# On the mechanical prototypes of fundamental hydrodynamic invariants and slow manifolds 

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#### Abstract

Arnol'd's group-theoretical concept of generalized rigid body includes the Euler equations of motion of the classical gyroscope and ideal homogeneous fluid as particular representatives. Here, this concept is extended to motion in force fields with a scalar or vector potential and in a Coriolis force field. The concepts of generalized heavy top and generalized MHD system are introduced. As particular cases, they include, on the one hand, the Euler - Poisson equations of the classical heavy top and the Kirchhoff equations of motion of a solid body in a potential flow of an ideal incompressible fluid and, on the other hand, the Oberbeck - Boussinesq equations of motion of a heavy fluid and MHD equations. On this basis, mechanical prototypes are constructed for all known fundamental hydrodynamic invariants and global geophysical flows, including a prototype of the general atmospheric circulation.


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## 1. Introduction

In 1879 , the British hydrodynamicist Greenhill [1] noted that the Euler equations of motion of a rigid body with a fixed point strictly describe the flow of an ideal incompressible homogeneous fluid inside an ellipsoid in the class of spatially linear velocity fields. This fact was later widely applied to studying the motions of solids with cavities filled with a liquid (see, e.g., [2-5]). However, we leave aside the utilitarian significance of Greenhill's work. Something else is important for us. Greenhill's result suggests that the mechanical and hydrodynamic Euler equations share some common fundamental symmetries that should manifest themselves in common properties of their solutions.

Almost a century later this idea was realized by V I Arnol'd [6] (see also more easily available publications [7, 8]). He has constructed equations of motion of a rigid body for an arbitrary Lie group, which include the mechanical and hydrodynamic Euler equations as special cases. This construction, called by Arnol'd the generalized rigid body (GRB), allowed him to give mechanical interpretations to the Kelvin circulation theorem and the Rayleigh theorem on the stability of two-dimensional ideal incompressible flows without inflection points of the velocity profile. It turned out that the conservation law for angular momentum is in a sense
a mechanical prototype of the Kelvin theorem, and the hydrodynamic Rayleigh theorem corresponds to the mechanical Euler theorems on the stability of rotations about the longest and shortest principal axes of inertia. These results, slightly opening the mysteries of the Universe and, furthermore, giving aesthetic pleasure, in their turn suggest that there are other nontrivial peculiarities of mechanical systems, which can be carried over to hydrodynamic entities, at least if global-scale flows are considered.

This article is completely dedicated to the problem touched on. In the exposition of the material we will need invoking the main concepts of the theory of Lie groups and their representations, without which a strict proof of the analogies between hydrodynamic and mechanical systems becomes virtually impossible. Unfortunately, this may make the article difficult to read for some. Therefore, in Section 2, the invariant form of the Euler equations of motion of a classical gyroscope is prefaced with a description of its motion and derivation of the corresponding equations in terms of the well-known (for physicists) group of rotations of threedimensional Euclidean space, which is then easily extended to an arbitrary Lie group. It is relevant to note as well that the modern mathematicians have fairly long exploited the above theory to investigate fundamental hydrodynamic problems (see, e.g., the above-mentioned book [8], in which such an approach is applied to studying the structural properties of solutions of the hydrodynamic equations). Thus, it seems timely to introduce more insistently the group-theoretical concepts and approaches in theoretical fluid mechanics, which, as a matter of fact, is part of field theory based precisely on these concepts.

We extend the GRB concept to motions in external force fields of scalar and vector potentials, as well as in the Coriolis force field. Finally, we formulate the concepts of generalized heavy top and generalized magnetohydrodynamic system (GMHDS), which include, on the one hand, the mechanical Euler-Poisson equations of a heavy top and the Kirchhoff equations of motion of a rigid body in a potential flow of an ideal incompressible homogeneous fluid [9] and, on other hand, the Oberbeck - Boussinesq equations of motion of a heavy fluid and magnetohydrodynamic equations as some particular cases. It is relevant to emphasize that the EulerPoisson and Kirchhoff equations, as the Euler equations, allow dual mechanical and hydrodynamic interpretations. They strictly describe the motions of an ideal heavy fluid and an ideal liquid conductor inside an ellipsoid in the class of spatially linear velocity fields and corresponding thermal and magnetic fields.

On the basis of the introduced concepts, it has been shown [10] that the mechanical 'primogenitors' of potential vorticity (PV) [11-13] (in the book by Landau \& Lifshitz [14] this quantity is called the Ertel invariant) and hydrodynamic and magnetohydrodynamic helicities [15-17] are the projection of the angular momentum of a heavy top onto the direction of the force of gravity, the squared angular momentum of a classical gyroscope, and the projection of the total angular momentum in a solid-fluid system onto the direction of the total momentum, respectively.

The Kelvin theorem for the magnetic vector potential and the invariance of the magnetic helicity [17] follow from the immobility of the magnetic field with respect to the fluid, as the constancy of the angular momentum of a classical gyroscope in space implies the conservation of this quantity squared in the frame of reference 'frozen' in the body.

The next step includes the construction of some mechanical prototypes for slow manifolds of rotating fluids. The choice of a rotating fluid is not arbitrary since the quasigeostrophic approximation for its equations of motion naturally singles out, on a global scale, the flows that are expected to bear most similarity with corresponding motions of a rigid body with a fixed point. The problem of the existence and stability of the quasigeostrophic manifolds, first touched on by Obukhov [13], is actively discussed in the modern hydrodynamic literature (see, e.g., [18-21] and references therein). In Sections 9 and 10, the geostrophic trajectories of the model equations are compared with their exact solutions, which coincide with the exact particular solutions of the original hydrodynamic equations.

