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Application of cybernetic methods in physics

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Abstract. Basic aspects of the subject and methodology for a new and rapidly developing area of research that has emerged at the intersection of physics and control theory (cybernetics) and emphasizes the application of cybernetic methods to the study of physical systems are reviewed. Speed-gradient and Hamiltonian solutions for energy control problems in conservative and dissipative systems are presented. Application examples such as the Kapitza pendulum, controlled overcoming of a potential barrier, and controlling coupled oscillators and molecular systems are presented. A speed-gradient approach to modeling the dynamics of physical systems is discussed.

1. Introduction. Physics and cybernetics

The exchange of matter and exchange of energy are the two types of interaction between a system and the external medium commonly studied in physics. In the second half of the 20th century, attention focussed on a third type of interaction, the exchange of information. The relation between information and other characteristics of a physical system has been discussed in the works of Szillard [1], Gabor [2], Brillouin [3], R L Stratonovich [4, 5], and others [6 – 8]. In 1994, *Uspekhi Fizicheskikh Nauk* published an article by B B Kadomtsev, "Dynamics and information" [9], in which the properties of physical systems capable of exchanging information with the external medium were analyzed in detail (in Ref. [9], such systems were called informationally open). The exchange of information substantially enriches the class of possible behaviors of the system and poses new problems, whose solutions have been recently studied by many researchers (e.g., see Refs [10 – 13]). In particular, researchers have been studying aspects of transferring and processing information in quantum systems. The results of these studies have been used to develop the principles regarding the design and functioning of quantum computers [13, 14].

At the same time, Kadomtsev concluded, first in his article [9] and later in his book [10], that the study of the relationships between the dynamics and information in physical systems constitutes only the first step; further studies must allow not only for statistical laws of information transfer but also for the goals and ways of using such transfer in the system. In particular, he noted that "one should go further and study the hierarchization of structures, the formation of structures with

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memory, the feasibility of acquiring information from outside, storing it in the memory, and using information to control and process the stored information with the aim of achieving optimal control'' [9].

The decade that has passed since the publication of article [9] has proven the validity of this prognosis. The number of publications devoted to aspects of control in physical systems or, in a broader sense, to the application of cybernetic methods in physics has soared. The present review is devoted to a discussion of some results in this area of research.

Cybernetics has a well-defined date of birth: the day in 1948 when book [15] by US mathematician Norbert Wiener (1894 – 1964) was first published. Wiener defined cybernetics as ''the science of control and communications in the animal and machine.'' Today, cybernetics is understood as control theory in a broad sense, while cybernetic methods incorporate not only control methods but also methods for estimating variables and parameters of systems, methods of filtering, optimization, pattern recognition, etc.

During the last decades, control theory has rapidly developed in connection with engineering. Nevertheless, cybernetic terms rarely appeared in the leading physics journals and the effect of cybernetics on physics has been practically nil. This comes as no surprise, because the two sciences are very different: physics, and mechanics in particular, is a classical *descriptive* science, while cybernetics (control theory) is ''in a certain sense a paradigm of *prescriptive* sciences'' [16]. This means that while the goal of physics is to study and *describe systems*, the goal of cybernetics is to transform the systems with the aim of forming prescribed behavior.

We note that although automatic and automated systems of measurement and control have been used in experimental physical research for a long time and a modern physical experiment cannot be conducted without automatics, control usually plays an auxiliary role, ensuring that the experimental parameters are kept within prescribed limits. Here, there is certainly no new relationship between physics and control theory, even when the use of cybernetic methods reveals new theoretical results and essentially new physical effects.

The situation changed dramatically in the 1990s with the beginning of the rapid development of two new areas of research: 'controlling chaos' and 'controlling quantum systems'. The history of controlling chaos is especially graphic. In 1990, *Physical Review Letters* published an article by E Ott, C Grebogi, and J A Yorke of the University of Maryland (USA), which was titled ''Controlling chaos'' [17]. The response to the article was a real upsurge in publications. According to the *Science Citation Index*, by the beginning of the 21st century, the overall yearly number of publications in this field exceeded 400 papers, with the total number exceeding 3000. In their paper, Ott et al. stated that even weak control in the form of feedback applied to a nonlinear (chaotically vibrating) system may dramatically alter the dynamics and properties of the system, e.g., chaotic motion may transform into periodic. This paper caused an avalanche of publications in which experimentally, and very often via computer simulation, it was shown how control (with or without feedback) can affect the behavior of various real and model physical systems. The method became known as the Ott – Grebogi – Yorke (OGY) method, and the number of citations of Ref. [17] in 2002 exceeded 1300. Most papers on this topic are published in physics journals, and the authors of many studies are physicists. Therefore, there is no doubt that the new area of research belongs to physics. The development of control methods for chaotic processes has been stimulated by the new demands of the emerging applications in laser and chemical technologies, communication techniques, biology, and medicine.

We note that in the 1980s, in papers published by researchers from Moscow State University, it was shown that chaotic processes can be transformed into periodic processes by applying an external harmonic excitation to the system, which can be interpreted as feedforward control in the system (e.g., see Refs [18 – 24]). Unfortunately, there was no substantial reaction to these publications in the scientific community.

Strange as it might seem, in many works, the tools of modern nonlinear control theory have been used very sparingly, although the key role of nonlinearity of the system in such phenomena has always been stressed. The explanation is simple: the problems encountered often differ from the traditional problems of automatic control — instead of the classical control goals, such as taking the path to a certain point (the regulation problem) and moving it closer to a specified motion (the program-control problem and the tracking problem), one sets goals that are less rigid, namely, creating modes with partially specified properties, qualitatively changing the phase portraits of systems, and synchronizing chaotic oscillations. On the other hand, more rigorous restrictions are imposed on the 'weakness' of the control action, corresponding to the physically obvious requirement that human intervention into the natural course of the investigated process be minimal. Later, it turned out that such statements are important and interesting not only in relation to chaotic systems but also for a broader class of problems of oscillatory-process control [25]. The next natural step that was taken dealt with the statement of the general problem of studying the properties of a physical system that can be created or altered by applying (weak) feedback to the system [26 – 28]. There is an increasing number of publications on employing the methods of cybernetics — control theory — in the search for new physical effects in various areas of physics and mechanics: active control of vibration and noise, optimal control of thermodynamic systems, control of particle beams in accelerators, and plasma stabilization in problems of controlled nuclear fusion.

The last decade has seen an upsurge in studies in the field of controlling molecular and quantum systems. We believe this area of research was permeated by the ideas of control before any other area. The history of ideas of controlling molecular systems goes back to the Middle Ages, when alchemists looked for ways to influence chemical reactions in their attempts to transform lead and mercury into gold. The next important step was taken by British physicist James Clerk Maxwell, who in 1871 invented a hypothetical creature (the Maxwell demon) capable of measuring the velocities of individual molecules of a gas in a vessel and of directing fast molecules into one part of the vessel and slow molecules into another part. 20th-century physicists referred to the Maxwell demon when they studied the relationship between energy and information [1, 3, 4, 7, 29]. The understanding that any measurements and calculations require resources and the expenditure of these resources must be minimized [8, 14] led

to the idea of designing quantum computers [13, 14]. Experimental implementations of the idea of the Maxwell demon have been discussed in Refs [30, 31].

By the end of the 1970s, there appeared the first statements and solutions of problems dealing with controlling quantum systems, based on methods of control theory; in particular, criteria were set for the controllability of quantum systems [32]. In the 1980s and 1990s, advances in laser technology led to the design of lasers capable of generating pulses of coherent radiation whose length was roughly several femtoseconds. The pulse length of such a laser is comparable to the period of the natural vibrations of molecules, which (at least theoretically) makes a femtosecond laser a device for controlling the behavior of individual atoms and molecules. The development of new technologies has stimulated rapid growth in research on coherent control of molecular systems based on classical and quantum models [33, 34]. The number of publications in the area of control of quantum systems alone exceeds 600 papers per annum. The use of the methods of control theory has opened new horizons in studying and changing the motion of atoms and molecules and determines the means and, obviously, the natural limits of interference with the intricate natural processes occurring in the microworld.

The usefulness of cybernetic methods in physics has been a topic for a long time. For instance, in Turchin's book *The Phenomenon of Science*, written in the 1960s, there is the following passage: "Physicochemical, biological, and social phenomena are described in terms cybernetic concepts with equal success" [35]. In our opinion, however, no unified description of the various applications of cybernetic methods in physics has been done so far. The goal of the present review is to focus attention on the rapidly developing field of studies of physical systems by cybernetic methods and to formulate and illustrate, using examples, the various general principles that underlie such research.

2. Peculiarities of using cybernetic methods in physics

For brevity, by *cybernetic physics* we understand the area of research that studies physical systems using cybernetic methods. Below, we describe the main features of the subject of this area of research, which includes the models and problems in controlling physical systems and the methods based on those of control theory.

2.1 Models of the controlled system dynamics

The formal statement of any control problem begins with the selection of a model of the dynamics of the controlled system and the model of the control goal. Even if the controlled system model is not given or is unknown, it must be defined in one way or another. The difference between cybernetic models and traditional dynamical models used in physics and mechanics is that in the former, the inputs and outputs of the system are specified explicitly, because this is needed in order to build feedbacks. Several classes of controlled system models have been considered in the literature on the control of physical systems. Here, we limit ourselves to the often encountered models with lumped parameters, models described by ordinary differential equations (ODEs) in the state space,

$$\dot{x} = F(x, u)\,, \tag{2.1}$$

where $x = x(t)$ is an n-dimensional vectors of the variables of the controlled system state,[1] $\dot{x} = \mathrm{d}x/\mathrm{d}t$; and $u = u(t)$ is an m-dimensional vector of inputs (controlling variables). We let the components of a state vector be denoted by $x_1, \ldots, x_n$ and the components of the control action vector by $u_1, \ldots, u_m$. Thus, equation of state (2.1) is simply a compact notation for the ODE system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)\,, \qquad i = 1, 2, \ldots, n\,. \tag{2.2}$$

Models of type (2.1) can be used to describe two physically distinct controlled system classes.

(1) *Objects with coordinate control*. The input variables are certain physical quantities, such as forces, moments, electric and magnetic field strengths, etc. For instance, the model of a controlled oscillator (pendulum) is given by

$$J\ddot{\varphi} + \varrho\dot{\varphi} + mgl\sin\varphi = u\,, \tag{2.3}$$

where $\varphi = \varphi(t)$ is the angle of deflection of the pendulum from the vertical line (output quantity); $u = u(t)$ is the controlling torque (input quantity); and J, m, l, g, and ϱ are, respectively, the moment of inertia, mass, length of the pendulum, acceleration of gravity, and friction coefficient. Equation (2.3) can be reduced to (2.1), with the state vector of the form $x = (\varphi, \dot{\varphi})^{\mathrm{T}}$.

(2) *Objects with parametric control*. The input parameters are variations of the physical parameters of the system, e.g., $u(t) = p - p_0$, where p_0 is the nominal value of the physical parameter p. For instance, we suppose that the pendulum is controlled by slowly changing its length. Model (2.3) then becomes

$$J\ddot{\varphi} + \varrho\dot{\varphi} + mg\bigl[l_0 + u(t)\bigr]\sin\varphi = 0\,, \tag{2.4}$$

where l_0 is the initial length of the pendulum.

The authors of some publications prefer to speak of the above two variants of controlled system as entirely different controlled systems. However, from the standpoint of cybernetics, this difference is unessential when we are dealing with processes described by nonlinear models of type (2.1). Examining the cases of coordinate and parametric control separately is meaningful only in a linear controlled system model, because a linear system with a feedback linear in the coordinates retains its linearity, while the same system with a feedback linear in the parameters does not retain its linearity (becomes bilinear) and requires the use of more complex methods for analysis and synthesis.

In addition to being capable of describing the dynamics, an controlled system model must incorporate the description of measurements (of observable quantities). Let l variables $y_1, \ldots, y_l$ be available for observation. We call these variables the system outputs or observables, and assume that the effect of the measurements on the controlled system dynamics and

[1] The following notation is adopted in this review: $x \in \mathcal{R}^n$ is a real-valued n-dimensional vector (column); $x = \mathrm{col}\,(x_1, x_2, \ldots, x_n)$ is a column vector with components $x_1, x_2, \ldots, x_n$;

$$|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$$

is the Euclidean norm of x; if X is a vector or matrix, X^{T} is the result of transposition (in particular, if X is a column vector, X^{T} is a row vector); we also let I_n denote the identity $n \times n$ matrix.

the measurement noise are negligible.[2] Then, the measurement can be described by an l-dimensional vector function $y = h(x)$.

2.2 Control goals

It has proved convenient to classify control problems according to the type of control goal.

2.2.1 Regulation (stabilization). Regulation is understood to be the reduction of the vector of the state variables of the system, $x(t)$, or the vector of output variables, $y(t)$, to certain equilibrium states x_* and y_*, respectively. In stating the problem, the time it takes to attain the goal as such is not considered; instead, the control goals are specified as limit relations

$$\lim_{t\to\infty} x(t) = x_* \tag{2.5}$$

or

$$\lim_{t\to\infty} y(t) = y_* . \tag{2.6}$$

Under limited disturbances, the attainment of goals (2.5) and (2.6) is usually impossible, and they are therefore replaced by estimates for the upper bound on the limit error,

$$\overline{\lim_{t\to\infty}} \left|x(t) - x_*\right| \leqslant \Delta \tag{2.7}$$

or

$$\overline{\lim_{t\to\infty}} \left|y(t) - y_*\right| \leqslant \Delta , \tag{2.8}$$

where Δ is the magnitude (parameter) of the admissible error. When random disturbances or interferences act on the controlled system, the usual approach is to consider averaged goal conditions of the type

$$\overline{\lim_{t\to\infty}} \mathrm{M}\left|x(t) - x_*\right| \leqslant \Delta \tag{2.9}$$

or

$$\overline{\lim_{t\to\infty}} \mathrm{M}\left|y(t) - y_*\right| \leqslant \Delta , \tag{2.10}$$

where M denotes the expectation value.

Attaining goals (2.5) – (2.10) becomes more complicated if the desired equilibrium state x_* is unstable in the absence of control. This is typical of problems of control of chaotic systems. It may also be possible that without control, the state x_* is not an equilibrium one, but this case introduces no additional difficulties because the control action does not then disappear as the path approaches the point x_*.

2.2.2 Tracking. In tracking problems, one is asked to 'move' the controlled system state variable vector $x(t)$ or the output vector $y(t)$ closer to the desired time functions $x_*(t)$ or $y_*(t)$, respectively, i.e., the following control goals are set:

$$\lim_{t\to\infty} \left[x(t) - x_*(t)\right] = 0 , \tag{2.11}$$

$$\lim_{t\to\infty} \left[y(t) - y_*(t)\right] = 0 . \tag{2.12}$$

[2] Generally speaking, this assumption does not hold for processes occurring in the microworld, in particular for quantum mechanical properties, because a macroscopic measuring device can strongly affect the microscopic system and may even destroy such a system. Each problem of this type must be examined separately.

The desired output $y_*(t)$ can be interpreted as the *goal*, or *command signal*. The function $x_*(t)$ or $y_*(t)$ can be specified as an explicit function of time or can be measured as the process develops. It may also be defined in terms of the motion of another, auxiliary, system, called the *reference model* or the *goal model*. In this case, the problem of finding the regulator that ensures the attainment of goal (2.11) or (2.12) is called the *control problem with a reference model*. The typical problem of controlling chaos, the stabilization of an unstable periodic solution (orbit), also belongs to tracking problems where $x_*(t)$ is a T-periodic solution of a free ($u(t) = 0$) system (2.1) with the initial condition $x_*(0) = x_{*0}$, i.e., $x_*(t + T) = x_*(t)$ for all $t \geqslant 0$.

2.2.3 Excitation (swing-up, spin-up, speeding-up) of oscillations. In problems associated with the excitation of oscillations, it is assumed that the system is initially at rest and must be set into oscillatory motion with given characteristics, where the path along which the tip of the phase vector of the system moves is not specified in advance or is unknown or its shape has no effect on the attainment of the goal. Such problems are widely known from electrical engineering, radio engineering, acoustics, and laser and vibrational technologies, where the process of generation of periodic oscillations must be set into motion. This class of problems also incorporates problems of dissociation and ionization of molecular systems, escape from a potential well, chaotization, and other problems associated with an increase in energy that can lead to a phase transition in the system. Formally, such problems are reduced to tracking problems, but the desired motions in this case become irregular and the goal trajectory $x_*(t)$ can be specified only partially.

