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Cosmological branes and macroscopic extra dimensions

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Contents

Abstract. **The idea of adding extra dimensions to the physical world — thus making the observable universe a timelike surface (or brane) embedded in a higher-dimensional space–time — is briefly reviewed, which is believed to hold serious promise for solving fundamental problems concerning the hierarchy of physical interactions and the cosmological constant. Brane localization of massless gravitons is discussed as a mechanism leading to the effective four-dimensional Einstein gravity theory on the brane in the low-energy limit. It is shown that this mechanism is a corollary of the AdS/CFT correspondence principle well-known from string theory. Inflation and other cosmological evolution scenarios induced by the local and nonlocal structures of the effective action of the gravitational brane are considered, as are the effects that enable the developing gravitational-wave astronomy to be used in the search for extra dimensions. Finally, a new approach to the cosmological constant and cosmological acceleration problems is discussed, which involves variable local and nonlocal gravitational 'constants' arising in the infrared modifications of the Einstein theory that incorporate brane-induced gravity models and models of massive gravitons.**

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1. Introduction

Today, the modern quantum theory of the microworld and the theory of the macroworld or cosmology have begun to overlap both at the fundamental, theoretical level and at the level of experimental observations. Perhaps the most illustrative example of this process is provided by the theory of cosmological inflation, which, on the one hand, resolves the known problems of the standard cosmological Big Bang scenario and, on the other hand, explains details of the development of the large-scale space – time structure observed with an increasingly higher accuracy in spacecraft, astronomical, and aerostatic experiments during the last decade. The quantum theory of cosmological perturbations is fundamentally exploited to predict the formation of the low-frequency region in the spectrum of microwave cosmic background or relic radiation.

But the theory of cosmological perturbations is a semiphenomenological one because it is based on the quantum theory of physical fields in a curved space – time and actually concerns its lowest (in terms of quantization) or tree-level approximation; it does not explain the origin of the inflation stage of cosmological expansion. Expansion mechanisms in the framework of the inflation theory also remain essentially classical or built up within the framework of semiphenomenological quantum models. Attempts to include modern cosmology into the basic physical theory of high energies lead to the idea of a multidimensional space – time, the subject matter of the present review.

The idea of multidimensional space – time is not new. It was first suggested by Nordström in 1914 [1] and forestalled the formulation of the general relativity theory in the form of the scalar gravity theory as a component of the Maxwell electrodynamics in a five-dimensional space – time. This idea was further developed in the works of T Kaluza and O Klein [2] that laid the foundation of the so-called Kaluza – Klein theory. Much later, in the 1980s, the Kaluza – Klein five-dimensional approach was extensively applied to the analysis of multidimensional superstring theories and their phenomenology. An important element of this approach was the qualitative explanation of the fact that extra dimensions (in the case of their compactification on a certain scale) are unobservable in the low-energy region lying below this scale.

The concept of multidimensional fundamental space – time became actually necessary only in the framework of the superstring theory, which is now universally accepted to be the most promising theory of high energies unifying quantum gravity and gauge field theory. The reason is that the superstring theory and its low-energy manifestations can be consistently formulated only in the distinguished dimensions of the fundamental space – time, $D = 10$ and $D = 11$, whereas other dimensionalities are simply forbidden. In this situation, as in the Kaluza – Klein scenario, the four-dimensionality of the observable world is achieved by compactification of extra dimensions on an energy scale unattainable in the framework of sub-Planckian physics.

Motivation for introducing extra dimensions also comes from the hierarchy problem in high-energy physics and cosmology. The problem is to explain the extensive area of energy desert separating the electroweak interaction scale of the order of 1 TeV and the Planck scale of quantum gravity, 10^{19} GeV. The hierarchy problem acquires special importance in the cosmological context because it reflects a huge gap (120 orders of magnitude) between the quantum-gravity Planckian scale and the observed cosmological constant scale.

To clarify the aforesaid, we note that the universe is characterized by a number of basic cosmological parameters, such as the mean matter density ρ, the anisotropy of the microwave background $\Delta T/T$, and the cosmological density parameter Ω measured in units of the critical density of the expanding universe:

$$\rho \simeq 10^{-29} \text{ g cm}^{-3} , \tag{1.1}$$

$$\frac{\Delta T}{T} \simeq 10^{-5} , \tag{1.2}$$

$$\frac{\rho}{\rho_{\text{crit}}} \equiv \Omega \simeq 1 . \tag{1.3}$$

The critical density is expressed in terms of the Hubble constant H, i.e., the logarithmic derivative of the cosmological scale factor $a(t)$ with respect to the observer proper time t,

$$\rho_{\text{crit}} = \frac{3M_{\text{P}}^2 H^2}{8\pi} , \quad H = \frac{\dot{a}}{a} , \tag{1.4}$$

where $M_{\text{P}}^2 = 1/G$ is the Planck mass squared, the inverse of the gravitational constant value.

A most important recent discovery resulting from a combination of differently designed experiments to observe supernovae, cosmic microwave anisotropy, and microlensing is the contemporary cosmological acceleration [3 – 5] corresponding to the approximately 70% content of the total matter density in the universe due to a special component (with the equation of state close to $p = -\rho$) of a dark energy interpreted as the effective cosmological constant Λ:

$$\rho = \rho_{\text{m}} + \rho_{\Lambda} , \tag{1.5}$$

$$\Omega_{\Lambda} \simeq 0.73 . \tag{1.6}$$

The cosmological constant turns out to be immeasurably smaller than possible values of the vacuum energy for the known models of the fundamental quantum theory, covering the range from the electroweak coupling to quantum gravity and string theory. The density of the vacuum energy is determined by the energy scale of the respective model; it is of the order of $\rho_{\text{EW}} \sim 1 \text{ TeV}^4$ for the electroweak theory and $\rho_{\text{P}} \sim M_{\text{P}}^4$ for quantum gravity with the Planck mass $M_{\text{P}} \sim 10^{19}$ GeV. For this reason, the vacuum energy of the current cosmological expansion falls 56 and 120 orders of magnitude behind the predictions of these models, respectively:

$$\frac{\rho_{\Lambda}}{\rho_{\text{EW}}} \sim 10^{-56} , \quad \frac{\rho_{\Lambda}}{\rho_{\text{P}}} \sim 10^{-120} . \tag{1.7}$$

The paradox related to this value of the cosmological constant (provided, of course, that the observed dark energy is correctly interpreted as a fundamental constant or the vacuum energy) lies in the fact that it differs from zero in spite of its smallness.

The discovery of the cosmological acceleration has drastically changed the status of the cosmological constant problem. In the past, all efforts in fundamental physics were concentrated on the construction of a model with zero vacuum energy. Now, the correct model must explain a nonzero value, which, on the one hand, is immeasurably small in comparison with the vacuum energies of electroweak interactions, Grand Unification, and Planck gravity but, on the other hand, actually predominates the total matter density in the universe. All this reflects the formerly unprecedented interweaving of problems of fundamental microphysics, phenomenology, and cosmology. Interestingly, the extra dimension concept appears to be very fruitful in light of these problems and also undergoes a radical change that distinguishes it from the old Kaluza – Klein approach [6].

The principal difference between the two concepts lies in the fact that extra space – time dimensions can be macroscopic and even noncompact despite the four-dimensional nature of the directly observable physical world. The four-dimensionality is achieved by means of matter localization in a multidimensional space – time (in its bulk) on its four-dimensional submanifolds called branes.[1] The resulting properties of the interaction allow describing such a fundamentally multidimensional model in effectively four-dimensional terms.

The dynamics of brane geometry and the brane localization of matter constitutes brane cosmology, which has come to take the place of the traditional cosmology of the four-dimensional world. It turns out that extra dimensions in the framework of this concept open up new prospects for the solution of the hierarchy problem, including modification of the extent of the energy 'desert' lying between the electroweak coupling and Planck gravity physics, establishment of a closer relation with the string theory in the form of the so-called

[1] In what follows, in the Russian version of this paper, the multidimensional part of space – time is referred to as 'объем (volume)'. 'Bulk' is the equivalent term widely used in the international English-language literature.

AdS/CFT correspondence principle, construction of new inflation mechanisms in the early universe, and, finally, development of models for solving the cosmological constant problem encompassing both the hierarchy and cosmological acceleration.

The present review is designed to briefly discuss this new trend in high (and, as shown below, ultra-low) energy physics and cosmology. We start from the comparison of the Kaluza–Klein picture and the idea of matter (e.g., graviton) localization on four-dimensional submanifolds of the space–time. Thereafter, we evaluate the size and the number of extra dimensions consistent with on-going experiments and ensuing from the simplest multidimensional Arkani-Hamed–Dimopoulos–Dvali (ADD) model and demonstrate mechanisms underlying changes in fundamental constants of the theory within the framework of the brane concept.

Next, we consider the mechanism of brane localization of gravitons in the Randall–Sundrum model demonstrating the consistency of the observed four-dimensional space–time with the presence of noncompact extra dimensions. It turns out that this mechanism is a corollary of the AdS/CFT correspondence principle well-known from string theory, implying the duality (equivalence) of the supergravity theory formulated in the background of the multidimensional anti-de Sitter space and the conformal field theory in the form of a supersymmetric Yang–Mills model at the boundary of this space.

The discussion of this consistency is followed by the consideration of the brane effective action in the two-brane Randall–Sundrum model, whose nonlocal nature reflects the AdS/CFT correspondence principle. We further demonstrate the inflation scenario on the brane induced by repulsing branes and associated with the phase transition between ultra-local and essentially nonlocal phases of the theory. Applications of the theory are illustrated by potentially observable effects of gravitational-wave oscillations and gravitational echo, which may be useful for the study of extra dimensions by the methods of gravitational-wave astronomy. The review is concluded by a brief discussion of brane cosmological models with a variable local gravitational ‘constant’ and nonlocal modifications of the Einstein theory that may serve as a new mechanism for the solution of the cosmological constant problem. The Dvali–Gabadadze–Porrati model and other theories of brane-induced gravity are considered as potential sources of such modifications.

2. The Kaluza–Klein picture and localization of matter

To illustrate the difference between the Kaluza–Klein (KK) picture and the concept of matter localization on a brane, we consider a five-dimensional space–time of cylindrical form with the fifth coordinate y compactified to a circle of length $2\pi L$ (Fig. 1). The complete set of coordinates covering the space is denoted as

$$X^A = (x^\mu, y)\,, \quad A = 0,1,2,3,5\,, \quad \mu = 0,1,2,3\,, \quad -\infty < x < \infty\,, \quad 0 < y < 2\pi L\,. \tag{2.1}$$

The coordinates x^μ span a flat four-dimensional space–time with the Lorentzian metric $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)$. The full metric has the form

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{d}y^2\,. \tag{2.2}$$

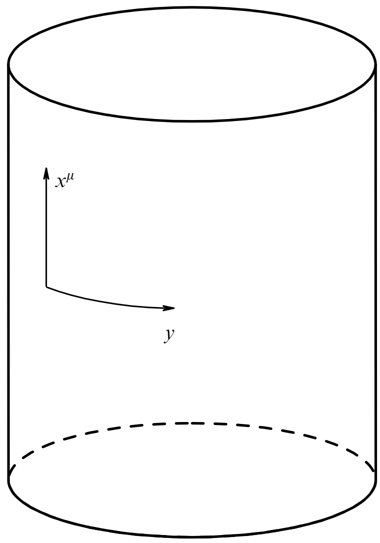

Figure 1. Five-dimensional space–time of the cylindrical type in the Kaluza–Klein picture with the fifth coordinate y compactified to a circle.

The massless scalar field in such a space satisfies the equation involving the five-dimensional d'Alembertian $\Box_5 = \eta^{\mu\nu}\partial_\mu\partial_\nu + \partial_y^2$,

$$\Box_5\,\phi(x,y) = 0\,. \tag{2.3}$$

Its expansion in discrete Fourier harmonics periodic on the circle and in the continuum of plane waves in the x-space,

$$\phi_p(x,y) = \exp\left(\mathrm{i}p_\mu x^\mu + \frac{\mathrm{i}ny}{L}\right), \quad n = 0, \pm 1, \ldots, \tag{2.4}$$

leads to the massive dispersion equation for the four-dimensional momentum p^μ of each nth harmonic:

$$p^2 + m_n^2 = 0\,, \quad p^2 \equiv p_\mu p^\mu\,, \tag{2.5}$$

$$m_n^2 = \frac{n^2}{L^2}\,. \tag{2.6}$$

Thus, a massless (from the five-dimensional standpoint) field is a tower of four-dimensional massive KK-modes with $|n| \geqslant 1$ built up over the zero massless mode $n = 0$ and having discrete mass spectrum (2.6) determined by the compactification scale L of the extra dimension.

Evidently, the first massive level of the KK-spectrum cannot be excited and the corresponding compact dimension cannot be observed at the energy scale $E < 1/L$. Therefore, sufficiently small extra dimensions are invisible for the observer bounded from above in the energy scale. Because early multidimensional supergravity theories postulated the Planckian compactification scale ($L \sim 1/M_\mathrm{P} \sim 10^{-33}$ cm), direct observation of extra dimensions was possible only at the Planckian energy scale ($M_\mathrm{P} \sim 10^{19}$ GeV). This automatically ensured the effective four-dimensionality of the sub-Planckian physics. Moreover, no massive KK-partners of ordinary particles of the standard model have thus far been discovered in modern accelerators, even at 100 GeV. Therefore, the size of the extra dimension must satisfy the constraint $L^{-1} > 100$ GeV or be smaller than 10^{-17} cm (in conventional units of length).

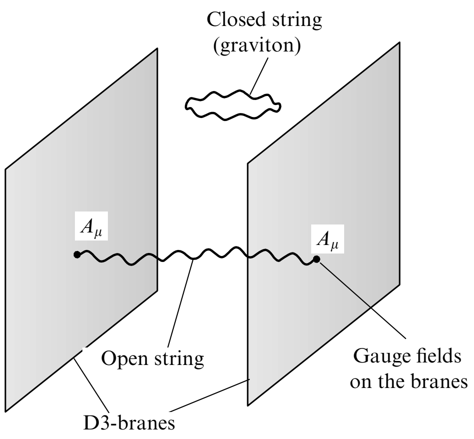

Figure 2. Schematic of a compactification of the multidimensional space – time alternative to the Kaluza – Klein picture. The gauge fields of matter associated with the ends of *open* strings are fundamentally four-dimensional objects localized on D3-branes (nonperturbative bound states of the strings) and have no KK-partners. The graviton, being a low-energy approximation of a *closed* string, can propagate in the multidimensional bulk.

Such a result seems to exclude the possibility of any multidimensional space – time with macroscopic extra dimensions. However, there is a radically different concept of multidimensionality, besides the standard Kaluza – Klein picture, based on the localization of matter on four-dimensional submanifolds, i.e., branes embedded in a multidimensional bulk. The main difference between this concept and the Kaluza – Klein approach consists in the fact that the ordinary matter fields are localized on branes and are four-dimensional rather than multidimensional objects at the fundamental level, whereas the gravitational field freely lives and propagates in the multidimensional bulk. This makes it possible to satisfy conditions at which the multidimensional gravitational field is also localized on the brane and becomes effectively four-dimensional in the low-energy region in spite of the macroscopic and even infinite extent of extra dimensions. As a by-product of this set-up, Newton's gravitational constant G_4 (or the Planckian scale of quantum gravity $M_{\rm P}^2 = G_4^{-1}$) ceases to be a fundamental quantity and is now determined by the combination of the fundamental D-dimensional gravitational constant G_D and the scale of an extra dimension L (it is shown below that it may tend to infinity).

Such a scheme of matter and gravity localization on the brane is motivated, on the one hand, by the very first works in which it was proposed for the fermion matter in the background of a kink, i.e., a solution of the nonlinear equation for a self-interacting scalar field that describes a domain wall [7, 8]. On the other hand, this picture ensues from the low-energy superstring theory, in which branes emerge as the bound states (Dp-branes) of open strings. They are in fact $(p+1)$-dimensional time-like surfaces on which the ends of open strings are localized. The ends of open strings carry gauge fields that, at the fundamental level, are $(p+1)$-dimensional objects residing on the branes (Fig. 2). This explains why gauge fields do not live in the bulk and have no KK-partners.

On the contrary, closed strings, which are known to describe the spin-2 field, can freely propagate in the bulk and therefore allow free propagation of 10-dimensional gravitons.[2] This accounts for the disparate roles of gravity and matter fields in the new picture of extra dimensions. In this picture, only the four-dimensional massless graviton has a tower of massive KK-partners. However, the small compactification scale L has nothing to do with the fact that these KK-partners remain invisible in low-energy experiments. The reason is only that the massless zero mode undergoes localization on the brane, whereas wave packets of massive KK-partners are expelled into the bulk and outside the brane. Therefore, they are weakly coupled to four-dimensional matter. In this way, the compactification scale and the energy scale below which the theory is effectively four-dimensional can be made independent.

It turns out that the properties of graviton localization reflect the so-called AdS/CFT correspondence principle in the field theory of type IIB superstrings and are essentially based on the presence of curvature of the multidimensional bulk (its AdS-character). However, consideration of the Randall – Sundrum model, a carrier of these properties, should be preceded by discussion of a simpler variant, the ADD model [9] with a flat background space, which allows formulating simple experimental constraints on the parameters of extra dimensions and suggests ways to address the hierarchy problem both in the sector of matter fields and in the gravitational sector.

3. The ADD model: size (and number) of extra dimensions

In the Arkani-Hamed – Dimopoulos – Dvali (ADD) model [9], the four-dimensional gravitational constant

$$G_4 = \frac{1}{M_{\rm P}^2} \tag{3.1}$$

is not fundamental. On the contrary, fundamental is the D-dimensional gravitational action

$$S = \frac{1}{16\pi G_D} \int {\rm d}^D X \, G^{1/2} R(G_{AB}) \,, \quad D = 4 + N \,, \tag{3.2}$$

with the D-dimensional gravitational constant expressed through the fundamental energy scale of M-theory that is essentially different from the Planck scale $M_{\rm P}$:

$$G_D = \frac{1}{M^{D-2}} \,. \tag{3.3}$$

It is supposed that the D-dimensional gravitational field is coupled to four-dimensional matter localized on the brane of the codimension $N = D - 4$; in this case, the characteristic size of extra dimensions is finite and equals L.

If one confines oneself to the low-energy approximation, to which the main contribution is made by zero modes of the gravitational field unrelated to the additional coordinates y, the integral over y in multidimensional integral (3.2) is factored through the internal space volume:

$$\int {\rm d}^D X \equiv \int {\rm d}^N y \int {\rm d}^4 x = L^N \int {\rm d}^4 x \,. \tag{3.4}$$

[2] The low-energy string theory also contains the dilaton scalar field and the fields of forms living in a 10-dimensional space. However, they are not taken into consideration in the present simplified scheme of the relation between the physics of extra dimensions and the phenomenology of string D-branes.

The effective action for the zero mode of the gravitational field represented by the four-dimensional metric $g_{\mu\nu}(x)$ then takes the form of the Einstein action,

$$S_{\rm eff}[g_{\mu\nu}] = \frac{L^N M^{N+2}}{16\pi} \int {\rm d}^4 x\, g^{1/2} R(g)\,, \qquad (3.5)$$

with the gravitational constant G_4 and the corresponding Planck mass

$$G_4 = \frac{G_D}{L^N}\,, \quad M_{\rm P} = M(ML)^{N/2}\,. \qquad (3.6)$$

Thus, the observed Planckian scale of the gravity theory is a derivative from the fundamental $(D = 4 + N)$-dimensional scale and the size of extra dimensions. This property allows us to radically change the approach to the hierarchy problem and transfer, e.g., the fundamental gravitational scale from the Planckian region to a region of substantially lower energies. For example, the scale of multidimensional gravity may be chosen in the electroweak interaction region or slightly above it, $M \sim 1$ TeV. It is worth noting that in accordance with formula (3.6), this requires a rather large internal space unfeasible in the framework of the standard Kaluza–Klein scheme, where it would be necessary to introduce sufficiently light KK-partners of matter particles forbidden by modern collider experiments. However, KK-partners of matter are completely nonexistent in the framework of the concept of brane-localized matter; hence, there is no such experimental constraint.

Possible limitations on the size of extra dimensions ensue from the gravitational sector (because only gravitons are allowed to propagate in the multidimensional bulk). These limitations largely follow from the Cavendish-type experiments conducted to verify Newton's law of attraction between test masses. This law has been fairly well substantiated within the scope of celestial mechanics but is bounded from below in the millimeter range. As follows from high-precision table-top experiments [10], the law of gravitational attraction has been established down to distances of about 0.2 mm. However, deviations at smaller distances cannot be experimentally excluded. This means that, unlike collider-imposed constraints in the framework of the KK-approach, $L < 10^{-17}$ cm, we have a much more moderate estimate,

$$L < 0.2\ {\rm mm}\,. \qquad (3.7)$$

Estimate (3.7) can be used to impose constraints on the number of extra dimensions. It follows from (3.6) that the compactification scale L is expressed through the fundamental scale M and brane codimension N as

$$L = M^{-1}\left(\frac{M_{\rm P}}{M}\right)^{2/N} \simeq 10^{32/N-17}\ {\rm cm}\,, \qquad (3.8)$$

where it is assumed that $M \sim 1$ TeV. Therefore, the following estimates on L hold for the three lowest codimensions of the brane:

$$N = 1\,, \quad L \sim 10^{15}\ {\rm cm}\,,$$
$$N = 2\,, \quad L \sim 10^{-1}\ {\rm cm}\,, \qquad (3.9)$$
$$N = 3\,, \quad L \sim 10^{-6}\ {\rm cm}\,.$$

It is straightforward to see that the first case is altogether excluded by the findings of celestial and planetary mechanics. The second case lies in the millimeter range of modern high-precision experiments [10]. The last case appears to be hardly attainable in the near future despite rapid progress in the enhancement of accuracy of experiments designed to verify the Newtonian law of gravity. In other words, one extra dimension in the ADD model is already excluded, and the substantiation of this model with two-dimensional internal space is currently in order.

4. The Randall–Sundrum model

In the preceding section, we relaxed the constraint on the size of extra dimensions by shifting it to the millimeter range. Here, we demonstrate that it can be actually infinite in the presence of a nonzero curvature in the bulk. We consider the Randall–Sundrum model [11, 12] describing a five-dimensional gravitational field having the cosmological term Λ_5 and coupled to a four-dimensional brane:

$$\begin{aligned} &S\big[G_{AB}(X), \psi(x)\big] \\ &\quad = \frac{1}{16\pi G_5} \int {\rm d}^5 X\, G^{1/2}\big[{}^5R(G_{AB}) - 2\Lambda_5\big] \\ &\quad + \int {\rm d}^4 x\, g^{1/2}\left(\frac{1}{8\pi G_5}\,[K] - \sigma + L_{\rm m}(g_{\alpha\beta}, \psi, \partial\psi)\right). \end{aligned} \qquad (4.1)$$

The brane has a tension σ and is populated by a four-dimensional matter field ψ with the Lagrangian $L_{\rm m}(g_{\alpha\beta}, \psi, \partial\psi)$ and the induced metric $g_{\mu\nu}(x)$.