The equations of motion of a normal gyroscope (without gravity) in the Coriolis field are treated as a model of barotropic flows of a rotating fluid. At small Rossby numbers, it describes slow precessions of the top in the direction opposite to the direction of the general rotation, the projection of the angular momentum onto the direction of the general rotation being approximately invariant up to the Rossby number squared. These features of the motion of the top resemble the behavior of planetary waves transferring angular momentum in the westward direction or the approximate invariance of the quasigeostrophic PV of global barotropic geophysical flows expressed by the CharneyObukhov equation, which is also valid up to the Rossby number squared.

The stability of a slow baroclinic manifold with respect to global disturbances of quasigeostrophic equilibrium is illustrated by comparison of the exact and quasigeostrophic model solutions. The quasigeostrophic approximation for the equations of motion of a heavy top in the Coriolis force field is shown in Section 9 to coincide with the mechanical Euler equations written in terms of the vertical vorticity and thermal wind components, i.e., the main characteristics of global baroclinic geophysical flows, and the stratification parameter appears in the equations as a given one (similarly to the case of the quasigeostrophic equations of motion of a baroclinic atmosphere). The principle result is that the exact phase trajectories 'feel' the existence of the slow manifold even when the motion becomes strongly ageostrophic because of large-amplitude inertial-gravitational oscillations. The motion resembles the interaction of the Rossby and iner-tial-gravitational waves in a reduced shallow water model [19]. The slow manifold itself is a sort of two-sided curved mirror from which the trajectories are reflected as long as the stratification parameter does not exceed a certain critical value. When the parameter reaches the critical value, the trajectories pierce and smash the mirror, and chaos arises.

Section 10 is devoted to the investigation of a 'toy' general atmospheric circulation (GAC), which is described by the equations of motion of a hydrodynamic heavy top in a Coriolis force field, with the inclusion of dissipation, a horizontally inhomogeneous external heat drive (imitating the equator-pole temperature difference), and the slope of the general rotation axis relative to gravity. The dissipation is taken into account by the inclusion of a linear (in velocity) friction simulating the influence of the Ekman boundary layer, while the external drive is calculated by Newton's formula according to which the heat influx is proportional to the deviation of temperature from some background distribution.

The stationary quasigeostrophic solutions of the model equations are shown to reproduce the energy cycle and the stability properties (including the asymptotics of the lower branch of the stability diagram) of the fundamental Hadley and Rossby regimes observed under natural conditions and in laboratory analogs of the GAC. The quasigeostrophic approximation of the model taking into account the sloping effect coincides with the slightly generalized, widely known three-component Lorenz system [22] (see also Ref. [18], where the quasigeostrophic approximation for the reduced viscous shallow water equations is shown to be exactly the stochastic Lorenz model). Therefore, special attention in Section 10 is given to the comparison of quasigeostrophic strange attractors with the corresponding exact solutions of the primary model equations. The sloping effect is indeed found to favor (under certain conditions) the stochastization and, consequently, the unpredictability of the phase trajectories. It is worth noting that the mechanism of stochastization depends on the sign of the sloping effect (the system is not invariant with respect to the sign of the angle between the gravity and the general rotation axis) and may have an origin other than the Lorenz strange attractor, if the sloping effect is negative. It is relevant to emphasize that the enumerated properties of the global geophysical flows were obtained exclusively from symmetry considerations on the basis of only one toy model of a rotating heavy fluid without concrete definition of the boundary conditions, geometry, or other features of the system.

## 2. Arnol'd's construction of a generalized rigid body

The motion of a rigid body with a fixed point can be treated in terms of the rotations of a coordinate system (CS) 'frozen' in the body with respect to the CS immobile in the space. The origins of both CSs coincide with the fixed point. The totality of such rotations forms an $\mathrm{SO}(3)$ group that can be considered a space of generalized coordinates. It can be said in such cases that the configuration space of a rigid body is a group of rotations of the Euclidean space. In terms of the


Figure 1. The Eulerian angles $\theta, \varphi, \psi$ defining the orientation of the axes $x_{1}$, $x_{2}, x_{3}$ of the coordinate system moving with respect to the immobile coordinate system $(X, Y, Z) ; 0 N$ is the nodal line [23].

Eulerian angles $\theta, \varphi, \psi$ (Fig. 1), any rotation $g \in \mathrm{SO}(3)$ is specified by the orthogonal matrix

$$
g(\theta, \varphi, \psi)=B_{\varphi} C_{\theta} B_{\psi}
$$

where $B_{\varphi}$ and $C_{\theta}$ are the matrices describing rotations around immobile axes $Z$ and $X$ by angles $\varphi$ and $\theta$, respectively; $B_{\psi}=B_{\varphi=\psi}$,

$$
\begin{aligned}
B_{\varphi} & =\left(\begin{array}{ccc}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{array}\right), \\
C_{\theta} & =\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{array}\right) .
\end{aligned}
$$

Let us remember that the Euler equations of motion of a classical gyroscope can be written through the angular velocities that are specified in terms of the Eulerian angles by the following formulas [23] (project the angular velocities $\dot{\theta}, \dot{\varphi}, \dot{\psi}$ onto the moving axes $x_{1}, x_{2}, x_{3}$; see Fig. 1):

$$
\begin{aligned}
& \boldsymbol{\omega}=\omega_{1} \mathbf{e}_{1}+\omega_{2} \mathbf{e}_{2}+\omega_{3} \mathbf{e}_{3}, \\
& \omega_{1}=\dot{\varphi} \sin \theta \sin \psi+\dot{\theta} \cos \psi, \\
& \omega_{2}=\dot{\varphi} \sin \theta \cos \psi-\dot{\theta} \sin \psi, \\
& \omega_{3}=\dot{\varphi} \cos \theta+\dot{\psi} .
\end{aligned}
$$