The problems of exciting oscillations can often be reformulated via a scalar *goal function* $G(x)$ by specifying the control goal as attaining the limit relation

$$\lim_{t\to\infty} G\big(x(t)\big) = G_* \tag{2.13}$$

or an inequality for the lower bound on the goal function,

$$\underline{\lim_{t\to\infty}}\, G\big(x(t)\big) \geqslant G_* . \tag{2.14}$$

Often, the goal function is simply the total energy of the free system, $H(x)$.

2.2.4 Synchronization. By synchronization, we mean coincidence or convergence of the state variables of two or more systems or the matched variation of some of the quantitative characteristics of the systems. A synchronization problem differs from a control problem with a reference model, because it allows for temporal shifts between the matched variables. These shifts may be either constant or tending to constants (asymptotic phases). Moreover, in many synchronization problems, the links between the systems are bidirectional, which means that the limit mode in the system (a synchronous solution) is not known in advance.

A common feature of excitation and synchronization control problems is that the desired behavior is not uniquely fixed — its characteristics are specified only partially. For instance, in the oscillation excitation problem, restrictions are imposed only on the size of the oscillation amplitude, while the frequency and shape of the oscillation may vary within

certain limits. The main requirement in synchronization problems is often the coincidence or matching of the oscillations of all subsystems, while the characteristics of the motion of each subsystem may vary within broad limits.

If the required relation is established only asymptotically, as $t \to \infty$, one can speak of asymptotic synchronization. But if synchronization in a control-free system (i.e., $u = 0$) is absent or the synchronous mode is either unstable or has a very narrow attraction range, the synchronization control problem can be formulated as a problem of finding the control action that ensures a synchronous mode. Synchronization then acts as a control goal. For instance, a goal corresponding to ensuring asymptotic synchronization of the state vectors (phase coordinates) of two systems can be written as

$$\lim_{t\to\infty}\left[x_1(t) - x_2(t)\right] = 0\,. \tag{2.15}$$

This relation expresses the condition required for the convergence of the states $x(t) = \left(x_1(t), x_2(t)\right)$ in the united space of the states of the two system to the diagonal set $x_1 = x_2$. Often, it has proven convenient to write goal conditions (2.11)–(2.13) or (2.15) in terms of an appropriate goal function $Q(x, t)$ in the form of a limit relation:

$$\lim_{t\to\infty} Q\big(x(t), t\big) = 0\,. \tag{2.16}$$

For instance, to reduce goal (2.15) to (2.16), one can use the quadratic goal function $Q(x) = |x_1 - x_2|^2$. Instead of the Euclidean norm, another norm can be used to specify the same goal, for instance, the quadratic goal function

$$Q(x, t) = \left[x - x_*(t)\right]^{\mathrm{T}} \Gamma \left[x - x_*(t)\right],$$

where Γ is a positive definite symmetric matrix. Goal functions corresponding to other types of synchronization (frequency, phase, extremum, etc.) can be found in Refs [28, 36, 37].

2.2.5 Modification of limit sets (attractors) of systems. This class of goals incorporates particular goal types such as

- a change in the type of equilibrium (e.g., the transformation of an unstable equilibrium position into a stable one, or vice versa);
- a change in the type of the limit set (e.g., the transformation of a limit cycle into a chaotic attractor or vice versa, a change in the fractal dimension of the limiting set); and
- a change in the position and type of a bifurcation point in the system's parameter space (e.g., see Ref. [38]).

Goals of this type often correspond to phase transitions. In many works on chaotic mode control, the quantitative characteristics of the desired motion are not assumed to be predetermined; instead, the desired qualitative type of the limit set (attractor) is specified. For instance, it may be required to transform chaotic irregular oscillations into periodic or quasiperiodic oscillations. If it is necessary to specify the desired degree of irregularity quantitatively, the goal functions can be formed in terms of the well-known characteristics of chaos, such as the Lyapunov exponents, fractal dimensions, and entropy (see reviews of works on controlling chaos in Refs [28, 39, 40]).

In addition to the main control goal, additional goals or restrictions may be specified, one of which is the requirement that the control goal be attained through weak (low-power or cost-effective) control. The requirement that the control be weak is important for physical problems, because it means that external actions do not destroy the intrinsic properties inherent in the physical system, in other words, do not 'assault' the system.

Whether or not the control goal is attained may depend on the way the initial conditions of the system are specified. If the goal is attained with any initial conditions, one can speak of global attainability. Otherwise, the initial condition must be specified or a set of initial conditions Ω must be specified such that the goal is attained for every solution $x(t)$ of system (2.1) with control for initial conditions belonging to this set, i.e., at $x(0) = x_0 \in \Omega$.

2.3 Control algorithms

In physics, one often speaks of controlling a system when a certain parameter of the system (or its model) is specified. This parameter is known as the input, bifurcation, or control parameter, and its variation leads to changes in a certain characteristic of the system's behavior, known as the output parameter. One then speaks of the controllability of the system if under variations of the input parameter within its range of admissible values, the range of the output parameter encompasses the values corresponding to the desired operation modes of the system.

Rigorously, we cannot speak of control within this problem setup. There is only the possibility of achieving a given value of the output parameter for a constant value of the input parameter. Actually, feeding a calculated but time-constant action to the system may not lead to the desired goal. For example, we consider the problem of stabilizing the unstable equilibrium at $\varphi = \pi$ of the pendulum in (2.3), with the control action assumed to be constant. The equilibrium condition $\varphi = \pi$ implies that $u(t) = 0$. But because of the instability of the equilibrium at $\varphi = \pi$, very small deviations in the initial conditions or very small disturbances may lead to large deviations in the control goal.

Control that is a function of time presents many more possibilities. If the control action (variable or parameter) depends solely on time, $u = u(t)$, such action is called program action and the way the control is achieved is called program control, open loop control, or feedforward control. Actually, program control may also depend on the parameters and initial conditions of the control system, $u(t) = U(t, x_0)$. The possibility of a qualitative variation of the system's dynamics when control is specified in the form of a high-frequency time function was first discovered in experiments conducted by P L Kapitza at the end of the 1940s. He found that the upper, unstable, equilibrium position of a pendulum becomes stable if the pendulum suspension axis is made to vibrate in the vertical direction with a high frequency [41] (also see Refs [42, 43]). Similar ideas formed the basis for a branch of control theory known as vibrational control [44, 45].

A substantial number of papers have been devoted to the study of vibration-induced or noise-induced effects, which can be interpreted as control problems with a controlling function depending solely on time. Such effects include vibration-induced (vibrational) resonance [46], vibration-induced transport (vibrotransportation) [43], stochastic resonance [47], noise-induced transitions [48–50], stochastic ratchets [49–51], and synchronization by periodic action of an external or random force [52].

The principle of phase stability operating in accelerators, proposed independently by V I Veksler and E McMillan in 1944 – 1945, can be seen as control by action with a slowly varying frequency. Such a method of control (sometimes called the dynamic autoresonance [53] or chirping [54]) makes it possible to accumulate the energy of a nonlinear system while the system remains in the resonant mode, and at present is used to excite atoms [55], plasma [53], molecular systems [54], hydrodynamic solitons [56], etc. We also note that averaging is used to study such systems [57].

Control action in which one uses the results of measurements of the system state or outputs (observables) in calculating $u(t)$ presents even greater possibilities. Such control is called *state feedback*,

$$u(t) = U\big(x(t)\big)\,, \tag{2.17}$$

or *output feedback*,

$$u(t) = U\big(y(t)\big)\,. \tag{2.18}$$

Often, all three types of control — constant, program, and feedback — are present in physical problems. Because feedback control requires that the measurement of quantities needed for the control design be available (which is often not the case), the study of the properties of the control system begins with the study of the lowest form of control, constant control, then the possibilities of open loop control (program control) are studied, and finally feedback control (if such a problem setup is possible) is studied.

A typical formulation of a control problem that takes the physics of the problem into account is as follows.

Find all possible types of the system's behavior attainable through the use of controlling functions whose norm does not exceed a fixed (*small*) *value and, possibly, with given restrictions applied.*

In solving this problem, it may prove useful to solve an auxiliary problem more suited for control theory:

Find a controlling function (*or a feedback law*) *with a minimal norm that ensures the given behavior of the system* (*the specified control goal*).

2.4 Methods of designing control algorithms

The methods of cybernetic physics are based on the well-developed methods of control theory: linear, nonlinear, optimal, and adaptive control; identification (reconstruction) of parameters and filtering and estimating the states (parameters); and optimization of systems. Usually, some of the parameters of the physical system are unknown, while others are inaccessible for measurements, i.e., according to the terminology used in control theory, we are forced to synthesize the control under conditions of uncertainty. To solve such problems, methods of robust and adaptive control have been developed.

The above methods form the basis of courses in automatic control theory in engineering departments of universities, and numerous books on this subject have been written (e.g., see Refs [58 – 61]).

We briefly discuss a quite general approach to designing control algorithms for nonlinear systems, an approach often mentioned in the present review, known as the speed-gradient (SG) method [62]. The method is used to solve problems of controlling continuous-time systems in which the control goal is specified in terms of a goal function. We describe the design of such algorithms by calculating the speed gradient for the continuous nonstationary system

$$\dot{x} = F(x, u, t) \tag{2.19}$$

with the control goal defined in (2.16), where $Q(x, t) \geqslant 0$ is a smooth goal function.

To design the algorithm, we first calculate the scalar function $\dot{Q} = \omega(x, u, t)$, the speed of variation of the quantity $Q = Q\big(x(t), t\big)$; along the trajectories of Eqn (2.19),

$$\omega(x, u, t) = \frac{\partial Q(x, t)}{\partial t} + \big[\nabla_x Q(x, t)\big]^{\mathrm{T}} F(x, u, t)\,.$$

We then find the gradient of $\omega(x, u, t)$ with respect to the input variables:

$$\nabla_u \omega(x, u, t) = \left(\frac{\partial \omega}{\partial u}\right)^{\mathrm{T}} = \left(\frac{\partial F}{\partial u}\right)^{\mathrm{T}} \nabla_x Q(x, t)\,.$$

Finally, the following differential equation specifies the algorithm of changing $u(t)$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\Gamma\, \nabla_u \omega(x, u, t)\,, \tag{2.20}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a positive definite symmetric matrix, for example, $\Gamma = \mathrm{diag}\,\{\gamma_1, \ldots, \gamma_m\}$, $\gamma_i > 0$. It is only natural to call algorithm (2.20) the *speed-gradient algorithm* (SGA), because the variation in $u(t)$ is proportional to the speed gradient of Q.

The origin of algorithm (2.20) can be explained as follows. To attain control goal (2.16), it is advisable to vary $u(t)$ such that $Q\big(x(t), t\big)$ decreases. But because $Q\big(x(t), t\big)$ is independent of $u(t)$, it is difficult to find the direction of such decrease (in particular, this is due to the need to find the sensitivity functions). Instead, we decrease $\dot{Q}$ by ensuring that $\dot{Q} < 0$, which is the condition that $Q\big(x(t), t\big)$ decreases. The function $\dot{Q} = \omega(x, u, t)$ explicitly depends on u, which makes it possible to design algorithm (2.20).

To illustrate the above, we write the SGA for the problem of controlling a system that is linear in its inputs,

$$\dot{x} = A(x, t) + B(x, t)u\,, \tag{2.21}$$

where $A(x, t)$ is an n-vector and $B(x, t)$ is an $n \times m$ matrix. Equation (2.21) can also be written as

$$\dot{x} = A(x, t) + \sum_{i=1}^{m} B_i(x, t) u_i\,, \tag{2.22}$$

where the u_i are the components of the vector $u \in \mathcal{R}^m$, and the $B_i(x, t)$ are the columns of the matrix $B(x, t)$, $B_i(x, t) \in \mathcal{R}^n$. Let the goal function be

$$Q(x, t) = \frac{1}{2}\big[y - y_*(t)\big]^{\mathrm{T}} P\big[y - y_*(t)\big]\,, \tag{2.23}$$

where $y = G(x, t) \in \mathcal{R}^l$; $y_*(t)$ is the reference signal (the desired output path), $y_*(t) \in \mathcal{R}^l$; $G(x, t)$ is a smooth vector function, and P is a positive definite symmetric $l \times l$ matrix. The speed of variation of $Q\big(x(t), t\big)$ is

$$\omega(x, u, t) = \big[y - y_*(t)\big]^{\mathrm{T}} P\big[CA(x, t) + CB(x, t)u - \dot{y}_*(t)\big]\,, \tag{2.24}$$

where $C = C(x,t) = \partial G(x,t)/\partial x$, and the speed gradient and SGA become, respectively,

$$\nabla_u \omega(x,u,t) = B(x,t)^{\mathrm{T}} C^{\mathrm{T}} P\big[y - y_*(t)\big]\,, \tag{2.25}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\Gamma B(x,t)^{\mathrm{T}} C^{\mathrm{T}}\, P\big[y - y_*(t)\big]\,. \tag{2.26}$$

The gain matrix Γ is often chosen in the form of a diagonal matrix ($\Gamma = \mathrm{diag}\,\{\gamma_i\}$) or a scalar matrix ($\Gamma = \gamma I$), where γ_i and γ are positive numbers. Algorithm (2.26) with $B(x,t) = \mathrm{const}$ and $C(x,t) = \mathrm{const}$ constitutes the well-known integrated control law.

A similar approach can be used in the design and generalization of another classical control law, the proportional law. The algorithm is known as the *speed-gradient algorithm in finite form*,

$$u(t) = u_0 - \Gamma \nabla_u \omega\big(x(t), u(t), t\big)\,, \tag{2.27}$$

where u_0 is an initial (reference) control value (it is usually assumed that $u_0 = 0$). Algorithms of a more general structure are also used,

$$u(t) = u_0 - \gamma \psi\big(x(t), u(t), t\big)\,, \tag{2.28}$$

where $\gamma > 0$ is the scalar step factor (gain) and the vector function $\psi(x,u,t)$ satisfies the *pseudogradient condition*

$$\psi(x,u,t)^{\mathrm{T}} \nabla_u \omega(x,u,t) \geqslant 0\,. \tag{2.29}$$

An algorithm of type (2.28) is known as the *speed-pseudo-gradient algorithm*. A particular case of this algorithm is the *sign*, or *relay*, algorithm

$$u(t) = u_0 - \gamma\, \mathrm{sign}\, \nabla_u \omega\big(x(t), u(t), t\big)\,, \tag{2.30}$$

where the 'sign' of a vector is understood to apply to each component separately: for $x = \mathrm{col}\,(x_1, \ldots, x_m)$, we have

$$\mathrm{sign}\, x = \mathrm{col}\,(\mathrm{sign}\, x_1, \ldots, \mathrm{sign}\, x_m)\,.$$

Making a meaningful and justified selection of SGAs requires verifying the applicability of such algorithms. The applicability conditions for various cases can be found in Refs [60, 62]. The main conditions are the convexity of the function $\omega(x,u,t)$ in u, and the existence of 'ideal control', a vector u_* such that $\omega(x,u_*,t) \leqslant 0$ for all values of x (the attainability condition).

The speed-gradient algorithm is closely related to the Lyapunov function $V(x)$, the system's state function, which decreases along the path. The Lyapunov function is an abstract analog of physical characteristics such as energy and entropy. It is important that the Lyapunov function can be used not only for the analysis but also for the synthesis of systems, i.e., for solution of the inverse problems. In particular, we have the finite form of SG algorithms if we take the goal function $V(x) = Q(x)$. The differential form of SG algorithms is obtained if we assume that $V(x,u) = Q(x) + 0.5(u - u_*)^{\mathrm{T}} \Gamma^{-1}(u - u_*)$, where u_* is the desired ('ideal') value of the controlling variables.

2.5 Results

A considerable part of the results in traditional areas of physics is represented (or can be represented) in the form of conservation laws, which reflect the fundamental laws of nature and state that certain quantities do not change as the system evolves. In studying systems with control, the results must show to what extent the evolution of the system can be changed by control. Hence, the results in cybernetic physics are formulated *not as conservation laws but as* ***transformation laws*** determining the class of possible types of behavior attainable by control from a given class (the limit of control).