From the four-dimensional standpoint, the brane tension can be regarded as a four-dimensional cosmological term. We assume that in the coordinates $X^A = (x^\mu, y)$, the brane is a time-like plane, $X^5 \equiv y = 0$ (this is the choice of coordinate gauge, because the brane can always be placed at this point along the fifth coordinate by a coordinate transformation). Then, the induced metric $g_{\mu\nu}(x) = G_{\mu\nu}(x, 0)$. We further assume that the entire five-dimensional space is Z_2-symmetric with respect to the brane plane, i.e., that the space–time on the right of the brane (at $y > 0$) can be obtained by mirror reflection of its half-space at $y < 0$. This property can be formulated within the normal Gaussian coordinate system $(G_{5\mu}(X) = 0)$ as the parity of functions of the remaining nonzero metric coefficients:

$$G_{\mu\nu}(x, y) = G_{\mu\nu}(x, -y)\,, \quad G_{55}(x, y) = G_{55}(x, -y) = 1\,.$$

In terms of the five-dimensional space, the brane is a delta-shaped distribution of matter and tension, and therefore the solution of the corresponding multidimensional Einstein equations cannot be smooth, that is, derivatives of metric coefficients normal to the brane undergo a jump. Specifically, a jump in the trace of the brane extrinsic curvature approached from the right and the left, $[K] = K(y = 0^+) - K(y = 0^-)$, is nonzero and gives rise to an additional surface term called the Gibbons–Hawking action [13]. The introduction of this term is necessary for the validity of the variational procedure for the total five-dimensional action.

The variational procedure leads to the Einstein equations in the five-dimensional bulk and the Israel matching conditions on the brane surface [14], which are used to express the jump of the extrinsic brane curvature $K_{\mu\nu}$ as

$$[K^{\mu\nu} - g^{\mu\nu} K] = 8\pi G_5 S^{\mu\nu} \qquad (4.2)$$

through the total energy-momentum tensor of the four-dimensional brane action $S^{\mu\nu}$ in (4.1), including both the contribution of matter, $T^{\mu\nu}$, and that of tension,

$$\frac{1}{2}\, g^{1/2}\, S^{\mu\nu} = \frac{\delta}{\delta g_{\mu\nu}} \int \mathrm{d}^4 x\, g^{1/2} \big(-\sigma + L_{\mathrm{m}}(g_{\alpha\beta}, \psi, \partial\psi) \big)\,, \quad (4.3)$$

$$S^{\mu\nu} = -\sigma g^{\mu\nu} + T^{\mu\nu}\,. \quad (4.4)$$

By virtue of Z_2-symmetry, the extrinsic curvatures on the right and the left of the brane have opposite signs and, in the normal Gaussian system of coordinates, are given by

$$K_{\mu\nu}\Big|_{y=\pm 0} = -\frac{1}{2}\, \frac{\mathrm{d} g_{\mu\nu}}{\mathrm{d} y}\Big|_{y=\pm 0} \quad (4.5)$$

(this sign convention corresponds to the definition of the extrinsic curvature as the derivative of the four-dimensional metric along the normal to the brane directed outside the half-space $y > 0$ and inside the half-space $y < 0$).

It turns out that in the absence of matter on the brane, there is a simple solution of the complete system of the Einstein equations and boundary matching conditions in the form of a piecewise smooth metric,

$$\mathrm{d}s^2 = \mathrm{d}y^2 + \exp\left(-\frac{2|y|}{l} \right) \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \quad (4.6)$$

(where $\eta_{\mu\nu}$ is the flat Minkowski metric) under the condition of a fine tuning between the *negative* five-dimensional cosmological constant and the positive brane tension:

$$\Lambda_5 = -\frac{6}{l^2}\,, \qquad \sigma = \frac{3}{4\pi G_5 l}\,. \quad (4.7)$$

Locally, solution (4.6) describes the geometry of the anti-de Sitter (AdS) space, that is, a homogeneous space with a constant negative curvature. Globally, it is a gluing of two Z_2-symmetric regions of the AdS space extending in the Poincaré coordinates (x^μ, y) between the brane $y = 0$ and their corresponding horizons $y = \pm\infty$.

A priori, such a construction looks artificial, but it actually has a sufficiently strong motivation coming from the string theory in the form of the so-called Horava – Witten model [16] that suggests the solution of the problem of chiral fermions in the field approximation of string theory by means of compactification of a multidimensional space on an orbifold. In a simplified five-dimensional analog of this model, the fifth coordinate y is compactified to a circle of a finite length $2d$ and ranges the values $-d \leqslant y \leqslant d$. The points d and $-d$ are identified, and therefore the points $y = 0$ and $|y| = d$ parameterize diametrically opposite points on such a circle. Furthermore, the pairs of points y and $-y$ (and the corresponding fields in the bulk) are also identified, which results in an orbifold, i.e., a manifold lacking smoothness at two fixed points, $y = 0$ and $|y| = d$. At these points, two branes are introduced that can be populated by chiral fermion matter induced from the multidimensional bulk (Fig. 3).

It turns out that metric (4.6) is a self-consistent solution of the five-dimensional Einstein equations and the Israel matching conditions on an orbifold with two branes, provided their tensions $\sigma_\pm$ coincide up to a sign and are again fine-tuned to the negative cosmological constant in the bulk:

$$\sigma_+ = -\sigma_- = \frac{3}{4\pi G_5 l}\,. \quad (4.8)$$

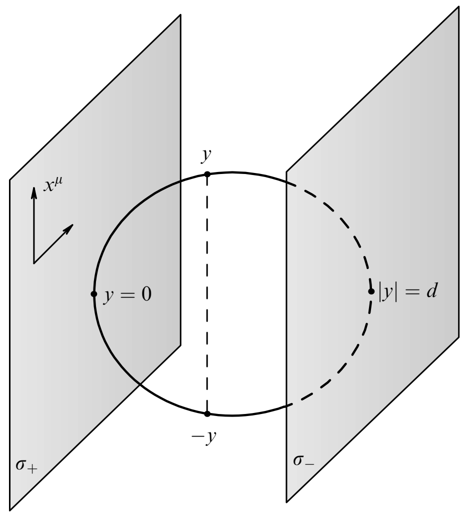

Figure 3. The qualitative picture of compactification of the fifth dimension on an orbifold in the Horava – Witten model underlying the Randall – Sundrum model with two branes having tensions $\sigma_\pm$ of opposite signs. Z_2-identification of points y and $-y$ is shown by the dashed line.

Of course, the gravitational action of such a system is given by expression (4.1) containing two brane surface integrals with the corresponding tensions $\sigma_\pm$ and jumps of the extrinsic curvature. A specific case of the one-brane Randall – Sundrum model is obtained in the formal limit as $d \to \infty$.

4.1 The brane hierarchy problem

Thus, the introduction of the piecewise smooth AdS space is motivated by the phenomenological Horava – Witten model. Moreover, the curved character of the multidimensional space in this model opens up new prospects for the solution of the hierarchy problem in the particle phenomenology. Specifically, it allows the observed scale of electroweak and other gauge interactions (by analogy to the gravitational scale) to be made derivative from the properties of an extra dimension. To show this, a negative-tension brane is assumed to be populated by the Higgs field $H(x)$ with the action

$$S_{\mathrm{brane}} = \int \mathrm{d}^4 x\, g^{1/2} \big(g^{\mu\nu}\, \partial_\mu H\, \partial_\nu H - \lambda (H^2 - v^2)^2 \big)\,, \quad (4.9)$$

where the parameter v gives the fundamental scale of spontaneous symmetry breaking. This field is minimally coupled to the induced metric

$$g_{\mu\nu} = \exp\left(-\frac{2|y|}{l} \right) \eta_{\mu\nu}\,,$$

depending on the point along the fifth coordinate at which the brane is localized. Due to the scale factor

$$a(y) = \exp\left(-\frac{2|y|}{l} \right)\,, \quad (4.10)$$

whereby this metric in the five-dimensional bulk differs from the flat one, the initial field H is not canonically normalized: the coefficient of its kinetic term in the action is different from unity.

Canonical normalization is restored for the field

$$\bar{H} = \exp\left(-\frac{|y|}{l}\right) H\,, \tag{4.11}$$

in terms of which action (4.9) takes the form of a Higgs action,

$$S_{\text{brane}} = \int \mathrm{d}^4 x\, \left(-\eta^{\mu\nu}\,\partial_\mu \bar{H}\,\partial_\nu \bar{H} - \lambda(\bar{H}^2 - \bar{v}^2)^2\right), \tag{4.12}$$

with a different effective scale of symmetry breaking:

$$\bar{v} = \exp\left(-\frac{|y|}{l}\right) v\,. \tag{4.13}$$

This scale is exponentially small in units of length of the extra dimension. Therefore, even a small variation of the probe-brane position in the bulk leads to an exponentially strong change in the energy scale of the theory from the standpoint of a four-dimensional observer residing on this brane. This gives a key to the solution of the hierarchy problem in particle phenomenology [11], providing additional evidence of the value of the Randall–Sundrum model. [3]

However, there is an important issue concerning the gravitational interaction. Specifically, does the effective theory remain four-dimensional in the gravitational sector and, if it does, at which distance scale? We see shortly that this scale is determined by the curvature radius of the AdS space, l, rather than by the size of the extra dimension, d; the four-dimensional gravity then remains valid for distances exceeding l. This property is underlain by the phenomenon of localization of the zero mode of the gravitational field, which is discussed in the next subsection.

4.2 Brane localization of gravitons

The simplest test of the four-dimensionality of a theory in the gravitational sector consists of checking Newton's law of attraction between two massive sources localized on a given brane. For this, we consider an extreme situation of a noncompact extra dimension ($0 \leqslant |y| < \infty$) and construct a linearized gravity theory in the background of the Randall–Sundrum solution,

$$g_{\mu\nu} = \exp\left(-\frac{2|y|}{l}\right)\eta_{\mu\nu} + h_{\mu\nu}(x, y)\,, \tag{4.14}$$

where $h_{\mu\nu}(x, y)$ are the nonzero components of metric perturbations in the so-called Randall–Sundrum gauge [4] [12, 17, 19, 20]:

$$h_{55} = h_{5\mu} = 0\,, \quad \partial^\mu h_{\mu\nu} = h^\mu_\mu = 0\,. \tag{4.15}$$

We first consider the vacuum (in the absence of matter on the brane) linearized Einstein equations and Israel matching conditions. In this gauge, they assume the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{4}{l^2} + \frac{\Box}{a^2(y)}\right) h_{\mu\nu}(x, y) = 0\,, \tag{4.16}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} + \frac{2}{l}\right) h_{\mu\nu}(x, y)\bigg|_{y=0} = 0\,, \tag{4.17}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the four-dimensional d'Alembertian.

We separate the variables in Eqns (4.16) and (4.17) by expanding metric perturbations with respect to plane four-dimensional waves,

$$h(x, y) = \frac{1}{\sqrt{z}}\,\varphi_m(z)\exp(\mathrm{i}px)\,, \qquad p^2 = -m^2\,, \tag{4.18}$$

where the coefficient function of the fifth coordinate $\varphi_m(z)$ is written in terms of the new variable

$$z = (\operatorname{sign} y)\, l \exp\frac{|y|}{l}\,, \quad |z| \geqslant l\,, \tag{4.19}$$

and the variable separation parameter m plays the role of mass of the KK-modes of the gravitational field. By virtue of the linearized Einstein equations for perturbations (4.18),

$$\frac{z^{3/2}}{l^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \Box - \frac{15}{4z^2}\right)\varphi_m(z)\exp(\mathrm{i}px) = 0\,, \tag{4.20}$$

the square of this mass is an eigenvalue of the stationary Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{15}{4z^2}\right)\varphi_m(z) = m^2\varphi_m(z) \tag{4.21}$$

on the half-line $z \geqslant l$ with the potential

$$V(z) = \frac{15}{4z^2}\,, \tag{4.22}$$

which is qualitatively depicted in Fig. 4.

Owing to the Z_2-symmetry of the problem, the eigenfunctions are continued evenly to the half-line $z \leqslant -l$: $\varphi_m(-z) = \varphi_m(z)$. This makes it possible to consider the problem only at $z \geqslant l$. The boundary condition for the Sturm–Liouville problem at $z = l$ emerges from the linearized Israel matching condition and has the form of the generalized Neumann condition:

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{3}{2l}\right)\varphi_m(z)\bigg|_{z=0} = 0\,. \tag{4.23}$$

A notable property of such a boundary problem is the presence of a discrete bound level or the zero mode with $m = 0$ in its spectrum. Also, the problem contains a continuous spectrum of positive masses that starts from zero ($m > 0$).

It is easy to see that the zero mode

$$\varphi_0(z) = \left(\frac{1}{l}\right)^{1/2}\left(\frac{l}{z}\right)^{3/2} \tag{4.24}$$

[3] We note that this inference is a simplified illustration of the mechanism of an exponential hierarchy between the Planckian and electroweak scales because only their ratio in the effective theory on the brane is determined by the brane position in the multidimensional space. Complete derivation must include analysis of the graviton kinetic term when its coupling to matter is normalized to unity (see Refs [11, 6] for the details). Outside the framework of such a derivation, the metric on a homogeneous brane can always be locally converted to unity by a scale transformation of the coordinates. The true nonlocal character of this mechanism is demonstrated in Section 5.1, where it is emphasized that low-energy modes on a positive-tension brane are shifted to the ultraviolet region on the other brane.

[4] This gauge is actually a combination of the gauge of the normal Gaussian coordinate system and corollaries of a part of the linearized Einstein equations (constraints) allowing metric perturbations to be chosen as transverse-traceless ones [20].

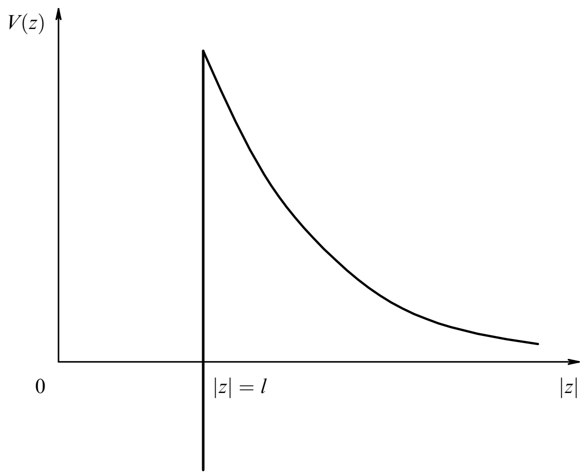

Figure 4. Potential of the stationary Schrödinger equation for the Kaluza – Klein modes of a five-dimensional graviton as a function of the modulus of the fifth coordinate, Eqn (4.19). The brane is localized at $|z| = l$ and the vertical line denotes the negative delta-shaped contribution to the potential that involves the normalizable zero mode of the massless graviton localized on the brane.

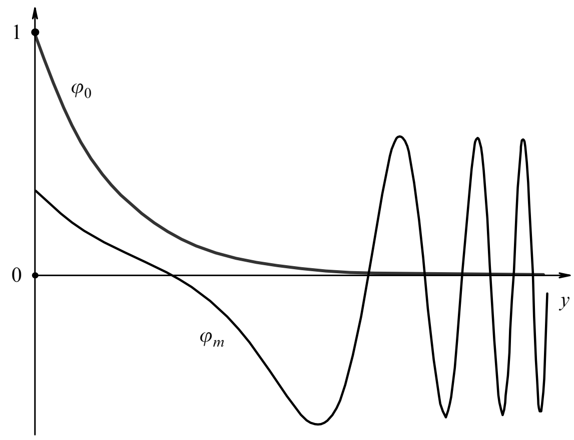

Figure 5. Behavior of the massless zero mode of the graviton φ_0 and its massive Kaluza – Klein partners φ_m $(m > 0)$ as functions of the fifth coordinate y in the neighborhood of the brane localized at $y = 0$. The plots are constructed for the values $m = 0.3$ and $l = 1$. For small masses, the KK-partners of the graviton are expelled into the bulk outside the brane.

satisfies Eqns (4.21) and (4.23) and can be normalized,

$$2\int_l^\infty \mathrm{d}z\, \varphi_0^2(z) = 1\,, \tag{4.25}$$

while massive KK-modes are expressed through the linear combinations of the second-order Bessel and Neumann functions

$$\varphi_m(z) = \left(\frac{mz}{2}\right)^{1/2} \frac{Y_1(ml)\, J_2(mz) - J_1(ml)\, Y_2(mz)}{\left(J_1^2(ml) + Y_1^2(ml)\right)^{1/2}}\,, \tag{4.26}$$

and can be normalized to the delta-function in the continuous spectrum: [5]

$$2\int_l^\infty \mathrm{d}z\, \varphi_m(z)\, \varphi_{m'}(z) = \delta(m - m')\,. \tag{4.27}$$

The appearance of a bound state in the positive potential (4.22) seems unnatural; nevertheless, it has a qualitative explanation. The fact is that boundary condition (4.23) can be simulated in the form of a negative delta-shaped contribution to this potential at the brane localization point $|z| = l$. As a result, it acquires a 'volcano'-like form (see Fig. 4) and its delta-shaped crater becomes a receptacle for the bound zero-energy level.

The qualitative behaviors of the zero mode and massive KK-modes are quite different. The zero mode is concentrated in the vicinity of $z \sim l$ near the brane, while the massive modes recede to infinity along the fifth coordinate in the form of oscillating standing waves. [6] Moreover, it follows from the asymptotic form of cylindrical functions of a small argument that the behavior of modes (4.26) on the brane in the small-mass region is given by

$$\varphi_m(l) \sim \left(\frac{ml}{2}\right)^{1/2}, \quad m \to 0\,. \tag{4.28}$$

In other words, infrared modes of the continuous Kaluza – Klein spectrum, unlike the zero mode, are 'expelled' from the brane (Fig. 5). This phenomenon is called localization of massless gravitons; it is responsible for the restoration of the effective four-dimensional gravity theory. The carrier of the four-dimensional interaction on the brane is a brane-localized massless graviton. The tower of its massive KK-partners is weakly coupled to matter on the brane, which accounts for only weak corrections in the low-energy region due to extra dimensions [12, 15].

To see this, we consider a nonrelativistic law of attraction between two particles localized on a brane. It is given by the Green's function of the four-dimensional Laplacian, i.e., the spatial part of the operator of the wave equation for five-dimensional gravitons (4.20),

$${}^4\Delta = \frac{z^{3/2}}{l^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + {}^3\Delta - \frac{15}{4z^2}\right) z^{1/2}\,, \tag{4.29}$$

acting in the bulk spatial coordinates $\mathbf{X} = (\mathbf{x}, y)$. (In its turn, $\mathbf{x} = x^m$ $(m = 1, 2, 3)$ denotes a set of spatial coordinates on the brane and ${}^3\Delta = \delta^{mn}\,\partial_m\partial_n$ is the three-dimensional 'flat' Laplacian in these coordinates.) The attraction potential in question is proportional to the intrabrane (from brane to brane) Green's function of an elliptic operator in the case where both its points lie on the brane,

$$G_5 D(\mathbf{x}, \mathbf{x}') = G_5\, \frac{1}{{}^4\Delta}\, \delta(\mathbf{X}, \mathbf{X}')\bigg|_{y = y' = 0}\,, \tag{4.30}$$

and to the five-dimensional coupling constant.

The four-dimensional delta-function in (4.30),

$$\delta(\mathbf{X}, \mathbf{X}') \equiv {}^3\delta(\mathbf{x}, \mathbf{x}')\, \delta(y - y') = {}^3\delta(\mathbf{x}, \mathbf{x}')\, \delta(z - z')\, \frac{z}{l}\,, \tag{4.31}$$

[5] The coefficient 2 in (4.27) effectively takes the contribution of integration over $z \leqslant -l$ and the Z_2-symmetry of the modes into account.

[6] The requirement of nonsingularity $\varphi_m(z)$ at infinity excludes imaginary values of the mass and therefore forbids tachyonic modes in the KK-spectrum with $m^2 < 0$.

can be expanded in the complete set of plane waves in the $\mathbf{x}$-space and the complete set of z-harmonics that includes the discrete zero mode and the continuous spectrum of massive modes:

$$\delta(z - z') = \sum_m \varphi_m(z)\,\varphi_m(z') \equiv \varphi_0(z)\,\varphi_0(z') + \int_0^\infty \mathrm{d}m\;\varphi_m(z)\,\varphi_m(z')\,. \tag{4.32}$$

Substituting this expansion in (4.30) and taking into account that $\varphi_m(z)$ are the eigenfunctions of problem (4.21) leads to

$$G_5 D(\mathbf{x}, \mathbf{x}') = \frac{G_5 l}{(2\pi)^3}\int \mathrm{d}^3\mathbf{p}\,\sum_m \frac{\exp\left(\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}')\right)}{\mathbf{p}^2 + m^2}\,\frac{\varphi_m(z)\,\varphi_m(z')}{\sqrt{zz'}}\bigg|_{z=z'=l} = -\frac{G_5}{4\pi r}\,\varphi_0^2(l) - \frac{G_5}{4\pi r}\int_0^\infty \mathrm{d}m\;\varphi_m^2(l)\exp(-mr)\,, \tag{4.33}$$

$$r = |\mathbf{x} - \mathbf{x}'|\,.$$

Furthermore, taking (4.24) and (4.28) into consideration and integrating over masses in the large-distance asymptotic regime finally yields

$$G_5 D(\mathbf{x}, \mathbf{x}') = -\frac{G_4}{4\pi r}\left(1 + \frac{l^2}{2r^2} + \ldots\right), \qquad r \gg l\,, \tag{4.34}$$

where G_4 plays the role of the effective four-dimensional gravitational constant:

$$G_4 = \frac{G_5}{l}\,. \tag{4.35}$$

It can be concluded that up to small corrections caused by the contribution of massive KK-modes, the attraction between bodies on a brane in the large-distance region ($r \gg l$) is governed by the four-dimensional Newton law. In this case, as with the ADD model in (3.6), the effective gravitational constant is given by a combination of the fundamental five-dimensional constant and the extra-dimension scale. The role of this extra dimension is played, as mentioned earlier, by the cosmological radius of the five-dimensional AdS geometry. These properties are guaranteed by two important aspects of the AdS bulk in the Randall–Sundrum model, i.e., brane localization of the graviton zero mode and pushing its light KK-partners outside the brane. The zero mode reproduces the four-dimensional law, while the smallness of the amplitudes of light massive modes (4.28) accounts for their small contribution despite the continuity of their spectra and the absence of a gap separating them from the discrete massless state $m = 0$ [12, 15].