Here, $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$ are the basis vectors of the moving CS.
Projecting $\dot{\theta}, \dot{\varphi}, \dot{\psi}$ onto the immobile axes $X, Y, Z$ yields the components of the angular velocity in the space:

$$
\begin{aligned}
& \boldsymbol{\Omega}=\Omega_{1} \mathbf{e}_{x}+\Omega_{2} \mathbf{e}_{y}+\Omega_{3} \mathbf{e}_{z} \\
& \Omega_{1}=\dot{\psi} \sin \theta \sin \varphi+\dot{\theta} \cos \varphi \\
& \Omega_{2}=-\dot{\psi} \sin \theta \cos \varphi+\dot{\theta} \sin \varphi \\
& \Omega_{3}=\dot{\psi} \cos \theta+\dot{\varphi}
\end{aligned}
$$

where $\mathbf{e}_{x}, \mathbf{e}_{y}, \mathbf{e}_{z}$ are the basis vectors of the immobile CS.
The space of the vectors of the three-dimensional oriented Euclidean space is isomorphic to the space of third-rank skew-symmetric matrices. In tensor notations, this isomorphism is described by the formulas $a_{i j}=-\varepsilon_{i j k} a_{k}, a_{i}=-\varepsilon_{i j k} a_{j k} / 2$ $(i, j, k=1,2,3)$, according to which there is the following one-to-one correspondence:

$$
\mathbf{a}=\left(a_{1}, a_{2}, a_{3}\right) \Leftrightarrow\left(\begin{array}{ccc}
0 & -a_{3} & a_{2} \\
a_{3} & 0 & -a_{1} \\
-a_{2} & a_{1} & 0
\end{array}\right)=a,
$$

with

$$
\mathbf{a} \times \mathbf{b} \Leftrightarrow[a, b], \mathbf{a} \cdot \mathbf{b}=-\frac{1}{2} \operatorname{tr}(a * b)
$$

here, [, ] means the commutation operation, the dot • and the multiplication sign $\times$ denote the operators of the normal scalar and cross multiplication of vectors in three-dimensional Euclidean space, respectively, and the asterisk * denotes the ordinary multiplication of matrices.

Thus, by the angular rotational velocities of a rigid body, both vectors of the three-dimensional space and third-rank skew-symmetric matrices can be meant. Now, it is easy to verify by direct calculation that, if $g=g(t)$ is a trajectory of the rigid body in the configuration space, then

$$
g^{-1} \dot{g} \doteq \mathrm{~L}_{g^{-1}} \dot{g}=\omega \Leftrightarrow \boldsymbol{\omega}, \quad \dot{g} g^{-1} \doteq \mathrm{R}_{g^{-1}} \dot{g}=\Omega \Leftrightarrow \boldsymbol{\Omega},
$$

i.e., the angular velocities in the body and space are defined by the linear mappings $\mathrm{L}_{g^{-1}}$ and $\mathrm{R}_{g-1}$ of the vector $\dot{g}$ of the tangent space $\mathrm{T}_{g}$ of the group $\mathrm{SO}(3)$ at the point $g$ onto the Lie algebra of the $\mathrm{SO}(3)$ group (tangent space $\mathrm{T}_{e} \doteq \hat{g}$ at the group unit); they are called the left and right translations, respectively. It is easy to check that the angular velocities themselves are related as follows:

$$
\Omega=\mathrm{R}_{g^{-1}} \mathrm{~L}_{g} \omega \doteq \mathrm{Ad}_{g} \omega
$$

where $\mathrm{Ad}_{g}$ is called the adjoint representation of the $\mathrm{SO}(3)$ group; $\mathrm{Ad}_{g h}=\operatorname{Ad}_{g} \mathrm{Ad}_{h}$ for every $g$, and $h \in \mathrm{SO}(3)$.

The penultimate formulas can be used as the definitions of angular velocities in the body and space, respectively, for the classical gyroscope. To construct a dynamical system similar to the Euler top on an arbitrary Lie group, these formulas should be written in an invariant form, valid for the Lie group, whose elements are not necessarily linear operators. In this case,

$$
\begin{equation*}
\omega=\mathrm{L}_{g^{-1} *} \dot{g} \in \hat{g}, \quad \Omega=\mathrm{R}_{g^{-1} *} \dot{g} \in \hat{g}, \tag{2.1}
\end{equation*}
$$

where $\mathrm{L}_{g *}$ and $\mathrm{R}_{g *}$ are the linear mappings of the tangent spaces induced by the left and right translations, $\mathrm{L}_{g}$ and $\mathrm{R}_{g}$, and $\hat{g}$ is the Lie algebra (tangent space at the group unit) of the group $G$. Now, the adjoint representation that maps the Lie algebra onto itself can be written in the form

$$
\begin{align*}
& \operatorname{Ad}_{g}=\left(\mathrm{R}_{g^{-1}} \mathrm{~L}_{g}\right)_{* e}: \hat{g} \rightarrow \hat{g}, \\
& \Omega=\operatorname{Ad}_{g} \omega \tag{2.2}
\end{align*}
$$

Further consideration of the motion characteristics of the classical gyroscope will be conducted in an invariant form valid for an arbitrary Lie group.