An example of a transformation law is the following principle proposed in the seminal paper by Ott–Grebogi–Yorke (the OGY law) [17]:

Each controllable chaotic motion can be transformed into a periodic motion by an arbitrarily small control.

The term 'controllable' in the above law indicates that the problem can be solved in principle. Conditions sufficient for controllability depend on the problem and constitute a topic for a separate investigation. Other examples of transformation laws can be found in Sections 3 and 4.

Summarizing, we note once more that the subject of cybernetic physics is the study of physical systems in the presence of direct and/or reverse (feedback) links with the surrounding medium. This makes it similar to the theory of open systems [63]. What makes it different from the theory of open systems is that feedback is not assumed to be fixed but must be synthesized. In the process of synthesizing feedback, one uses methods developed in cybernetics.

3. Control of conservative systems

One of the most important quantities in physics is energy, which is not only the main invariant of a system and the key to a description of a system on the basis of the Hamiltonian formalism but also a measure of interaction between different systems. The equations of dynamics in the Hamiltonian form are used to describe quite different physical systems and phenomena, from celestial bodies to molecular ensembles. Hence, it is only natural to begin the study of the fundamental laws of transformation of the properties of systems via control with the energy transformation laws. In this section, we assume that the investigated system is conservative, i.e., we ignore losses and dissipation. Then, in free motion (i.e., in the absence of external forces), the system's energy is an invariant. Hence, the statement of the problem of transferring the system from one energy level to another by weak (ideally, arbitrarily weak) control makes sense.

For brevity, we limit ourselves to examining control problems in which the mathematical model of the system is given in the *Hamiltonian form*,

$$\dot{q}_i = \frac{\partial H(q,p,u)}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H(q,p,u)}{\partial q_i}\,, \qquad i = 1, \ldots, n\,, \tag{3.1}$$

where n is the number of the degrees of freedom; $q = \mathrm{col}\,(q_1, \ldots, q_n)$ and $p = \mathrm{col}\,(p_1, \ldots, p_n)$ are vectors of generalized coordinates and generalized momenta, which form the state vector of the system, $x = \mathrm{col}\,(q,p)$; $H = H(q,p,u)$ is the Hamiltonian of the controlled system; and $u(t)$ is the dimensionless input (the vector of external generalized forces), $u(t) \in \mathcal{R}^m$.

The result below can be extended to systems that are more general than Hamiltonian systems, including systems whose models involve the controlling variables nonlinearly [64].

In the vector form, model (3.1) can be written as

$$\dot q = \nabla_p H(q,p,u)\,, \qquad \dot p = -\nabla_q H(q,p,u)\,. \tag{3.2}$$

We examine the problem of approaching a given energy level H_* of a free (uncontrolled) system, i.e., specify the control goal as

$$\lim_{t\to\infty} H_0\big(q(t),\,p(t)\big) = H_*\,, \tag{3.3}$$

where $H_0(q,p) = H(q,p,0)$ is the Hamiltonian of the free system described by Eqn (3.1) with $u=0$. We assume that the Hamiltonian is linear in control,

$$H(q,p,u) = H_0(q,p) + H_1(q,p)^{\mathrm T}u\,,$$

where $H_0(q,p)$ is the Hamiltonian of the free system and $H_1(q,p)$ is an m-dimensional vector whose components are the so-called *interaction Hamiltonians*.

To solve the problem, we use the SG method (see Section 2.4). We introduce the goal function

$$Q(x) = \frac{1}{2}\big(H_0(q,p) - H_*\big)^2 \tag{3.4}$$

with $x = \operatorname{col}(q,p)$. The control goal in (3.3) then becomes

$$\lim_{t\to\infty} Q\big(x(t)\big) = 0\,. \tag{3.5}$$

To apply the SG method, we calculate the speed (rate) of variation of the goal function caused by the control of the system,

$$\dot Q = (H_0 - H_*)\left(\frac{\partial H_0}{\partial q}\,\dot q + \frac{\partial H_0}{\partial p}\,\dot p\right) = (H_0 - H_*)\{H_0, H_1\}u\,, \tag{3.6}$$

and then calculate the speed gradient in u, $\nabla_u \dot Q = (H - H_*)\{H_0, H_1\}^{\mathrm T}$, where $\{H_0, H_1\}$ is the Poisson bracket[3] of the Hamiltonians H_0 and H_1. We can write the SGA in a finite form, e.g., in the linear and relay variants:

$$u = -\gamma(H_0 - H_*)\{H_0, H_1\}^{\mathrm T}\,, \tag{3.7}$$

$$u = -\gamma\,\operatorname{sign}\big[(H_0 - H_*)\{H_0, H_1\}^{\mathrm T}\big]\,, \tag{3.8}$$

where $\gamma > 0$ is the gain. Other variants of the general algorithm can also be employed by choosing ψ in the relation

$$u = -\psi\big[(H_0 - H_*)\{H_0, H_1\}^{\mathrm T}\big] \tag{3.9}$$

to be a vector function with values in $\mathcal{R}^m$ and to satisfy the condition $\psi(z)^{\mathrm T}z > 0$ for $z \in \mathcal{R}^m$, $z \neq 0$ (*the strict pseudogradient condition*).

The possibilities of changing the properties of a controllable Hamiltonian system by controlling it are determined by the properties of the closed system built. The following sufficient conditions for attaining goal (3.3) have been obtained in Refs [25, 28].

If the first and second partial derivatives of H_0 and H_1 in the set $\Omega_0 = \{x\colon Q(x) \leqslant Q_0\}$ are bounded for a certain $Q_0 > 0$ and the function $\psi(z)$ in (3.9) *is continuous and satisfies the strong pseudogradient condition $\psi(z)^{\mathrm T}z > 0$ for $z \in \mathcal{R}^m$, $z \neq 0$, then algorithm* (3.9) *for system* (3.2) *with the initial condition $x(0) \in \Omega_0$ ensures that $u(t) \to 0$ as $t \to \infty$ and, moreover, either goal* (3.3) *is attained on the path $x(t)$ or the convergence $\{H_0, H_1\}(x(t)) \to 0$ is ensured on the same path as $t \to \infty$.*

In addition, let the following auxiliary conditions be satisfied:

A1. *For each $c \neq H_*$, there exists a positive ε such that every nonempty connected subset of the set*

$$D_{\varepsilon,c} = \big\{x : \big|\{H_0(x), H_1(x)\}\big| \leqslant \varepsilon\,,\ \big|H_0(x) - c\big| \leqslant \varepsilon\big\} \cap \Omega_0$$

is bounded.

A2. *The largest invariant set $M \subset D_0$ of a free system (i.e., the set of the entire trajectories of system* (3.2) *at $u = 0$ contained in D_0), where $D_0 = \big\{x : \big\{H_0(x), H_1(x)\big\} = 0\big\} \cap \Omega_0$, consists of not more than a countable number of isolated points without finite accumulation points.*

Then every solution of system (3.2), (3.9) *either ensures the attainability of goal* (3.3) *or tends to a certain point belonging to D_0 and corresponding to equilibrium of the free system. Furthermore, the set of initial conditions at which the solution of system* (3.2), (3.9) *tends to an unstable equilibrium of the system has the measure null.*

Let the Hamiltonian of a controlled system have the form $H(q,p,u) = H_0(q,p) + H_1(q,p)^{\mathrm T}u$ with

$$H_0(q,p) = \frac{1}{2}\,p^{\mathrm T}A(q)^{-1}p + \Pi(q)\,, \qquad H_1(q,p) = q\,, \tag{3.10}$$

where q and p are n-dimensional generalized coordinates, $A(q)$ is a positive definite symmetric kinetic-energy matrix, and $\Pi(q) \geqslant 0$ is the potential energy. We then have $p = A(q)\dot q$, and the equations of the system can be reduced to the Lagrangian form

$$\frac{\mathrm d}{\mathrm dt}\big(A(q)\dot q\big) - \frac{1}{2}\,\dot q^{\mathrm T}\,\frac{\partial}{\partial q}\big(A(q)\big)\dot q + \nabla_q \Pi(q) = u\,. \tag{3.11}$$

In terms of the coordinates $(q,\dot q)$, the energy becomes

$$H_0(q,\dot q) = \frac{1}{2}\,\dot q^{\mathrm T}A(q)\dot q + \Pi(q)\,. \tag{3.12}$$

The free system has equilibrium states of the form $(0,\bar q)$, where $\bar q$ is the stationary (critical) point of the potential $\Pi(q)$ ($\nabla\Pi(\bar q) = 0$). We assume that all stationary points of the function $\Pi(q)$ are isolated. The above result then implies that if the initial energy layer

$$\Omega_0 = \big\{(q,p)\colon\ H_0 \leqslant H_0(q,p) \leqslant H_*\big\} \quad \text{for} \quad H_0 \leqslant H_*$$

[3] We recall that the Poisson bracket of smooth functions $f(q,p)$ and $g(q,p)$ is the sum

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\,\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\,\frac{\partial g}{\partial q_i}\right).$$

If f and g are vector functions whose respective dimensions are l and m, the Poisson bracket can be defined component-wise and is the $l \times m$ matrix

$$\{f,g\} = \sum_{i=1}^{n}\left[\frac{\partial f}{\partial q_i}\left(\frac{\partial g}{\partial p_i}\right)^{\mathrm T} - \frac{\partial f}{\partial p_i}\left(\frac{\partial g}{\partial q_i}\right)^{\mathrm T}\right].$$

In particular, if f is a scalar and g an m-dimensional column vector, $\{f,g\}$ is an m-dimensional row vector (covector). Similarly, if f and g are vectors with the respective dimensions l and m, then $\{f,g\}$ is an $l \times m$ matrix.

or

$$\Omega_0 = \{(q,p):\ H_* \leqslant H_0(q,p) \leqslant H_0\} \quad \text{for} \quad H_0 \geqslant H_*$$

does not contain minima of the potential $\Pi(q)$, then goal (3.3) is attained for almost all solutions. Moreover, if the matrix $A(q)$ is positive definite, i.e., if $p^{\mathrm{T}}A(q)p \geqslant \mu|p|^2$ for a positive μ and all $p \in \mathcal{R}^n$, it can be easily shown that almost all solutions of a closed system approach the set

$$S = \{(q,p):\ H_0(q,p) = H_*\}.$$

Example. For the simple pendulum model, the free-system Hamiltonian is

$$H_0(q,p) = \frac{p^2}{2J} + mgl(1-\cos q), \tag{3.13}$$

where $q(t)$ is the angular coordinate ($q(t) \in \mathcal{R}^1$), p is the system's momentum, J is the moment of inertia with respect to the rotation axis, and l is the distance between the rotation axis and the pendulum's center of mass. If we choose the control action as the torque applied to the suspension axis, the equations of motion in the Hamiltonian form can be written as

$$\dot q = J^{-1}p, \qquad \dot p = -mgl\sin q + u(t), \tag{3.14}$$

where $u(t)$ is the controlling torque. Equation (3.14) implies that $p = J\dot q$, and the interaction Hamiltonian has the form $H_1(q,p) = q$. Goal (3.3) corresponds to the stabilization of the pendulum in its lowest position (for $H_* = 0$), the swinging of the pendulum to its amplitude

$$q_* = \arccos\left(1 - \frac{H_*}{mgl}\right)$$

(for $0 < H_* < 2mgl$), and the rotation of the pendulum (for $H_* > 2mgl$). The value $H_* = 2mgl$ is the energy corresponding to the motion along the separatrix, which is the set consisting of a countable number of smooth curves separating the regions of oscillatory and rotational motion in the phase plane. For pendulum (3.14), speed-gradient algorithms (3.7) and (3.8) acquire the simple form

$$\dot u = -\gamma(H_0 - H_*)\dot q, \tag{3.15}$$

$$u = -\gamma\,\mathrm{sign}\left[(H_0 - H_*)\dot q\right]. \tag{3.16}$$

The above result for the case of controlling the pendulum energy implies that if the initial energy layer between the levels H_0 and H_* does not contain equilibrium states, the level H_* is to be reached for all initial conditions, while if the initial layer contains only the unstable equilibrium states $(\pi(2k+1), 0)$, $k = \pm1, \pm2, \ldots$, goal (3.3) is to be attained at almost all initial conditions.

In particular, for any Hamiltonian system that satisfies conditions A1 and A2 (see above), goal (3.3) is attained when an arbitrarily weak control action is applied to the system and for almost all initial conditions if the stationary points of the potential $\Pi(q)$ are isolated and the initial energy layer contains no stable equilibrium points of the system, i.e., if $H_* > \sup_{\bar q}\Pi(\bar q)$, where the upper bound is taken over all local minima of $\Pi(q)$ that are in the initial connected component of the energy layer Ω_0. Indeed, the right-hand sides of the closed system are bounded in the region Ω_0, and control can be made as small as desired by selecting a small value of the gain γ. Of course, a decrease in γ makes the time needed to attain the goal longer, but the goal is still attained.

By interpreting conditions A1 and A2 as the sufficient conditions for the controllability of the system in energy, we can formulate the results in Refs [25, 28] as a transformation law of the system by feedback.

If a system is controllable in energy, the value of the free-system energy can be changed by an arbitrary quantity via arbitrarily small feedback.

The above results can be extended to problems in which the attainable goals are more complicated and, in particular, to the problem of stabilization of several invariants (first integrals of motion) of a free system at given levels [61, 64].

Let k scalar invariants $G_i(q,p)$, $i = 1, \ldots, k$, be specified, i.e., $\{H_0, G_i\} \equiv 0$, $i = 1, \ldots, k$, and let the control goal be chosen in the form

$$\lim_{t\to\infty} G_i\big(p(t), q(t)\big) = G_i^*, \qquad i = 1, \ldots, k, \tag{3.17}$$

where G_i^* are fixed numbers. We introduce the goal function

$$Q(q,p) = \frac{1}{2}\big(G(q,p) - G^*\big)^{\mathrm{T}} R\big(G(q,p) - G^*\big), \tag{3.18}$$

where

$$G(q,p) = \mathrm{col}\left\{G_1(q,p), \ldots, G_k(q,p)\right\},$$

$$G^* = \mathrm{col}\left\{G_1^*, \ldots, G_k^*\right\},$$

and the $k \times k$ matrix $R = R^{\mathrm{T}}$ is positive definite and symmetric. Then the SGA built according to goals (3.17) and (3.18) can be written as

$$u = -\gamma\{\overline H, Q\} = -\gamma\{\overline H, G\}R\big(G(q,p) - G^*\big), \tag{3.19}$$

where $\overline H$ is a column vector with the entries given by the interaction Hamiltonians H_j, $j = 1, \ldots, m$, i.e., the bracket $\{\overline H, Q\}$ is a column vector with the entries $\{H_j, Q\}$ and the bracket $\{\overline H, G\}$ is an $m \times k$ matrix function with the entries $\{H_j, G_i\}$. We can consider a broader class of algorithms,

$$u = -\psi\big(\{\overline H, Q\}\big), \tag{3.20}$$

where $\psi(y)$ is a vector function that satisfies the condition $\psi(z)^{\mathrm{T}}z > 0$ for $z \neq 0$.

In Refs [61, 64], the controllability conditions were derived for the Hamiltonian system

$$\begin{aligned} \dot q &= \nabla_p H_0(q,p) + \sum_{j=1}^{m} \nabla_p H_j\, u_j, \\ \dot p &= -\nabla_q H_0(q,p) - \sum_{j=1}^{m} \nabla_q H_j\, u_j. \end{aligned} \tag{3.21}$$

The results of these works can be interpreted as follows:

If a system is controllable with respect to a set of invariants, the values of these invariants can be changed by arbitrary quantities by arbitrarily weak feedback of form (3.19).