4.3 The role of radion

A similar situation is realized with the relativistic interaction law. As in the previous formulas, the retarded five-dimensional brane-to-brane propagator of the problem in (4.16) and (4.17) is essentially given by the four-dimensional propagator with small corrections,

$$D^{\mathrm{ret}}(X, X')\Big|_{y=y'=0} = \frac{1}{l}\,\frac{1}{\Box_4}\,\delta(x, x')\Big|^{\mathrm{ret}}, \quad (x - x')^2 \gg l^2\,. \tag{4.36}$$

But in the relativistic region, the tensor structure of gravitational potentials acquires importance, and they no longer reduce to its Newton 00-component.

The retarded potential arising from the redistribution of matter $T_{\mu\nu}$ on the brane is obtained by the action of the four-dimensional propagator on the right-hand side of the Israel matching conditions in the presence of matter,

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} + \frac{2}{l}\right) h_{\mu\nu}(x, y)\Big|_{y=0} = -8\pi G_5\left(T_{\mu\nu} - \frac{1}{3}\,\eta_{\mu\nu} T\right) \tag{4.37}$$

(a solution of the wave problem with the Neumann boundary condition at the time-like boundary of a five-dimensional space). With (4.36) taken into account in the long-wave approximation, this implies that

$$h_{\mu\nu}(x) \simeq -16\pi G_4\,\frac{1}{\Box}\left(T_{\mu\nu} - \frac{1}{3}\,\eta_{\mu\nu} T\right)(x)\,. \tag{4.38}$$

This cannot be a correct generally relativistic law of gravitational wave emission because the coefficient $1/3$ on the right-hand side of the equation is wrong. In the four-dimensional Einstein gravity, this coefficient equals $1/2$.

As shown in Refs [17, 18], such a discrepancy is resolved if it is borne in mind that this metric is not induced on the brane and the left-hand side of boundary condition (4.37) must contain a metric in the *normal Gaussian system of coordinates* $h^{\mathrm{NG}}_{\mu\nu}(x, y)$ instead of the metric $h_{\mu\nu}(x, y)$ in Randall–Sundrum gauge (4.15). In this gauge, the brane is not the plane $y = 0$ and its embedding in the (x^μ, y) space is given by a set of a scalar field (radion) $\Pi(x)$,

$$y = \Pi(x)\,, \tag{4.39}$$

and a vector field $\xi^\mu(x)$ (which distinguishes the four-dimensional coordinates on the brane from coordinates in the bulk). Hence [17],

$$h^{\mathrm{NG}}_{\mu\nu}(x, y) = h_{\mu\nu}(x, y) + \frac{2}{l}\,\eta_{\mu\nu} a^2(y)\,\Pi(x) + l\,\partial_\mu\partial_\nu \Pi(x) + a^2(y)\,(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)(x)\,. \tag{4.40}$$

Substituting expression (4.40), instead of $h_{\mu\nu}(x, y)$, into the left-hand side of Eqn (4.37) and calculating the trace, owing to the tracelessness of $h_{\mu\nu}$, we obtain the equation for $\Pi(x)$:

$$\Box\,\Pi(x) = \frac{8\pi G_5}{6}\,T(x)\,. \tag{4.41}$$

This equation indicates that the radion field responsible for brane embedding in the bulk is determined by the trace of the matter stress tensor that 'bends' the brane.

Equation (4.40) implies that the brane-induced metric differs from (4.38) by the conformal term

$$\frac{2}{l}\,\eta_{\mu\nu}\,a^2(y)\,\Pi(x)$$

and by a set of pure-gauge terms of the form $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. The latter can be eliminated by the proper choice of the four-dimensional system of coordinates, whereas the right-hand side of the equation for $h^{\mathrm{NG}}_{\mu\nu}(x, y)$ is modified, due to the conformal term with radion field (4.41), by an additional contribution:

$$h_{\mu\nu}^{\rm NG}(x,y) = h_{\mu\nu}(x,y) + \frac{8\pi G_5}{3l}\, a^2(y)\,\eta_{\mu\nu}\,\frac{1}{\Box}\,T(x)\,. \qquad (4.42)$$

Therefore, the right-hand side of the equation for the induced metric $h_{\mu\nu}^{\rm ind}(x) = h_{\mu\nu}^{\rm NG}(x,0)$ acquires the generally relativistic combination of the energy-momentum tensor and its trace with the coefficient 1/2:

$$h_{\mu\nu}^{\rm ind}(x) \simeq -16\pi G_4\,\frac{1}{\Box}\left(T_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}T\right)(x)\,. \qquad (4.43)$$

It is easy to show that the conservation law for the matter stress tensor (which holds in the zero-mode approximation because the leakage of matter into the bulk is mediated by massive KK-modes) ensures the harmonic gauge of the metric in this equation:

$$\partial^\nu h_{\mu\nu}^{\rm ind} - \frac{1}{2}\,\partial_\mu h^{\rm ind} = 0\,.$$

Thus, the radion contribution restores the correct four-dimensional law of propagation of long-wave gravitational waves emitted by matter on the brane. Evidently, this law, by analogy to Newton's law, holds for wavelengths larger than the AdS background scale l.

4.4 The AdS/CFT correspondence principle and graviton localization

Localization of gravitons in the Randall – Sundrum model is a manifestation of the so-called AdS/CFT correspondence principle in string theory [22 – 24]. This principle establishes a duality relation (equivalence) between the theories formulated in the bulk and on its boundary and in the opposite coupling (weak and strong) regimes, respectively [21]. Specifically, a remarkable property of this duality is that the semiclassical weakly coupled theory in a multidimensional bulk gives rise on its boundary to a quantum theory in the strong coupling regime.

The AdS/CFT correspondence was suggested as a hypothesis for the IIB supergravity theory in a ten-dimensional space (which is a direct product $\mathrm{AdS}_5\times \mathrm{S}^5$ of the five-dimensional anti-de Sitter space and the five dimensional sphere) and for the four-dimensional conformally invariant $\mathrm{U}(N)$ Yang – Mills theory with extended $\mathcal{N}=4$ supersymmetry defined on the four-dimensional boundary of AdS_5. Parameters of these two theories including five-dimensional gravitational G_5 and cosmological $\Lambda_5 = -6/l^2$ constants on the supergravity side and the 't Hooft constant $\lambda \equiv g_{\rm YM}^2 N$ on the Yang – Mills theory side (with the coupling constant $g_{\rm YM}$) in the limit of large λ and N are related by

$$\frac{l^3}{G_5} = \frac{2N^2}{\pi}\,, \quad \lambda \equiv g_{\rm YM}^2 N = \left(\frac{l}{l_{\rm s}}\right)^4, \quad \lambda\to\infty\,, \quad N\to\infty \qquad (4.44)$$

($l_{\rm s}$ is the string length in the IIB superstring theory that generates the ten-dimensional supergravity in the field-theory limit).

It can be seen that limit (4.44) establishes a correspondence between the low-energy small-curvature regime in the supergravity theory $\Lambda_5 = -6/l^2 \to 0$ and the nonperturbative strong coupling regime $\lambda\to\infty$ in conformal field theory. The quantitative verification of this correspondence is very difficult due to its nonperturbative nature and was originally performed in the framework of the perturbation theory only for a distinguished class of supersymmetry-protected correlators. For this reason, the present discussion is confined to the demonstration that the tree approximation for a multidimensional theory in the bulk can give rise to essentially quantum contributions of the dual theory formulated on the boundary; specifically, it may involve the phenomenon of graviton localization on the brane (interpreted as the boundary of the AdS bulk).

For this purpose, the AdS/CFT correspondence should be formulated in terms of the effective action induced on the brane (the boundary of a multidimensional space) and obtained by integration over fields in the bulk. For simplicity, we work in the Euclidean variant of the theory linked to the theory in the Lorentzian space – time by the standard Wick rotation [7] such that the Lorentzian and Euclidean actions are related by the transformation $\mathrm{i}S^{\rm L} = -S^{\rm E}$.

The effective action as a functional of the four-dimensional metric $g_{\mu\nu}(x)$ is derived from the five-dimensional theory with the action $S_5\big[G_{AB}(X)\big]$ by means of functional integration over a class of five-dimensional metrics $G_{AB}(X)$ that induce the following four-dimensional metric on the boundary of the five-dimensional manifold ∂M_5:

$$\exp\left(-S_{\rm eff}\big[g_{\mu\nu}(x)\big]\right) = \int {\rm D}G_{AB}(X)\,\exp\left(-S_5\big[G_{AB}(X)\big]\right)\Big|_{G_{\mu\nu}(\partial M_5)=g_{\mu\nu}(x)}\,. \qquad (4.45)$$

In the tree-level approximation, this action reduces to the five-dimensional action

$$S_{\rm eff}[g_{\mu\nu}] = S_5\big[G_{AB}[g_{\mu\nu}]\big] + O\left(\frac{1}{N^2}\right), \qquad (4.46)$$

calculated at the solution $G_{AB}[g_{\mu\nu}]$ of the classical gravitational equations of motion in the bulk with the boundary conditions at the boundary in the form of the fixed induced metric:

$$\frac{\delta S_5[G_{AB}]}{\delta G_{AB}(X)} = 0\,, \qquad (4.47)$$

$$G_{\mu\nu}(\partial M_5) = g_{\mu\nu}(x)\,. \qquad (4.48)$$

According to (4.44), $1/N^2 \sim G_5/l^3 \to 0$ plays the role of the Planck constant of the semiclassical expansion.

It follows from the AdS/CFT hypothesis that the result of the calculations is the functional

$$S_5\big[G_{AB}[g_{\mu\nu}]\big] = \Gamma_4[g_{\mu\nu}]\,. \qquad (4.49)$$

This functional is the quantum effective action of the four-dimensional theory whose qualitative structure begins with the Einstein term incorporating the effective gravitational

[7] Consideration of the AdS/CFT correspondence in the Euclidean space is dictated by the fact that it is based on the Graham – Fefferman mathematical construction for the reconstruction of the solution of the Einstein equations in the bulk from the metric asymtptotics near its boundary (being moved to conformal infinity), which is also originally formulated and strictly proven in the Euclidean signature [25].

constant G_4 and the typical loop logarithmic contribution quadratic in the curvature:

$$\Gamma_4[g_{\mu\nu}] \sim \int d^4x\, g^{1/2}\left(-\frac{1}{16\pi G_4}\, R + \beta\, R_{\mu\nu} \ln \frac{\Box}{\mu^2}\, R^{\mu\nu} + \ldots\right), \tag{4.50}$$

with a certain specific value of the ultraviolet cut-off μ^2 and beta function β related to the conformal anomaly of the theory.

An essential feature of correspondence (4.49) is that the ultraviolet-finite tree functional proves to be equal to the quantum effective action of the local theory that requires renormalization and shows an explicit cut-off dependence. It turns out that its ultraviolet divergences on the right-hand side of relation (4.49) are regulated by the position of the four-dimensional boundary tending to infinity of the five-dimensional AdS space. In other words, the infrared cut-off in tree supergravity ensures ultraviolet renormalization of the quantum conformal theory at the boundary. Such a duality of ultraviolet–infrared renormalizations is achieved by introducing a number of counterterms that are responsible for graviton localization in the Randall–Sundrum model. In the bosonic sector of the theory, it occurs as is schematically described below.

The AdS/CFT correspondence can be formulated in the effective action language as the equality

$$\exp\left(-W_{\rm SUGRA}[g_{\mu\nu}]\right) = \exp\left(-W_{\rm CFT}[g_{\mu\nu}]\right) \tag{4.51}$$

of two different generating functionals [22, 23]:

$$\exp\left(-W_{\rm SUGRA}[g_{\mu\nu}]\right) \equiv \int DG_{AB}\, \exp\left(-S_{\rm grav}[G_{AB}]\right)\Big|_{G_{\mu\nu}\to g_{\mu\nu},\, y\to-\infty}, \tag{4.52}$$

$$\exp\left(-W_{\rm CFT}[g_{\mu\nu}]\right) \equiv \int D\phi\, \exp\left(-S_{\rm CFT}[\phi,\, g_{\mu\nu}]\right). \tag{4.53}$$

Functionals (4.52) and (4.53) are defined, respectively, for the gravitational theory in the asymptotically de Sitter bulk M_5 [parameterized by Poincaré coordinates $X^A = (x^\mu, y)$] and for the conformal field theory on its boundary

$$\partial M_5:\ \ y = \text{const} \to -\infty\,, \tag{4.54}$$

parameterized by the coordinates x^μ. The four-dimensional conformal Yang–Mills field denoted symbolically by ϕ 'lives' in the background metric $g_{\mu\nu}(x)$ at boundary (4.54). The same metric serves as the boundary condition for a class of five-dimensional metrics over which the integration is performed in (4.52).

The action of the field $\phi = A_\mu$ at the boundary (bosonic part of the supersymmetric model) is constructed in accordance with the standard rules of generally covariant minimal coupling to the metric,

$$S_{\rm CFT}[\phi, g_{\mu\nu}] = \frac{1}{4g_{\rm YM}^2}\int_{\partial M_5} d^4x\, g^{1/2}\,{\rm Tr}\, F_{\mu\nu}^2\,, \tag{4.55}$$

whereas the gravitational action in the bulk requires a more sophisticated procedure dictated by the infrared properties of the AdS space. The fact is that at the tree level, upon substitution in the Einstein–Hilbert action (with the Gibbons–Hawking surface term) in accordance with (4.46),

$$S_5[G_{AB}] = -\frac{1}{16\pi G_5}\int_{M_5} d^5X\, G^{1/2}\left({}^5R(G_{AB}) + \frac{12}{l^2}\right) - \frac{1}{8\pi G_5}\int_{\partial M_5} d^4x\, g^{1/2} K\,, \tag{4.56}$$

the solution of the Einstein equations with the given induced metric on the boundary already leads to infinities as the boundary is moved to the asymptotic region of the AdS bulk.

For elimination of the infinities, the complete gravitational action in (4.52),

$$S_{\rm grav}[G_{AB}] = S_5[G_{AB}] + S_1[g_{\mu\nu}] + S_2[g_{\mu\nu}] + S_3^\epsilon[g_{\mu\nu}]\,, \tag{4.57}$$

must contain, besides (4.56), a set of three counterterms — functions of the metric induced at the boundary, of the zeroth, first, and second order in the curvature [25–27]:

$$S_1[g_{\mu\nu}] = \frac{3}{8\pi G_5 l}\int_{\partial M_5} d^4x\, g^{1/2}\,, \tag{4.58}$$

$$S_2[g_{\mu\nu}] = \frac{l}{32\pi G_5}\int_{\partial M_5} d^4x\, g^{1/2} R\,, \tag{4.59}$$

$$S_3^\epsilon[g_{\mu\nu}] = -\ln\epsilon\,\frac{l^3}{64\pi G_5}\int_{\partial M_5} d^4x\, g^{1/2}\left(R_{\mu\nu}^2 - \frac{1}{3}\, R^2\right). \tag{4.60}$$

Counterterm (4.60) explicitly contains a logarithmically divergent factor $\ln\epsilon \to -\infty$ corresponding to the displacement of the boundary to infinity:

$$\epsilon \to 0 \leftrightarrow y \to -\infty\,, \tag{4.61}$$

while its coefficient coincides, owing to relations (4.44), with the beta function in the $\mathcal{N} = 4$ SU(N) superconformal Yang–Mills field theory (also defined by the conformal anomaly of the model) [28]:

$$\frac{l^3}{64\pi G_5} = \frac{N^2}{32\pi^2} = \beta_{\rm CFT}\,. \tag{4.62}$$

It is important that the coefficient in cosmological term (4.58) coincides with half the brane tension in Randall–Sundrum model (4.7), $\sigma/2$, and the coefficient in Einstein term (4.59) is determined by the effective four-dimensional gravitational constant (4.35), $1/32\pi G_4$.

We now apply the AdS/CFT duality relation (4.51) to the Randall–Sundrum quantum model. In the quantum version of this model, the entire five-dimensional volume is divided by the brane into two half-spaces, $M_5 = M_5^+ \cup M_5^-$, with the five-dimensional metrics $G_{AB}^\pm$ that induce the same four-dimensional metric $G_{\mu\nu}^\pm(\partial M_5^\pm) = g_{\mu\nu}(x)$ on the brane $\Sigma = \partial M_5^\pm$ (the boundary between them) (Fig. 6). We note that unlike the background Randall–Sundrum solution, the quantum metrics $G_{AB}^\pm$, as independently integrated fields, do not satisfy the Z_2-symmetry condition.

The complete gravitational action in Randall–Sundrum model (4.1) is the sum of five-dimensional Hilbert–Einstein actions (4.56) on these half-spaces $S_5[G_{AB}^\pm]$ and the four-dimensional action of the brane that consists of the term with tension written as the doubled value of the counterterm $2S_1[g_{\mu\nu}]$ in (4.58):

$$S_{\rm RS}[G_{AB}] = S_5[G_{AB}^+] + S_5[G_{AB}^-] + 2S_1[g_{\mu\nu}]\,. \tag{4.63}$$

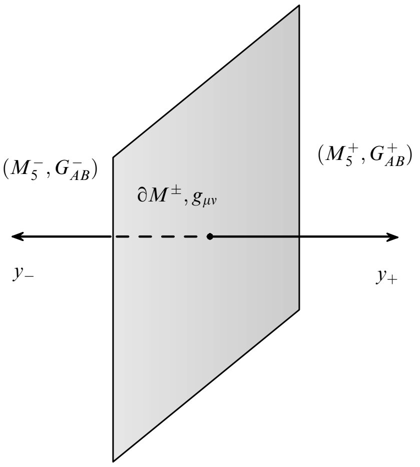

Figure 6. The picture of five-dimensional space – time in a quantum variant of the Randall – Sundrum model. The five-dimensional bulk $M_5 = M_5^+ \cup M_5^-$ is divided by the brane $\Sigma = \partial M_5^{\pm}$ into two half-spaces with independent five-dimensional metrics $G_{AB}^{\pm}$ that induce the same four-dimensional metric $G_{\mu\nu}^{\pm}(\partial M_5^{\pm}) = g_{\mu\nu}(x)$ on the brane, the argument of the effective brane action.

(We do not include the brane matter action here. In what follows, it always enters the complete effective action additively and is not involved in integration over the five-dimensional metric.)

Thus, the effective brane action in the Randall – Sundrum model is defined by the functional integral

$$\exp\left(-S_{\rm eff}\left[g_{\mu\nu}(x)\right]\right) = \int \mathrm{D}G_{AB}^+\,\mathrm{D}G_{AB}^-\,\exp\left(-S_{\rm RS}\left[G_{AB}(X)\right]\right)\bigg|_{G_{\mu\nu}^{\pm}(\Sigma)=g_{\mu\nu}(x)}. \qquad (4.64)$$

Because action (4.63) is the sum of the contributions of two half-spaces, the complete metric integral is factored into the product of two identical integrals over metrics $G_{AB}^{\pm}$ that account for the effective Z_2-symmetry:

$$\int \mathrm{D}G_{AB}^+\,\mathrm{D}G_{AB}^-\,\exp\left(-S_5[G_{AB}^+] - S_5[G_{AB}^-]\right) = \left(\int \mathrm{D}G_{AB}\,\exp\left(-S_5[G_{AB}]\right)\right)^2. \qquad (4.65)$$

The squared integral in (4.65) is taken over a class of five-dimensional metrics with the fixed induced metric at the boundary of the asymptotically AdS space; this exactly corresponds to the statement of the problem in the formulation of the AdS/CFT correspondence. Therefore, in accordance with (4.51) – (4.53), integral (4.65) is expressed in terms of the effective action of the conformal field theory:

$$\int \mathrm{D}G_{AB}\,\exp\left(-S_5[G_{AB}]\right) = \exp\left(S_1+S_2+S_3\right)\int \mathrm{D}G_{AB}\,\exp\left(-S_{\rm grav}[G_{AB}]\right) = \exp\left(S_1+S_2+S_3-W_{\rm CFT}^{\epsilon}\right). \qquad (4.66)$$

By taking the square of this expression in (4.65) and substituting the result in (4.64), it can be shown that the contributions of the four-dimensional cosmological term $2S_1$ cancel each other and the effective brane action takes the form

$$S_{\rm eff} = -2S_2 + 2W_{\rm CFT}^{\epsilon} - 2S_3. \qquad (4.67)$$

In accordance with (4.59), the contribution of the counterterm $-2S_2$ to (4.67) gives rise to the exactly Einsteinian term with the effective gravitational constant in (4.35). As regards the effective action of the superconformal Yang – Mills field, it contains neither cosmological nor Einsteinian terms owing to supersymmetry; in the low-energy region of small curvatures, it starts with the logarithmically divergent term ($\epsilon \to 0$),

$$2W_{\rm CFT}^{\epsilon}\left[g_{\mu\nu}(x)\right] = \int \mathrm{d}^4x\, g^{1/2}\left[2\beta_{\rm CFT}\,R_{\mu\nu}\left(-\ln\epsilon + \ln(l^2\Box)\right)R^{\mu\nu} + \ldots\right], \qquad (4.68)$$

quadratic in the Ricci tensor and accompanied by a logarithmically nonlocal form factor. The coefficient at this term is determined either by the beta function or by the conformal anomaly of the theory, Eqn (4.62), and coincides with the coefficient in counterterm (4.60).

Consequently, if the ultraviolet regularization parameter ϵ in (4.68) is identified with the infrared regularization parameter in (4.60), the difference $2W_{\rm CFT}^{\epsilon} - 2S_3$ entering (4.67) is finite. In this case, the effective brane action of the Randall – Sundrum model in the low-energy region assumes the final form [8]

$$S_{\rm eff}\left[g_{\mu\nu}(x)\right] = \int \mathrm{d}^4x\, g^{1/2}\left(-\frac{1}{16\pi G_4}\,R + \frac{l^2}{32\pi G_4}\,R_{\mu\nu}\ln(l^2\Box)\,R^{\mu\nu} + \ldots\right). \qquad (4.69)$$

To conclude, the Randall – Sundrum model characterized by constants (G_5, l) is dual to the four-dimensional Einstein gravity with Newton's constant $G_4 = G_5/l$ coupled to a regularized superconformal field theory. This finding confirms the localization of the massless graviton in the five-dimensional Randall – Sundrum model: its effective action on a brane in the long-wave region (or small-curvature region) reproduces the four-dimensional Einstein theory with effective constant (4.35) (in the absence of a cosmological term) and short-distance corrections noticeable in the scale range $1/\sqrt{\Box} \ll l$. It can be shown that the nonlocal logarithmic corrections in (4.69) give rise to the corrections for Newton's law obtained in (4.34) by taking KK-modes of the five-dimensional model into consideration [18].

