial' surface S use should be made of the overall procedure described in the previous section. We emphasize, however, that although the generalization of the notion of group velocity to the case of inhomogeneous waves is nonexistent, it is quite frequently employed when $|\mathrm{Re}\,\mathbf{k}| \gg |\mathrm{Im}\,\mathbf{k}|$. We assume this condition to be fulfilled and expand $D(\mathbf{k})$ in terms of $\operatorname{Im}\mathbf{k}$ correct to second order to obtain

$$D(\mathbf{k}) \approx D(\operatorname{Re}\mathbf{k}) - \frac{1}{2}\operatorname{Im}k_i \operatorname{Im}k_j \frac{\partial^2 D}{\partial \operatorname{Re}k_i \,\partial \operatorname{Re}k_j} + \mathrm{i}\operatorname{Im}k_i \frac{\partial D}{\partial \operatorname{Re}k_i} = 0\,. \tag{16}$$

From the condition that the imaginary part of expression (16) is equal to zero it follows that the propagation direction of a weakly inhomogeneous wave ($|\operatorname{Re}\mathbf{k}| \gg |\operatorname{Im}\mathbf{k}|$) is close to the direction in which its amplitude remains constant [5]. In isotropic media $\mathbf{V}_{\mathrm{gr}} \parallel \mathbf{k}$, and therefore the constant phase lines and constant amplitude lines should intersect at an angle close to the right one. In the simplest case of waves described by the Helmholtz equation, these lines are orthogonal for arbitrary values of $\operatorname{Im}\mathbf{k}$ [see expression (11)].

We have considered the procedure of extending the complex wave vector. To continue the amplitude factor $E(\mathbf{r})$, advantage should be taken of relationship (9) with complex values of $\mathbf{k}$, $\mathbf{e}$, and E.

In equations (15), as in equalities (5), the quantity D may be replaced with the frequency ω defined by relationship (2) (see the note at the end of the previous section).

4. Generalized geometrical optics of Gaussian wave beams

Electromagnetic waves are quite often injected into experimental devices in the form of narrow wave beams. In the analysis of such beams, the complex eikonal is conveniently expanded into a power series of deflections from the beam axis:

$$\psi(\mathbf{r}) = \psi(s) + k_i(s)\xi_i + \frac{1}{2}\varkappa_{ij}(s)\xi_i\xi_j\,. \tag{17}$$

Here, the following notation was introduced:

$$\xi_j = \big(x_i - x_i(s)\big)(\delta_{ij} - l_i l_j)\,, \qquad \mathbf{l} = \frac{\partial D/\partial \mathbf{k}}{|\partial D/\partial \mathbf{k}|}$$

is a unit vector tangent to the beam axis, and s is the distance along the axis. The beam axis is defined as the ray trajectory corresponding to the highest intensity.

In an anisotropic medium, the group velocity and wave vector directions may not coincide, and therefore a term linear in ξ_i is present in expansion (17). The real part of the quantities $\varkappa_{ij}$ characterizes the curvature of the beam wavefront, which may be related, for instance, to its focusing, while the imaginary part describes the intensity distribution over the beam cross section. Wave beams with an intensity distribution defined by expression (17) are termed Gaussian or Gaussian-like. Gaussian beams owe their spatial boundedness to the fact that $\operatorname{Im}k_i = \xi_j \operatorname{Im}\varkappa_{ij} \neq 0$ off the axis.

The task of extending the electromagnetic field from the initial surface S, which was discussed in the previous sections, necessitates consideration of the set of ray trajectories. In the case of a Gaussian beam, it will suffice to consider only one axial ray trajectory as well as the evolution of the quantities $\varkappa_{ij}$ along it. Knowing them is sufficient to describe the ray trajectories passing in the vicinity of the beam axis. The scheme of geometrical optics was augmented with the inclusion of the equations for the $\varkappa_{ij}$ quantities in Ref. [4].

The $\varkappa_{ij}$ quantities have the significance of on-axis derivatives of the wave vector:

$$\varkappa_{ij} = \frac{\partial k_i}{\partial x_j}\bigg|_{\xi=0}\,.$$

The equation for them is derived by setting equal to zero the total second spatial derivative of the dispersion relation:

$$\left(\frac{\partial D}{\partial k_k}\frac{\partial}{\partial x_k}\right)\varkappa_{ij} = -\frac{\partial^2 D}{\partial x_i\,\partial x_j} - \varkappa_{ik}\frac{\partial^2 D}{\partial x_j\,\partial k_k} - \varkappa_{jk}\frac{\partial^2 D}{\partial x_i\,\partial k_k} - \varkappa_{ik}\varkappa_{jl}\frac{\partial^2 D}{\partial k_k\,\partial k_l}\,.$$

This equation can be represented as the directional derivative along the traveling direction of rays and can supplement the system of equations (5):

$$\frac{\mathrm{d}}{\mathrm{d}s}\varkappa_{ij} = -\left|\frac{\partial D}{\partial \mathbf{k}}\right|^{-1}\left(\frac{\partial^2 D}{\partial x_i\,\partial x_j} + \varkappa_{ik}\frac{\partial^2 D}{\partial x_j\,\partial k_k} + \varkappa_{jk}\frac{\partial^2 D}{\partial x_i\,\partial k_k} + \varkappa_{ik}\varkappa_{jl}\frac{\partial^2 D}{\partial k_k\,\partial k_l}\right). \tag{18}$$

Here, the derivatives of the quantity D are calculated on the beam axis and are therefore real, unlike the quantities $\varkappa_{ij}$. Since both the quantity D and the derivatives $\mathrm{d}D/\mathrm{d}x_i$ vanish in the ray trajectories, by multiplying Eqn (18) into the factor

$$-\left|\frac{\partial D}{\partial \mathbf{k}}\right|\left(\frac{\partial D}{\partial \omega}\right)^{-1}$$

it is possible to move from the spatial derivative in this equation to the temporal one. In doing so, the quantity S in Eqn (18) is replaced by the frequency.