4. Control of dissipative systems

4.1 Excitability index

We examine Hamiltonian systems with dissipation,[4]

$$\dot{q}_i = \frac{\partial H(q,p,u)}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H(q,p,u)}{\partial q_i} - R_i(q,p)\,, \quad i = 1,\ldots,n\,, \tag{4.1}$$

where $q = \mathrm{col}\,(q_1,\ldots,q_n)$ and $p = \mathrm{col}\,(p_1,\ldots,p_n)$ are the vectors of generalized coordinates and generalized momenta that form the system's state vector $x = \mathrm{col}\,(q,p)$; $H = H(q,p,u)$ is the Hamiltonian of the controlled system; $u(t)$ is the dimensionless input (the vector of the external generalized forces), $u(t) \in \mathcal{R}^m$; and $R(q,p) = \mathrm{col}\,\big(R_1(q,p),\ldots,R_n(q,p)\big)$ is the dissipation function, which satisfies the condition

$$R(q,p)^{\mathrm{T}}\,\frac{\partial H_0(q,p)}{\partial p} \geqslant 0\,, \tag{4.2}$$

where $H_0(q,p) = H(q,p,0)$ is the free-system energy. Inequality (4.2) means that dissipation of energy occurs during free motions of the system: $\dot{H}_0 \leqslant 0$.

Clearly, dissipation hinders controlling the system's energy. It is therefore interesting to estimate the possibility of transforming the energy of systems with given levels of control and dissipation. It is especially interesting to study the case of small dissipation (weakly damped systems), for which the oscillatory behavior of processes in the system and the presence of resonance phenomena are characteristic features. We estimate the limits of possible energy transformations with given levels of control and dissipation and the possibility of initiating resonant modes via feedback control for linear-input Hamiltonian systems with dissipation, systems with the Hamiltonian $H(q,p,u) = H_0(q,p) + H_1(q,p)u$.

To analyze the variation of the system characteristics as a result of control, we must introduce a quantitative measure of the limits of such variation, which may depend on the selection of the system input and output and on the acceptable value of the control action. For definiteness, we take the free-system energy $E(x) = H_0(q,p)$ as the output, while the input is measured by its maximum (in time) absolute value. The rate of variation of the energy of the free system with dissipation is given by the expression

$$\dot{H}_0 = \{H_0, H_1\}u - R(q,p)^{\mathrm{T}}\,\frac{\partial H_0}{\partial p}\,,$$

i.e., in the absence of control, $E(x)$ has a tendency to decrease, characterized by the dissipation rate

$$\varrho(x) = R(q,p)^{\mathrm{T}}\,\frac{\partial H_0}{\partial p}\,.$$

Hence, the characteristic describing the degree of increase of $E\big(x(t)\big)$ caused by control is important. Such a characteristic determines the measure of excitability of motion (oscillations) in the system and may be called the *excitability index*.

[4] The results for a broader class of systems can be found in Ref. [28].

To find the excitability index, one should calculate the maximum value of $E\big(x(t)\big)$ asymptotically attainable under limited control, i.e., as $t \to \infty$. But computer experiments have shown that there may be no limit as $t \to \infty$. Hence, we consider the upper and lower limits and say that the functions $\chi_E^+(\gamma)$ and $\chi_E^-(\gamma)$, defined for $\gamma \geqslant 0$ as

$$\chi_E^+(\gamma) = \overline{\lim_{t\to\infty}} \sup_{\substack{|u(\cdot)|\leqslant\gamma \\ x(0)=0}} E\big(x(t)\big)\,, \tag{4.3}$$

$$\chi_E^-(\gamma) = \underline{\lim_{t\to\infty}} \sup_{\substack{|u(\cdot)|\leqslant\gamma \\ x(0)=0}} E\big(x(t)\big)\,, \tag{4.4}$$

are *the upper and lower excitability indices of system* (4.1) *with respect to the output* $E(x)$.

The excitability index can be measured in experiments in the same way as the ordinary frequency response of a linear system. In contrast to measurements of the frequency response, where a harmonic signal with a constant amplitude and varying frequency is fed to the system's input, the amplitude (level) of the input signal is varied in measurements of the excitability index and the input signal is specified as feedback. From the results in Ref. [28] (p. 60), it follows that if the output $E(x)$ and the dissipation rate $\varrho(x)$ for certain positive α_0, α_1, ϱ_0, ϱ_1, and d satisfy the relations

$$\alpha_0|y|^2 \leqslant E(x) \leqslant \alpha_1|y|^2 + d\,, \tag{4.5}$$

$$\varrho_0|y|^2 \leqslant \varrho(x) \leqslant \varrho_1|y|^2\,, \tag{4.6}$$

where $y = \{H_0, H_1\}$, and the set

$$\Omega^- = \left\{ x\colon\ \{H_0,H_1\} = 0\,,\ E(x) < \alpha_0\left(\frac{\gamma}{\varrho_1}\right)^2 \right\}$$

does not contain the entire paths of free system (4.1) (at $u = 0$), then the excitability indices $\chi_E^+(\gamma)$ and $\chi_E^-(\gamma)$ with respect to $E(x)$ satisfy the inequalities

$$\alpha_0\left(\frac{\gamma}{\varrho_1}\right)^2 \leqslant \chi_E^-(\gamma) \leqslant \chi_E^+(\gamma) \leqslant m\alpha_1\left(\frac{\gamma}{\varrho_0}\right)^2 + d\,, \tag{4.7}$$

while the excitability indices $\chi_y^+(\gamma)$ and $\chi_y^-(\gamma)$ with respect to y satisfy the inequalities

$$\sqrt{\alpha_0}\,\frac{\gamma}{\varrho_1} \leqslant \chi_y^-(\gamma) \leqslant \chi_y^+(\gamma) \leqslant (m\alpha_1)^{1/2}\,\frac{\gamma}{\varrho_0} + \sqrt{d}\,. \tag{4.8}$$

The lower estimate is then attained for

$$u(t) = \gamma\,\mathrm{sign}\,y(t)\,. \tag{4.9}$$

We note that control law (4.9), which places a lower bound on the excitability indices, is independent of the parameters of the controlled system: the potential energy, kinetic energy, and dissipation functions.

The above estimates can be applied to controlled mechanical systems described by equations in the Lagrangian form. Inequalities (4.5) then act as conditions of uniform nonsingularity and boundedness of the kinetic-energy matrix $A(q)$ (a standard assumption for mechanical systems) and the bounded nature of the potential-energy function. Inequalities (4.6) correspond to viscous friction that grows not faster than a linear function.

Inequalities (4.7) may be interpreted as the laws of the energy transformation by limited control for systems with dissipation. For instance, let $R(p) = \varrho\dot{q}$ and $\varrho \to 0$. Then, *for*

a controlled Lagrangian or Hamiltonian system with small dissipation of degree ϱ, the level of energy attainable by control of a level γ is of the order of $(\gamma/\varrho)^2$.

We reconsider the pendulum in the example in Section 3 and allow for friction with a coefficient ϱ. The above estimates for $n = 1$ can be written as

$$0.5\left(\frac{\gamma}{\varrho}\right)^2 \leqslant \overline{H} \leqslant \left(\frac{\gamma}{\varrho}\right)^2 + 2\omega_0^2, \tag{4.10}$$

and the law of control (4.9) becomes $u = \gamma \operatorname{sign}(\dot{q})$.

Inequalities (4.7) and (4.8) make it possible to estimate the degree of excitability and the resonance properties of nonlinear systems by supplying additional quantitative information concerning the dynamical characteristics of such systems.

4.2 Feedback resonance

Using the above general results, we can establish some properties of physical systems through action with feedback in the measurements. As an example, we describe resonance with feedback.

Resonance plays an important role in physics and technology, causing both useful and dangerous effects. Apparently, the first to study and describe the phenomenon of resonance was Galileo Galilei, who in his treatise *Dialogues Concerning Two New Sciences*, published in 1638, wrote [67]: "... a pendulum at rest, even a very heavy one, can be set in motion (and a very noticeable one) by simply blowing on it if we hold our breath when the pendulum is moving toward us and blow when it is moving away, at the proper moment of its swinging motion."

For oscillatory systems, the resonant mode corresponds to excitation of significant oscillations of the system by applying a weak control action and, hence, by ensuring the most effective transfer of energy from the exciting system to the excited one. The laws of resonance have been thoroughly studied for linear systems. But for nonlinear oscillatory systems, even the very definition of the resonance concept requires refining. We now discuss this aspect in greater detail.

Let a controlled nonlinear oscillator with one degree of freedom be described by the equation

$$\ddot{\varphi} + \Pi'(\varphi) = u, \tag{4.11}$$

where $\varphi = \varphi(t)$ is a scalar phase coordinate, $u = u(t)$ is the scalar control action, and $\Pi(\varphi) \geqslant 0$ is the potential. The state of system (4.11) is represented by the vector $x = \operatorname{col}(\varphi, \dot{\varphi})$, and the total energy of the system is $H(\varphi, \dot{\varphi}) = \dot{\varphi}^2/2 + \Pi(\varphi)$. We ask the question: To what extent can an arbitrarily weak external action $u(t)$ change the path of system (4.11)?

It is well-known that for a quadratic potential $\Pi(\varphi) = \omega_0^2\varphi^2/2$, i.e., for a harmonic oscillator described by the linear equation $\ddot{\varphi} + \omega_0^2\varphi = u$, a harmonic external action $u(t) = \gamma \sin \omega t$ with $\omega = \omega_0$ and an arbitrarily small amplitude γ leads to the occurrence of unbounded solutions, e.g., of the form

$$\varphi(t) = -\frac{\gamma t}{2\omega_0}\cos \omega t,$$

which is commonly called the resonance.

But the dynamics of nonlinear systems is more complicated. Even for a simple pendulum, driven oscillations may have a more complicated, irregular form [65, 66]. The complexity of creating and studying resonant modes in nonlinear systems can be explained by the fact that the frequency in such systems is amplitude-dependent. It is only natural to think that oscillations in a nonlinear system can be generated by varying the frequency of the external action as a function of the amplitude of oscillations. This means that $u(t)$ must depend on $\varphi(t)$, which actually means that the scalar control action is formed by feedback. The results in Section 3 show that action of type (4.9) makes it possible to achieve any given level of energy, i.e., to create something similar to a resonant mode in the system.

We now introduce losses (dissipation) of the viscous-friction type into system (4.11), i.e., instead of (4.11), we consider the equation

$$\ddot{\varphi} + \varrho\dot{\varphi} + \Pi'(\varphi) = u, \tag{4.12}$$

where $\varrho > 0$ is the dissipation factor. For linear systems of type (4.12) [with $\Pi(\varphi) = \omega_0^2\varphi^2/2$] what is known as resonance is the mode in which the amplitude of oscillations is at its maximum, and this occurs for the action $u(t) = \gamma \sin \omega t$ with $\omega^2 = \omega_0^2 - \varrho^2/4$. At small values of ϱ, the oscillations in system (4.12) with $u(t) = \gamma \sin \omega t$ then have the amplitude $\bar{A} \approx \gamma/(\varrho\omega_0)$ and the energy (averaged over one period) $0.5(\gamma/\varrho)^2$. The oscillation amplitude for nonlinear oscillator (4.12) with the action in (3.15) or (3.16) can also become very large. The estimates in Section 4.1 imply that the energy attained in systems (3.16) and (4.12) is no lower than

$$\overline{H} = \frac{1}{2}\left(\frac{\gamma}{\varrho}\right)^2 \tag{4.13}$$

if the parameters in law (3.16) are selected such that $H_* \geqslant \overline{H}$. Thus, we can say that feedback (3.15) [or (3.16)] creates a resonant mode in nonlinear system (4.12), with the energy of the mode (in the particular case of a harmonic oscillator) being no lower than the oscillation energy of the harmonic excited at the resonance frequency. We call this phenomenon [26, 27] the *feedback resonance*, or f-resonance.

It must be noted that the understanding of resonance in physics has remained practically the same since the time of Galileo. In most works on the subject, the input action is usually assumed to be harmonic (periodic, at the most). In the book by Andronov et al. [68], whose first edition appeared in 1938, the authors introduced the concept of autoresonance as a 'resonance generated by a force caused by the motion of the system', i.e., they pointed to the possibility of actions in the form of feedback. However, in Ref. [68], they considered only the case of a linear system of the second order with a relay in feedback and only made estimates of the size of the limit cycles. The system was assumed to be closed, i.e., what was actually studied was the *internal* resonance in the system, which probably explains the origin of the term 'autoresonance'.

Autoresonant modes in nonlinear systems were analyzed in Refs [69, 70]. In particular, the researchers proposed using SGAs to tune the parameters of a nonlinear system to guarantee that the mode was resonant [70]. Other researchers assumed the excitation to be either a periodic function, with only an 'adiabatic' frequency variation allowed (i.e., a variation that is slow compared to the fundamental tone of the oscillations) [53, 71] or a stochastic action (stochastic resonance; see Refs [47, 49, 50]). The phenomenon of feedback resonance described above occurs when an *external*

action that changes its spectrum in the course of the process is applied to the system.

It is interesting that Galileo's description of resonance does not contradict the existence of feedback. In addition, it suggests how feedback can be used to force the pendulum into the resonant mode: one simply needs to blow "at the proper moment of its swinging motion."

5. Examples

5.1 Controlled Kapitza pendulum

The mathematical model of a pendulum controlled by changing the position of the suspension point is given by

$$J\ddot{\varphi} + \varrho\dot{\varphi} + mgl\sin\varphi = mlu\sin\varphi\,, \tag{5.1}$$

where $\varphi = \varphi(t)$ is the angle of the pendulum's deflection from the lower vertical position, $u = u(t)$ is the vertical acceleration of the suspension point (which is the control action), $J = ml^2$ is the pendulum's moment of inertia, and $\varrho \geqslant 0$ is the friction coefficient.

Kapitza examined the harmonic law of displacement of the suspension point and experimentally established that the pendulum is stabilized near its upper, unstable, equilibrium position. Many theoretical studies, done before and after Kapitza's experiments (e.g., see Refs [43, 72]), have shown that stabilization of unstable equilibrium as the suspension point moves with a frequency ω and an amplitude A, for which $u(t)$ assumes the form $u(t) = A\omega^2\sin\omega t$, sets in at high frequencies ω, i.e., when the input action in (5.1) is large or, more precisely, when the condition $A\omega > J\omega_0^2$ is satisfied, where $\omega_0 = \sqrt{2g/l}$ is the frequency of small oscillations of the pendulum about the lower equilibrium position (see Refs [41, 43, 73]). The displacement of the suspension point may then remain small, which makes the effect even more paradoxical. Thus, stabilization of unstable equilibrium by a high-frequency harmonic action is possible, but requires the application of considerable force.

Can a similar behavior of the Kapitza pendulum be achieved with a smaller amplitude of $u(t)$ when feedback is used in the law of vibrations of a suspension axis?

An approach that is common in automatic control theory based on linearization of the system's model does not work here. Indeed, linearization produces good results only near the equilibrium position or a certain path, but we are interested in a global solution, applicable in the entire state space of the pendulum.

We set an auxiliary control goal,

$$\lim_{t\to\infty} H(t) = H_*\,, \tag{5.2}$$

where H is the pendulum's total energy,

$$H = \frac{J}{2}(\dot{\varphi})^2 + mgl(1-\cos\varphi)\,. \tag{5.3}$$

Goal (5.2) differs somewhat from the ordinary goals in control theory, regulating and tracking. It looks more like the goal of a person sitting on a swing. A similar problem may emerge in the actuation of a vibration device, the design of a walking robot, the operation of a pendulum clock, etc. Human wisdom suggests that swinging requires much weaker forces than keeping the pendulum (or the hand of a robot) in a fixed position. Can a small force bring a heavy swing to its upper position?

For the goal function, we take the standard deviation $Q = (H - H_*)^2/2$ of the pendulum's total energy from the desired value H_*. Using the SG method, we obtain the simple algorithms

$$u = -\gamma(H - H_*)\dot{\varphi}\sin\varphi\,, \tag{5.4}$$

$$u = -\gamma\,\mathrm{sign}\left[(H - H_*)\dot{\varphi}\sin\varphi\right]. \tag{5.5}$$

We take algorithm (5.5) and select the value of the pendulum energy in the upper position, $H_* = 2mgl$, as the desired energy level. Then the results in Section 4.1 suggest that the energy level attained in systems (5.1) and (5.2) is no lower than

$$\overline{H} = \frac{1}{2}\left(\frac{\gamma}{\varrho}\right)^2, \tag{5.6}$$

which means that the level $H_* = 2mgl$ is reached for $\gamma > 2\varrho\omega_0$. In particular, at $\varrho = 0$, stabilization of the energy level surface $H = H_*$ is reached even with a very small control amplitude γ. In the event of underdamping (ϱ is small), the control can be weak (γ is small).