5. The two-brane Randall – Sundrum model

Today, the AdS/CFT correspondence principle is a hypothesis that is very difficult to verify because of the nonperturbative nature of the problem in the strong coupling region. As noted above, the AdS/CFT principle was first suggested based on purely algebraic symmetry considerations [21] and verified in the framework of the perturbation theory for the lowest correlation functions protected by supersymmetry from radiation corrections [29]. In recent years, this principle

[8] We recall that in this section, we work with the Euclidean signature, which accounts for the negative scalar curvature coefficient in the Euclidean action.

has been confirmed using exactly solvable string models in a plane-wave metric [30] and a special class of composite operators forming an integrable system [31].

In any case, the status of the AdS/CFT correspondence principle and the scope of its applicability remain to be clarified. For this reason, we disregard it in this section and instead reproduce the results of the preceding section by means of direct calculations in the framework of the perturbation theory in powers of the space – time curvature. Specifically, we consider the effective brane action in the two-brane Randall – Sundrum model, compute it in the form of a covariant curvature expansion, and demonstrate that at a finite interbrane distance, this model generates, in the low-energy limit, a gravitational theory of the Brans – Dicke type; this theory suggests an inflation mechanism with a nonminimally coupled inflaton, i.e., a radion field describing interbrane distance dynamics. We show that this model is rich in phase transitions as branes are moving apart, and this process may be associated with the evolution of the early and modern universe (from the standpoint of the brane paradigm in cosmology).

We recall that in accordance with this paradigm, our observable low-energy world is a four-dimensional brane embedded in a multidimensional space with macroscopic extra dimensions. Details of this embedding and interactions of the brane with other potential brane worlds govern the evolution of our universe. Brane localization of massless gravitons and restoration of the four-dimensional Einstein gravity theory eliminate the main contradiction that existed in the framework of the old Kaluza – Klein picture, that is, the impossibility of observing extra dimensions.

5.1 Effective action in the two-brane model

We consider the two-brane Randall – Sundrum model briefly described in Section 4. The action of this model is given by

$$S\big[G_{AB}(X)\big] = \frac{1}{16\pi G_5}\int \mathrm{d}^5 X\, G^{1/2}\big({}^5R(G) - 2\Lambda_5\big) + \sum_I \int_{\Sigma_I} \mathrm{d}^4 x\, g^{1/2}\left(\frac{1}{8\pi G_5}[K] - \sigma_I\right), \tag{5.1}$$

where the index $I = \pm$ labels two branes with tensions $\sigma_\pm$ and $[K]$ is a jump of the extrinsic curvature trace in the Gibbons – Hawking surface term associated with either side of each brane.

We recall that the branes are localized at antipodal points of a circle in the fifth dimension parameterized by the fifth coordinate:

$$y = y_\pm\,, \quad y_+ = 0\,, \quad |y_-| = d\,.$$

Z_2-symmetry identifies points y and $-y$ on the circle and leaves the points $y_\pm$ fixed. When brane tensions are chosen depending on the negative cosmological ($\Lambda_5 = -6/l^2$) and the five-dimensional gravitational (G_5) constants in agreement with (4.7) and (4.8), this model (in the absence of matter on the branes and in the bulk) allows a solution with AdS metric (4.6) in the bulk and the conformally flat metric $a^2(y_\pm)\eta_{\mu\nu}$ on both branes.

The metric on the negative-tension brane contains the scale factor $a^2(d) = \exp(-2d/l)$, which, as mentioned earlier, allows solving the hierarchy problem [11]. With a proper fine tuning of brane tensions (4.7) and (4.8), this solution exists for an arbitrary distance d between the branes; the two conformally flat branes remain in equilibrium.

We now consider the case where the brane-induced negative-tension metrics differ from the background values by small perturbations,

$$g^\pm_{\mu\nu}(x) = a^2_\pm \eta_{\mu\nu} + h^\pm_{\mu\nu}(x)\,, \tag{5.2}$$

$$a_+ = 1\,, \quad a_- = \exp\left(-\frac{2d}{l}\right) \equiv a\,, \tag{5.3}$$

which induce a perturbed solution of the Einstein equations in the bulk,

$$\mathrm{d}s^2 = \mathrm{d}y^2 + \exp\left(-\frac{2|y|}{l}\right)\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + h_{AB}(x,y)\,\mathrm{d}x^A\,\mathrm{d}x^B\,. \tag{5.4}$$

We then calculate the tree-level effective action on the branes in the quadratic approximation in $h^\pm_{\mu\nu}(x)$. This can be achieved by solving a linearized variant of boundary value problem (4.47), (4.48) generalized to the case of two boundaries with metrics (5.2) and by substituting this solution into five-dimensional action (5.1) in the above approximation.

The resulting action is invariant under two four-dimensional differomorphisms on the branes that in the linearized approximation amount to the transformations of metric perturbations

$$h^\pm_{\mu\nu} \to h^\pm_{\mu\nu} + \partial_\mu \xi^\pm_\nu + \partial_\nu \xi^\pm_\mu \tag{5.5}$$

with two independent vector fields $\xi^\pm_\mu = \xi^\pm_\mu(x)$. Consequently, the action can be expressed in terms of tensor invariants of the transformations or linearized Ricci tensors for $h^\pm_{\mu\nu}(x)$,

$$R^\pm_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_\lambda\partial_\mu h^\lambda_\nu + \partial_\lambda\partial_\nu h^\lambda_\mu - \partial_\mu\partial_\nu h\right)^\pm\,, \tag{5.6}$$

on the flat four-dimensional background of both branes. The result obtained with the help of this procedure in [32] or by functional integration of the effective four-dimensional equations of motion for metric perturbations on the branes [20] is given by the space – time integral of a nonlocal 2×2 quadratic form:

$$S_{\mathrm{eff}}[g^\pm_{\mu\nu}] = \frac{1}{16\pi G_4}\int \mathrm{d}^4 x\left[\mathbf{R}^{\mathrm{T}}_{\mu\nu}\,\frac{2\mathbf{F}(\Box)}{l^2\Box^2}\,\mathbf{R}^{\mu\nu} - \frac{1}{6}\,\mathbf{R}^{\mathrm{T}}\,\frac{2\mathbf{F}_1(\Box)}{l^2\Box^2}\,\mathbf{R}\right]. \tag{5.7}$$

Here, G_4 is the effective four-dimensional gravitational constant ($G_4 = G_5/l$), and $\mathbf{R}^{\mu\nu}$ and $\mathbf{R}^{\mathrm{T}}_{\mu\nu}$ are two-dimensional columns

$$\mathbf{R}_{\mu\nu} = \begin{bmatrix} R^+_{\mu\nu}(x) \\ R^-_{\mu\nu}(x) \end{bmatrix} \tag{5.8}$$

and rows

$$\mathbf{R}^{\mathrm{T}}_{\mu\nu} = \big[R^+_{\mu\nu}(x)\ R^-_{\mu\nu}(x)\big] \tag{5.9}$$

of Ricci curvatures associated with the two branes.

The nonlocal form factors in Eqn (5.7) are 2×2 matrix-valued functions of the d'Alembertian $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ acting on the curvatures and expressed through the fundamental operator

$$\mathbf{F}(\Box) = -\frac{1}{J_2^+ Y_2^- - J_2^- Y_2^+} \times \begin{bmatrix} \sqrt{\Box}\, z_+ u_+(z_-) & -\dfrac{2}{\pi} \\ -\dfrac{2}{\pi} & \sqrt{\Box}\, z_- u_-(z_+) \end{bmatrix}, \tag{5.10}$$

which is constructed in terms of the basis functions of the variable $z > 0$ [replacing the fifth coordinate y in accordance with (4.19)]

$$u_\pm(z) = Y_1^\pm J_2(z\sqrt{\Box}) - J_1^\pm Y_2(z\sqrt{\Box}), \tag{5.11}$$

$$J_\nu^\pm \equiv J_\nu(z_\pm\sqrt{\Box}), \quad Y_\nu^\pm \equiv Y_\nu(z_\pm\sqrt{\Box}). \tag{5.12}$$

These functions are in turn composed of cylindrical Bessel, J_ν, and Neumann, Y_ν, $\nu = 1, 2$ functions of $z\sqrt{\Box}$.[9] The second form factor is given by

$$\mathbf{F}_1(\Box) = 2\mathbf{F}(\Box) - l^2\Box\, \mathrm{diag}\left[-\frac{1}{2}, \frac{1}{2a^2}\right]. \tag{5.13}$$

Zero modes (basis functions) of nonlocal operators correspond to propagating modes of the theory. For example, in the transverse-traceless (graviton) sector, the modes $\mathbf{v}_n(x) = \mathbf{v}_{n\mu\nu}(x)$ are determined by the properties of the operator $\mathbf{F}(\Box)$. They can be represented as a pair of two-dimensional columns, whose dimensionality corresponds to the number of branes,

$$\mathbf{v}_n(x) = \begin{bmatrix} v_n^+(x) \\ v_n^-(x) \end{bmatrix}, \tag{5.14}$$

and the modes satisfy the equation

$$\mathbf{F}(\Box)\,\mathbf{v}_n = 0. \tag{5.15}$$

The condition of the existence of null vectors of the matrix-valued operator

$$\det \mathbf{F}(\Box) = 0 \tag{5.16}$$

serves to determine masses m_n of the propagating modes,

$$(\Box - m_n^2)\,\mathbf{v}_n(x) = 0, \tag{5.17}$$

which are therefore the roots $\Box = m_n^2$ of characteristic equation (5.16). For operator (5.10), this equation has the form

$$\det \mathbf{F}(\Box) \sim \Box(Y_1^- J_1^+ - Y_1^+ J_1^-) = 0 \tag{5.18}$$

and gives the discrete spectrum of KK-modes in the theory[10] starting from a massless mode with $\Box = m_0^2 \equiv 0$. In this case, the KK-mass spectrum is discrete because the fifth dimension is compact.

Equation (5.18) guarantees the existence of a massless graviton in the theory spectrum, and its localization on the positive-tension brane means the restoration of the four-dimensional Einstein gravity in the low-energy region. This can be demonstrated by considering the long-wave limit for nonlocal form factors in effective action (5.7) satisfying the constraints

$$l\sqrt{\Box} \ll 1, \quad \frac{l\sqrt{\Box}}{a} \ll 1. \tag{5.19}$$

The first condition defines the low-energy region on the positive-tension brane (the so-called Planckian brane) and the second a similar region on the other brane. (It should be borne in mind that the energy of physical modes on branes is determined using their four-dimensional metric: $\sqrt{g_\pm^{\mu\nu}\partial_\mu\partial_\nu} = \sqrt{\Box}/a_\pm$.)

The results of analysis reported in Ref. [20] illustrate the difficulty of the description of the low-energy theory in terms of two fields. The difficulty occurs because the low-energy region on the Planckian brane, Σ_+, corresponds to high energies on Σ_-, especially in the limit of infinite interbrane distance $a = \exp(-d/l) \to 0$. We therefore postpone the discussion of the two-field effective action until Section 5.4, where gravitational-wave effects from matter sources on both branes are considered. In the meantime, we turn to the reduced effective action in terms of a single metric field of the Planckian brane. This gives us the opportunity to explicitly demonstrate the restoration of the four-dimensional Brans – Dicke theory on the brane (the Einstein theory nonminimally coupled to an additional scalar field), freely analyze the limit as $a \to 0$, and show the realization of the AdS/CFT correspondence in this limit.

5.2 Local and nonlocal phases of the model

We consider the reduction of the brane effective action obtained by averaging over the degrees of freedom on the positive-tension brane (or functional integration over its four-dimensional metric). The use of this procedure is justified by the simple fact that the brane is invisible for an observer residing on the Planckian brane.

In the tree limit, the reduction $S_{\mathrm{eff}}[g_{\mu\nu}^\pm] \to S_{\mathrm{red}}[g_{\mu\nu}^+]$ is equivalent to the elimination of fields on the negative brane in terms of the fields on the positive-tension brane,

$$S_{\mathrm{red}}[g_{\mu\nu}^+] = S_{\mathrm{eff}}\big[g_{\mu\nu}^+, g_{\mu\nu}^-[g_{\mu\nu}^+]\big], \quad g_{\mu\nu}^- = g_{\mu\nu}^-[g_{\mu\nu}^+], \tag{5.20}$$

as the solution of their corresponding equations of motion

$$\frac{\delta S_{\mathrm{eff}}[g_{\mu\nu}^\pm]}{\delta g_{\mu\nu}^-(x)} = 0. \tag{5.21}$$

The result of such reduction is given below for two energy regions, one corresponding to (5.19) and the other to the case of a large distance between the branes when the second inequality in (5.19) is violated.

For a small or finite interbrane distance in the energy region (5.19), the reduced action is given by

$$S_{\mathrm{red}}[g_{\mu\nu}, \varphi] = \int d^4x\, g^{1/2}\left[\left(\frac{1}{16\pi G_4} - \frac{1}{12}\,\varphi^2\right)R + \frac{1}{2}\,\varphi\,\Box\varphi + \frac{l^2\,\varkappa(\varphi)}{32\pi G_4}\,C^2_{\mu\nu\alpha\beta}\right] \tag{5.22}$$

[9] We note that functions (5.11) are basis functions of the operator of gravitational perturbations (4.20) satisfying the linearized Israel matching conditions on the corresponding branes. The function $u_+(z)$ coincides with eigenfunction (4.26) up to normalization when $\sqrt{\Box}$ is identified with its value on the mass shell, the KK-mode mass m.

[10] We note that the left-hand side of Eqn (5.18) is the Wronskian of harmonics $u_\pm(z)$ in (5.11). Its equality to zero means a linear dependence of these two functions or the existence of a harmonic in the space of the fifth coordinate that at the same time satisfies homogeneous boundary conditions on both branes and is a physical mode in the two-brane Randall – Sundrum model.

in terms of the induced metric $g^+_{\mu\nu} \equiv g_{\mu\nu}$, the additional scalar field $\varphi(x)$, and the local coefficient function of this field at the squared Weyl tensor

$$\varkappa(\varphi) = \frac{1}{4}\left[\ln\frac{1}{a^2} - (1-a^2) - \frac{1}{2}(1-a^2)^2\right]_{a^2 = 4\pi G_4\varphi^2/3} . \quad (5.23)$$

The scalar field $\varphi(x)$ emerges from Eqn (5.3) as a result of promoting the modular variable in the model, i.e., the parameter of the interbrane distance d [or $a = \exp(-d/l)$], to a dynamical field in accordance with the relation

$$\varphi(x) = \sqrt{\frac{3}{4\pi G_4}}\exp\left(-\frac{d}{l} + \frac{\Pi(x)}{l}\right), \quad (5.24)$$

where $\Pi(x)$ is the local four-dimensional radion field describing the deviation of brane embedding in the Randall–Sundrum system of coordinates from $y = y_+$ to $y = y_+ - \Pi(x)$ (see the discussion in Section 4.3 and Refs [33, 20]).[11] Clearly, action (5.22) represents the Einstein gravity theory nonminimally coupled to the scalar φ of the Brans–Dicke type that describes the local x-dependent interbrane distance and short-distance corrections in terms of the squared Weyl tensor $C^2_{\mu\nu\alpha\beta}$ with local (but φ-dependent) coefficient (5.23).

For a larger interbrane distance corresponding to the high-energy region on the invisible brane,

$$l\sqrt{\Box} \ll 1, \qquad \frac{l\sqrt{\Box}}{a} \gg 1, \quad (5.25)$$

reduced action (5.20) has a different form [20],

$$S_{\rm red}[g_{\mu\nu}] = \frac{1}{16\pi G_4}\int d^4x\, g^{1/2}\left[R + \frac{l^2}{2}\, C_{\mu\nu\alpha\beta}\, k(\Box)\, C^{\mu\nu\alpha\beta}\right], \quad (5.26)$$

$$k(\Box) = \frac{1}{4}\left(\ln\frac{4}{l^2(-\Box)} - \mathbf{C}\right), \quad (5.27)$$

where $\mathbf{C}$ is the Euler constant. The radion decouples from gravity, and the quadratic Weyl term becomes nonlocal, with a logarithmic form factor characteristic of the AdS/CFT correspondence phenomenon, an imitation of quantum logarithms of the conformal field theory on the brane by the tree (super)gravity action calculated in the bulk.

Transition from local phase (5.22) to nonlocal one (5.26) represents a renormalization-group flow (AdS-flow) interpolating between the limits of small and large interbrane distances. The scalar field φ, starting from the value

$$\varphi = \left(\frac{3}{4\pi G_4}\right)^{1/2}, \quad (5.28)$$

i.e., from the point of coincident branes $a = 1$ at which the effective Planck mass [the overall coefficient at the scalar curvature in (5.22)] vanishes, tends to zero as the branes recede from each other, because $a \to 0$ as $d \to \infty$.

The field condensate φ is given by a coefficient function $\varkappa(\varphi)$ that looks like an effective potential of the Coleman–Weinberg type, logarithmic in $G_4\varphi^2 = \varphi^2/M_{\rm P}^2$, Eqn (5.23). It further delocalizes into the logarithmic form factor $k(\Box)$ of the quadratic Weyl term. The leading logarithmic term in the coefficient function

$$\varkappa(\varphi) \sim \frac{1}{4}\ln\frac{M_{\rm P}^2}{\varphi^2}, \quad \varphi \to 0, \quad (5.29)$$

does not grow infinitely but is instead saturated by the logarithm of the gravitational radiation scale characterized by space–time inhomogeneity of the Weyl tensor, i.e., the logarithmic nonlocality of the form factor

$$k(\Box) \sim \frac{1}{4}\ln\frac{4}{l^2\Box}. \quad (5.30)$$

The physics of this transition is obvious: the tower of massive KK-modes that are infinitely heavy at the initial point of coincident branes becomes very light as $a \to 0$. Its spectrum becomes practically continuous and its cumulative effect is expressed in the form of a logarithmic nonlocality characteristic of the AdS/CFT correspondence.

5.3 Scenario of diverging branes and brane inflation

An interesting question is whether a given AdS-flow can be realized at the dynamical level as a physical process of branes receding (or moving together) and have valuable cosmological applications. One dynamical mechanism was proposed in Ref. [33] by introducing a weak detuning between brane tensions (4.7) and (4.8), resulting in the appearance of a small positive cosmological term [34] in effective action (5.22); this term can generate brane inflation [35–37, 23, 24].

If the excess part of the Planckian brane tension is denoted by $\sigma_{\rm e}$,

$$\sigma = \frac{3}{4\pi G_5 l} + \sigma_{\rm e}, \quad (5.31)$$

the metric–radion part of action (5.22) takes the form

$$S_{\rm eff}[g_{\mu\nu}, \varphi] = \int d^4x\, g^{1/2}\left\{\left(\frac{1}{16\pi G_4} - \frac{1}{12}\varphi^2\right)R + \frac{1}{2}\varphi\,\Box\varphi - \sigma_{\rm e}\right\} \quad (5.32)$$

(we here omit the contribution of Weyl corrections). In this form, the model does not contain a good scalar potential that could directly generate radion evolution by making it roll down from the potential wall, e.g., induce inflation. But a nonminimal curvature coupling of the field φ allows inflationary applications of this model.

To analyze the inflationary applications, we pass to the Einstein parameterization of the action in terms of a new conformally equivalent metric and a new scalar field $(\bar g_{\mu\nu}, \phi)$ [38]:

$$g_{\mu\nu} = \cosh^2\left[\left(\frac{4\pi G_4}{3}\right)^{1/2}\phi\right]\bar g_{\mu\nu}, \quad (5.33)$$

$$\varphi = \left(\frac{3}{4\pi G_4}\right)^{1/2}\tanh\left[\left(\frac{4\pi G_4}{3}\right)^{1/2}\phi\right]. \quad (5.34)$$

In this parameterization, the infinite range of variation of the new scalar field, $|\phi| < \infty$, covers the range of changes of the radion field φ, $|\varphi| \leqslant (3/4\pi G_4)^{1/2}$, or $a \leqslant 1$, in which the effective φ-dependent gravitational constant in action (5.32)

[11] The radion field $\Pi(x)$ is absent from the initial two-brane action (5.7) only because it was put there on its mass shell $\Box\Pi + lR/6 = 0$. The expression for the effective action outside the radion mass shell given in Ref. [20] contains its kinetic term that leads, in terms of field $\varphi(x)$, to Eqn (5.22).

is positive: [12]

$$\frac{1}{16\pi G_4(\varphi)} = \frac{1}{16\pi G_4} - \frac{1}{12}\,\varphi^2\,. \tag{5.35}$$

This guarantees stability of the theory in the new variables.

In the Einstein parameterization, unlike in the initial one, the action describes a theory with a minimally coupled field ϕ that has a monotonically growing potential [due to the conformal factor relating the two metrics in (5.33)]:

$$\bar{S}_{\rm eff}[\bar{g}_{\mu\nu}, \phi] = \int {\rm d}^4x\, \bar{g}^{1/2}\left\{\frac{1}{16\pi G_4}\,\bar{R} + \frac{1}{2}\,\phi\,\bar{\Box}\phi - V(\phi)\right\}, \tag{5.36}$$

$$V(\phi) = \sigma_{\rm e}\cosh^4\left[\left(\frac{4\pi G_4}{3}\right)^{1/2}\phi\right]. \tag{5.37}$$

The positive minimum of the potential $V(\phi)$ at $\phi = 0$ corresponds to infinite interbrane distance, whereas the infinite value of $V(\phi)$ describes the coincident brane limit with (5.28). This means that the branes repulse each other, that is, they recede as the radion field rolls down the potential barrier. The visible brane with the excessive positive tension is curved and, in the slow roll-down regime, has a quasi-de Sitter geometry embedded in the five-dimensional AdS-space. At large initial values of ϕ (small interbrane distance), this potential may prove too steep for the slow-roll regime, but it can maintain a power-law inflation. In contrast, at small ϕ ($\phi \ll \sqrt{3/4\pi G_4}$, large distance), the slow roll-down conditions are fairly well satisfied.

Thus, the radion mode in such a two-brane model may be a candidate for the role of the inflation-generating inflaton. Moreover, at a later stage [$V(0) = \sigma_{\rm e}$], the residual cosmological constant can be interpreted as dark energy responsible for the current cosmological acceleration.

Unfortunately, such a model has a number of drawbacks. One of them is unfavorable conditions for the reheating of the universe as it comes out of the inflation stage; these conditions are attributable to the overlapping of the inflation and cosmological acceleration stages that leave no room for the oscillatory stage in the inflaton evolution [39]. Another problem is related to the impossibility of extrapolating the phase of local effective action (5.22) to large interbrane distances where the AdS/CFT correspondence principle comes into force and the action becomes a purely Einsteinian one with small nonlocal corrections (5.26). Perhaps this phase transition may be helpful in solving the problem of exiting the inflation and reheating the universe, but it requires a more detailed study.