The kinetic energy of a rigid body is independent of its spatial location and is specified by a quadratic positive definite function of the angular velocities in the body:

$$
E=\frac{1}{2} I_{i j} \omega_{i} \omega_{j} \doteq \frac{1}{2}(I \omega, \omega),
$$

where $I$ is a time-independent symmetric matrix called the tensor of inertia momenta with respect to the moving CS. This form determines a left-invariant metric for the whole Lie group according to the relationships

$$
\begin{equation*}
E=\frac{1}{2}\langle\omega, \omega\rangle_{e} \doteq \frac{1}{2}(I \omega, \omega) \doteq \frac{1}{2}\langle\dot{g}, \dot{g}\rangle_{g} \tag{2.3}
\end{equation*}
$$

where $\langle$,$\rangle denotes scalar multiplication on the appropriate$ tangent spaces. Specifying the scalar product on the Lie algebra is equivalent to specifying an isomorphic mapping of the Lie algebra $\hat{g}$ into the Lie coalgebra $\hat{g}^{*}$ (the space of linear forms of Lie algebra elements). This mapping is done by the operator $I: \hat{g} \rightarrow \hat{g}^{*}$. Therefore, the kinetic momentum, or angular momentum, $m=I \omega$ should be considered an element of the Lie coalgebra, and the quantity $(I \omega, \omega)=(m, \omega)$ is the value of the linear form $m$ at $\omega$.

Since the kinetic energy is independent of the choice of CS, the following equalities should be valid:

$$
(m, \omega)=(M, \Omega)=\left(M, \operatorname{Ad}_{g} \omega\right) \doteq\left(\operatorname{Ad}_{g}^{*} M, \omega\right)
$$

Here, the operator $\operatorname{Ad}_{g}^{*}: \hat{g}^{*} \rightarrow \hat{g}^{*}$ is dual to $\operatorname{Ad}_{g}$ and realizes the oncoming mapping induced with the adjoint representa-


Figure 2. Diagram of the mappings acting in the tangent and cotangent spaces of an arbitrary Lie group $G[8] . \mathrm{T}_{g} G$ and $\mathrm{T}_{g}^{*} G$ are the tangent and cotangent spaces at the point $g$ of the Lie group $G$. Other notation is given in the text.
tion [cf. (2.2)]:

$$
\begin{equation*}
m=\operatorname{Ad}_{g}^{*} M, \quad \text { or } \quad M=\operatorname{Ad}_{g_{-1}}^{*} m \tag{2.4}
\end{equation*}
$$

The operator $\mathrm{Ad}_{g}^{*}$ is called the coadjoint representation of the Lie group $G ; \mathrm{Ad}_{g h}^{*}=\mathrm{Ad}_{h}^{*} \mathrm{Ad}_{g}^{*}$ for every $g$ and $h \in G$. The diagram of the operators [8] acting in the tangent and cotangent spaces is shown in Fig. 2.

The derivative of the adjoint representation at the group unity in the direction of the vector $\xi \in \hat{g}$, denoted as $\mathrm{ad}_{\xi}$, is defined by the formula

$$
\begin{equation*}
\operatorname{ad}_{\xi}=\left.\frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{Ad}_{\exp \left(t^{\xi}\right)}\right|_{t=0}, \tag{2.5}
\end{equation*}
$$

where $\exp (t \xi)$ is a one-parameter group given by the vector $\xi \in \hat{g}$. From this, it follows that, for any $\xi, \eta \in \hat{g}$,

$$
\begin{equation*}
\operatorname{ad}_{\xi} \eta=[\xi, \eta], \tag{2.6}
\end{equation*}
$$

where [, ] is the operation of commutation in the Lie algebra. Proof:

$$
\begin{aligned}
& \left.\frac{\mathrm{d}}{\mathrm{~d} t} \mathrm{Ad}_{\exp (t \xi)} \eta\right|_{t=0}=\left.\frac{\mathrm{d}}{\mathrm{~d} t} \exp (t \xi) \eta \exp (-t \xi)\right|_{t=0} \\
& \quad=\xi \eta-\eta \xi=[\xi, \eta]
\end{aligned}
$$

The derivative of the coadjoint representation at the group unity in the direction of the vector $\xi$

$$
\begin{equation*}
\operatorname{ad}_{\xi}^{*}=\left.\frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{Ad}_{\exp (t \xi)}^{*}\right|_{t=0} \tag{2.7}
\end{equation*}
$$

is an operator dual to $\mathrm{ad}_{\xi}$, i.e.,

$$
\begin{equation*}
\left(\mathrm{ad}_{\xi}^{*} a, \eta\right)=\left(a, \mathrm{ad}_{\xi} \eta\right) \tag{2.8}
\end{equation*}
$$

for every $a \in \hat{g}^{*}$ and $\xi, \eta \in \hat{g}$. By introducing the notation $\mathrm{ad}_{\xi}^{*} a \doteq\{\xi, a\}$, the last formula can be rewritten as

$$
\begin{equation*}
(\{\xi, a\}, \eta)=(a,[\xi, \eta]) \tag{2.8a}
\end{equation*}
$$

Now the equations of motion of a rigid body whose configuration space is an arbitrary Lie group can be obtained using the kinetic energy (2.3) as a Lagrangian function and
applying the least action principle. According to it, the GRB moves along a geodesic, which is not surprising because the body moves inertially. However, to derive the equations of motion, we will act somewhat differently. The least action principle applied to the immobile CS means that the kinetic momentum $M$ is fixed with respect to the space (see, e.g., Ref. [23])

$$
\begin{equation*}
\dot{M}=0 . \tag{2.9}
\end{equation*}
$$

From this, the equation of GRB motion in the moving CS immediately follows. Indeed, differentiating (2.4) with respect to time in the direction of the body trajectory and taking into account (2.9), we obtain

$$
\begin{aligned}
0= & \left.\frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{Ad} g_{g^{-1}(t)}^{*} m\right|_{t=0}=\left.\frac{\mathrm{d}}{\mathrm{~d} t} \mathrm{Ad}_{\exp (-t \omega)}^{*}\right|_{t=0} m \\
& +\left.\mathrm{Ad}_{\exp (-t \omega)}^{*}\right|_{t=0} \dot{m} .
\end{aligned}
$$