That the desired level of energy is reached does not necessarily mean that stabilization of the equilibrium corresponding to this level is reached. But as shown in Refs [64, 74], if $\varrho = 0$, algorithm (5.5) at $H_* = 2mgl$ ensures the convergence $H\big(\varphi(t), \dot{\varphi}(t)\big) \to H_*$ and the convergence $\big(\varphi(t), \dot{\varphi}(t)\big) \to (\pi, 0)$ as $t \to \infty$ for almost all initial conditions, with $\gamma > 0$ as small as desired.

The problem of controlling a pendulum by moving its suspension point has an interesting feature. Because the control action $u(t)$ is the acceleration, the general properties of an SGA imply that $u(t) \to 0$ as $t \to \infty$. It is unclear, however, what happens to the speed and position of the suspension point. The formal model suggests that the speed and position of the suspension point may even increase indefinitely, which strips the problem of all practical importance.

Following Ref. [74], we describe a modification of the control algorithms that is free from the above defect. For this, we introduce the extended goal function

$$Q_1 = Q + \frac{1}{2}\,z^{\mathrm{T}}Pz\,, \tag{5.7}$$

where $z = \mathrm{col}\,(\zeta, \dot{\zeta})$, with ζ and $\dot{\zeta}$ being the height and speed of the suspension point, respectively, and $P = P^{\mathrm{T}} \geqslant 0$ is a positive semidefinite weight matrix. Then, the equality $\ddot{\zeta} = u$ may be interpreted as an additional equation of motion, i.e., the system transforms into one with two degrees of freedom and the state $x = \mathrm{col}\,(q, \dot{q}, \zeta, \dot{\zeta})$.

Using the SG method, we arrive at the control algorithm

$$u = -\gamma(H_0 - H_*)\dot{q}\sin q - \mu\dot{\zeta} - \nu\zeta\,, \tag{5.8}$$

where $\gamma > 0$, $\mu > 0$, and $\nu > 0$ are the gains.

The results in Section 3 cannot be used to study this system because the control goal is not specified in terms of the system energy. Nevertheless, using the general results in Refs [27, 61], one can show that the new control goal is attained and $\zeta(t) \to \mathrm{const}$ for almost all initial conditions at $\nu = 0$. If $\mu > 0$ and $\nu > 0$, the system exhibits a stronger

property, $\zeta(t) \to 0$, i.e., the deviation of the suspension point from the initial position asymptotically tends to zero.

Similar argument leads to the swinging algorithm in the case where the suspension point moves horizontally or obliquely. Additional difficulties may emerge because of the incompleteness of or inaccuracies in the measurements, e.g., if the angular velocity $\dot{\varphi}(t)$ cannot be measured. Incompleteness of control may also have a negative effect, e.g., when the inertia of an engine rotating the pendulum cannot be ignored and the dynamics of the controlled system is described by the equations

$$J\ddot{\varphi} + mgl\sin\varphi = mlu\sin\varphi\,, \qquad T\ddot{u} + u = v\,, \tag{5.9}$$

where $v = v(t)$ is the new control signal. Indeed, the control action $v(t)$ is not present on the right-hand side of the first equation in (5.9) and the speed gradient vanishes.

The modern theory of nonlinear and adaptive control has a big arsenal of approaches that can be used to overcome the above difficulties [60, 61].

5.2 The problem of escape from a potential well

The problem of escape from a potential well (or of overcoming a barrier) under the action of external forces is encountered in many areas of physics and mechanics, beginning with the 1930s [75]. Sometimes, the escape is an undesirable effect (buckling of membranes and shells or capsizing of vessels and vehicles), while in other cases escape is useful and necessary. Overcoming a potential barrier often corresponds to a phase transition in the physical system. In all cases, certain conditions that guarantee the presence or absence of the overcoming of the barrier must be met. The majority of researchers study the case of a typical external action, harmonic or noise-type [46, 49, 50]. But how weak can the action causing the escape be?

In many works, this phenomenon is studied for nonlinear oscillators with one degree of freedom described by the equation

$$\ddot{\varphi} + \varrho\dot{\varphi} + \Pi'(\varphi) = u\,, \tag{5.10}$$

where $\varrho > 0$. For instance, the minimum amplitude of the harmonic action

$$u(t) = \gamma\sin\omega t \tag{5.11}$$

causing escape of the solution of (5.10) from the potential well was found in Ref. [76] via computer simulation involving two typical potentials: $\Pi(\varphi) = \varphi^2/2 - \varphi^3/3$ (sometimes called the Helmholtz potential) and $\Pi(\varphi) = \varphi^2/2 - \varphi^4/4$ (which corresponds to the Duffing equation and has two potential wells that are symmetric with respect to the zero point).

In particular, it was shown in Ref. [76] that with the Duffing potential, escape inevitably occurs in system (5.10) and (5.11) at $\varrho = 0.25$ for $\gamma > 0.212$ and $\omega \approx 1.07$, while no escape occurs for $\gamma < 0.212$ and any frequencies of the external action (Fig. 1a with $\gamma = 0.211$ and $\omega = 1.08$). [5]

But can escape be induced at mush smaller action amplitudes by creating conditions needed for feedback resonance? Selecting $\overline{H}$ as the height of the potential barrier

[5] Here and in what follows, the values of all variables are given in relative units.

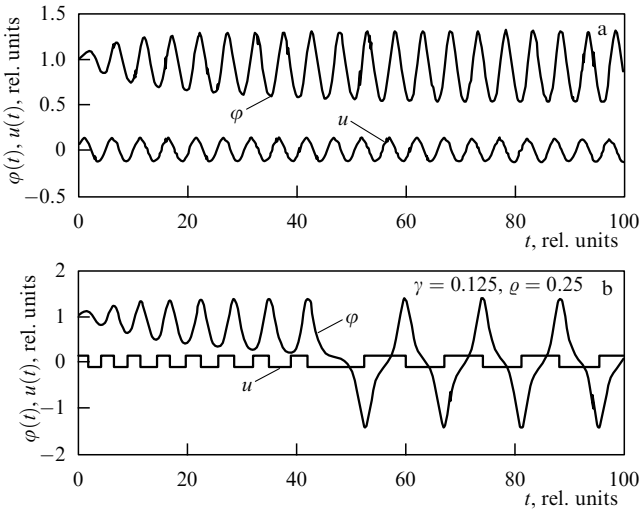

Figure 1. Ejection from a potential well for the Duffing system: (a) harmonic excitation, (b) excitation by the SG algorithm.

and solving (4.13) for γ, we arrive at expression (4.9), which guarantees escape. For instance, for the Duffing equation, $\overline{H} = 0.25$, and hence $\gamma = 0.1767$, which is 83% of the value found in Ref. [76]. For systems with one degree of freedom, law (4.9) becomes

$$u(t) = \gamma\,\mathrm{sign}\,\dot{\varphi}\,. \tag{5.12}$$

We note that both law (5.12) and law (4.9) are independent of the type of the potential $\Pi(\varphi)$ and, hence, can be used to generate the resonant mode in any oscillator described by model (5.10).

Computer simulation has shown that escape occurs at even smaller values of the amplitude of input signal (5.12) (e.g., see Fig. 1b with $\gamma = 0.125$). Generally, the power gain obtained when the system is excited by feedback is inversely proportional to the degree of dissipation and can be made as large as desired for lightly damped systems.

5.3 Control of synchronization of two oscillators

We examine the problem of synchronizing the oscillations of two coupled one-dimensional oscillators with one degree of freedom, e.g., simple pendulums. Such a model is encountered in the description of various physical and mechanical problems (e.g., see Ref. [66]). If we assume that dissipation is linear, the system of two coupled oscillators is described by the equations

$$\begin{aligned}\ddot{\varphi}_1(t) + \varrho\dot{\varphi}_1 + \Pi'\big(\varphi_1(t)\big) &= k\big(\varphi_2(t) - \varphi_1(t)\big) + u(t)\,,\\ \ddot{\varphi}_2(t) + \varrho\dot{\varphi}_2 + \Pi'\big(\varphi_2(t)\big) &= k\big(\varphi_1(t) - \varphi_2(t)\big)\,,\end{aligned} \tag{5.13}$$

where $\varphi_i(t)$ is the generalized coordinate of the ith oscillator (e.g., the angle of swing, or rotation angle), $i = 1, 2$; $u(t)$ is the control action (e.g., the external force torque expressed in units of angular acceleration); ϱ is the friction coefficient (the degree of dissipation); k is the coupling constant (e.g., the spring constant); and $\Pi(\varphi)$ is the potential [e.g., for the pendulum, $\Pi(\varphi) = \omega_0^2(1 - \cos\varphi)$].

We introduce the system's state vector

$$x(t) = \mathrm{col}\,(\varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2) \in \mathcal{R}^4\,.$$

The total energy $H(x)$ of system (5.13) with the coupling energy taken into account is given by the expression

$$H(x) = \frac{\dot\varphi_1^2 + \dot\varphi_2^2}{2} + \Pi(\varphi_1) + \Pi(\varphi_2) + \frac{k(\varphi_2 - \varphi_1)^2}{2}\,. \qquad (5.14)$$

We examine the problem of exciting synchronous antiphase oscillations in this system with a given amplitude by introducing additional limited feedback. Such a problem can be interpreted as the problem of attaining a given level of energy of the system with an additional requirement that the oscillators be in antiphase. We synthesize the control algorithm by the SG method by introducing the partial goal functions

$$Q_\varphi(\dot\varphi_1, \dot\varphi_2) = \frac{\delta_\varphi^2}{2}\,, \qquad Q_H(x) = \frac{\big(H(x) - H^*\big)^2}{2}\,, \qquad (5.15)$$

where $\delta_\varphi = \dot\varphi_1 + \dot\varphi_2$, $H\big(x(t)\big)$ is the total energy of the system, and H^* is its fixed value.

The minimum value of Q_φ corresponds to the requirement that the oscillations are in antiphase (for small initial phases $\varphi_1(0)$ and $\varphi_2(0)$ at least, the identity $Q_\varphi(\dot\varphi_1, \dot\varphi_2) \equiv 0$ holds only when $\dot\varphi_1 \equiv -\dot\varphi_2$). Minimization of Q_H means that the desired oscillation amplitude has been attained. The total goal function $Q(x)$ is defined here as a weighted sum of the partial goal functions,

$$Q(x) = \alpha Q_\varphi(\dot\varphi_1, \dot\varphi_2) + (1 - \alpha)\, Q_H(x)\,, \qquad (5.16)$$

where α is a weight factor, $0 \leqslant \alpha \leqslant 1$. Calculating the speed gradient, we arrive at the control algorithm

$$u(t) = -\gamma\big[\alpha\delta_\varphi(t) + (1 - \alpha)\,\delta_H(t)\big]\,\dot\varphi_1(t)\,, \qquad (5.17)$$

where $\delta_\varphi(t) = \dot\varphi_1(t) + \dot\varphi_2(t)$, $\delta_H(t) = H_t - H^*$, and $\gamma > 0$ is the gain.

The results in Section 3 cannot be directly applied to the given problem because δ_φ is not an invariant of the system even when $\varrho = 0$ (it retains its value in the motions of the free system only on the goal set). Hence, we are confronted with the problem of analytically studying the attainability of the goal in systems (5.13) and (5.17). At the same time, computational experiments have shown that synchronization algorithm (5.17) works.

Here are some results of modeling the process of excitation and synchronization of oscillations by algorithm (5.17) in the system consisting of two identical pendulums. The following values of the parameters were chosen: $k = 5$, $\omega_0 = 0.4\pi$, $\gamma = 0.8$, $\alpha = 0.7$, and $H^* = 4.0$. All initial conditions were assumed to be zero conditions with one exception, $\varphi_2(0) = 0.05\pi$.

We begin with the case where $\varrho = 0$. Figures 2–5 show that after a certain transient processes, both pendulums perform oscillations with opposite phases and the two partial goal functions approach the desired values. The times of the transient process are the same for both H_t and Q_φ (about 20 rel. units). The relation between the times of transient processes in H_t and in Q_φ can be changed by varying the coefficient α. The amplitude of the control action may be made as small as desired by reducing the gain γ.

In the presence of dissipation, synchronization of the pendulums at a given energy level can also be achieved, but the size of control cannot be arbitrarily reduced. Figure 6 with the results of modeling at $\varrho = 0.05$ shows that as the same

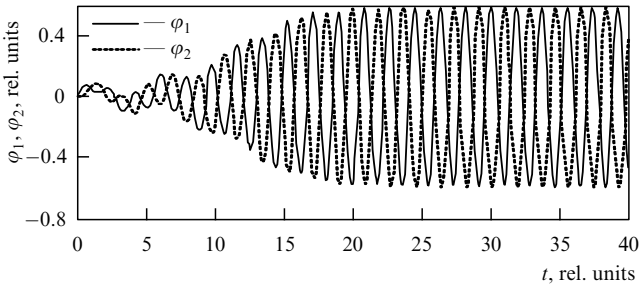

Figure 2. Excitation of antiphase oscillations of pendulums.

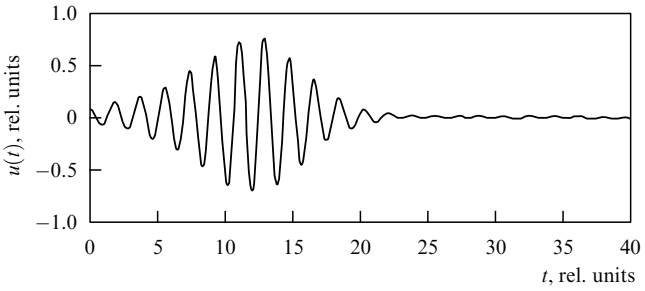

Figure 3. Control signal $u(t)$.

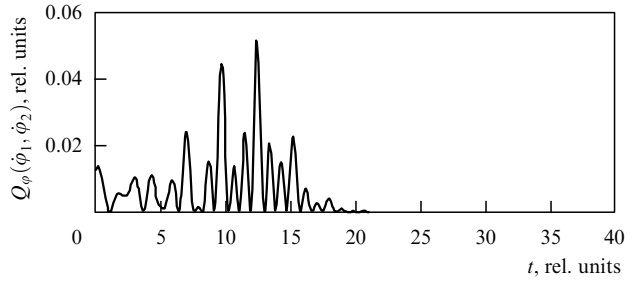

Figure 4. Dynamics of the synchronization goal function $Q_\varphi\big(\dot\varphi_1(t), \dot\varphi_2(t)\big)$.

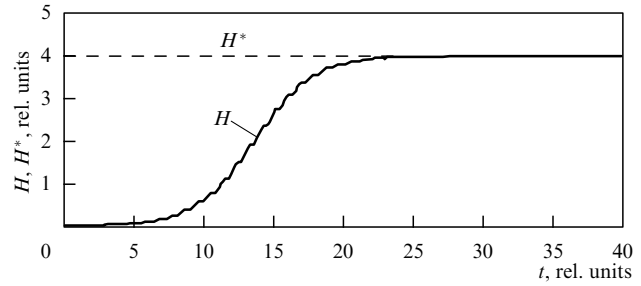

Figure 5. Dynamics of the oscillation energy H_t.

energy level $H^* = 4.0$ as in the case without dissipation is attained, the amplitude of the control action approaches the value

$$u_\infty = \varlimsup_{t \to \infty} \big|u(t)\big| \approx 0.5\,.$$

Calculations by formula (4.1) yield a result of the same order of magnitude, i.e., there is good agreement between theory and experiment. We note when the sign in the expression for the synchronization error is reversed, i.e., $\delta_\varphi = \dot\varphi_1 - \dot\varphi_2$, in-phase synchronization may be achieved [36].