Also, it should be borne in mind that the introduction of an additional ultra-small tension $\sigma_{\rm e}$ in Eqn (5.31) comparable to the contemporary dark energy scale raises the problem of superfine tuning and thus necessitates consideration of alternative mechanisms of brane dynamics. Interestingly, the mechanism of brane repulsion can also be based on the presence of the Weyl term in the ultralocal and nonlocal phases of the theory, Eqns (5.22) and (5.26). When the brane universe is filled with gravitational radiation, this term may be positive and, for small distances, give rise to the interbrane potential

$$-\frac{l^2}{32\pi G_4}\,\varkappa(\varphi)\,C^2_{\mu\nu\alpha\beta}\,. \tag{5.38}$$

In the case of coincident branes ($a = 1$), this potential has a maximum because the coefficient $\varkappa(\varphi)$ given by Eqn (5.23) is strictly positive. However, the repulsive force between the branes is very small and vanishes at $a = 1$ because the behavior of $\varkappa(\varphi)$ at the brane junction point is given by [13]

$$\varkappa(\varphi) \sim \frac{(1-a^2)^3}{12}\,.$$

Unfortunately, this potential for $C^2_{\mu\nu\alpha\beta} > 0$ is negative and as such cannot maintain inflation even though it realizes interbrane interaction in the presence of gravitational radiation on the brane.

5.4 Gravitational-wave oscillations and massive gravitons

We consider the effects of gravitational massive KK-modes in the two-brane Randall – Sundrum model in the presence of matter on both the visible (Planckian) and the 'invisible' branes [40]. For example, one such effect allows the second brane to be seen by the methods of gravitational-wave astronomy. This is the sole possible way to directly observe a multidimensional space in the framework of the brane concept that forbids light propagation in extra dimensions. This method is based on the effect of gravitational oscillations analogous to neutrino oscillations and unavoidable in any other model involving quanta of different masses.

Gravitational radiation from brane-localized sources detected by an observer residing, e.g., on the Planckian brane, can be investigated using the formalism of the two-brane effective action developed in Section 5. Further discussion is confined to the transverse-traceless components of gravitational perturbations $h^{\pm}_{\mu\nu}(x)$ describing radiation from covariantly conserved sources on the corresponding branes, $\Sigma_{\pm}$. Then, action (5.7) quadratic in fields can be rewritten in the 2×2 quadratic form

$$S_{\rm eff}[h^{\pm}_{\mu\nu}] = \int {\rm d}^4x\left(\frac{1}{32\pi G_4}\,\mathbf{h}^{\rm T}\,\frac{\mathbf{F}(\Box)}{l^2}\,\mathbf{h} + \frac{1}{2}\,\mathbf{h}^{\rm T}\mathbf{T}\right), \tag{5.39}$$

in terms of two-dimensional columns of metric perturbations and the energy-momentum tensors $T^{\pm}_{\mu\nu}(x)$ of brane-localized matter sources

$$\mathbf{h} = \begin{bmatrix} h^+(x) \\ h^-(x) \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} T^+(x) \\ T^-(x) \end{bmatrix}. \tag{5.40}$$

The superscript 'T' indicates transposition of columns into rows (hereinafter, tensor indices are omitted).

The kernel of quadratic form (5.39) is given by nonlocal operator (5.10) that enters only the transverse traceless sector of the model. [The overall power of the d'Alembertian $\Box$ differs from (5.10) because the action is expressed directly in terms of metric perturbations rather than of linearized Ricci curvatures (5.6).]

The linear equations for the gravitational potentials $\mathbf{h}(x)$ following from this action can be solved in terms of the

[12] The instability region $a > 1$ can be excluded from this consideration because its boundary $a = 1$ corresponds to the coincident brane limit.

[13] Interestingly, expression (5.23) is the logarithmic term $\ln(1/a^2)$ with the first two terms in the Taylor series subtracted at the point $a^2 = 1$.

retarded Green's function of the operator $\mathbf{F}(\Box)$:

$$\mathbf{h}(x) = -8\pi G_4 l^2 \, \mathbf{G}_{\rm ret}(\Box) \, \mathbf{T}(x) \,, \tag{5.41}$$

$$\mathbf{F}(\Box) \, \mathbf{G}_{\rm ret}(\Box) = \mathbf{I} \,. \tag{5.42}$$

Here, the Green's function is represented in the matrix-valued operator form as a function of $\Box$. For this function, it is possible to write the spectral decomposition in standard Green's functions of a massless graviton and its massive KK-partners whose masses are given by Eqn (5.16):

$$\mathbf{G}(\Box) = \sum_{n=0} \frac{\mathbf{v}_n \mathbf{v}_n^{\rm T}}{\Box - m_n^2} \,. \tag{5.43}$$

The isotopic structure of residues at the poles $\Box = m_n^2$ in Eqn (5.43) is determined by the direct product of polarization functions of the basis functions $\mathbf{v}(x)$ of the operator $\mathbf{F}(\Box)$ that satisfy Eqn (5.15). In the two-brane problem, they are two-dimensional columns $\mathbf{v}_n$ in the expression for the basis function in the form

$$\mathbf{v}_n(x) = (2p^0)^{-1/2} \exp{(\mathrm{i}px)} \, \mathbf{v}_n \,, \qquad p^2 + m^2 = 0 \,.$$

These columns are null vectors of the operator matrix $\mathbf{F}(m_n^2)$,

$$\mathbf{F}(m_n^2) \, \mathbf{v}_n = 0 \,, \tag{5.44}$$

and satisfy the normalization conditions [40]

$$\mathbf{v}_n^{\rm T} \, \frac{\mathrm{d}\mathbf{F}(m^2)}{\mathrm{d}m^2} \, \mathbf{v}_n \bigg|_{m^2 = m_n^2} = 1 \,. \tag{5.45}$$

We are interested in gravitational radiation at low frequencies and large distances from the sources when it is possible to use the low-energy limit on the Planckian brane $l\sqrt{\Box} \ll 1$, but the operator argument $l\sqrt{\Box}/a$ can assume any value by virtue of the smallness of the parameter $a = \exp{(-d/l)}$ (large interbrane distances). In this limit, due to the known small-argument asymptotic form of the Bessel functions, operator (5.10) has the approximate form [20]

$$\mathbf{F}(\Box) \approx \frac{l^2 \Box}{2} \begin{bmatrix} 1 & J_2^{-1}(l\sqrt{\Box}/a) \\ J_2^{-1}(l\sqrt{\Box}/a) & -\dfrac{2}{l\sqrt{\Box} a} \dfrac{J_1(l\sqrt{\Box}/a)}{J_2(l\sqrt{\Box}/a)} \end{bmatrix} . \tag{5.46}$$

In accordance with Eqn (5.16), expression (5.46) in the above approximation ($a \ll 1$) immediately yields the mass spectrum of KK-modes (determined by the roots of the first-order Bessel function) and their polarization vectors:

$$\mathbf{v}_0 = \frac{\sqrt{2}}{l} \begin{bmatrix} 1 \\ a^2 \end{bmatrix} , \quad m_0 = 0 \,, \tag{5.47}$$

$$\mathbf{v}_n = \frac{\sqrt{2}\, a}{l} \begin{bmatrix} J_2^{-1}(j_n) \\ -1 \end{bmatrix} , \quad m_n = \frac{a}{l} \, j_n \,, \quad J_1(j_n) = 0 \,, \quad n > 0 \,. \tag{5.48}$$

These expressions are used here to construct the Green's function (5.43) and find gravitational radiation from the local sources on the branes.

We confine ourselves to the observation of a gravitational signal from two distributed sources $T^{\pm}$ on the 'visible' brane Σ_+. For simplicity, we consider only frequencies below the mass threshold of the second massive mode m_2. Then, only the first two terms in spectral expansion (5.43), with $n = 0$ and $n = 1$, contribute to the signal. Using the structure $\mathbf{v}_{0,1}$ in (5.48), we find the expression for this signal

$$h^+(x) = -16\pi G_4 \, \frac{1}{\Box}\bigg|_{\rm ret} \big(T^+(x) + a^2 T^-(x)\big)$$
$$- \, 16\pi G_4 \, \frac{1}{\Box - m_1^2}\bigg|_{\rm ret} \left(\frac{a^2}{\mathcal{J}^2} \, T^+(x) - \frac{a^2}{\mathcal{J}} \, T^-(x) \right) , \tag{5.49}$$

where $\mathcal{J} \equiv J_2(lm_1/a) \simeq 0.403$.

We now consider local astrophysical sources at the point $\mathbf{x} = 0$ on both branes with equal intensities and frequencies:

$$T^{\pm}(t, \mathbf{x}) = \mu \exp{(-\mathrm{i}\omega t)} \, \delta(\mathbf{x}) \,. \tag{5.50}$$

If the frequency of the source exceeds the mass threshold of the first massive mode ($\omega > m_1$), then both modes, massive and massless, become excited and produce long-distance gravitational waves. At a distance r from the source, its signal on each brane consists of the superposition of spherical waves of massless and massive quanta. Such a superposition on the brane Σ_+ is given by the sum of contributions from the sources on Σ_+ and Σ_-,

$$h^+[T^+] = A \exp{(-\mathrm{i}\omega t)}$$
$$\times \left(\exp{(\mathrm{i}\omega r)} + \frac{a^2}{\mathcal{J}^2} \exp{\left[\mathrm{i}(\omega^2 - m_1^2)^{1/2} r\right]} \right) , \tag{5.51}$$

$$h^+[T^-] = A a^2 \exp{(-\mathrm{i}\omega t)}$$
$$\times \left(\exp{(\mathrm{i}\omega r)} - \frac{1}{\mathcal{J}} \exp{\left[\mathrm{i}(\omega^2 - m_1^2)^{1/2} r\right]} \right) , \tag{5.52}$$

where $A = 4G_4\mu/r$ is the amplitude of the massless wave from source (5.50).

The amplitudes detected by gravitational-wave interferometers depend on the absolute values of expressions (5.51) and (5.52):

$$\left| h^+[T^+] \right| = \mathcal{A}^+ \left[1 - \frac{4a^2\mathcal{J}^2}{(\mathcal{J}^2 + a^2)^2} \sin^2 \frac{\pi r}{L} \right]^{1/2} , \tag{5.53}$$

$$\left| h^+[T^-] \right| = \mathcal{A}^- \left[1 + \frac{4\mathcal{J}}{(\mathcal{J} - 1)^2} \sin^2 \frac{\pi r}{L} \right]^{1/2} . \tag{5.54}$$

It can be seen that they are modulated over the radial variable by oscillations with the wavelength

$$L = 2\pi \left(\omega - \sqrt{\omega^2 - m_1^2} \right)^{-1} \simeq \frac{2\pi}{m_1} \,, \tag{5.55}$$

where the approximate equality corresponds to $m_1 \ll \omega$.

The coefficients of amplitudes (5.53) and (5.54) are given by the expressions

$$\mathcal{A}^+ = \left(1 + \frac{a^2}{\mathcal{J}^2} \right)^2 A \approx A \,, \tag{5.56}$$

$$\mathcal{A}^- = \left(\frac{1}{\mathcal{J}} - 1 \right) a^2 A \approx 2.2 a^2 A \,, \tag{5.57}$$

valid in the limit $a \ll 1$.

Thus, the amplitudes of waves from both sources undergo oscillations, although to different degrees. These oscillations result from the interference of field quantum waves with different masses analogous to neutrino oscillations. This phenomenon parametrically depends on the distance between the branes, i.e., the radion, and corresponds to what actually deserves the name radiation-induced gravitational oscillations (RIGOs). The oscillating part of the amplitude of a gravitational wave from T^+, (5.53), is suppressed in the range of large interbrane distances by the small factor $a^2 \ll 1$. In contrast, oscillations of the amplitude of a signal from T^- in (5.54) are comparable with the signal itself regardless of the distance between the branes.

For RIGOs to be assessed, it should be borne in mind that the length of these oscillations in terms of the curvature radius l of the AdS space and the scale factor a of the hidden brane is given by

$$L = \frac{2\pi}{j_1}\frac{l}{a} \approx 1.6\,\frac{l}{a}\,, \tag{5.58}$$

where $j_1 \approx 3.831$ is the first root of J_1. In other words, the oscillation length is inversely proportional to a.

On the other hand, oscillations become observable when their length is comparable to the shoulder length of the gravitational-wave detector. For earth-based LIGO-type interferometers, such a requirement corresponds to $L \sim 10^3$ m. By combining this parameter with the maximum curvature radius of the AdS space l estimated from Cavendish-type table-top experiments in the submillimeter range ($l \leqslant 10^{-4}$ m) [10], it is possible to find the upper limit of the scale factor for the hidden brane, $a \leqslant 10^{-6}$, at which oscillations become observable. Unfortunately, substitution of this value into the ratio of amplitudes (5.56) and (5.57) gives the estimate

$$\frac{\mathcal{A}^-}{\mathcal{A}^+} \leqslant 10^{-14}\,. \tag{5.59}$$

This means that the amplitude of a wave coming from the source on the hidden brane undergoing sufficiently long and detectable oscillations is markedly suppressed in comparison with that of a gravitational wave from the analogous source on the Planckian brane.

If a strongly oscillating wave is to compare with a weakly oscillating signal emitted by matter on the brane being considered, it must be generated by a source 14 orders of magnitude more powerful. Evidently, this makes RIGO detection impracticable in the near future. Nevertheless, it was suggested in [40] that brane compactification mechanisms in M-theory may give rise to a large condensate of gauge superpartners η on the hidden brane [41], which determines the quadrupole moment of cosmic strings $\mu \sim \eta^2$ [42]. This can lead to the production of strong gravitational waves on the hidden brane readily compensating for the effect of suppressive factor (5.59).

In principle, the RIGO-effect is a common feature of all multidimensional models because they always involve amplitude modulation of gravitational waves composed of a mixture of massless and massive modes. In traditional extra-dimension models, the mass of the first KK-mode is so large that it can never be produced by an astrophysical source nor can it lead to macroscopic oscillations.

In contrast, the geometry of curved extra dimensions allows KK-modes to lie so low that they may undergo observable oscillations. By way of example, gravitational waves from sources on the hidden brane lead to strong oscillations in our world. It appears reasonable to suggest, without postulating strong sources on the hidden brane, the existence of other mechanisms combining small KK-masses and strong modification of gravitational signals in models with extra dimensions. One of such models is that of the gravitational echo effect, which specifically colors gravitational radiation from a source on our brane due to its reflection in the bulk [43].

5.5 The gravitational echo effect

In this section, we consider specific features of the relativistic propagation of a signal in space–time with a compact fifth coordinate. This problem is interesting from the standpoint of gravitational-wave astronomy as a tool for searching for extra dimensions. We show that the compactness of extra dimensions and the presence of an alternative brane give rise to the gravitational echo phenomenon [43] that modifies the retarded potentials of (quasi)point-like sources in a peculiar way.

For simplicity, we consider a flat five-dimensional space–time with the fifth coordinate ranging within finite limits $0 < y < L$ that can be identified with two branes, as in the Randall–Sundrum model. (In all other aspects, the model under consideration bears more likeness to the ADD model with the extra dimension size L.) We impose the Neumann boundary conditions on the branes to simulate the Israel matching conditions. Moreover, the tensor and gauge structures of gravitational perturbations are omitted under the assumption that their equations of motion have the form of equations of a massless scalar field. This simplification does not affect propagation patterns of relativistic signals in a five-dimensional space having time-like boundaries.

Thus, the five-dimensional field $\Phi(X)$ coupled to the source $J(X)$ with the five-dimensional gravitational constant satisfies the equation

$$\Box_5\,\Phi(X) = -G_5\,J(X) \tag{5.60}$$

in the five-dimensional space and the Neumann boundary conditions

$$\partial_y\Phi(X)\Big|_{y=0} = \partial_y\Phi(X)\Big|_{y=L} = 0\,. \tag{5.61}$$

on the branes.

For a brane-localized source, $J(X) = j(x)\,\delta(y)$, the solution of the problem on the same brane

$$\Phi(X)\Big|_{y=0} = G_5\int \mathrm{d}^4x'\,D(x-x')\,j(x') \tag{5.62}$$

is given by the interbrane propagator

$$D(x) = -\frac{1}{\Box_5}\,\delta(X)\Big|_{y=0}\,, \tag{5.63}$$

which can be constructed from the five-dimensional propagator in a boundless infinite space,

$$D_5(X) \equiv -\frac{1}{\Box_5}\,\delta(X) = \frac{\mathrm{i}}{8\pi^2}\,\frac{1}{\left((X)^2\right)^{3/2}}\,, \tag{5.64}$$

by the image-based method. [14]

[14] Here, $X^2 = -(x^0)^2 + \mathbf{x}^2 + y^2$ is the Lorentzian interval, and the rule for bypassing singularities during integration of the Green's function kernel is given by the boundary conditions for physical time x^0 (it is formulated below for the retarded Green's function).

As shown in Ref. [43] for the Neumann boundary conditions, the answer obtained with the image-based method has the form of the sum of contributions from the source and its images localized at an infinite sequence of points X_n:

$$D(x) = 2\sum_{n=-\infty}^{\infty} D_5(X - X_n)\Big|_{X=(x,0)}, \tag{5.65}$$

$$X_n = (0,\, 2nL)\,, \quad (X - X_n)^2 = x^2 + 4n^2L^2\,. \tag{5.66}$$

(This is also substantiated in [43] in terms of the KK-representation and momentum representation.)

In the region of large intervals, $(x - x')^2 \gg L^2$, the sum in (5.66) can be replaced by an integral, calculating which gives

$$D(x, x') \simeq \frac{\mathrm{i}}{8\pi^2 L}\int_{-\infty}^{+\infty} \mathrm{d}y\, \frac{1}{\left((x - x')^2 + y^2\right)^{3/2}} = \frac{\mathrm{i}}{4\pi^2 L}\,\frac{1}{(x - x')^2}\,, \tag{5.67}$$

and thus leads to the restoration of the four-dimensional Green's function [see Eqn (4.36), in which the role of the effective size of the extra dimension is played by l]. Practically speaking, this method amounts to obtaining the Green's function in a space from the Green's function in the space whose dimension is raised by one. In particular, it allows finding Newton's potential on the brane produced by a point-like source at the origin of spatial coordinates by integrating Eqn (5.65) over time.

Integration of (5.65) by means of the Wick rotation to $x^4 = \mathrm{i}x^0$ gives

$$V(r) = G_5\int_{-\infty}^{\infty} \mathrm{d}x^0\, D(\mathbf{x}, x^0) = \frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} \frac{1}{r^2 + 4n^2L^2}\,, \qquad r^2 = \mathbf{x}^2\,, \tag{5.68}$$

or

$$V(r) = \frac{G_4}{4\pi r}\left(1 + \frac{2}{\exp(\pi r/L) - 1}\right), \tag{5.69}$$

where G_4 is the four-dimensional gravitational constant [cf. Eqn (4.35)]:

$$G_4 = \frac{G_5}{L}\,. \tag{5.70}$$

It is clear that at large distances ($r \gg L$), this result reproduces the ordinary Newton potential with the Yukawa-type corrections in the four-dimensional world. At small distances, it turns into the five-dimensional law $1/r^2$.

Unlike Newton's potential, the retarded potential from a source on the brane is derived from (5.62) by substituting the retarded propagator $D^{\mathrm{ret}}(x - x')$. This can be done, term by term, in the sum in (5.65) using the kernel of the five-dimensional retarded Green's function obtained from the imaginary part of the Feynman propagator [which follows from (5.64)] by means of the iε-prescription $X^2 \to X^2 + \mathrm{i}\varepsilon$ [43]:

$$D_5^{\mathrm{ret}}(r, t) = \frac{1}{4\pi^2}\,\mathrm{Re}\,\frac{\mathrm{i}}{(r^2 - t^2 + \mathrm{i}\varepsilon)^{3/2}} = -\frac{1}{4\pi^2 r}\,\frac{\partial}{\partial r}\,\frac{\theta(t - r)}{(t^2 - r^2)^{1/2}}\,. \tag{5.71}$$

The result is

$$D^{\mathrm{ret}}(t, \mathbf{x}) = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty} \frac{1}{r}\,\frac{\partial}{\partial r}\,\frac{\theta\left(t - (r^2 + 4n^2L^2)^{1/2}\right)}{(t^2 - r^2 - 4n^2L^2)^{1/2}}\,, \tag{5.72}$$

which is essentially different from the four-dimensional case in two respects. First, the retarded Green's function (5.71) is supported inside the entire future light cone of the point-like source at $r = 0$, $t = 0$ (which corresponds to the absence of the Huygens principle in an odd-dimensional space–time). For this reason, the sum in (5.72) involves the contributions of all images of the original source belonging to the inside of the light cone of the observation point. Second, the retarded function has a root singularity on the light cone.

We consider the retarded potential from a brane source

$$J(x, y) = f(t)\,\delta^{(3)}(\mathbf{x})\,\delta(y) \tag{5.73}$$

with an arbitrary time dependence $f(t)$. Taking (5.72) into consideration, we have

$$\Phi(t, r) = -\frac{G_5}{2\pi^2} \times \sum_{n=-\infty}^{\infty} \frac{1}{r_n}\,\frac{\partial}{\partial r_n}\int_{-\infty}^{t - r_n} \mathrm{d}t'\, \frac{f(t')}{\left((t - t')^2 - r_n^2\right)^{1/2}}\Bigg|_{r_n = (r^2 + 4n^2L^2)^{1/2}}\,. \tag{5.74}$$

In particular, for the function $f(t)$ describing a constant-amplitude pulse of finite duration T,

$$f(t) = \theta(t) - \theta(t - T)\,, \tag{5.75}$$

expression (5.74) takes the following form after integration over time:

$$\Phi(t, \mathbf{x}) = \frac{G_5 t}{16\pi^2 L^3}\,\theta(t - r)\,I(\alpha, \beta) - (t \to t - T)\,. \tag{5.76}$$

The function $I(\alpha, \beta)$ of the parameters

$$\alpha = \frac{(t^2 - r^2)^{1/2}}{2L}\,, \quad \beta = \frac{r}{2L} \tag{5.77}$$

is given by the finite sum

$$I(\alpha, \beta) = \sum_{n=-[\alpha]}^{[\alpha]} \frac{1}{(\beta^2 + n^2)(\alpha^2 - n^2)^{1/2}}\,, \quad m < \alpha\,, \tag{5.78}$$

where $[\alpha]$ denotes the integer part of α.

We first discuss the contribution of the front edge of the pulse, i.e., the first term in Eqn (5.75). When t reaches the radius r at which the observer is localized, the observer begins to receive a signal encoded in the first term of Eqn (5.76). Because this term contains the integer part of the parameter α ($n = [\alpha]$), its contribution is a discontinuous function of time. Each time the parameter α becomes an integer, a new pair of terms at $n = \pm[\alpha]$ appears in sum (5.78) and the answer suffers a jump. This corresponds to the arrival of signals from new images at $y = \pm 2[\alpha]L$.