Equation (18), in combination with the equations of conventional geometrical optics (5), make up a system which completely defines the spatial evolution of Gaussian beams.

It is pertinent to note that Eqn (18) was first obtained by Bernshtein and Fridlend [11], who analyzed with its aid the evolution of uniform wave fields and therefore the quantities $\varkappa_{ij}$ were assumed in Ref. [11] to be real. Under this assumption, it is possible to investigate the effects of wave beam focusing – defocusing with the aid of Eqn (18). As noted in Ref. [4], owing to the specific character of the spatial dependence of Gaussian wave beams their evolution can also be analyzed with the aid of equation (18) in which the $\varkappa_{ij}$ quantities should be taken to be complex.

To be precise, only those $\varkappa_{ij}$ quantities of the entire set are complex which define the transverse spatial structure of the Gaussian beam. At the same time, the quantities

$$\frac{\mathrm{d}k_i}{\mathrm{d}s} = l_j\frac{\mathrm{d}k_i}{\mathrm{d}x_j}$$

characterizing the variation of the wave vector on the beam axis should be real in the absence of irreversible processes. This last proposition is not evident, because equation (18) at first glance links all the $\varkappa_{ij}$ quantities. From this equation, however, for the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}k_i}{\mathrm{d}s}\right)$$

it is possible to obtain the following expression

$$\frac{d^2 k_i}{ds^2} = -\left|\frac{dD}{d\mathbf{k}}\right|^{-1} \left(\hat{L}\,\frac{\partial D}{\partial x_i} + \frac{dk_i}{ds}\,\hat{L}\left|\frac{\partial D}{\partial \mathbf{k}}\right|\right), \qquad (19)$$

where the operator $\hat{L}$ is defined as

$$\hat{L} = \frac{\partial}{\partial s} + \frac{dk_i}{ds}\,\frac{\partial}{\partial k_i}\,.$$

From expression (19) it follows that the quantities dk_i/ds make up a closed collection, and since Im $dk_i/ds = 0$ at the initial point of a ray trajectory in accordance with the equations of geometrical optics, this relation will subsequently be fulfilled as well.

The electric field amplitude on the axis of a Gaussian beam can be found from the condition that the beam energy flux is constant:

$$V_{\text{gr}} W = \text{const}\,,$$

where

$$W = \int dS_{\perp}\, w$$

(the integral is taken over the beam cross section), and

$$w = \frac{1}{8\pi}\, e_i e_j^* \,\frac{\partial}{\partial \omega}\, \omega \varepsilon_{ij} |E|^2$$

is the energy density. Simple calculations yield

$$W = \frac{1}{16\pi}\, e_i e_j^* \,\frac{\partial}{\partial \omega}\, \omega \varepsilon_{ij} |E|^2 \,\frac{1}{(\varkappa_{11}\varkappa_{22} - \varkappa_{12}^2)^{1/2}}\,,$$

where

$$\varkappa_{11} = h_{1i} h_{1j}\, \text{Im}\, \varkappa_{ij}\,,$$
$$\varkappa_{12} = h_{1i} h_{2j}\, \text{Im}\, \varkappa_{ij}\,,$$
$$\varkappa_{22} = h_{2i} h_{2j}\, \text{Im}\, \varkappa_{ij}\,,$$

and the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ in combination with vector $\mathbf{l}$ make up an orthonormal triad.

The Gaussian wave beam is a convenient object to analyze with the aid of a parabolic equation. The Gaussian distribution of electric field is described by the simplest (largest-scale) solution of the parabolic equation [see, for instance, expression (13)].

In the case of an arbitrary nonuniform anisotropic medium, the use of the parabolic equation turns out to be rather arduous. Pereverzev [7] produced the system of equations for the parameters of the simplest solution of the parabolic equation, the system equivalent to equation (18). Propagation of the Gaussian beam in a nonuniform anisotropic medium was considered in detail by Permitin and Smirnov [13]. The even simpler case of a uniform isotropic medium was discussed in the previous section. The Gaussian beam in such a medium is characterized by a single parameter

$$\varkappa_{xx} = \frac{k}{y - \mathrm{i}a}\,, \quad \text{where} \quad k = \frac{\omega}{c}\,\sqrt{\varepsilon}$$

[see expression (13)].

It is easily seen that this expression satisfies Eqn (18). Indeed, $D = -k_x^2 - k_y^2 + k^2$ in the case under consideration and equation (18) takes on the form

$$\frac{d\varkappa_{xx}}{dy} = -\frac{\varkappa_{xx}^2}{k}\,.$$

Therefore, the above-discussed approaches to the investigation of Gaussian beam evolution are equivalent.

5. Conclusion

The foregoing consideration reveals that the dispersion relation derived as the solvability condition of the system of Maxwell equations contains all the information about the spatial structure of short-wave electromagnetic waves. This information may be extracted with recourse to the GO formalism. Generalizing GO to inhomogeneous waves characterized by a complex wave vector permits us to take into account the diffraction phenomenon.

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