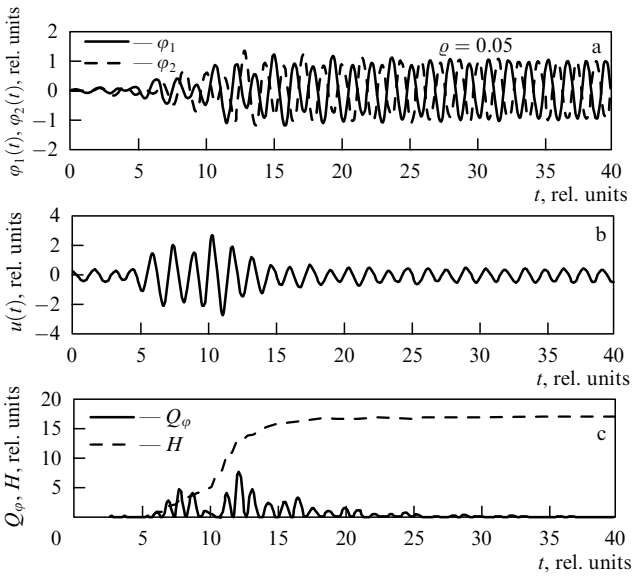

Figure 6. Synchronization processes in the presence of dissipation.

6. Control of distributed systems

6.1 Problems and methods of control in distributed systems

In many respects, the methods of controlling oscillations in distributed (spatial-temporal) systems are based on the ideas developed for lumped systems. Moreover, many researchers use finite-dimensional models of the control system of ODE systems to synthesize control. Such models can be obtained by discretization over the space of distributed models described by partial differential equations or as a set of ODE describing separate spatial elements (cells) or by discarding the 'tail' in the expansion over the basis in the initial infinite-dimensional state space (Bubnov–Galerkin methods). In this section, we discuss the first two variants and assume that the cells interact with each other via 'bonds' that reflect the spatial structure of the entire system, often called an array or a lattice.

A typical representative of the class of models of physicochemical processes is the so-called reaction–diffusion equation

$$\frac{\partial x}{\partial t} = \varepsilon \Delta x + F(x, u)\,, \tag{6.1}$$

where $x = x(r, t)$ is the function (possibly, a vector function) of the spatial variables $r \in D \subset \mathcal{R}^n$ and time t that determines the state of the system;

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial r_i^2}$$

is the Laplace operator, which specifies the diffusion type of the spatial interaction of the elements; and ε is the diffusion coefficient. Usually, the boundary conditions are either periodic, $x(a, t) = x(b, t)$ at $D = [a, b] \subset \mathcal{R}^1$, or describe the absence of a flux through the boundary,

$$\left(\frac{\partial x}{\partial r}\right)\Big|_{r=a} = \left(\frac{\partial x}{\partial r}\right)\Big|_{r=b} = 0\,.$$

When equation (6.1) is discretized over the space, the set D is replaced by a finite number of points (nodes) r_i, $i = 1, 2, \ldots, N$, to each of which there corresponds a state variable $x_i(t)$. The dynamics of the $x_i(t)$ is determined by both the internal dynamics of $F(x_i, u_i)$ and the interactions with the neighboring nodes. For instance, if the space is one-dimensional, $r \in [a, b]$, and the interactions are of a diffusion nature, we obtain the systems of equations

$$\dot{x}_i = \varepsilon(x_{i-1} - 2x_i + x_{i+1}) + F(x_i, u_i)\,, \qquad i = 1, 2, \ldots, N-1\,. \tag{6.2}$$

If the boundary conditions are periodic, an additional constraint $x_0(t) = x_N(t)$ is specified, while in the absence of flux through the boundaries, additional constraints $x_0(t) = x_1(t)$ and $x_{N-1}(t) = x_N(t)$ are imposed. The case of zero boundary conditions is often considered: $x_i(t) = 0$ for $i \leqslant 0$ and for $i \geqslant N$. In many works, the systems are also discretized in time, which leads to what is known as *coupled-map models*, or cellular automaton models:

$$x_i(n+1) = x_i(n) + \varepsilon\big[x_{i-1}(n) - 2x_i(n) + x_{i+1}(n)\big] + hF\big(x_i(n), u_i(n)\big)\,, \quad i = 1, \ldots, N-1\,, \quad n = 0, 1, 2, \ldots\,. \tag{6.3}$$

In models (6.1) and (6.3), control affects the dynamics of each cell, which corresponds to the case of spatial (field) control. Another class of problems, *boundary control*, emerges when the right-hand sides of (6.2) and (6.3) are control-independent, i.e., $F(x, u) \equiv F(x)$, but control enters the boundary cell equations, $\dot{x}_0 = \varepsilon(x_1 - x_0) + F_0(x, u)$ (for periodic boundary conditions). The most general systems are the spatially inhomogeneous systems described in the one-dimensional case by the model

$$\begin{aligned} &\dot{x}_i = F_i(x_i, x_{i-1}, x_{i+1}, u)\,, \qquad i = 1, 2, \ldots, N-1\,,\\ &\dot{x}_0 = F_0(x_0, x_1, u)\,,\\ &\dot{x}_N = F_N(x_N, x_{N-1}, u)\,. \end{aligned} \tag{6.4}$$

Various types of the interaction of oscillations in neighboring cells are considered as the control goal, in addition to ordinary goals, which are the same as in lumped systems. As in lumped systems, the attainment of the goal does not determine the process in the system in full. An interesting research direction is to determine the possible types of behavior of systems with and without control.

The specific control goals in distributed systems are:

- stabilization of a given uniform (homogeneous) or spatially periodic stationary field (standing wave);
- stabilization of a given spatially periodic nonstationary field (traveling wave);
- generation or destruction of a helical wave (for the number of spatial dimensions no smaller than two);
- generation or destruction of a given nonuniform field (a contrast structure, clusters, patterns); and
- control of self-organization and disorganization of systems.

Problems of controlling distributed systems have been discussed in control theory since the 1960s (e.g., see Ref. [77]). However, judging by publications, the interest in this topic increased considerably only in the mid-1990s in connection with the interest in controlling chaos in distributed systems. The methods that were used in the first works on controlling

chaos in distributed systems were basically those that had been developed for lumped systems: the OGY method, delayed feedback, etc. [78]. In the works that followed, other approaches were also studied (numerically, as a rule).

For instance, the authors of Ref. [79] examined a one-dimensional array with $N = 100$ cells that were described by a logistic map with $F(x, u) = 1 - \alpha x^2 + u$, where the parameter α is specified such that the oscillations in each cell at $u \equiv 0$ are chaotic. Computational experiments have shown that at sufficiently large values of gain γ, the local feedback

$$u_i(n) = \gamma\left[x_i(n) - \frac{1}{N+1}\sum_{j=0}^{N} x_j(n-1)\right], \quad i = 1, 2, \ldots, N-1 \tag{6.5}$$

stabilizes a spatially homogeneous distribution, $x_i \equiv x_*$, $i = 0, 1, 2, \ldots, N$. For smaller values of γ, an inhomogeneous distribution consisting of several homogeneity clusters sets in, with each cell oscillation in the periodic mode. Similar behavior of the system was observed in the presence of local feedback in the discrepancy (error),

$$u_i(n) = \gamma\big(x_i(n) - x_*\big), \tag{6.6}$$

and when 'global' feedback (which reflects reality more adequately) is introduced. Such feedback depends on the observable mean values of the variables,

$$u_i(n) = -\frac{\gamma}{N+1}\sum_{j=0}^{N}\big[x_j(n) - x_j(n-1)\big] \tag{6.7}$$

or

$$u_i(n) = -\frac{\gamma}{N+1}\sum_{j=0}^{N} x_j(n) - x_*. \tag{6.8}$$

The results were substantiated in Ref. [80].

The case of pinning control of a one-dimensional lattice, where control acts only on each pth cell described by the Lorentz system, was considered in Ref. [81]. The researchers showed that it is possible to approach a spatially homogeneous (coherent) motion that is nevertheless chaotic, provided that control varies in time in a discrete manner according to law (6.6) with $\gamma = 1$ and is applied to the first equation in the Lorentz system. In Ref. [82], this result was extended to a two-dimensional lattice of Lorentz systems by using integral feedback, which the researchers called adaptive. Similar results were obtained in Ref. [83] for the complex Ginzburg – Landau equation

$$\dot{A} = A + (1 + \mathrm{i}\mu_1)\frac{\partial^2 A}{\partial r^2} - (1 + \mathrm{i}\mu_2)|A|^2 A \tag{6.9}$$

and in Ref. [84] for the Swift – Hohenberg equation, which describes the dynamics of lasers. The Ginzburg – Landau equation with pinning control applied to a finite number of points has been studied in Refs [85, 86]. The equation describes a broad range of phenomena in laser physics, hydrodynamics, chemical turbulence, etc. and can represent various types of complex behavior. As a result of computational experiments, the authors of Ref. [85] found the largest distance between the points to which control is applied that ensures the attainment of the control goal. A similar result with boundary control was obtained in Ref. [86].

It was shown in Ref. [87] that solutions of the Kuramoto – Sivashinsky equation

$$\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^4\varphi}{\partial r^4} = u \tag{6.10}$$

can be stabilized by periodic delayed feedback in the speed $u = \varepsilon' \partial\varphi(t-\tau)/\partial t$, where τ is the *delay time*.

Pinning control was used in Ref. [88] to stabilize the zero solution ($x_i(t) \equiv 0$) of the system of coupled oscillators with diffusion – gradient bonds,

$$\dot{x}_i = f(x_i) + \frac{\varepsilon}{2}(x_{i-1} - 2x_i + x_{i+1}) + \frac{\rho}{2}(x_{i-1} - x_{i+1}) + u_i, \tag{6.11}$$

and to solve the Ginzburg – Landau equation with the initial condition corresponding to the chaotic mode. The researchers used linear feedback with large gain in each lth oscillator. The stability of the closed system was analyzed by using models that were linearized near the goal solution.

The minimum number density of local control points and their optimum arrangement in a one-dimensional array of coupled logistic systems, $f(x) = ax(1-x)$ in (6.11), with stabilization by linear feedback, have been determined in [89]. In Ref. [90], a method for stabilizing an unstable spatially homogeneous solution of the reaction – diffusion equation was developed and its applicability was demonstrated using the complex Kuramoto – Tsuzuki equation. A method of suppressing chaos and helical waves in the Maxwell – Bloch equations with diffraction bonds by a weak spatial perturbation was proposed in Ref. [91].

Interesting problems of 'cluster synchronization' in two- and three-dimensional arrays of nonlinear oscillators have been examined in Refs [92 – 95]. Lyapunov functions of a special type were used to find the conditions needed for the partitioning of the array into a given number of compact clusters of oscillators that vibrate synchronously. The researchers found that as the degree of the interrelationship increases, the number of clusters decreases to unity, i.e., the entire array becomes synchronized. Although control is not present explicitly in the above papers, the results of these works can be interpreted as a choice (synthesis) of the coupling constant k that ensures the given degree of clusterization in the system.

6.2 Energy control in the sine-Gordon and Frenkel – Kontorova models

We now show how the SG method can be used to control systems of the sine-Gordon type according to energy criteria. We introduce the notation

$$x_t = \frac{\partial x}{\partial t}, \quad x_{tt} = \frac{\partial^2 x}{\partial t^2},$$
$$x_{r_i} = \frac{\partial x}{\partial r_i}, \quad x_{r_i t} = \frac{\partial^2 x}{\partial r_i\,\partial t}, \quad x_{r_i r_i} = \frac{\partial^2 x}{\partial r_i^2}$$

and examine a system described by the sine-Gordon equation

$$Jx_{tt} = k\Delta x - E\sin x, \tag{6.12}$$

where $x = x(r, t)$ is the system's state function, $r \in X \subset \mathcal{R}^n$, $\Delta x = \sum_{i=1}^{n} x_{r_i r_i}$, J and k are the parameters of the system, and $E = E(t)$ is the external action (e.g., the external electric field strength). We assume that $E = E_0 + u(t)$, where E_0 is the

base level of the field and $u(t)$ is the control action. System (6.12) can be interpreted as a model of an array of diffusion-coupled oscillators (pendulums, liquid crystals, etc.), each of which is placed at the point r_i. Then, $x(r_i, t)$ is the rotation angle (angle of swing) of an oscillator.

We now pose the problem of bringing the energy values of the free system

$$H = \frac{1}{2}\int_X \left[J\left(\frac{\partial x}{\partial t}\right)^2 + k|\nabla_r x|^2 + 2E_0(1-\cos x)\right] \mathrm{d}r \quad (6.13)$$

to the given level H_*, which corresponds to the control goal

$$\lim_{t\to\infty} H(t) = H_* . \quad (6.14)$$

Introducing the function $V(t) = \big(H(t) - H_*\big)^2/2$ and examining the relation

$$\dot V = \frac{\mathrm{d}V}{\mathrm{d}t} = -u(t)\big(H(t) - H_*\big)\int_X x_t \sin x \,\mathrm{d}r , \quad (6.15)$$

we find that $\dot V \leqslant 0$ at

$$u(t) = \gamma\big(H(t) - H_*\big)\int_X x_t \sin x \,\mathrm{d}r , \quad (6.16)$$

i.e., action (6.16) brings the system to the goal.

We examine the spatially one-dimensional and spatially discrete problem,

$$J\ddot x_j = \frac{k}{h^2}(x_{j+1} - 2x_j + x_{j-1}) - \big(E_0 + u(t)\big)\sin x_j , \quad (6.17)$$
$$j = 1, 2, \ldots, N ,$$

which corresponds to the continuous system

$$Jx_{tt} = kx_{rr} - \big(E_0 + u(t)\big)\sin x \quad (6.18)$$

in the set $X = [a, b]$, where $x_j = x\big[a + j(b-a)/(N+1)\big]$.

System (6.17) is simply the controlled version of the classical *Frenkel–Kontorova model* proposed in 1939 (e.g., see Ref. [66]). Many researchers have studied this problem.

The common approach to studying system (6.18) without control is to impose either zero boundary conditions $x(a, t) = x(b, t) = 0$, which correspond to

$$x_0(t) \equiv x_{N+1}(t) \equiv 0 \quad (6.19)$$

in discrete model (6.17), or conditions of zero flux through the boundary $x_r\big|_{r=a} = x_r\big|_{r=b} = 0$, which are given by

$$x_0 = x_1 , \qquad x_N = x_{N+1} \quad (6.20)$$

in (6.17).

The problem of controlling the chain's energy can be solved on the basis of the results in Section 3. The energy control algorithm obtained by the SG method is

$$u(t) = \gamma\big(H(t) - H_*\big)\sum_{j=1}^{N} \dot x_j \sin x_j , \quad (6.21)$$

with $\gamma > 0$. The conditions needed for the attainability of a goal of this type (see Section 3) suggest that control goal (6.14) in systems (6.17) and (6.21) is attained if the energy layer between the levels $H(0)$ and H_* contains no equilibrium states of the system satisfying the conditions $\sin x_j = 0$, $j = 1, \ldots, N$.

We note that in the particular case where $N = 2$ with boundary conditions (6.20), the system becomes

$$\begin{aligned} J\ddot x_1 &= \frac{2k}{h^2}(x_2 - x_1) - \big(E_0 + u(t)\big)\sin x_1 , \\ J\ddot x_2 &= \frac{2k}{h^2}(x_1 - x_2) - \big(E_0 + u(t)\big)\sin x_2 . \end{aligned} \quad (6.22)$$

In this case, the problem of controlling the energy becomes similar to the problem of synchronizing two pendulums, examined in Section 5.3.

The discrete analog of control algorithm (6.16) is

$$u(t) = \gamma\big(H(t) - H_*\big)(x_2 - x_1)(\sin x_1 - \sin x_2) , \quad (6.23)$$

where $H(t)$ is defined as the discrete representation of (6.13):

$$H = \frac{J}{2}(\dot x_1^2 + \dot x_2^2) + \frac{k}{2}(x_1 - x_2)^2 + E_0(2 - \cos x_1 - \cos x_2) . \quad (6.24)$$

The constructed control algorithms can be used to study the properties of nonlinear oscillatory systems in various problems, including problems of controlling the vibrations of oscillating particles, of the orientation of particles in a given direction, etc.

The problem of controlling the excitation and synchronization of oscillations in a chain consisting of N simple pendulums connected in series is examined in Ref. [36] (for the particular case of $N = 2$, see Section 5.3).

7. Control of molecular and quantum systems

7.1 Laser control of molecular dynamics

Problems associated with controlling processes in the microworld have a rich history. The Maxwell demon has already been examined in Section 1. In the 20th century, various problems of controlling chemical technology processes (within what became known as chemical cybernetics, which was being rapidly developed in the 1960s and 1970s), nuclear reactors [62, 97], particle beams [98], processes in solids via laser radiation [99], etc., were studied.