Moreover, the signals are singular [have the form $(\alpha - [\alpha])^{-1/2}$] directly at $\alpha = [\alpha]$ in correspondence with the singularity structure of the retarded Green's function in five dimensions. This singularity is not very strong and can be smoothed by averaging over time intervals between consecutive singular peaks. In the course of time, more and more new signals of the gravitational echo come from the new images of the source. As a result, the signal detected on the brane assumes the form of a sequence of singular peaks. Evidently, the observer's detector is unable to resolve individual peaks. Therefore, it is reasonable to average them over time as described in the paragraphs below.

Because of late-time restoration of the four-dimensional signal (as contributions from a large number of images accumulate), it is necessary to first find the asymptotic form of sum (5.78) at $\alpha \gg 1$, as was done in Ref. [43], and then average it over the time interval between the peaks. In the last singular term ($n = [\alpha]$) and the subleading term in the $\alpha^{-1/2}$ of the asymptotic form, this procedure amounts to taking averages of the form

$$\left\langle \left(\alpha - [\alpha]\right)^{-1/2} \right\rangle \equiv \int_{[\alpha]}^{[\alpha]+1} \frac{d\alpha}{\left(\alpha - [\alpha]\right)^{1/2}} = 2\,, \tag{5.79}$$

$$\left\langle \left(\alpha - [\alpha] + 1\right)^{1/2} \right\rangle \equiv \int_{[\alpha]}^{[\alpha]+1} d\alpha \left(\alpha - [\alpha] + 1\right)^{1/2} = \frac{2}{3}\left(2^{3/2} - 1\right). \tag{5.80}$$

Hence,

$$\left\langle I(\alpha, \beta) \right\rangle = \frac{\pi}{\beta} \frac{\coth(\pi\beta)}{(\alpha^2 + \beta^2)^{1/2}} - \frac{2^{3/2}}{\alpha^{1/2}(\alpha^2 + \beta^2)} \frac{2^{5/2} - 5}{3}\,, \quad \alpha \gg 1\,, \tag{5.81}$$

which, by virtue of (5.76), finally gives

$$\left\langle \Phi(t, \mathbf{x}) \right\rangle = V(r)\,\theta(t - r) - \frac{G_4}{3\pi^2 t} \left(\frac{L^2}{t^2 - r^2}\right)^{1/4} \times (2^{5/2} - 5)\,\theta(t - r) - (t \to t - T)\,, \quad t^2 - r^2 \gg L^2\,, \tag{5.82}$$

where $V(r)$ is exactly the Newton potential (5.69), the subtracted term at $t - T$ describes the contribution of the back front of pulse (5.68), and the second term represents corrections due to the contribution of echo signals from a large number of images.[15]

At $t > T + r$, signals from both the forefront and the back front of the pulse reach the observer, which leads to the cancellation of the first Newtonian term. In four dimensions, this would result in the absence of a signal at $t > T + r$ (the observer would see only the passage of the finite pulse of length T with a delay r). A more interesting situation takes place in five dimensions; specifically, the entire area inside the light cone contributes to the signal, which never vanishes abruptly. Even after the burning candle dies, its fading light continues to reach the observer, coming from progressively more distant images of the source. The 'tail' of the signal originates from the second term in Eqn (5.82) and at later

[15] We note that this contribution arises from the subleading term in $\alpha^{-1/2}$ of the asymptotic form that contains averaging (5.80).

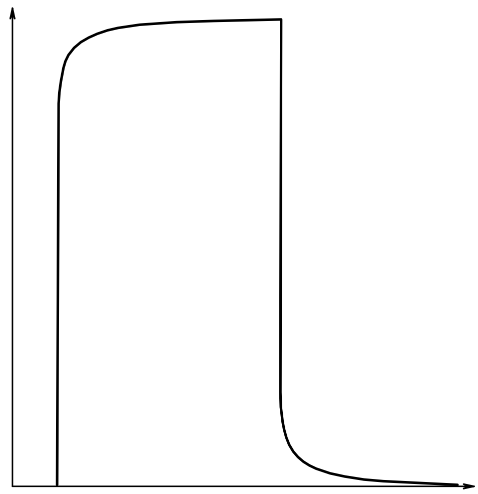

Figure 7. Typical shape of a signal: forefront screening and tail signal following the back front.

times $t \gg r$ and $t \gg T$ assumes the following form that decays in accordance with a power-like law:

$$\left\langle \Phi(t, \mathbf{x}) \right\rangle_{\rm tail} = (2^{5/2} - 5)\, \frac{G_4}{2\pi^2} \frac{L^{1/2}\, T}{t^{5/2}}\,. \tag{5.83}$$

It is essential that at $t \gg r$, the tail signal is independent of the distance from the source.

Another interesting effect is the distortion of the signal front edge. At $r < t < r + T$, when the back front of the pulse has not yet reached the observer, the second term in (5.82) is responsible, due to its negative sign, for the partial screening of the purely four-dimensional part of the signal (its resultant form is represented in Fig. 7). The correction for $\Phi_0(t, \mathbf{x}) = G_4/4\pi r$ has the form

$$\left\langle \Phi(t, \mathbf{x}) \right\rangle \simeq \Phi_0(t, \mathbf{x}) + \Phi_1(t, \mathbf{x})\,, \tag{5.84}$$

$$\Phi_1(t, \mathbf{x}) = -\frac{2}{3\pi} \frac{2^{5/2} - 5}{\alpha^{1/2}(t)}\, \Phi_0(t, \mathbf{x})\,, \quad r < t < r + T\,, \quad \alpha(t) \gg 1\,, \tag{5.85}$$

where the parameter $\alpha(t) = (t^2 - r^2)^{1/2}/2L$ has the meaning of the number of peaks that have already passed through the observer by the moment of observation t. (We disregard the difference of α from $[\alpha]$ at $\alpha \gg 1$.)

The number $\alpha(t)$ is assumed to be sufficiently large in order to guarantee formation of the four-dimensional part [the first term in (5.85)]. At the same time, it should be small enough to ensure the relation $t \sim r$. The minimal value of $\alpha(t)$ is eventually determined by the resolving power of the observer's detector. It must not necessarily recognize a single peak but must be able to resolve the time interval between the arrival of the front edge and the moment of observation t. A similar situation takes place at the back front: the tail signal for a t value close to $r + T$ largely depends on the number of peaks that come up immediately after the initial source ceases to emit the signal [43].

A similar computation of the retarded potential from a periodic source of frequency ω

$$f(t) = \exp(i\omega t) \tag{5.86}$$

is much simpler. The calculation of the integral in Eqn (5.74) using the formula

$$\int_{-\infty}^{t-r} \frac{dt' \exp(i\omega t')}{\sqrt{(t-t')^2 - r^2}} = \frac{-i\pi}{2} \exp(i\omega t)\, H_0^{(2)}(\omega r) \qquad (5.87)$$

gives a sum that can also be reduced to an integral in the limit $\omega L \ll 1$ [43].

As a result, the leading contribution

$$\Phi(r,t) \simeq \Phi_0(r,t) = \frac{G_4}{8\pi r} \exp\big(i\omega(t-r)\big) \qquad (5.88)$$

turns out to be a four-dimensional retarded spherical wave, and the correction for it in the region of large distances ($\omega r \gg 1$) has the form

$$\Phi_1(r,t) = L\sqrt{\frac{2\omega}{\pi r}} \exp\left(\frac{i\pi}{4}\right) \Phi_0(r,t)\,. \qquad (5.89)$$

We note the phase shift of this expression with respect to the four-dimensional signal in (5.88). Now, amplitude (5.89) also depends on the signal frequency. These two factors allow the effect of extra dimension to be distinguished from the purely four-dimensional part of the signal. In other words, the five-dimensionality of space – time is manifest as the frequency-dependent enhancement of the amplitude $\left|\Phi_0(r,t) + \Phi_1(r,t)\right|$ of the signal from a periodic source.

Thus, there are at least two potentially observable effects of the compact fifth dimension, viz. residual luminosity from a source having a finite lifetime T and enhancement of the signal amplitude from a periodic source. In the four-dimensional theory, the residual luminosity is nonexistent in principle; therefore, its observation would give clear evidence in support of the reality of extra dimensions. The relative magnitude of this effect with respect to the Newton's potential amplitude at a distance r from the source $\Phi_0(t,r) = G_4/4\pi r$ at a time instant $t \geqslant r$ is limited by the estimate

$$\frac{\Phi_{\rm tail}}{\Phi_0} \sim \frac{L^{1/2} T}{r^{3/2}}\,. \qquad (5.90)$$

The effect of enhancement of periodic signal (5.89) involves a phase shift, which can also be used in observations. In accordance with Eqn (5.89), the relative value of this effect behaves as

$$\frac{\Phi_1}{\Phi_0} \sim \frac{L}{\sqrt{\lambda r}}\,, \qquad (5.91)$$

in terms of the signal wavelength $\lambda = 2\pi/\omega$ (the speed of light in our units is $c = 1$).

The effects of residual luminosity from the source and amplification of the periodic signal suggest the possibility, in principle, of observing extra dimensions by the methods of gravitational-wave astronomy, although these effects are very weak. Estimates obtained for the LIGO and LISA detectors (based on the data in [44, 45]) can be interpreted as follows. For the LISA interferometer operated in the frequency range $10^{-4} - 1$ Hz, a source (e.g., supernova explosion) some 10 Mps from the earth producing a signal of duration $T \sim 10^4$ s (associated with the lower boundary of the frequency range) would give, in accordance with (5.90), $\Phi_{\rm tail}/\Phi_0 \sim 10^{-25}$, assuming the scale $L \sim 10^{-1}$ cm for the size of the extra dimension. The LIGO detector operates at higher frequencies ($1 - 10^4$ Hz) and is therefore more suitable for the observation of the periodic signal amplification effect. For the radiation frequency $\omega \sim 100$ Hz (e.g., from a binary stellar system located 10 Mps from the earth), the estimate $\Phi_1/\Phi_0 \sim 10^{-18}$ can be found from (5.91).

Thus, the resultant effect is too small to be experimentally observed either at present or in the near future. It should be noted, however, that tail radiation (5.83) at later times is independent of the distance from the source. This implies the feasibility of a collective effect due to the superposition of a large number of different sources. The resulting signal is then proportional to the number of sources in our part of the universe and can be distinguished in agreement with the characteristic decay law $t^{-5/2}$.

There is another time scale interval that can hopefully be employed for the detection of extra dimensions. It corresponds to (5.84), (5.85), i.e., the region of transition from the moment of arrival of the front edge signal at $t = r$ to later times $t > r$. In this regime, the value of r is significantly higher, allowing a large number $\alpha(t)$ of signal peaks to be generated. This number must be large enough to guarantee the formation of a four-dimensional signal (5.84). (It should be recalled that this signal is completely restored at infinite t as the cumulative effect of an infinite number of source images.) However, the number $\alpha(t)$ must not be so large as to enable a gravitational antenna to resolve the interval $\Delta t = t - r$ between the arrival of the front edge and the moment of observation.

The correction for the four-dimensional part of signal (5.85) is suppressed by the factor $\alpha^{-1/2}(t)$. It follows from (5.77) that the lower limit of $\alpha(t)$ is related to the time-resolving power of the gravitational antenna, $\Delta t = t - r$, as $\alpha_{\rm min} \sim (\Delta t/L)^{1/2}(r/L)^{1/2}$. This gives $\alpha_{\rm min}^{-1/2}(t) \sim 10^{-8}$ (at the frequency $\omega \sim \Delta t^{-1} \sim 10^4$ Hz inherent in the LIGO detector) and thus improves previous estimates and opens up some new prospects for gravitational-wave astronomy of extra dimensions.

6. The cosmological constant problem and brane cosmology

In this section, we return to the problems of the cosmological constant and cosmological acceleration that provided (as mentioned in Section 1) strong motivation for the study of brane models with extra dimensions. The solution of these problems actually embraces the scope of possible mechanisms extending far beyond the brane concept. We therefore confine ourselves to the consideration of only one type of these mechanisms represented by a set of modifications of the Einstein theory in the far infrared region characterized by the horizon scale (inverse Hubble constant):

$$\frac{1}{H_0} \sim 10^{28}\ \text{cm} \sim (10^{-33}\ \text{eV})^{-1}\,.$$

Once again, the class of infrared modifications is broader than the possibilities of the brane paradigm. Therefore, we first dwell on these modifications in the framework of the simplest brane models and then briefly discuss the general nonlocal mechanism of a *new* solution of the cosmological constant problem based on a partial violation of brane graviton localization. In this case, a low-energy graviton proves to be metastable, which leads to the modification of the theory not only in the small-distance region but also in the region of ultralarge scales comparable to the cosmological horizon.

6.1 New mechanism of the small cosmological constant

What is the *new* solution to the cosmological constant problem? It should be recalled that the crux of the problem is in the huge discrepancy between the very low mean density of energy in the universe, $\mathcal{E} \sim 10^{-29}$ g cm^{-3} $\sim (10^{-5}$ eV$)^4$, which generates (in accordance with the Einstein equations) the current cosmological acceleration with the Hubble constant H_0,

$$H_0^2 \sim G\mathcal{E}\,, \tag{6.1}$$

and the vacuum energy scale of all fundamental field theory models, from the electroweak theory $\mathcal{E} \sim (1\ \text{TeV})^4$ to quantum gravity $\mathcal{E} \sim (10^{19}\ \text{GeV})^4$.

Former attempts to address the problem aimed at constructing models with zero vacuum energy were largely based on supersymmetry forbidding renormalization of the cosmological constant. The mechanism underlying such models based on mutual annihilation of the contributions from particles and their superpartners ceases to work in a phase with spontaneously broken supersymmetry [46] and loses sense in the framework of the cosmological acceleration phenomenon, in which the effective value of the cosmological constant is very small but differs from zero.

An alternative solution of the problem should probably be sought in the scalar curvature sector rather than in the cosmological term sector of the Einstein – Hilbert action. The smallness of H_0^2 may ensue from the smallness of the proportionality coefficient, i.e., the gravitational constant G, and not from the small value of $\mathcal{E}$. In other words, the small value of the Hubble constant in cosmological acceleration is attributable to the vacuum energy being weakly gravitating rather than to it being too small. What distinguishes the vacuum energy from other local sources of the gravitational field at distance scales much smaller than cosmological ones is the degree of space – time homogeneity $\mathcal{E}$. It is supposed that the vacuum energy does not cluster, is practically homogeneous at the horizon scale

$$\frac{\nabla\mathcal{E}}{\mathcal{E}} \sim H_0\,, \tag{6.2}$$

and gravitates with its long-distance gravitational constant $G_{\text{LD}} \ll G_{\text{P}}$, which is much smaller than the Planck constant determining 'everyday' gravitational physics at the scale of galaxies, planetary systems, Cavendish-type submillimeter experiments, etc.:

$$H_0^2 \sim G_{\text{LD}}\mathcal{E} \ll G_{\text{P}}\mathcal{E}\,.$$

This idea, probably formulated for the first time in Ref. [47], implies that the fundamental constant in the Einstein equations should be promoted to the level of a nonlocal operator, which, for the sake of covariance, can be regarded as a function of the d'Alembertian interpolating between the Planckian scale of the gravitational constant and its long-distance value, [16]

$$G \to G(\Box)\,, \quad G_{\text{P}} > G(\Box) > G_{\text{LD}}\,. \tag{6.3}$$

[16] The idea of a scale-dependent gravitational constant was also suggested in Ref. [48], but it was not formulated there in terms of a nonlocal operator.

We note that the mechanism of a scale-dependent gravitational constant in the form of nonlocality is not the sole one. The notion of scale includes not only the degree of space – time inhomogeneity but also the field amplitude. For this reason, the infrared modification of the theory can also be based on the gravitational 'constant', analogous to (5.35), locally dependent on distinguished physical fields, a variety of the so-called quintessence [46, 49].

Such a mechanism is realized, for example, in brane cosmological models of the Randall – Sundrum type. However, it is less universal, being dependent on the behavior of a specific quintessence field. In contrast, the nonlocal substitution mechanism in (6.3) leads to the modified Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = 8\pi G(\Box)\,T_{\mu\nu}\,, \tag{6.4}$$

in which the gravitational strength of the matter source $T_{\mu\nu} = T_{\mu\nu}(x)$ is determined, regardless of its field content, by the character of the inhomogeneous x-dependence. Therefore, consideration of nonlocal modifications like (6.3) and (6.4) should be preceded by a discussion of the cosmological application of the Randall – Sundrum model with a variable *local* gravitational constant; it is to be followed by an examination of the Dvali – Gabadadze – Porrati model [50], which suggests both the cosmological acceleration mechanism and the aforementioned nonlocal mechanism.

6.2 Brane cosmology of the Randall – Sundrum model

Brane cosmological models are described by the elegant formalism of effective four-dimensional equations of motion that fairly well discriminates between a part of dynamic quantities formulated in terms of local fields on the brane and objects showing nonlocal field dependence in a multidimensional bulk [51]. We confine ourselves to the case of a vacuum bulk populated only by the contribution of the cosmological constant.

We recall that for the five-dimensional brane system in (4.1), the four-dimensional Lagrangian of matter and metric on the brane generates, as a source in five-dimensional Einstein equations, the total surface stress tensor that includes both the brane tension itself and the contributions of matter and possible invariants of intrinsic curvature (4.3). These equations form the boundary value problem

$$R_{AB} - \frac{1}{2}\,G_{AB}\,R^{(5)} = -\Lambda_5\,G_{AB}\,, \tag{6.5}$$

$$K_{\mu\nu} - Kg_{\mu\nu} = 4\pi G_5 S_{\mu\nu}\,, \tag{6.6}$$

where Israel matching conditions (4.2) take account of the Z_2-symmetry of the five-dimensional metric.

Components of five-dimensional Einstein equation (6.5) confined to the brane contain metric derivatives, up to the second order inclusive, tangential and normal to the brane. In accordance with Israel matching equations (6.6), the normal first-order derivative is expressed through another brane object, the brane stress tensor. Only the normal second-order derivative remains undefined and requires the solution of equations in the bulk.

It turns out that this contribution can be explicitly disentangled and separated from the remaining part, which can be formulated in closed four-dimensional terms. This is achieved by the projection of Eqn (6.5) on the brane surface and the use of the Gauss – Codazzi equations. These equations allow decomposing the projections of the five-dimensional curvature tensor into the sum of the four-dimensional curvature tensor and a quadratic expression in the extrinsic curvature $K_{\mu\nu}$ (and covariant derivatives of $K_{\mu\nu}$). The substitution of the extrinsic curvature in terms of the brane stress tensor gives the equation

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -\frac{1}{2}\, g_{\mu\nu}\, \Lambda_5 + (8\pi G_5)^2\, \Pi_{\mu\nu} - \mathcal{E}_{\mu\nu}\,, \qquad (6.7)$$

where all Ricci curvatures are constructed from the brane metric. It has the form of the usual Einstein equation with a nontrivial source on the right-hand side containing the cosmological term, along with a quadratic combination of the stress tensor $\Pi_{\mu\nu}$ and mixed projection of the five-dimensional Weyl tensor ${}^5C_{ACBD}$ (on the brane and its normal n^A):

$$\Pi_{\mu\nu} = -\frac{1}{4}\, S_{\mu\alpha}S^{\alpha}_{\nu} + \frac{1}{12}\, SS_{\mu\nu} + \frac{1}{8}\left(S^2_{\alpha\beta} - \frac{1}{3}\, S^2\right) g_{\mu\nu}\,, \quad (6.8)$$

$$\mathcal{E}_{\mu\nu} = {}^5C_{A\mu B\nu}\, n^A n^B\,. \qquad (6.9)$$

It is the Weyl term that contains nonlocal information about the bulk that can be comprehensively found only by solving five-dimensional equations. It is shown below, however, that symmetry considerations can substantially simplify the structure of this term and establish asymptotic regimes in which its contribution is inessential. Its general properties include tracelessness and the transformation law

$$\mathcal{E}^{\mu}_{\mu} = 0\,, \qquad (6.10)$$

$$\nabla^{\mu}\mathcal{E}_{\mu\nu} = (8\pi G_5)^2\, \nabla^{\mu}\Pi_{\mu\nu}\,, \qquad (6.11)$$

dictated by the transverse character of the Einstein tensor in (6.7). Therefore, when the right-hand side of (6.11) vanishes, the traceless symmetric tensor $\mathcal{E}_{\mu\nu}$ plays the role of conserved radiation energy – momentum tensor. That is why it is usually called the contribution of *dark* radiation.

As a consequence of the Codazzi equation $\nabla^{\nu}K_{\nu\mu} - \nabla_{\mu}K = {}^5R_{5\mu}$ in the absence of energy flow from the brane into the bulk ($T_{5\mu} = 0$), in view of (6.6), the conservation law for the total stress tensor (4.3) is also satisfied,

$$\nabla^{\mu}S_{\mu\nu} = 0\,, \qquad (6.12)$$

which essentially decouples the equations of brane matter dynamics and dark radiation dynamics.

The nonreducible (traceless) character of the dark radiation tensor suggests that it does not contribute to the cosmological term in effective equations (6.7). We note that only half of the five-dimensional cosmological constant contributes to this term; however, in a model with brane tension (4.4) on the right-hand side of Eqn (6.7), it is supplemented due to σ and becomes

$$-\frac{1}{2}\, g_{\mu\nu}\, \Lambda_5 + (8\pi G_5)^2\, \Pi_{\mu\nu} = -\Lambda_4^{\rm eff}\, g_{\mu\nu} + \ldots\,, \qquad (6.13)$$

$$\Lambda_4^{\rm eff} = \frac{1}{2}\, \Lambda_5 + \frac{(8\pi G_5\sigma)^2}{12}\,. \qquad (6.14)$$

In particular, for the tension $\sigma = 3/(4\pi G_5 l)$ in the Randall – Sundrum model with the negative cosmological constant ($\Lambda_5 = -6/l^2$), its effective four-dimensional cosmological constant is zero, $\Lambda_4^{\rm eff} = 0$; in the absence of dark radiation, this guarantees the existence of a flat brane in the piecewise smooth AdS bulk.

The contribution of the Weyl term can be analyzed in a cosmological problem by explicitly solving the five-dimensional Einstein equations and Israel matching conditions. For this, the metric in the bulk should be taken in the form reflecting homogeneity of the problem with respect to the brane spatial coordinates and other time-like sections $y = \text{const}$:

$$\mathrm{d}s^2 = -N^2(t,y)\,\mathrm{d}t^2 + a^2(t,y)\,\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + b^2(t,y)\,\mathrm{d}y^2\,. \qquad (6.15)$$

Here, $a(t,y)$ is the scale factor of these sections with the spatial metric γ_{ij} of constant positive, negative, or zero curvature ($k = \pm 1, 0$), $N(t,y)$ is the lapse function, and $b(t,y)$ is the 'pace' function for the fifth coordinate.