Then, according to (2.7) $-(2.8 \mathrm{a})$,

$$
\begin{equation*}
\dot{m}=-\mathrm{ad}_{-\omega}^{*} m=\mathrm{ad}_{\omega}^{*} m=\{\omega, m\} . \tag{2.10}
\end{equation*}
$$

In terms of angular velocity, the equation of motion in the body takes the form

$$
\begin{equation*}
\dot{\omega}=B(\omega, \omega), \tag{2.11}
\end{equation*}
$$

where the bilinear function $B(\zeta, \eta)$ is given by the formula [7, 8]

$$
\begin{equation*}
\langle[\zeta, \eta], \xi\rangle=\langle B(\xi, \zeta), \eta\rangle \tag{2.12}
\end{equation*}
$$

for any $\zeta, \eta, \xi \in \hat{g}$.
Proof of (2.11):

$$
\begin{aligned}
\langle\dot{\omega}, \xi\rangle & =(\dot{m}, \xi)=(\{\omega, m\}, \xi)=(m,[\omega, \xi]) \\
& =\langle\omega,[\omega, \xi]\rangle=\langle B(\omega, \omega), \xi\rangle
\end{aligned}
$$

for any $\xi \in \hat{g}$.
As applied to $\mathrm{SO}(3)$, equations (2.10) coincide with the Euler equations of motion of the classical gyroscope. Indeed, by using the above-mentioned isomorphism of the spaces of skew-symmetric third-rank matrices and vectors of the threedimensional Euclidean oriented space, it is easy to show that

$$
(a, \zeta)=\mathbf{a} \cdot \zeta
$$

for any $a \in \hat{g}^{*}$ and $\zeta \in \hat{g}$. Let $\zeta$ be an arbitrary skewsymmetric third-rank matrix [an element of the Lie algebra of group $\mathrm{SO}(3)]$. Then, according to (2.10),

$$
(\dot{m}, \zeta)=(\{\omega, m\}, \zeta)=(m,[\omega, \zeta]) .
$$

In terms of the vectors of the three-dimensional Euclidean space, the last equality can be written as

$$
\dot{\mathbf{m}} \cdot \zeta=\mathbf{m} \cdot(\boldsymbol{\omega} \times \zeta)=(\mathbf{m} \times \boldsymbol{\omega}) \cdot \zeta .
$$

Hence,

$$
\begin{equation*}
\dot{\mathbf{m}}=\mathbf{m} \times \boldsymbol{\omega}, \quad \mathbf{m}=I \boldsymbol{\omega} \tag{2.13}
\end{equation*}
$$

because of the arbitrariness of $\zeta$.

Important remark. In the case of a right-invariant metric, the commutator and all related linear operations change their signs. From the physical viewpoint, a passage from the leftinvariant to a right-invariant metric means that the moving and immobile coordinate systems exchange their roles, i.e., the equations of GRB motion should now be written not with respect to the body but with respect to the space:

$$
\begin{gather*}
\dot{M}=-\{\Omega, M\}  \tag{2.14}\\
\dot{\Omega}=-B(\Omega, \Omega) . \tag{2.15}
\end{gather*}
$$

## 3. What is Kelvin's circulation theorem and the helicity invariant?

By construction, the energy (2.3) is the first integral of the Euler equations of motion (2.11) or (2.10). The immobility of the kinetic momentum in the space [see (2.9)] means that every component of this vector in the Lie coalgebra $\hat{g}^{*}$ is preserved. This gives in addition $n$ independent first integrals of the GRB ( $n$ is the dimension of $\hat{g}^{*}$ ). Bearing in mind the application of the GRB concept to describing infinite dimensional dynamical systems, V I Arnol'd formulates the proposition (2.9) in an invariant form:

Theorem 1. The orbits of the coadjoint representation of a group in the dual space to the Lie algebra are invariant manifolds for a flow specified by the Euler equations in this space $[7,8] \dagger$.

Proof. The angular momentum $m(t)$ can be obtained from $M(t)$ under the action of the coadjoint representation, while $M$ is immobile in the space ( $\dot{M}=0$ ), Q.E.D.

Let us formulate another conclusion important for our purposes, which follows from the proposition (2.9) and can also be constituted in the form of a theorem. Note beforehand that, if an element $\xi \in \hat{g}$ is immobile in the space, i.e.,

$$
\begin{equation*}
\dot{\xi}_{\mathrm{s}}=0 \tag{3.1}
\end{equation*}
$$

then in a coordinate system frozen in the body (the subscripts s and c originate from 'space' and 'corpus', respectively)

$$
\begin{equation*}
\dot{\xi}_{\mathrm{c}}=\left[\xi_{\mathrm{c}}, \omega\right] . \tag{3.2}
\end{equation*}
$$

Proof. The elements of a Lie algebra $\xi_{\mathrm{s}}$ and $\xi_{\mathrm{c}}$ are associated via the relation $\xi_{\mathrm{s}}=\operatorname{Ad}_{g(t)} \xi_{\mathrm{c}}$. Hence,

$$
0=\dot{\xi}_{\mathrm{s}}=\left.\frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{Ad}_{\exp (t \omega)} \xi_{\mathrm{c}}\right|_{t=0}=\operatorname{ad}_{\omega} \xi_{\mathrm{c}}+\dot{\xi}_{\mathrm{c}}=\left[\omega, \xi_{\mathrm{c}}\right]+\dot{\xi}_{\mathrm{c}}
$$

Theorem 2. For each $\xi \in \hat{g}$ immobile in the space, the quantity

$$
\begin{equation*}
H_{\xi}=\left(m, \xi_{\mathrm{c}}\right) \tag{3.3}
\end{equation*}
$$

is the first integral for the flow given by the system (2.10), (3.2) at $\hat{g}^{*} \times \hat{g}$.