The main difficulties in controlling processes on the atomic–molecular level are related to the small spatial size of the controlled systems and the high rates of the processes in them. The average size of molecules (monomers) is around 10 nm, the average distance between the atoms in a molecule is 1 nm, and the average speed of the atoms and molecules at room temperature is $10^2 - 10^3$ m s^{-1} and the period of natural oscillations of the atoms in a molecule is $1 - 10^2$ fs. Building devices for measurement and control on such a spatial–temporal scale constitutes an extremely complicated scientific and technological problem. We note that the operation of the existing chemical and nuclear reactors is based on using natural moderation of fast processes. For instance, implementation of systems for controlling nuclear processes is possible only because the rate of neutron-physical processes is moderated substantially by the presence of what is known as delayed neutrons, whose motion is characterized by time constants of the order of several seconds to several dozen seconds. The speed of molecules involved in chemical

reactions decreases because of diffusion, which creates the prerequisites for controlling chemical technology processes. However, for intricate control problems, e.g., for breaking a strong chemical bond while leaving a weaker bond intact (this is known as selective chemistry), we need selective interference into processes with characteristic time constants in the femtosecond range. Until recently, no technological possibilities for controlling such rapid processes existed.

The situation changed at the end of the 1980s with the appearance of ultrafast femtosecond lasers that generated pulses with a length of several dozen femtoseconds (today, this figure amounts to several femtoseconds) and of ways of controlling the shape of the laser pulses by computer. A new area of chemistry came into being, femtochemistry, for which A Zewail was awarded the Nobel Prize in Chemistry in 1999 [100]. With the development of ways of using femtosecond lasers, the term 'femtosecond technology', or 'femtotechnology', emerged.

Several approaches to controlling molecular systems have been proposed. In Shapiro and Brumer's approach [101], control is based on the interference of two laser beams with different frequencies, amplitudes, and phases, which became known as the pump – dump scheme. Tannor and Rice [102] proposed using two-pulse pump – dump schemes in the temporal region. Later, to optimize the pulses, methods of optimal control, in particular, one based on Krotov's method (see Ref. [103]), were employed. Rabitz et al. [104 – 107] studied various variants of optimal control in the classical and quantum mechanical descriptions of the dynamics of molecular motion. Judson and Rabitz [105] proposed implementing adaptive laser control of chemical reactions combined with search optimization methods (genetic algorithms), an idea that was later repeatedly corroborated in experiments [108 – 110]. The current state of the problem can be examined by turning to the article collections [111, 112].

One of the simplest problems in this class is the problem of dissociation of a diatomic molecule [54, 113 – 115]. At the same time, the problem is typical and can be used to compare the advantages and drawbacks of various methods. The possibility of dissociation of hydrogen fluoride HF in the presence of a periodically varying field (monochromatic laser radiation) was studied numerically in Ref. [113]. Using a similar method, the same researchers (see Ref. [114]) studied dissociation by a two-frequency (bichromatic) action and found that the intensity of the dissociating field can be substantially reduced. In Ref. [54], the intensity of the dissociating field was estimated by chirping, which amounts to uniform variation of the frequency of the external action. The researchers showed that it is possible to reduce the field intensity required for dissociation even more.

New possibilities for changing the physicochemical state of matter emerge when there is nonperiodic action in the form of feedback. In Section 5.2, we used the problem of escape from a potential well to show that feedback makes it possible to reduce the intensity of the action needed for overcoming a potential barrier by several orders of magnitude. When the action is selected by the SG method, the intensity needed for attaining a certain level of energy proves to be inversely proportional to the degree of dissociation of the system (see Section 4.1), and for conservative systems, the appropriate effect can be achieved (at least in theory) by an excitation intensity as small as desired. It is therefore interesting to use feedback for synthesizing control of molecular systems.

The main problem with feedback control lies in the difficulties related to measurements of the state of the system and the implementation of the control action over time periods comparable to the period of natural vibrations of the molecule.

Various algorithms of feedback control, including algorithms of optimum control, were proposed in Refs [106, 107, 115] and other papers. The general feature of these algorithms is that they are used in a model of a molecular system to synthesize the control action as a function of time. In computational experiments, one can assume that all the necessary signals are measured and the synthesized algorithm is carried out in a computer. As a result, a controlling signal is generated as a function of time, and its implementation is done without measurements and feedback. What impedes the practical application of this approach is the very large number of indeterminancies: the initial state of the system is not known precisely and the constructed controlling function is calculated and implemented with errors. Finally, the very model of the molecule is imprecise, because its parameters have not been reliably determined and, furthermore, the choice between classical and quantum mechanical descriptions is often a subject of discussion.

In Section 7.2, we describe a new approach to the problem of the dissociation of a diatomic molecule based on the SG method with an energy goal function [116, 117]. The obtained algorithms are robust, because they are independent of the shape of the intermolecular interaction potential. They make it possible to achieve dissociation at a lower intensity of the controlling field compared to chirping and are simpler to synthesize and calculate compared to optimal control methods.

7.2 Controlled dissociation of a diatomic molecule

The idea of the approach is that the problem of controlling dissociation is posed as one of achieving a given level of energy of the molecule (the dissociation threshold). To simplify matters, we assume that the given level of energy is somewhat lower than the dissociation threshold, i.e., we begin with the *predissociation* problem. The goal function is formed as the square of the deviation of the current energy from the fixed value. The control algorithm is built according to the standard SG scheme (see Section 3) and is applied to the standard model of a molecule (a 'standard molecule') in the course of time T_1 sufficient for dissociation of the molecule. If the control signal generated in this way as a function of time is fed to a real molecular system, its action leads to the dissociation of only those molecules whose initial state is close to the initial state of the standard molecule x_0 (the dissociation zone). But if the control signal fed to the system is in the form of pulses of length T_1 with the distance between the pulses being large, some molecules approach the state x_0 during the pause between the pulses (in the course of their chaotic thermal motion), enter the dissociation zone, and are dissociated by the next control pulse. If the algorithm is sufficiently robust, we can expect that the fraction of the molecules that are in the dissociation zone is not very small and the process is a rapid one. We now examine this approach in greater detail, following Ref. [117].

We adopt the classical description of the dynamics of an individual molecule in the form of Hamiltonian model (3.1). The interatomic distance $r(t)$ acts as the coordinate in (3.1),

and the Hamiltonian is

$$H = \frac{p^2}{2m} + \Pi(r) + \mu(r)\,u\,, \tag{7.1}$$

where m is the mass of the molecule, $\Pi(r)$ is the interatomic interaction potential, $\mu(r)$ is the molecule's dipole moment, and $u = u(t)$ is the external controlling field. To describe the interatomic interaction, we use the Morse potential

$$\begin{aligned}\Pi(r) &= D\{1 - \exp[-\alpha(r-a)]\}^2 - D \\ &= D\{\exp[-2\alpha(r-a)] - 2\exp[-\alpha(r-a)]\}\,,\end{aligned} \tag{7.2}$$

where D is the coupling energy and a is the equilibrium interatomic spacing. The dipole moment is specified in the form [115, 118]

$$\mu(r) = Ar\exp(-\xi r^4)\,, \qquad \mu'(r) = A(1-4\xi r^4)\exp(-\xi r^4)\,, \tag{7.3}$$

where A and ξ are constant parameters. If ξa^4 is small compared to unity, the dipole moment can be approximated by the linear function $\mu(r) = Ar$, $\mu'(r) = A$. The equation of controlled systems in the Lagrangian form becomes

$$m\ddot{r} = 2\alpha D\{\exp[-2\alpha(r-a)] - \exp[-\alpha(r-a)]\} + Au(t)\,. \tag{7.4}$$

This description assumes that the molecular motion is one-dimensional and its axis is parallel to the force lines of the controlling external field, i.e., effects associated with a change in orientation and rotation of the molecules are ignored.

In formulating the control goal, we allow for the fact that as the total energy of the molecule approaches $\Pi_* = \lim_{r\to\infty}\Pi(r)$, dissociation becomes more and more likely. Obviously, in the case of Morse potential (7.6), we have $\Pi_* = 0$.

For the goal function, we take the standard deviation

$$Q(q,p) = (H_0(q,p) - H_*)^2/2\,,$$

where $H_0(q,p) = p^2/(2m) + \Pi(r)$ is the energy of a free molecule and H_* is the fixed energy value close to the dissociation threshold Π_*. Calculating the speed gradient in the same way as we did before, we arrive at the simple feedback laws

$$u = -E\big(H_0(q,p) - H_*\big)\,\dot{r}\,, \tag{7.5}$$

$$u = -E\,\mathrm{sign}\,\big(H_0(q,p) - H_*\big)\,\mathrm{sign}\,\dot{r}\,, \tag{7.6}$$

where $E > 0$, $\mathrm{sign}\,(H) = 1$ for $H > 0$, $\mathrm{sign}\,(H) = -1$ for $H < 0$, and $\mathrm{sign}\,(0) = 0$. For modeling, we use a simplified version of algorithm (7.6) obtained under the assumption that the molecule's energy is always lower than the dissociation threshold H_*:

$$u = E\,\mathrm{sign}\,\dot{r}\,. \tag{7.7}$$

Algorithm (7.7) does not require the exact knowledge of the dissociation threshold H_* and can be used in other problems, e.g., to localize a molecule in the region of elevated energy (predissociation).

7.3 Simulation results for classical and quantum mechanical descriptions of molecules

Computational experiments with systems (7.4) and (7.7) were conducted with the numerical values of the parameters corresponding to the HF molecule [115, 118]: $m = 1732$, $D = 0.2101$, $\alpha = 1.22$, $a = 1.75$, $A = 0.4541$, $\xi = 0.0064$, and $E = 0.1$. The values are given in Hartree atomic units (a.u.). To calculate the control action, the initial conditions were chosen near the equilibrium state $r = a$ and $\dot{r} = 0$ (the standard molecule). The field intensity was chosen low: $E = 0.005$ a.u. The result of the calculations was the function $u(t)$ for $0 \leqslant t \leqslant T_1$.

The calculated control action $u(t)$ was applied to a model of an ensemble consisting of $N = 1000$ molecules. It was assumed that the molecules do not interact with each other or the boundary. The initial conditions for the molecules of the ensemble were specified as random values uniformly distributed over the surface of the fixed level of energy $H_0 = -0.8689D$. The control action was fed in the form of repetitive pulses with the pulse repetition period T_2 sufficiently long for the molecules to have time to 'spread' during the pause between the pulses. In experiments, $T_2 = 200T_0$, where T_0 is the period of small vibrations of the molecule near the equilibrium position. The fraction of the dissociated molecules (a percentage of the total number of molecules) was taken as the measure of the efficiency of control. Dissociation (more accurately, predissociation) was understood to occur when the molecule's energy was above the energy level $H_* = -0.1185D$.

The efficiency of the proposed algorithm was compared to that of the standard chirping algorithm $u(t) = E\cos(\phi_0 + \Omega_0 t - \varepsilon t^2/2)$. Figure 7a shows the time dependence of the fraction of the dissociated molecules under control by a linearly chirped field. The chirping rate ε (the rate of variation of the pulse frequency) was adjusted in the experiments to achieve the highest efficiency (the largest fraction of the dissociated molecules) and amounted to $\varepsilon = 0.01\Omega_0/T_0$ in the experiments. Figure 7b shows a similar dependence for the SG algorithm. The two figures clearly show that the efficiency of this algorithm is severalfold higher than the efficiency of linear chirping. It is important that the system with chirping is extremely sensitive to the chirping rate ε. The selection of the parameter ε requires much computer time and more exact knowledge of the parameters of the molecular Hamiltonian and the dipole moment than the efficient operation of algorithm (7.7) requires.

The interesting question about the meaningfulness of using classical description in modeling and controlling molecular processes is the cause of much discussion. The dynamics of a diatomic system are described more accurately, compared to the description provided by classical model (7.4), by a quantum mechanical model (more precisely, by a semiclassical model) represented by the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\,\frac{\partial\Psi}{\partial t} = \frac{\hbar^2}{2M}\,\frac{\partial^2\Psi}{\partial r^2} + \Pi(r)\Psi + Aru(t)\Psi\,, \tag{7.8}$$

where $\Psi = \Psi(t,r)$ is the wave function and $\Pi(r)$ is Morse potential (7.2). However, in many cases, classical calculations yield a result that is very close to reality. Hence, the results of modeling processes of controlled dissociation in the classical and quantum mechanical settings were numerically compared.

To numerically analyze quantum model (7.8), a finite-level approximation of the model was built by expanding the solution in eigenfunctions of the unperturbed Schrödinger equation with the Morse potential [119]. The control action and the modeling time were chosen the same as in the classical

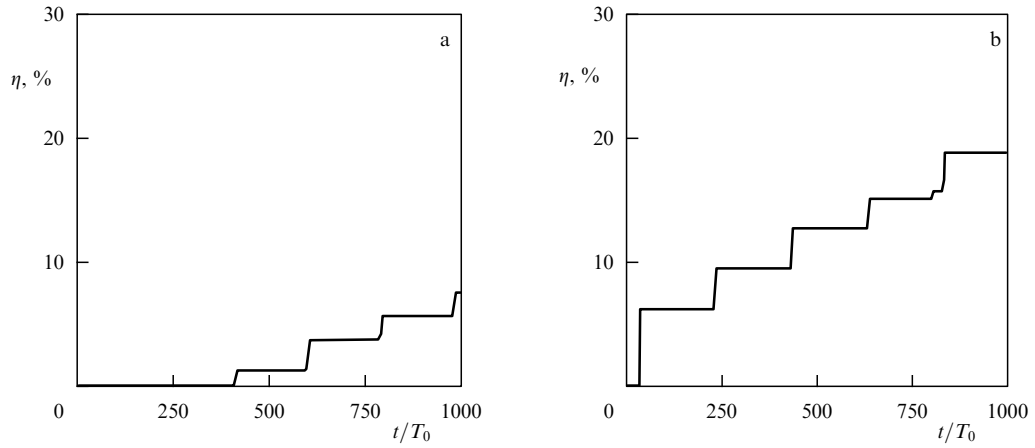

Figure 7. Controlled dissociation in a classical ensemble: (a) linearly chirped pulses and (b) pulses calculated by the SG method; η is the fraction of the dissociated molecules.

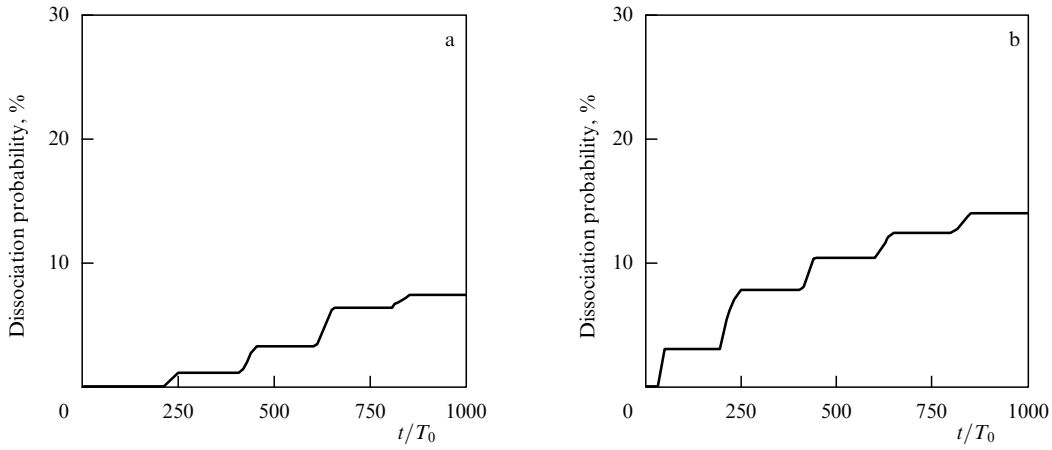

Figure 8. Dissociation probability for quantum mechanical modeling for $E = 0.005$ a.u.: (a) linearly chirped pulses and (b) pulses calculated by the SG method.

case. The initial state of the system was assumed pure and having an energy equal to the second energy level, while the dissociation threshold H_* corresponded to the 15th energy level of the HF molecule, just as in the classical case. The dissociation probability (the fraction of the dissociated molecules) was defined as the probability of states with the energy exceeding the dissociation threshold H_*.