We assume that the brane is located at $y = 0$ and, for the respective values of the parameter k, describes the spatially closed, open, and flat Friedmann universe. Assuming the presence of matter on the brane with the stress tensor of a spatially homogeneous ideal fluid,

$$T^{\mu}_{\nu} = \mathrm{diag}\,(-\rho, p, p, p)\,, \qquad (6.16)$$

it can be shown that the Einstein equations in the bulk have the integral of motion [52]

$$\frac{(a')^2}{a^2b^2} - \frac{(\dot{a})^2}{a^2N^2} - \frac{k}{a^2} + \frac{\Lambda_5}{6} + \frac{\mathcal{C}}{a^4} = 0\,, \qquad (6.17)$$

where the prime denotes differentiation with respect to the fifth coordinate ($a' \equiv \partial_y a$) and $\mathcal{C}$ is the integration constant. The existence of integral (6.17) is due to the absence of a matter flow from the brane into the bulk ($T_{5\mu} = 0$).

One of the matching conditions in (6.6),

$$\left.\frac{a'}{ab}\right|_{y=0} = -\frac{4\pi G_5}{3}\,\rho\,, \qquad (6.18)$$

allows finding a' on the brane. The substitution of this expression in (6.17) leads to the equation for the scale factor of the cosmological brane containing only brane quantities. It has the form of the generalized Friedmann equation for the Hubble constant H of the brane metric,

$$H^2 + \frac{k}{a^2} = \frac{\Lambda_4^{\rm eff}}{3} + \frac{8\pi G_4^{\rm eff}(\rho)}{3}\,\rho + \frac{\mathcal{C}}{a^4}\,, \qquad (6.19)$$

$$H \equiv \left.\frac{\dot{a}}{Na}\right|_{y=0}, \qquad (6.20)$$

with the effective cosmological constant (6.14), with the gravitational constant *locally depending on the density*,

$$G_4^{\rm eff}(\rho) = \frac{4\pi G_5^2\sigma}{3}\left(1 + \frac{\rho}{2\sigma}\right), \qquad (6.21)$$

and with the contribution of dark radiation. The dark radiation density decreases inversely proportionally to the fourth power of the scale factor; hence, it is inessential at the late stages of expansion.

The comparison with the 00-component of Eqn (6.7) indicates that dark radiation corresponds to the Weyl contribution $\mathcal{E}_{00} = -3\mathcal{C}/a^4$, whereas the quantity $\Pi_{00} = (\rho + \sigma)^2/12$ contributes to cosmological term (6.13) and gives rise to a matter source with a variable cosmological constant. It can be shown that dark radiation determines the deviation of geometry in the bulk from the purely (anti)-de Sitter one. In fact, the constant $\mathcal{C}$ plays the role of a Schwarzschild mass in the five-dimensional static Schwarzschild – (anti-)de Sitter solution with a moving spherical brane [53].

Thus, brane cosmology with a nonzero brane tension realizes the idea of a scale-dependent gravitational constant that can be very high at large positive matter densities and decreases in the course of expansion. Indeed, in accordance with the conservation law for the stress tensor of matter (6.16) [a corollary of (6.12)],

$$\dot{\rho} + 3(\rho + p)\,\frac{\dot{a}}{a} = 0\,, \tag{6.22}$$

the density ρ decreases with cosmological expansion for a broad class of the equations of state (e.g., $\rho \sim 1/a^{3(1+w)}$ for $p = w\rho$, $w \geqslant -1$). At later stages, the generalized Friedmann equation is converted to an equation of the standard model with an asymptotic value of the gravitational constant, $G_4^{\rm eff}(\rho) \to G_4^{\rm eff}(0)$, and the effective cosmological constant.

The phenomenology of the above model and its generalizations, including effects of graviton radiation from the brane into the bulk, are of great interest and have been studied in a number of works [54, 55]. However, in the context of the cosmological constant problem, this model reflects the old approach to its solution because it is actually based on a fine tuning of the five-dimensional vacuum energy Λ_5. The cosmological acceleration in this model is supported by the mechanism that is discussed, with all its drawbacks, in Section 7. This model should be treated as a high-energy modification of the Einstein theory, probably inadequate for the description of the contemporary accelerating universe. Below, we therefore consider a different model that appears to have a better chance to play the role of infrared modifications of the Einstein theory. It is the Dvali – Gabadadze – Porrati (DGP) model [50].

6.3 The brane-induced gravity model and cosmological acceleration

The action of the DGP model does not involve a cosmological term in the bulk or on the brane. Instead of the brane tension, it contains the four-dimensional Einstein term with the gravitational constant G_4 essentially different from the five-dimensional constant G_5:

$$S_{\rm DGP}\big[G_{AB}(X), \psi(x)\big] = \frac{1}{16\pi G_5}\int {\rm d}^5 X\, G^{1/2} R^5(G_{AB}) + \int {\rm d}^4 x\, g^{1/2}\left(\frac{[K]}{8\pi G_5} + \frac{R(g_{\mu\nu})}{16\pi G_4} + L_{\rm m}(g_{\alpha\beta}, \psi, \partial\psi)\right). \tag{6.23}$$

Such a term can be induced at the fundamental level by quantum effects in the bulk; therefore, the DGP model and its modifications are usually called brane-induced gravity models.

The model in question is interesting in that it suggests a simple mechanism of cosmological acceleration. Qualitatively, it is as follows. The DGP-brane tension contains the Einstein tensor

$$S_{\mu\nu} = -\frac{1}{8\pi G_4}\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right) + T_{\mu\nu}\,. \tag{6.24}$$

Therefore, in the absence of matter on the brane ($T_{\mu\nu} = 0$), effective equation (6.7) becomes quadratic in the Einstein tensor and, under the assumption of smallness of the Weyl contribution, allows a 'self-accelerating' solution.

Indeed, the use of the de Sitter ansatz for the metric with the effective Hubble constant H,

$$R_{\mu\nu} = 3H^2 g_{\mu\nu}\,,$$

gives

$$\Pi_{\mu\nu} = -3\left(\frac{H^2}{16\pi G_4}\right)^2 g_{\mu\nu}\,.$$

For $\mathcal{E}_{\mu\nu} \ll (G_5/G_4)^2 H^4 g_{\mu\nu}$, the equation becomes

$$H^2 - L^2 H^4 = 0\,,$$

i.e., gives rise to the de Sitter stage with the Hubble constant at the scale of the DGP model, $H = L^{-1}$, with

$$L = \frac{G_5}{2G_4} = \frac{M_{\rm P}^2}{2M_5^3} \equiv \frac{1}{m}\,. \tag{6.25}$$

Identification of the DGP scale with the current size of the universe horizon $L \sim 10^{28}$ cm allows interpreting this phase of evolution as cosmological acceleration.

Such a conclusion can be substantiated by once again using cosmological ansatz for metric (6.15) and integral of motion (6.17). In the presence of the induced Einstein term, the matching condition on the brane is modified by the four-dimensional curvature,

$$\left.\frac{a'}{ab}\right|_{y=0} = -\frac{8\pi G_5}{3}\,\rho + 2L\left(H^2 + \frac{k}{a^2}\right). \tag{6.26}$$

Its substitution in Eqn (6.17) leads to the new Friedmann equation [56]

$$\epsilon m\sqrt{H^2 + \frac{k}{a^2} - \frac{\mathcal{C}}{a^4}} = H^2 + \frac{k}{a^2} - \frac{8\pi G_4}{3}\,\rho\,, \tag{6.27}$$

where $\epsilon = \pm 1$ is the sign factor, $\epsilon = {\rm sign}\, a'$.

For later stages of the cosmological acceleration ($a \to \infty$), the contribution of dark radiation tends to zero, $\mathcal{C}/a^4 \to 0$, which results in essential simplification of the equation,

$$\sqrt{H^2 + \frac{k}{a^2}} = \frac{\epsilon m}{2} + \sqrt{\frac{8\pi G_4}{3}\,\rho + \frac{m^2}{4}}\,. \tag{6.28}$$

This indicates that the evolution in the DGP model begins at large matter densities $[\rho \gg m^2/G_4 = (M_{\rm P}\, m)^2]$ from the phase of the standard four-dimensional Friedmann model:

$$H^2 + \frac{k}{a^2} \simeq \frac{8\pi G_4}{3}\,\rho\,. \tag{6.29}$$

If the matter density decreases much below the DGP scale

$$\rho \sim (M_{\rm P}\, m)^2\,, \tag{6.30}$$

the evolution critically depends on the initial conditions, i.e., the sign of the normal derivative of the scale factor

$$\epsilon = {\rm sign}\left(H^2 + \frac{k}{a^2} - \frac{8\pi G_4 \rho}{3}\right)$$

[see matching condition (6.26)]. For the negative sign, the brane universe enters the so-called *five-dimensional* dynamical phase with the Hubble constant tending to zero in accordance with the law

$$H^2+\frac{k}{a^2}\simeq\left(\frac{4\pi G_5}{3}\,\rho\right)^2\to 0\,,\qquad H^2+\frac{k}{a^2}<\frac{8\pi G_4}{3}\,\rho\,. \tag{6.31}$$

Finally, for the positive sign, the universe enters the cosmological acceleration phase at late times with the asymptotic value of the Hubble constant at the DGP scale [56, 57]:

$$H^2+\frac{k}{a^2}\simeq m^2+\frac{16\pi G_4}{3}\,\rho\to m^2\,. \tag{6.32}$$

On the background of this acceleration, low-density matter gravitates with Newton's constant, which is twice as large as the fundamental constant G_4.

7. Infrared modifications of the Einstein theory

The DGP model is an infrared modification of the Einstein theory characterized by two energy scales: Planckian, $M_{\rm P}\sim 10^{19}$ GeV, and cosmological, $m\sim 10^{-33}$ eV. The transition between them in cosmological evolution occurs in the intermediate submillimeter region (6.30):

$$(mM_{\rm P})^{1/2}\sim 10^{-3}\ {\rm eV}\sim(10^{-2}\ {\rm cm})^{-1}\,.$$

Therefore, the DGP model does not seemingly contradict the current table-top experiments of the Cavendish type. Unfortunately, the real situation for this model is more complicated and less favorable because it contains a sufficiently low strong-coupling scale at which nonlinear corrections for the Einstein theory may contradict gravitational experiments at significantly larger distances [58, 59]. The strong coupling problem does not appear to manifest itself in the cosmological context because the corresponding nonlinear equations are local, but it emerges in full in spatially inhomogeneous processes.

To analyze the problem, it is necessary to go beyond the framework of the cosmological ansatz and consider generic fields. Naturally, the nonlinearity makes this impossible to do in the context of an exact theory but not in the framework of the perturbation theory on a flat background: the flat background solution of the DGP model, unlike the Randall–Sundrum model, is quite admissible. It turns out that with such an approach, the model exhibits a nonlocal gravitational 'constant' mechanism of type (6.3), (6.4). We first consider a realization of this mechanism in the general form and thereafter demonstrate it by the example of the DGP model.

The idea of replacing the gravitational function with a nonlocal operator $G_{\rm P}\to G(\Box)$, a function of the covariant d'Alembertian $\Box=g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, implies, according to [47], modification of the left-hand side of the Einstein equations

$$\frac{M^2(\Box)}{16\pi}\left(R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}R\right)=\frac{1}{2}\,T_{\mu\nu}\,, \tag{7.1}$$

where the nonlocal Planck mass is a function of the dimensionless combination of d'Alembertian and an additional length scale L interpolating between the Planck constant for small-size matter sources (smaller than L) and the long-distance constant $G_{\rm LD}=G(0)$:

$$\frac{1}{G(\Box)}\equiv M^2(\Box)=M_{\rm P}^2\big(1+\mathcal{F}(L^2\Box)\big)\,. \tag{7.2}$$

If the function of $z=L^2\Box$ satisfies the conditions $\mathcal{F}(z)\to 0$ at $z\gg 1$ and $\mathcal{F}(z)\to\mathcal{F}(0)\gg 1$ as $z\to 0$, the infrared modifications are insignificantly small for the processes varying in space faster than $1/L$; vice versa, they are large for slower processes at wavelengths of the order of L or longer.

The straightforward problem with a set-up of this type arises from the fact that for any nontrivial operator $\mathcal{F}(L^2\Box)$, the left-hand side of Eqn (7.1) does not satisfy the Bianchi identities and cannot be obtained by varying a covariant action. Specifically, a naive attempt to modify the gravitational action in accordance with

$$M_{\rm P}^2\int{\rm d}x\,g^{1/2}R\to\int{\rm d}x\,g^{1/2}M^2(\Box)\,R=M^2(0)\int{\rm d}x\,g^{1/2}R \tag{7.3}$$

makes no sense because, as a result of integration by parts, the action of the covariant d'Alembertian (leftward) selects its zero mode and the nonlocal operator in all regimes reduces to its infrared value $M_{\rm P}^2(0)$.

The noncovariance problem can be circumvented by resorting to the weak-field approximation, where Eqn (7.1) is understood only as the first linear term of the perturbation expansion in powers of the curvature. Its left-hand side must include terms higher than linear in the curvatures, while the nonlocal gravitational action $S_{\rm NL}[g_{\mu\nu}]$ must generate modified equations in agreement with

$$\frac{\delta S_{\rm NL}[g]}{\delta g_{\mu\nu}(x)}=\frac{M^2(\Box)}{16\pi}\,g^{1/2}\left(R^{\mu\nu}-\frac{1}{2}\,g^{\mu\nu}R\right)+O[R_{\mu\nu}^2]\,. \tag{7.4}$$

To obtain the leading term $S_{\rm NL}[g_{\mu\nu}]$, Eqn (7.4) should be functionally integrated in the explicit form [60] using the technique of covariant curvature expansion [61]. This method allows converting the noncovariant expansion in powers of gravitational excitations $h_{\mu\nu}$ into a series with respect to the space–time curvature and its derivatives with covariant nonlocal coefficients.

The starting point is the expansion of the Ricci tensor

$$R_{\mu\nu}=-\frac{1}{2}\,\Box h_{\mu\nu}+\frac{1}{2}\,(\nabla_\mu F_\nu+\nabla_\nu F_\mu)+O[h_{\mu\nu}^2]\,, \tag{7.5}$$

where

$$F_\mu\equiv\nabla^\lambda h_{\mu\lambda}-\frac{1}{2}\,\nabla_\mu h$$

is the linearized de Donder–Fock gauge. Expansion (7.5) can be solved by iterations with respect to $h_{\mu\nu}$ in the form of a nonlocal expansion in powers of the curvature starting from

$$h_{\mu\nu}=-\frac{2}{\Box}\,R_{\mu\nu}+\nabla_\mu f_\nu+\nabla_\nu f_\mu+O[R_{\mu\nu}^2]\,. \tag{7.6}$$

Here, $\nabla_\mu f_\nu+\nabla_\nu f_\mu$ reflects the gauge freedom in the solution stemming from the terms with the harmonic gauge in (7.5).

The result of functional integration of Eqn (7.4) is a nonlocal action [60] starting from the squared curvature:

$$S_{\rm NL}[g_{\mu\nu}] = -\frac{1}{16\pi}\int {\rm d}x\, g^{1/2}\left\{\left(R^{\mu\nu}-\frac{1}{2}\,g^{\mu\nu}R\right)\frac{M^2(\Box)}{\Box}\,R_{\mu\nu}+O[R^3_{\mu\nu}]\right\}. \quad (7.7)$$

Interestingly, in the simplest case where $M^2(\Box) = M_{\rm P}^2 = {\rm const}$, action (7.7) must reproduce the Einstein – Hilbert action; at first sight, this looks unnatural because the action contains no term linear in the curvature.

This apparent paradox is explained by the fact that the Einstein action in an asymptotically flat space with the asymptotic metric

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}\,, \quad h_{\mu\nu} = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty\,,$$

involves the Gibbons – Hawking surface integral over space – time infinity:

$$S_{\rm E}[g_{\mu\nu}] = -\frac{M_{\rm P}^2}{16\pi}\int {\rm d}x\, g^{1/2}R(g) + \frac{M_{\rm P}^2}{16\pi}\int_{|x|\to\infty}{\rm d}\sigma^\mu\,(\partial^\nu h_{\mu\nu}-\partial_\mu h)\,. \quad (7.8)$$

The surface integral can be transformed into a bulk integral of the linear part in $h_{\mu\nu}$ of the scalar curvature $\partial^\mu(\partial^\nu h_{\mu\nu}-\partial_\mu h)$ and can then be covariantly expanded in powers of the curvature using Eqn (7.6). Up to and including quadratic terms, such an expansion has the form [60]

$$\int_{|x|\to\infty}{\rm d}\sigma^\mu\,(\partial^\nu h_{\mu\nu}-\partial_\mu h) = \int {\rm d}x\, g^{1/2}\left\{R-\left(R^{\mu\nu}-\frac{1}{2}\,g^{\mu\nu}R\right)\frac{1}{\Box}\,R_{\mu\nu}+\ldots\right\}. \quad (7.9)$$

Its substitution in Eqn (7.8) cancels the linear Ricci scalar terms, while the quadratic terms reproduce expression (7.7) with the numerical coefficient $M^2(\Box) = M_{\rm P}^2$, which may be taken outside the integrand. The result is a *nonlocal* form of the *local* Einstein action [33, 20, 60]. That its expansion begins with the curvature squared is in line with the massless spin-2 theory. The nonlocality of expression (7.7) is less trivial, being the payment for the explicit covariance of this expansion, in contrast to the local but explicitly noncovariant action in $h_{\mu\nu}$ for the symmetric spin-2 tensor field.

The question is to what extent the infrared modification in (7.7) can be regarded as a universal one. Evidently, in general, operator coefficients at the squares of the Ricci scalar and tensor can be different and the sought generalization takes the form

$$S_{\rm NL}[g_{\mu\nu}] = -\frac{1}{16\pi}\int {\rm d}x\, g^{1/2} \times\left\{R^{\mu\nu}\,\frac{M_1^2(\Box)}{\Box}\,R_{\mu\nu}-\frac{1}{2}\,R\,\frac{M_2^2(\Box)}{\Box}\,R+O[R^3_{\mu\nu}]\right\}, \quad (7.10)$$

where the two nonlocal Planck 'masses' tend to a common limit $M_{\rm P}$ only in the high-energy region ($\Box \gg 1/L$). Moreover, in the infrared limit of the theory, polarizations of the spin-2 field may exist along with additional degrees of freedom that are not taken into account in this expression. However, one can integrate over them under the natural assumption that the additional degrees of freedom do not directly interact with matter fields. The result is just additional contributions to $M_1^2(\Box)$ and $M_2^2(\Box)$ [as is the case with the radion field in Section 5; see the footnote after Eqn (5.4)]. The reduction is possible in practically all cases except for the impossibility of expressing the additional fields in terms of the metric from their equations of motion, i.e., when the additional fields enter the action in the linear form and play the role of Lagrangian multipliers at certain combinations of metric degrees of freedom. It is shown below that this occurs in infrared modifications of the Einstein theory such as the Pauli – Fierz theory and the DGP model.

7.1 The Pauli – Fierz model and the van Damm – Veltman – Zakharov problem

The simplest infrared modification of the Einstein theory is the Pauli – Fierz model of a free massive tensor field. It is described by the quadratic part of Einstein action (7.8) modified by a *noncovariant* mass term on the *flat* space background:

$$S_{\rm mass}[g_{\mu\nu}] = -\frac{M_{\rm P}^2}{16\pi}\int {\rm d}^4x\left(\frac{m^2}{4}\,h_{\mu\nu}^2-\frac{m^2}{4}\,h^2\right), \quad (7.11)$$

$$h_{\mu\nu} \equiv g_{\mu\nu}-\eta_{\mu\nu}\,, \quad h \equiv \eta^{\mu\nu}h_{\mu\nu}\,. \quad (7.12)$$

This is the sole combination of mass terms that guarantees the absence of ghosts in the theory.

In the presence of the conserved matter sources, the linear equations of motion in the Pauli – Fierz model are

$$R_{\mu\nu}-\frac{1}{2}\,\eta_{\mu\nu}R+\frac{m^2}{2}\,(h_{\mu\nu}-\eta_{\mu\nu}h) = 8\pi G_4 T_{\mu\nu}\,, \quad (7.13)$$

where $R_{\mu\nu}$ denotes the linear part of the Ricci tensor in the graviton field. Differentiating this equation and taking the linearized Bianchi identity and conservation of $T_{\mu\nu}$ into account yield the 'gauge' for $h_{\mu\nu}$:

$$\partial^\mu(h_{\mu\nu}-\eta_{\mu\nu}h) = 0\,, \quad (7.14)$$

which implies the vanishing of the linearized Ricci scalar,

$$R = \partial^\mu\partial^\nu h_{\mu\nu}-\Box h = 0\,. \quad (7.15)$$

(We note that Eqn (7.15) is satisfied even in the presence of a nonvanishing trace of the matter stress tensor.) As a result, the gravitational field generated by a matter source becomes

$$h_{\mu\nu} = -\frac{16\pi G_4}{\Box-m^2}\left(T_{\mu\nu}-\frac{1}{3}\,\eta_{\mu\nu}T\right)+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu \quad (7.16)$$

up to longitudinal terms making no contribution to the interaction with conserved sources. [17]

It is essential that the tensor nature of the massive graviton propagator in Eqn (7.16) is different from the case

[17] Because the Pauli – Fierz theory is not gauge invariant, the longitudinal part is fixed and determined by the vector

$$\xi_\mu = -\frac{8\pi G_4}{3m^2}\,\frac{1}{\Box-m^2}\,\partial_\mu T\,.$$

of Einstein's general relativity theory, that is, the trace of $T_{\mu\nu}$ enters the general relativity theory with the coefficient 1/2 and the Pauli–Fierz model with the coefficient 1/3. This discrepancy is preserved in the vanishing mass limit and underlies the so-called van Damm–Veltman–Zakharov problem [62] according to which the massless limit of the Pauli–Fierz model does not correspond to the Einstein theory of the massless graviton. Such a situation is phenomenologically unacceptable because, for an arbitrarily small graviton mass, it leads to a wrong deflection of light beams in the sun's field and abnormal movement of the perihelium of Mercury.

The root cause of the problem lies in the additional longitudinal degree of freedom absent in the Einstein theory and present in the Pauli–Fierz model irrespective of the graviton mass value. From the standpoint of the general scheme of infrared modifications of the theory, this degree of freedom must enter Eqn (7.10) in the form of a Lagrangian multiplier responsible for additional equation (7.15). Without this degree of freedom, the effective equations following from action (7.10) do not reproduce the tensor structure of (7.16) regardless of the choice of the operators $M_1^2(\Box)$ and $M_2^2(\Box)$.