Proof. According to (2.10) and (3.2),

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(m, \xi_{\mathrm{c}}\right) & =\left(\dot{m}, \xi_{\mathrm{c}}\right)+\left(m, \dot{\xi}_{\mathrm{c}}\right)=\left(\{\omega, m\}, \xi_{\mathrm{c}}\right)+\left(m,\left[\xi_{\mathrm{c}}, \omega\right]\right) \\
& =\left(m,\left[\omega, \xi_{\mathrm{c}}\right]\right)+\left(m,\left[\xi_{\mathrm{c}}, \omega\right]\right)=0 .
\end{aligned}
$$

[^1]Theorems 1 and 2 are equivalent for a GRB with a finitedimensional configuration space and lead to the same consequences. This is not the case for an infinite-dimensional Lie group. Consider for comparison two 'limiting' cases of the Lie groups acting in three-dimensional Euclidean space $\mathrm{SO}(3)$ and the group SDiff $D$ of diffeomorphisms preserving the volume element $\delta \mu(\mathbf{x})(\mathbf{x} \in D)$ in a bounded threedimensional domain $D$ with boundary $\partial D$.

In the first case (the configuration space of the ordinary gyroscope), the Lie algebra and its dual space can be identified with the physical space in which the body moves. (As a matter of fact, six spaces - $R^{3}, R^{3 *}, \hat{g}, \hat{g}^{*}$, tangent space $\mathrm{T} G_{g}$, and cotangent space $\mathrm{T} G_{g}^{*}$ at point $g \in \mathrm{SO}(3)$, which generally differ in their dimension, are identified. The tangent and cotangent spaces of the group $\mathrm{SO}(n)$ have the dimension $n(n-1) / 2 \neq n$ if $n>3$.)

In the kinetic momentum space, the coadjoint orbits of the mechanical top are spheres centered at the origin,

$$
\begin{equation*}
m_{1}^{2}+m_{2}^{2}+m_{3}^{2}=\text { const }, \tag{3.4}
\end{equation*}
$$

i.e., Theorem 1 means the conservation of the kinetic momentum squared for equations (2.13). The same follows from Theorem 2, since $(m, \xi)=\mathbf{m} \cdot \xi$ (the dot $\cdot$ denotes the usual scalar multiplication of vectors in the three-dimensional Euclidean space) because of the identification of the spaces, so that the kinetic momentum immobile in the space can be taken as $\xi$. Thus, $H_{\mathrm{m}}=\mathbf{m} \cdot \mathbf{m}=m_{1}^{2}+m_{2}^{2}+m_{3}^{2}=$ const, which coincides with (3.4).

The Lie algebra $\hat{g}(D)$ of the group SDiff $D$ consists of divergence-free vector fields in $D$ tangent to the boundary $\partial D$. The inner product on $\hat{g}(D)$ is defined by the formula

$$
\begin{equation*}
\langle\xi, \boldsymbol{\eta}\rangle=\int_{D} \xi \cdot \boldsymbol{\eta} \delta \mu(\mathbf{x}), \tag{3.5}
\end{equation*}
$$

where the dot • denotes the local scalar multiplication of the vector fields $\boldsymbol{\xi}=\boldsymbol{\xi}(\mathbf{x})$ and $\boldsymbol{\eta}=\boldsymbol{\eta}(\mathbf{x})$ (see Refs [7, 8]). The role of the commutator is played by the Poisson bracket taken with the opposite sign, $-\{\text {, }\}_{\mathrm{P}}$, i.e.,

$$
\begin{equation*}
[\boldsymbol{\xi}, \boldsymbol{\eta}]=-\{\boldsymbol{\xi}, \boldsymbol{\eta}\}_{P}=(\boldsymbol{\eta} \nabla) \xi-(\xi \nabla) \boldsymbol{\eta} . \tag{3.6}
\end{equation*}
$$

The energy of a unit-density fluid,

$$
\begin{equation*}
E=\frac{1}{2} \int_{D} \mathbf{u}^{2} \delta \mu, \tag{3.7}
\end{equation*}
$$

where the Eulerian velocity field $\mathbf{u}=\mathbf{u}(\mathbf{x}, t)$ measured with respect to the space plays the role of the angular velocity in the space [7, 8]. Indeed (here, I almost literally reproduce Arnol'd's reasoning), let the fluid flow fulfil the diffeomorphism $g(t)$ within the time $t$ and, at this instant, the velocity is given by the vector field $\mathbf{u}=\mathbf{u}(\mathbf{x}, t)$. Then the diffeomorphism is

$$
g(t+\tau)=\exp (\mathbf{u} \tau) g(t)+o(\tau)
$$

provided that $\tau$ is small, $\exp (\mathbf{u} \tau)$ being a one-parameter group defined by the field $\mathbf{u}$. By differentiating with respect to $\tau$, we find $\mathbf{u}=\dot{g} g^{-1}$, i.e., the velocity field $\mathbf{u}$ results from the vector $\dot{g}$ tangent to the group at the point $g$ by a right translation.