The results of quantum mechanical modeling are shown in Fig. 8. Clearly, the SGA ensures a 15% dissociation probability after five pulses, which is much higher than the dissociation probability for a chirped pulse and agrees with the results obtained in the classical case (10 – 12%).

8. Laws of control and laws of dynamics of systems

Below, we study the links between the laws of control in technical systems and the laws of dynamics of physical systems. We show that the methods used to synthesize control algorithms, e.g., the SG method, make it possible to derive the laws of dynamics of physical systems, provided that the proper goal function is chosen.

8.1 Variational principles. The speed-gradient principle

We turn to the class of open physical systems, for which the models of the dynamics are described by systems of differential equations

$$\dot{x} = f(x, u, t)\,, \tag{8.1}$$

where $x \in \mathcal{R}^n$ is the state vector of the system, u is the vector of the input (free) variables, and $t \geqslant 0$. The problem of modeling (building a model of) the system can be interpreted as the problem of finding the law of variation (evolution) of $u(t)$ that meets a certain criterion of 'naturalness' of the system's behavior and imparts properties observed in the real physical system to the model being constructed.

In physics, such problems are widely known. Variational principles of building model systems gained acceptance long ago. A variational principle usually presupposes that a certain integral functional (e.g., the action functional in the principle of least action [120]) has been specified. Minimization of the functional yields the possible paths actually taken by the system, $\{x(t), u(t)\}$, as points in the appropriate functional space. Well-developed tools of variational calculus are used to explicitly determine the law of dynamics of the system.

It is interesting that the variational approach formed the basis for an entire area of research in control theory, the *theory of optimal control*, in which minimization of a functional is used to find the law of controlling a technical system that is appropriate in the given sense. In turn, methods of optimal control (Bellman's dynamic programming, the

Pontryagin maximum principle, etc.), which constitute developments of the methods of classical variational calculus, can be employed in building models of the dynamics of mechanical [121], thermodynamic [122], and other systems.

In addition to integral principles, differential principles (i.e., principles that are local in time) have been proposed, among which are Gauss's principle of least constraint and the principle of minimum energy dissipation. As noted by Planck [123], local principles have some advantages over integral principles, because they do not make the current state and motion of the system depend on the later states and motions. Following Ref. [62], we can formulate one more local variational principle based on the SG method.

The speed-gradient principle. *Among all possible motions of a system, only those for which the input variables vary in proportion to the speed gradient of a certain 'goal' functional Q are realized.*

The speed-gradient principle gives the researcher a choice between two types of models of the system's dynamics:

(a) models that follow from SGAs in the differential form,

$$\dot{u} = -\Gamma \nabla_u \dot{Q}\,; \tag{8.2}$$

(b) models that follow from SGAs in the finite form,

$$u = -\Gamma \nabla_u \dot{Q}\,. \tag{8.3}$$

Here, $\dot{Q}$ is the rate of variation of the goal functional along a path of system (8.1).

We describe the way to use the principle in the simplest, but important, case where the class of the models of dynamics (8.1) is specified by the relation

$$\dot{x} = u\,. \tag{8.4}$$

This relation simply implies that we seek a law of variation of the rates of the state variables of the system. In accordance with the SG principle, we must first introduce a goal functional (function) $Q(x)$. The choice of $Q(x)$ must be based on the physics of the real system and reflect the presence in the system of a tendency to decrease the current value $Q\big(x(t)\big)$. After this is done, the law of dynamics can be immediately written in form (8.2) or (8.3).

Specifying the law of dynamics in form (8.2) generates second-order differential equations of motion, which are invariant under time reversal $t \to -t$, i.e., correspond to reversible processes. But if we choose finite form (8.3), we are usually dealing with irreversible processes.

In Section 8.2, we use examples to demonstrate how the introduced principle can be applied.

8.2 Examples of the speed-gradient laws of dynamics

8.2.1 Particle motion in a potential field. For the first example, we take the problem of describing the motion of a particle in a potential field. The coordinates of the point are the state variables, i.e., $x = \mathrm{col}\,(x_1, x_2, x_3)$. For the goal function, we take the potential of the field, $Q(x)$, and derive the SG law of motion in a differential form. We calculate the speed gradient

$$\dot{Q} = \big[\nabla_x Q(x)\big]^{\mathrm{T}} u\,, \qquad \nabla_u \dot{Q} = \nabla_x Q(x)\,.$$

By choosing the positive definite diagonal matrix Γ as $\Gamma = m^{-1} I_3$, where $m > 0$ is a parameter and I_3 is the 3×3 identity matrix, we arrive at Newton's classical law of dynamics $\dot{u} = -m^{-1}\nabla_x Q(x)$, or

$$m\ddot{x} = -\nabla_x Q(x)\,. \tag{8.5}$$

Here, the parameter m is interpreted as the particle mass.

This example allows far-reaching generalizations. For systems whose motion is caused by potential forces, the field potential may act as the goal function $Q(x)$ and the inertia matrix determines the gain matrix in the algorithm. If the inertial properties of the system are different at different points of the configuration space, the metric varies in the space of velocities (the controlling variables). In this way, we can build models of the dynamics of complex mechanical systems described by Lagrangian equations of the second kind.

The SG principle can also be used in building models of the dynamics of distributed systems, described in infinite-dimensional state spaces. In particular, x can be a vector in a Hilbert space $\mathcal{X}$ and $f(x, u, t)$ a nonlinear operator defined in a dense set $D_F \subset \mathcal{X}$ (here, the solutions of Eqn (8.1) are interpreted as generalized solutions).

8.2.2 The wave equation and the heat conduction equation. Let $x = x(r)$ be the field of temperatures or concentrations of matter defined on a certain domain $\Omega \subset \mathcal{R}^3$, $r = \mathrm{col}\,(r_1, r_2, r_3) \in \Omega$. For the goal functional, we take the measure of the field's inhomogeneity,

$$Q(x) = \frac{1}{2}\int_\Omega \big|\nabla_r x(r,t)\big|^2\,\mathrm{d}r\,, \tag{8.6}$$

where $\nabla_r x(r,t)$ is the spatial gradient of the field $x = x(r)$. Assuming zero boundary conditions for simplicity, we calculate the speed gradient of functional (8.6). From the Green's formula with zero boundary conditions, we obtain

$$\dot{Q} = \int_\Omega \big(\nabla_r x(r,t)\big)^{\mathrm{T}} \nabla_r u(r,t)\,\mathrm{d}r = -\int_\Omega \Delta x(r,t)\,u(r,t)\,\mathrm{d}r\,, \tag{8.7}$$

where

$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial r_i^2}$$

is the Laplace operator. Bearing in mind that the gradient of a dot product with respect to one factor is equal to the other factor,[6] we find that the SG operator in the given case is simply the Laplace operator, $\nabla_u \dot{Q} = -\Delta x(r,t)$. Hence, the speed-gradient law of evolution of the system in differential form (8.2) becomes

$$\frac{\partial^2 x(r,t)}{\partial t^2} = \gamma\,\Delta x(r,t)\,, \tag{8.8}$$

which is the wave equation.

If we choose the algorithm in finite form (8.3), the corresponding dynamics equation

$$\frac{\partial x(t)}{\partial t} = \gamma\,\Delta x(r,t) \tag{8.9}$$

coincides with the simplest heat conduction equation.

[6] To make this reasoning more meaningful, we must assume that both integrands belong to a Hilbert space, e.g., $L_2(\Omega)$, which, however, imposes no serious restrictions on the general nature of such reasoning.

8.2.3 Dynamics of a viscous fluid. Suppose that an infinite-dimensional state vector of a system is formed by two functions, $x = \mathrm{col}\,\big(v(r,t),\, p(r,t)\big)$, where $v(r,t)$ is the velocity of the three-dimensional flow of the liquid, $v(r,t) \in \mathcal{R}^3$, and $p(r,t)$ is the pressure. We introduce the goal functional as

$$Q = \int_\Omega p(r,t)\,\mathrm{d}r + v_0 \int_\Omega \left|\nabla_r v(r,t)\right|^2 \mathrm{d}r\,, \tag{8.10}$$

where $v_0 > 0$ is a weight factor. Calculating the speed gradient of functional (8.10) in u given by (8.4), we obtain $\nabla_u \dot{Q} = \nabla_r p - v_0 \Delta v$. Hence, the differential form of the SG law is simply the Navier–Stokes equation, which describes the motion of a viscous fluid,

$$\rho\,\frac{\partial v(r,t)}{\partial t} = -\nabla_r p(r,t) + v\,\Delta v(r,t)\,, \tag{8.11}$$

where $v = v_0\gamma^{-1}$ is the viscosity and $\rho = \gamma^{-1}$ is the density of the fluid.

Other examples of derivation of the dynamics equations for mechanical, electrical, and other systems can be found in Ref. [62]. The SG principle can be used to describe a broad class of physical systems under potential or dissipative forces. On the other hand, the principle must be modified if we want it to be applicable to systems that are in vortex motion, e.g., to mechanical systems subjected to gyroscopic forces.

We note once more that this principle is of a dual nature: the differential form of the SG law corresponds to reversible processes, while the finite form corresponds to irreversible processes. The choice between the two forms, just as the choice of the goal and of the goal functional, entirely pertains to the physical setting of the problem. In some cases, this choice is ambiguous: for instance, a process that is reversible on one time scale may be irreversible on other time scales. Thus, this principle does not free the physicist from the need to choose or build a model of the system — it only helps to reduce the number of options in taking the decision and in establishing the 'purposefulness' in the system's behavior.

8.3 Onsager relations

The speed-gradient principle offers a new angle on some well-known physical facts and phenomena. For instance, we derive a generalized variant of the well-known principle of the symmetry of kinetic coefficients (the Onsager principle) in thermodynamics [124, 125]. We take an isolated physical system whose state is characterized by a set of thermodynamic variables $\xi_1, \xi_2, \ldots, \xi_n$. We let $x_i = \xi_i - \xi_i^*$ denote the variations in the variables from the equilibrium values $\xi_1^*, \xi_2^*, \ldots, \xi_n^*$. Let the dynamics of the quantities $x_1, x_2, \ldots, x_n$ be described by the differential equations

$$\dot{x}_i = u_i(x_1, x_2, \ldots, x_n)\,, \qquad i = 1, 2, \ldots, n\,. \tag{8.12}$$

We linearize Eqns (8.12) near equilibrium:

$$\dot{x}_i = -\sum_{k=1}^{n} \lambda_{ik} x_k\,, \qquad i = 1, 2, \ldots, n\,. \tag{8.13}$$

The *Onsager principle* [124, 125] states that the kinetic coefficients λ_{ik} are symmetric, i.e.,

$$\lambda_{ik} = \lambda_{ki}\,, \qquad i, k = 1, 2, \ldots, n\,. \tag{8.14}$$

The Onsager principle does not hold for all systems. Its proof (e.g., see Refs [124, 125]) rests on additional postulates. In particular, it is assumed that the system is near thermodynamic equilibrium. Below, we give a new proof that shows that for speed-gradient systems, the generalized variant of the Onsager principle holds without any additional assumptions and does not require that the system's model be linearized in advance.

We must formulate this variant first. Clearly, for the linear model of the system in (8.13), symmetry relations (8.14) are equivalent to the identities

$$\frac{\partial u_i}{\partial x_k}(x_1, x_2, \ldots, x_n) \equiv \frac{\partial u_k}{\partial x_i}(x_1, x_2, \ldots, x_n)\,. \tag{8.15}$$

We say that the *generalized Onsager principle* holds if identities (8.15) hold for systems described by nonlinear equations (8.12).

It turns out that if there is a smooth function $Q(x)$ for which dynamic equations (8.12) can be derived from the SG principle in the finite form, then identities (8.15) hold for all $x_1, x_2, \ldots, x_n$, i.e., the generalized Onsager principle holds.

Indeed, Eqns (8.12) express the SG law for $Q(x)$, and hence their right-hand sides can be represented in the form $u_i = -\gamma\,\partial\dot{Q}/\partial u_i$, $i = 1, 2, \ldots, n$. Then, $u_i = -\gamma(\partial Q/\partial x_i)$ [because $\dot{Q} = (\nabla_x Q)^{\mathrm{T}} u$)] and

$$\frac{\partial u_i}{\partial x_k} = -\gamma\,\frac{\partial^2 Q}{\partial x_i\,\partial x_k} = \frac{\partial u_k}{\partial x_i}\,,$$

which implies identities (8.15).

Obviously, symmetry relations (8.14) constitute a particular case of (8.15) for linear dynamic equations. Thus, for systems obeying the SG principle, the generalized Onsager relations in (8.15) are true without the assumption that the dynamic equations are linear, i.e., not only near equilibrium. Because the above derivation is based on differentiation, the right-hand sides of (8.12) are assumed smooth. It would seem that this excludes problems with nonsmooth or discontinuous functions, i.e., problems involving the propagation of shock waves. But in such cases, we can use variants of the SG algorithms developed specially for nonsmooth problems, in which the gradient is replaced by the *subgradient* [62].

8.4 Dynamics and the goal

It would be interesting to compare the above approach with the results obtained by a British expert in control systems, H H Rosenbrock [126, 127], who found that the Schrödinger equation can be derived from the Hamilton–Jacobi–Bellman principle of optimality.

Although the approach to construct dynamic equations of physical systems on the basis of extremum principles is widely known, in physics it is not connected to the concept of a goal, which means the attainment of the extremum of the goal functional. This is an illustration of the difference in approaches in physics and in the engineering sciences, where optimality as the goal of building an artificial (technological) system is of paramount importance. On the other hand, the use of the concepts of goal and expediency of behavior has in the past raised objections from some physicists and other researchers in the natural sciences. Such an attitude was most strongly expressed by Einstein [128] in 1950: "For the scientist, there is only 'being,' but no wishing, no valuing, no good, no evil, no goal."

Rosenbrock criticises Einstein's point of view and adduces arguments in favor of the idea that the concept of a goal is natural for both animate and inanimate nature. He notes that the rejection of the concept of a goal is a reaction to the 17th-century conflict between the Church and the emerging science and is not relevant today. In the 20th and 21st centuries, machines that act 'purposefully' and embody the goals set by humans have spread to all parts of the world and have become common fixtures in our environment. This fact forces us to treat the concept of a goal more seriously in physics as a science that deals with the most general laws governing the systems in the environment, animate, inanimate, and artificial (built by humans). Rosenbrock writes [33, 127]: "As an example, living organisms exhibit clear purposes, and if the substrate of quantum mechanical particles from which life evolves is described as purposeless the question arises how can purpose arise from a purposeless substrate?"

Rosenbrock's viewpoint is in common with the conclusions drawn by Kadomtsev [10], who wrote: "...there are many common traits in the behavior of complex systems, organic and inorganic, and inorganic systems with a complex structure are also not simple both structurally and in the nature of their behavior," and "...the expediency and possibility of selecting a goal may be assumed a natural product of the development of complex systems with information-related behavior."

The local evolution principle described above on the basis of the speed gradient rests on the concept of a goal to a greater extent than the integral extremum principles. Hence, in the cases where the concept of a goal and a goal function emerge in a natural way, the local evolution principle may prove more convenient and useful for designing models of the dynamics of systems. Incidentally, it must be noted that the SG principle agrees with the well-known biological principle according to which organisms and populations develop in a way to ensure a maximum increase in their biomass [62, 129].

9. Conclusion

We have analyzed the main features of the problem of control in physical systems and described the general approaches to solving the problems of controlling a fundamental characteristic of systems, the energy. We have established the laws of energy transformation for the main classes of systems, conservative and dissipative. New statements of the problem required new concepts (the excitability index, for instance) and led to a description of new effects (feedback resonance). The examples in which the problems of controlling physical systems were solved show the fruitfulness of the new approaches to the study and control of phenomena of the microworld and the macroworld.

In a single journal publication, it is impossible to cover all the research done in the field that is the topic of this review, and hence many important applications of the methods of control theory in physics had to be ignored. I believe that modern physics education must include a course in cybernetic physics that will incorporate areas such as the optimization thermodynamics [122], particle-beam control [98], theory of dynamical materials [130], controlling chaos [39, 40], controlling synchronization [28, 36], and controlled nuclear fusion.

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