Indeed, it is easy to see that action (7.10) leads to the linear gravitational potential of the form

$$h_{\mu\nu} = -\frac{16\pi}{M_1^2(\Box)\Box}\left(T_{\mu\nu} - \frac{1}{2}\,\alpha(\Box)\,\eta_{\mu\nu}T\right) + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,, \tag{7.17}$$

where the operator coefficient

$$\alpha(\Box) = \frac{2M_2^2(\Box) - M_1^2(\Box)}{3M_2^2(\Box) - 2M_1^2(\Box)} \tag{7.18}$$

takes the Pauli–Fierz numerical value $\alpha_{\mathrm{PF}} = 2/3$ only in the singular limit $M_1^2(\Box) \to 0$. It is equally easy to see that the inclusion of (7.15) with the Lagrangian multiplier into (7.10) improves the situation and leads to the correct retarded potential (7.16) with the choice of nonlocal operators

$$M_1^2(\Box) = M_1^2(\Box) = \frac{1}{G_4}\,\frac{\Box - m^2}{\Box}\,. \tag{7.19}$$

7.2 The Dvali–Gabadadze–Porrati model and the strong coupling problem

The situation in the DGP model is analogous to that in the Pauli–Fierz model with the more infrared-soft mass term ($m^2 \to m\sqrt{-\Box}$), where the role of m is played by the scale in (6.25). To demonstrate this, an effective brane action is constructed by integrating over fields in the bulk in Eqn (6.23). For this, it is necessary, as was done in Section 5, to expand the initial action in gravitational perturbations,

$$G_{AB}(X) = \eta_{AB} + H_{AB}(X)\,, \tag{7.20}$$

solve the linear equations for $H_{AB}(X)$ in the bulk with the fixed boundary conditions on the brane, and substitute the result into the action quadratic in fields. For this purpose, a system of coordinates should be used in which the brane position is fixed by a constant value of the fifth coordinate $X^5 \equiv y = 0$.

Fixing the coordinate gauge in the bulk is achieved by adding a term quadratic in the linearized de Donder–Fock gauge conditions to action (6.23), which breaks gauge invariance:

$$S_{\mathrm{gauge}}[H_{AB}] = -\frac{M_5^3}{16\pi}\,\frac{1}{2}\int \mathrm{d}^5X\,\eta^{AB}F_AF_B\,, \tag{7.21}$$

$$F_A = \partial^B H_{AB} - \frac{1}{2}\,\partial_A H\,. \tag{7.22}$$

In this gauge, the equations of motion in the bulk assume the simplest form and give rise to the boundary value problem

$$\Box_5 H_{AB}(X) = 0\,, \tag{7.23}$$

$$H_{AB}(x,y)\Big|_{y=0} = h_{AB}(x)\,,\qquad h_{AB}(x) \equiv \big(h_{\mu\nu}(x), N_\mu(x), h_{55}(x)\big)\,. \tag{7.24}$$

The solution of problem (7.23), (7.24) is simpler compared to the Randall–Sundrum model with the curved background because $\Box_5 = \Box + \partial_y^2$ on the flat background. This solution, nonsingular at the infinity of the five-dimensional volume, can be written in the elegant, simple form

$$H_{AB}(x,y) = \exp\,(-y\Delta)\,h_{AB}(x)\,, \tag{7.25}$$

in terms of the auxiliary operator

$$\Delta = \sqrt{-\Box}\,. \tag{7.26}$$

(We treat the case of Lorentzian space–time as the analytic continuation from the Euclidean field theory in which the operator $\Box$ is negative definite, and hence $H_{AB}(x,y)$ vanishes as $y \to \infty$.)

Substituting the thus obtained solution into the five-dimensional part of the DGP-action in (6.23) (with the five-dimensional curvature and the Gibbons–Hawking surface integral) and taking gauge term (7.21) into consideration leads to [58]

$$S_5[G_{AB}] + S_{\mathrm{gauge}}[H_{AB}] = \frac{M_4^2}{16\pi}\,\frac{m}{4}\int \mathrm{d}^4x \times\left(-\tilde{h}^{\mu\nu}\Delta\tilde{h}_{\mu\nu} + \frac{1}{2}\,\tilde{h}\,\Delta\tilde{h} + \tilde{h}\Delta h_{55} - \frac{1}{2}\,h_{55}\Delta h_{55}\right), \tag{7.27}$$

where $m = 2M_5^3/M_4^2$ is the DGP gauge (6.25) and $\tilde{h}_{\mu\nu}$ is the combination of perturbation of the brane-induced metric and the shift functions in the fifth dimension $G_{5\mu} = N_\mu$:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{\Delta}\,(\partial_\mu N_\nu + \partial_\nu N_\mu)\,. \tag{7.28}$$

We note that combination (7.28) is gauge invariant under four-dimensional transformations

$$\delta_\xi h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,,\quad \delta_\xi N_\mu = -\Delta\xi_\mu\,,\quad \delta_\xi h_{55} = 0\,, \tag{7.29}$$

which, in their turn, are residual gauge transformations on the brane,

$$\delta_\Xi H_{AB} = \partial_A \Xi_B + \partial_B \Xi_A$$

with the vector field

$$\Xi^\mu(x,y) = \exp\,(-y\Delta)\,\xi^\mu(x)\,,\qquad \Xi^5(x,y) = 0 \tag{7.30}$$

leaving de Donder gauge (7.22) invariant and not displacing the brane from $y = 0$.

Thus, as expected, the effective action (7.27) induced from the bulk proves to be an invariant of four-dimensional transformations, but this invariance is actually realized with the use of Stueckelberg fields N_μ that are normally introduced by hand for covariantization of a gauge-noninvariant action. It was shown above that these fields can be eliminated in terms of metric variables by varying (7.27) with respect to N_μ and thereafter substituted into the action.

These procedures lead to

$$N_\mu = \frac{1}{\Delta}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\,\partial_\mu h - \frac{1}{2}\,\partial_\mu h_{55}\right) \tag{7.31}$$

and an explicitly invariant expression for $\tilde{h}_{\mu\nu}$ in terms of the linearized Ricci tensor:

$$\tilde{h}_{\mu\nu} = -2\,\frac{1}{\Box}\,R_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{\Box}\,h_{55}\,. \tag{7.32}$$

Their substitution yields

$$S_5 + S_{\text{gauge}} = \frac{M_4^2}{16\pi}\,m\int \mathrm{d}^4x \times\left(-R^{\mu\nu}\,\frac{\Delta}{\Box^2}\,R_{\mu\nu} + \frac{1}{2}\,R\,\frac{\Delta}{\Box^2}\,R - R\,\frac{\Delta}{\Box}\,h_{55}\right), \tag{7.33}$$

where the variable h_{55} cannot be eliminated in terms of the metric (this situation was discussed earlier).

Adding (7.33) to the four-dimensional part of the DGP action rewritten in nonlocal form (7.7) eventually gives the quadratic part of the effective action on the brane,

$$S^{\text{eff}}_{\text{DGP}}[g_{\mu\nu}] = -\frac{M_4^2}{16\pi}\int \mathrm{d}x\,g^{1/2} \times\left\{\left(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R\right)\frac{\Box - m\Delta}{\Box^2}\,R_{\mu\nu} + m\Pi R\right\}, \tag{7.34}$$

where the Lagrangian multiplier in front of the scalar curvature Π is related to the h_{55} component of the five-dimensional metric as

$$h_{55} = -2\Delta\Pi\,. \tag{7.35}$$

The variable Π was introduced in [58] as the longitudinal part of a function of the five-dimensional displacement: $N_\mu = \partial_\mu\Pi + N'_\mu$. It parameterizes brane bending as a five-dimensional diffeomorphism with the vector field

$$\Xi^A(x,y) = \delta_5^A \exp(-y\Delta)\,\Pi(x)\,, \quad \delta_\Xi H_{55} = 2\,\partial_y \Xi_5$$

that does not break the de Donder gauge in the bulk but displaces the brane by $\Xi(x,0) = \Pi(x)$ and therefore is not a symmetry of the action. The diffeomorphism does not manifest itself locally in the bulk, and all its effect is reduced to the contribution on the brane that starts from the local term $m\Pi R$ in the perturbation theory.

Thus, the DGP model on the brane is effectively described in the linear approximation by the Pauli – Fierz model with nonlocal mass term (7.33) generated from the bulk. Expression (7.33) is actually a covariant completion of the Pauli – Fierz term (7.11) with the nonlocal mass $\sqrt{m\Delta}$.[18] The gravitational potential from a matter source in this model has the form analogous to (7.17):

$$h_{\mu\nu} = -16\pi G_4\,\frac{1}{\Box - m\Delta}\left(T_{\mu\nu} - \frac{1}{3}\,\eta_{\mu\nu}T\right) + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \tag{7.36}$$

The long-distance action of matter on the brane in (7.36) is determined by the propagator that coincides with the four-dimensional one in the small-interval region ($|x| \ll L = 1/m$, $|\Box| \gg m^2$). In contrast, in the region of ultralarge distances, the long-distance action becomes five-dimensional:

$$\frac{1}{\Box - m\Delta}\,\delta(x) \simeq -\frac{1}{(2\pi)^4 m}\int \mathrm{d}^4p\,\frac{\exp(\mathrm{i}px)}{\sqrt{p^2}} = -\frac{2}{m}\,\frac{1}{(2\pi)^5}\int \mathrm{d}^5p\,\frac{\exp(\mathrm{i}px)}{p^2 + p_5^2} = \frac{2}{m}\,\frac{1}{\Box_5}\,\delta^{(5)}(X)\Big|_{y=0}\,, \quad |x| \gg \frac{1}{m}\,. \tag{7.37}$$

This phenomenon is usually interpreted as gravitational leakage into the bulk: a four-dimensional graviton is metastable and decays with the lifetime $L = 1/m$.

Unlike in the Randall – Sundrum model, localization of a four-dimensional graviton on the brane is not absolute, similarly to the first brane model by Gregory – Rubakov – Sibiryakov (GRS) [65], where the law of gravity and propagation of gravitational waves on the brane are generally relativistic in four dimensions only at intermediate distances and become five-dimensional at ultra-large distances. However, the GRS model suffers from ghost states with negative energy [66 – 68], whereas the DGP model is ghost-free (similarly to the Pauli – Fierz theory). Another property of gravitational potential (7.36) is its tensor dependence on $T_{\mu\nu}$: it does not coincide with the generally relativistic theory and corresponds to the Pauli – Fierz theory at all distance scales. It is noteworthy that the kinematic variable of brane embedding in the bulk, Π, is here analogous to the radion mode discussed in Section 4 in the context of the Randall – Sundrum model, where it ensures restoration of the correct tensor law (4.42), (4.43). However, the radion mode is unable to do the same in the DGP model, and the theory suffers from the van Damm – Veltman – Zakharov problem [62].

It turns out that the van Damm – Veltman – Zakharov problem is directly related to another difficulty intrinsic in the DGP model, that is, the presence of a rather low strong-coupling energy scale. As shown above, from the standpoint of cosmological evolution, the transition between the Einstein gravity phase and its infrared regime in the DGP model occurs in submillimeter range (6.30),

$$\Lambda_{\text{cross}} = \sqrt{mM_{\mathrm{P}}}\,, \tag{7.38}$$

or at the length scale $\Lambda^{-1}_{\text{cross}} \sim 10^{-2}$ cm.

However, in the perturbation theory that we have to use for the description of nonlocal processes, the analysis of higher-order quantities and their contribution to quantum effects acquires importance. In higher orders, the brane vibration mode Π gives rise to composite operators composed of powers of Π, N_ν, $h_{\mu\nu}$ and their derivatives that are suppressed by the inverse powers of M_{P} and m [69]. Their contribution becomes large at the strong coupling

[18] Covariant structures of the type such as the realization of the nonlocal cosmological ‘constant’ are also discussed in the renormalization theory context in Refs [63, 64].

scale [58]

$$\Lambda_{\rm strong} = (m^2 M_{\rm P})^{1/3} \sim (10^3\ {\rm km})^{-1}\,, \tag{7.39}$$

which is much below the submillimeter scale: $\Lambda_{\rm strong} \ll \Lambda_{\rm cross}$. The cause is that the kinetic term of the vibrational mode Π in (7.34), having the form $M_{\rm P}^2 m\,(\Pi R) \sim M_{\rm P}^2 m\,(\partial\Pi\,\partial h)$, originates exclusively from its mixing with the metric field and is small due to the smallness of m.

As a result, the Π-mode acquires a kinetic term of the form $M_{\rm P}^2 m^2 (\partial\Pi)^2$ upon diagonalization of the full kinetic term. Transition to the canonically normalized variable $\hat{\Pi}$, $\Pi = \hat{\Pi}/(mM_{\rm P})$, gives rise to negative powers of the small variable m of increasingly higher order in the expansion in powers of Π. Therefore, the composite operators of higher dimensions prove to be suppressed by factors having the form $1/M_{\rm P}^p m^q$, i.e., become essential at the energy scale $\Lambda_{p,q} = (M_{\rm P}^p m^q)^{1/(p+q)}$. As is shown by analysis in Ref. [58], the lowest scale occurs at the cubic term in Π and is given by expression (7.39). A similar situation takes place in the nonlinear Pauli–Fierz model (with the full Einstein term in the massless part of the action): the strong coupling scale in it equals $(m^4 M_{\rm P})^{1/5}$ [69]. [19]

Thus, the DGP model is characterized by the appearance of the hierarchy of length scales corresponding to the horizon scale, strong coupling scale, the scale of transition to the infrared phase, and finally the Planckian scale of quantum gravity,

$$L \gg L_{\rm strong} \gg L_{\rm cross} \gg L_{\rm P}\,. \tag{7.40}$$

The brane vibration mode begins to strongly interact at distances below $L_{\rm strong} \sim 1000$ km and violates agreement with the data of Cavendish-type experiments and celestial mechanics.

On the whole, the entire Einstein theory in which the cosmological phase occurs at $L_{\rm cross}$, in this hierarchy, falls out of the applicability range of the perturbation theory with wavelengths larger than $L_{\rm strong}$. Actually, quantum effects of the vibrational mode become strong at this scale, and the DGP model in the infrared region becomes sensitive to its ultraviolet behavior, which impairs its predictive value outside the framework of the fundamental theory.

8. Conclusion: problems and prospects

The physics of extra dimensions and brane cosmology suggests many new interesting mechanisms for the solution of the hierarchy problem, cosmological constant problem, and cosmological acceleration problem. These areas of research open up new opportunities for the semiphenomenological approach to the creation of a unified field theory, which is probably also approached from the fundamental side in the framework of string theory. Evidently, both fields come in contact via the AdS/CFT correspondence mechanism, which explains localization of a massless graviton on a brane and restoration of the Einstein theory of gravity in the brane world.

The Randall–Sundrum model, brane-induced gravity models, and other models of infrared modification of the Einstein theory remain too simple for a consistent explanation of the main fundamental problems in the early and late universe, but their combinations can help to describe it more realistically. In particular, the problem of restoration of the Einstein theory phase in the DGP model [with the correct four-dimensional tensor structure of the propagator in (7.36)] can be resolved by means of its synthesis with the Randall–Sundrum model.

When a brane has tension along with the Einstein term and the bulk has the negative cosmological constant related to the tension by expression (4.7), the gravitational potential in the range where the linear analysis is valid ($\Box \ll \Lambda^2_{\rm strong}$) is realized in form (7.17) with the operator functions [72]

$$M_{\rm I}^2(\Box) = M_4^2\left[1 + \frac{m\,K_1(l\Delta)}{\Delta\,K_2(l\Delta)}\right], \tag{8.1}$$

$$\alpha(\Box) = \frac{2}{2+lm} + \frac{2}{3}\,\frac{lm}{2+lm}\left[1 + \frac{K_1(l\Delta)}{l\Delta\,K_2(l\Delta)}\right], \tag{8.2}$$

in terms of Macdonald functions of the first and the second orders $K_{1,2}(x)$ and the curvature radius l of the background AdS bulk. In the range of distances $1/m \gg 1/\Delta \gg l$, the gravitational potential describes the four-dimensional generally relativistic gravity law with $\alpha(\Box) \simeq 1$ and the effective Planck mass [72]

$$M_{\rm P}^2 = M_4^2\left(1 + \frac{l}{2L}\right) \simeq M_4^2\,. \tag{8.3}$$

In other words, in the case of the horizon scale hierarchy and the bulk curvature radius $1/m = L \gg l$, such a model contains the Einstein gravity phase that does not suffer from the van Damm–Veltman–Zakharov problem. In fact, this is a generalization of the well-known result that such a problem is nonexistent for the free massive spin-2 field on the (A)dS-space background [73, 74] with the cosmological constant Λ in the limit $m^2/\Lambda \to 0$. Unfortunately, the synthesis of the Dvali–Gabadadze–Porrati and Randall–Sundrum models does not solve the problem of the low strong-coupling scale and the presence of ghost instabilities as shown in a broad class of models with nonfactorable geometry in the bulk and the induced Einstein term on one of the branes [68]. Specifically, the self-accelerating branch of the cosmological solution in the DGP model, Eqn (6.32), is unstable with respect to perturbations of the vibrational mode Π, which is a ghost mode on the background of the (quasi)-de Sitter solution [58, 79].

Another feasible generalization consists in an increase in the number of extra dimensions in the DGP model (brane co-dimension in the bulk) from 1 to $N = D - 4 \geqslant 2$ as dictated by the simplest ADD model (3.9) and can be realized in string theory [76, 77]. This generalization may be of value because it appears to contain an interesting qualitative mechanism for the solution of the cosmological constant problem [78]. When the vacuum energy on the brane becomes larger than the gravity scale in the bulk, $\mathcal{E} \gg M_D^4$ (where $G_D = (M_D)^{2-N}$ is the D-dimensional gravitational constant), it gives rise to an inflation of the brane world with the Hubble constant *decreasing* with the growth of $\mathcal{E}$ at $N > 2$:

$$H^2 \sim M_D^2\left(\frac{M_D^4}{\mathcal{E}}\right)^{1/(N-2)}. \tag{8.4}$$

[19] At the classical level, the strong coupling problem and the van Damm–Veltman–Zakharov problem in the nonlinear Pauli–Fierz model can probably be overcome by taking nonlinear terms into consideration [70]. However, at the quantum level, the theory retains the strong coupling scale [71] that can at maximum be raised to $(m^2 M_{\rm P})^{1/3}$ by including higher-order operators [69].

(Interestingly, this formula interpolates between the usual four-dimensional Friedmann evolution law $H^2 \sim \mathcal{E}$ at $N = 0$, the evolution law $H \sim \mathcal{E}$ in the five-dimensional DGP model, and the inverse dependence for $N > 2$.)

In the DGP model, the infrared modification scale analogous to (6.25) is

$$L = \frac{M_4}{M_D^2} \,. \tag{8.5}$$

A valuable property of this model is that the propagator in retarded potential (7.17) has the correct four-dimensional tensor structure with $\alpha = 1$ at intermediate distances $1/M_D \ll |x| \ll L$, for which the four-dimensional law of gravity is realized. It turns out, however, that the scalar sector of the metric on the brane contains a ghost of a tachyonic nature, whose negative contribution to the residue of the propagator restores its normal tensor nature [67]. The presence of tachyons and ghosts suggests classical and quantum instabilities, which questions the applicability of the model as a candidate for the consistent infrared modification of the Einstein theory [67].

Two approaches were proposed to circumvent this difficulty. It turns out that brane models with the codimension greater than one show an interesting phenomenon of infrared – ultraviolet mixing because the brane-to-brane propagator (the inverse of the quadratic part of the effective brane action) is singular and requires regularization [79]. Regularization of an ultraviolet nature essentially affects the behavior of the resultant effective theory in the infrared region.

In Ref. [67], regularization was carried out using the higher derivatives in the action that made the D-dimensional gravity in the bulk soft in the ultraviolet limit. An alternative approach was applied in [80], where regularization was in the form of brane smearing. (Moreover, the scalar of the four-dimensional curvature on the brane was replaced by the D-dimensional scalar, which is not forbidden by covariance considerations even though it looks somewhat unnatural from the geometric standpoint.) As a result, the DGP model proved to be free from ghosts and tachyons and its five-dimensional version even acquired the strong coupling scale $\Lambda_{\rm strong} = (M_5^7 m^2)^{1/9}$ substantially larger than (7.39) [81].

In another variant of the DGP model, with $N \geqslant 2$, the treatment of the poles of the Green's function from brane to brane was different from that in Ref. [67] and equivalent to a different choice of boundary conditions for the Green's function [82]. This resulted in the restoration of unitarity in the spectral representation of the propagator: the contribution of the ghost tachyon found in Ref. [67] was taken into consideration not via the standard Wick rotation from Euclidean theory but in the sense of the principal value, which does not make a nonunitary negative contribution to the spectral density (which appears to be an analog of the Lee and Wick prescription in the local field theory with ghosts [83]).

In this case, the propagator lost analyticity in the complex plane of the four-dimensional momentum, which signified the loss of causality. However, the loss of causality showed up only at the horizon scale $L = M_4/M_D^2$; therefore, this property remained acceptable from the standpoint of observable physics of intermediate distances. (We note that the idea of acausality in the form of fixing the asymptotically de Sitter boundary conditions in the remote future was suggested in [47] as a constituent part of an infrared modification of the Einstein theory with a nonlocal gravitational 'constant'.)

It is straightforward to see that brane models of the universe with large-size extra dimensions have far-reaching prospects for further development. It is worthwhile to note in conclusion that brane models, in spite of their exotic character, can greatly contribute to the explanation of the problems of the contemporary universe either by themselves or by prompting novel mechanisms beyond the framework of the brane concept proper. In particular, we note the mechanism of Lorentz invariance violation that can lead, for example, to the infrared modification of the Einstein theory by introducing a *Lorentz-noninvariant* mass term.

As shown in Ref. [84], such a variant of the Pauli – Fierz theory allows circumventing the ghost problem as well as the low strong-coupling scale problem and the van Damm – Veltman – Zakharov problem. In this model, the nonperturbative scale proves to be equal to $\Lambda_{\rm strong} = \sqrt{m M_{\rm P}}$ and consistent, despite hierarchy (7.40), with the submillimeter scale of the infrared – ultraviolet transition in the cosmological evolution of the DGP model, $\Lambda_{\rm cross} = L_{\rm cross}^{-1} \sim (0.1 \text{ mm})^{-1}$, such that its Einsteinian phase falls into the region of applicability of the perturbation theory. Another promising mechanism of infrared modification is represented by the so-called ghost condensation model [85] that can also include cosmological acceleration (albeit by the introduction of a residual cosmological term), demonstrates the Lorentz invariance violation [86], and may be treated as the Higgs gravity phase.

Collectively, these make the subject of brane physics and cosmology interminably attractive and potentially able to shed new light on some of the unsolved riddles of the early, contemporary, and future universe.

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