Therefore, $E$ defines a right-invariant metric on the whole group, and the equations of GRB motion with the configura-
tion space SDiffD, i.e., the equations of motion of an ideal incompressible homogeneous fluid can be written as

$$
\begin{equation*}
\dot{\mathbf{u}}=-B(\mathbf{u}, \mathbf{u}), \quad \operatorname{div} \mathbf{u}=0 \tag{3.8}
\end{equation*}
$$

The formula for $B(\mathbf{a}, \mathbf{b})$ is presented in Arnol'd's textbook [7] (see also Ref. [8]). Since its derivation is not quite trivial, it is worth carrying out the concrete calculations. To this end, we use the definition of a bilinear form $B$ [see (2.12)], which assumes the following form in this case:

$$
\begin{equation*}
\langle[\mathbf{a}, \mathbf{b}], \mathbf{c}\rangle=-\left\langle\{\mathbf{a}, \mathbf{b}\}_{\mathrm{P}}, \mathbf{c}\right\rangle=\langle B(\mathbf{c}, \mathbf{a}), \mathbf{b}\rangle \tag{3.9}
\end{equation*}
$$

for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \hat{g}(D)$. In addition, we need two formulas of vector analysis valid for divergence-free vector fields:

$$
\begin{align*}
& {[\mathbf{a}, \mathbf{b}]=(\mathbf{b} \nabla) \mathbf{a}-(\mathbf{a} \nabla) \mathbf{b}=\operatorname{rot}(\mathbf{a} \times \mathbf{b}),}  \tag{3.10}\\
& \operatorname{div}(\mathbf{a} \times \mathbf{b})=\mathbf{b} \operatorname{rot} \mathbf{a}-\mathbf{a} \operatorname{rot} \mathbf{b} . \tag{3.11}
\end{align*}
$$

Then, according to (3.9)-(3.11),

$$
\begin{aligned}
& \langle[\mathbf{a}, \mathbf{b}], \mathbf{c}\rangle=\langle\operatorname{rot}(\mathbf{a} \times \mathbf{b}), \mathbf{c}\rangle=\int_{D} \operatorname{rot}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \delta \mu \\
& \quad=\int_{D}(\mathbf{a} \times \mathbf{b}) \cdot \operatorname{rot} \mathbf{c} \delta \mu+\int_{D} \operatorname{div}[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \delta \mu \\
& \quad=\int_{D}(\mathbf{b} \times \operatorname{rot} \mathbf{c}) \cdot \mathbf{a} \delta \mu+\int_{\partial D}[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \cdot \delta \boldsymbol{\sigma} \\
& \quad=\int_{D}(\operatorname{rot} \mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \delta \mu+\int_{\partial D}[(\mathbf{a} \cdot \mathbf{c}) \mathbf{b}-(\mathbf{a} \cdot \mathbf{b}) \mathbf{c}] \cdot \delta \boldsymbol{\sigma} \\
& \quad=\langle B(\mathbf{c}, \mathbf{a}), \mathbf{b}\rangle .
\end{aligned}
$$

The last integral vanishes because of the tangency of $\mathbf{b}$ and $\mathbf{c}$ to the boundary $\partial D$. Therefore,

$$
\begin{equation*}
B(\mathbf{c}, \mathbf{a})=\operatorname{rot} \mathbf{c} \times \mathbf{a}+\operatorname{grad} \varphi \tag{3.12}
\end{equation*}
$$

for each $\mathbf{c}, \mathbf{a} \in \hat{g}(D)$, since, for a divergence-free vector field $\mathbf{b}$ tangent to boundary $\partial D$,

$$
\int_{D} \mathbf{b} \cdot \operatorname{grad} \varphi \delta \mu=\int_{D} \operatorname{div}(\varphi \mathbf{b}) \delta \mu=\int_{\partial D} \varphi \mathbf{b} \cdot \delta \boldsymbol{\sigma}=0 .
$$

Here, $\varphi$ is a gauge function determined by the condition $B(\mathbf{c}, \mathbf{a}) \in \hat{g}(D)$, i.e., $B(\mathbf{c}, \mathbf{a})$ is divergence-free and tangent to the boundary.

Thus, the first equation (3.8) coincides with the hydrodynamic Euler equation in the Bernoulli form,

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}=\mathbf{u} \times \operatorname{rot} \mathbf{u}-\operatorname{grad} \varphi, \tag{3.13}
\end{equation*}
$$

$\varphi$ being usually written as the Bernoulli function $\varphi=p+\mathbf{u}^{2} / 2$, where $p$ is the pressure.

Now the question is: what plays the role of kinetic momentum in fluid mechanics? To answer this question, strictly speaking, it is necessary beforehand to clarify what the hydrodynamic tensor of inertia momenta is and, moreover (if we want to pass from the description in the space to a description in the body, i.e., from the Eulerian to a Lagrangian description), what the kinetic momentum in the body $m=\operatorname{Ad}_{g}^{*} M$ is. Answers to those two questions are contained in the book [8] and require invoking additional
group-theoretical knowledge, which, in my opinion, is unjustified for consideration in this review because, first, the review is addressed to a wide physical and hydrodynamic readership and, second, we are only interested in the analogy between the mechanical and hydrodynamic invariants.

However, the first question can be answered avoiding the difficulty raised. It is enough to remember that, according to the theory of the GRB with a right-invariant metric, the kinetic momentum of the GRB is immobile with respect not to the space but to the body, i.e., in our case, with respect to the fluid. This means that the field lines of kinetic momentum, which are tangent to its vector field, are moving with the fluid, or they themselves are fluid, which is the same thing.

In order for the field lines of a divergence-free vector field to be fluid, it is necessary and sufficient that the vector field itself obey the Helmholtz equation (see [24]):

$$
\begin{equation*}
\frac{\partial \mathbf{M}}{\partial t}=(\mathbf{M} \nabla) \mathbf{u}-(\mathbf{u} \nabla) \mathbf{M}=\{\mathbf{M}, \mathbf{u}\}_{\mathrm{P}}=-[\mathbf{M}, \mathbf{u}] . \tag{3.14}
\end{equation*}
$$

The necessity can easily be proved. Let $\delta \mathbf{l}$ be an infinitesimal element of the fluid line (an element tangent to it). Then the variation rate of the length of this element is the velocity difference between its end points:

$$
\begin{equation*}
\frac{\mathrm{d} \delta \mathbf{I}}{\mathrm{~d} t}=(\delta \mathbf{I} \nabla) \mathbf{u}, \quad \text { or } \quad \frac{\partial \delta \mathbf{I}}{\partial t}=(\delta \mathbf{I} \nabla) \mathbf{u}-(\mathbf{u} \nabla) \delta \mathbf{I}=\{\delta \mathbf{I}, \mathbf{u}\}_{\mathrm{P}} \tag{3.14a}
\end{equation*}
$$

which is equivalent to (3.14). The proof of sufficiency will be reproduced below.

The only vector field satisfying the Helmholtz equation and consistent with the Euler equation (3.13) is the vorticity $\boldsymbol{\Omega}=\operatorname{rot} \mathbf{u}$, whose equation can be obtained by the application of the rot operation to (3.13):

$$
\begin{equation*}
\frac{\partial \boldsymbol{\Omega}}{\partial t}=\{\boldsymbol{\Omega}, \mathbf{u}\}_{\mathrm{P}} \tag{3.15}
\end{equation*}
$$

Thus, in fluid mechanics, the vorticity plays the role of the kinetic momentum, whose field lines, given by the equations

$$
\begin{equation*}
\frac{\mathrm{d} x}{\Omega_{x}}=\frac{\mathrm{d} y}{\Omega_{y}}=\frac{\mathrm{d} z}{\Omega_{z}}, \tag{3.16}
\end{equation*}
$$

are immobile with respect to the fluid.
The incompressibility of the fluid means the Lagrangian invariance of a volume element $\delta \mu=\delta \mathbf{I} \cdot \delta \boldsymbol{\sigma}$, where $\delta \mathbf{l}$ is a line fluid element and $\delta \boldsymbol{\sigma}$ is the area of an oriented surface element transversal to $\delta \mathbf{I}$ :

$$
\begin{equation*}
\frac{\mathrm{d} \delta \mu}{\mathrm{~d} t}=\frac{\mathrm{d}}{\mathrm{~d} t}(\delta \mathbf{l} \cdot \delta \boldsymbol{\sigma})=0 . \tag{3.17}
\end{equation*}
$$

Since both $\delta \mathbf{l}$ and $\boldsymbol{\Omega}$ satisfy the Helmholtz equation, the quantity

$$
\begin{equation*}
K=\boldsymbol{\Omega} \cdot \delta \boldsymbol{\sigma}=\oint_{C} \mathbf{u} \cdot \delta \mathbf{l} \tag{3.18}
\end{equation*}
$$

is a Lagrangian invariant ( $C$ is the boundary of $\delta \boldsymbol{\sigma}$ ). The invariance of (3.18) can be considered an infinitesimal formulation of Kelvin's circulation theorem. It is evident that, for an arbitrary reducible closed material contour $C$
bounding a surface $S$, the quantity

$$
\begin{equation*}
K=\int_{S} \boldsymbol{\Omega} \cdot \delta \boldsymbol{\sigma}=\oint_{C} \mathbf{u} \cdot \delta \mathbf{l}, \tag{3.19}
\end{equation*}
$$

is also a Lagrangian invariant by virtue of the Stokes theorem. However, it follows from the equations of motion that the velocity circulation is preserved along any closed material contour, including an irreducible one, for example, a contour encircling a hole in a multiply connected domain. Therefore, velocity circulation is a broader concept than vorticity; this is essential for defining the hydrodynamic inertia operator, which is discussed in the book [8].

The following is important. In terms of Theorem 1, the conservation of vorticity means that the image of the coadjoint orbit in the Lie algebra of group SDiff $D$ consists of velocity vector fields isovortical to the given field and is an invariant manifold for the flow described by the Euler equations of motion of an ideal, constant-density fluid. Thus, in fluid mechanics, Theorem 1 has the form of Kelvin's circulation theorem, which is a corollary of the immobility of the kinetic momentum rotu with respect to the fluid and of the invariance of the volume element.

Note that, by virtue of (3.14a) and (3.17), $\delta \boldsymbol{\sigma}$ is governed by the equation

$$
\begin{equation*}
\frac{\mathrm{d} \delta \boldsymbol{\sigma}}{\mathrm{~d} t}=-\delta \boldsymbol{\sigma} \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \tag{3.20}
\end{equation*}
$$

In fact, by substituting (3.14a) into (3.17) and passing to tensor notation, we obtain the following equality equivalent to (3.20):

$$
\delta l_{i}\left(\delta \sigma_{k} \frac{\partial u_{k}}{\partial x_{i}}+\frac{\mathrm{d} \delta \sigma_{i}}{\mathrm{~d} t}\right)=0
$$

(here, summation over dummy indices is implied).
Now, it is easy to prove that, in order for the field lines of a divergence-free vector field to be liquid, it is enough that the vector field itself satisfy the Helmholtz equation. Indeed, by the definition of the field lines [see (3.16)], an element $\delta \mathbf{I}$ of a field line is also governed by the Helmholtz equation. Let us consider, at the initial time, a volume element $\delta \mu=\delta \mathbf{l} \cdot \delta \boldsymbol{\sigma}$ centered at the point of intersection of a liquid surface $S$ and line $l$, where $\delta \boldsymbol{\sigma}$ is an element of the liquid surface $S$ transversal to the element $\delta \mathbf{I}$. Then, since $\delta \mathbf{I}$ and $\delta \boldsymbol{\sigma}$ satisfy (3.14a) and (3.20), respectively, $\delta \mu$ is the Lagrangian invariant and hence $\delta \mathbf{I}$ is a liquid element.

By the way, let us note that the necessary and sufficient condition of the immobility of the chosen surface with respect to the fluid (i.e., of the fact that the surface is liquid as well) is that any element $\delta \boldsymbol{\sigma}$ of the surface satisfy equation (3.20).

Now let us turn to Theorem 2, which, in the context of the case under consideration and in view of the metric right invariance and the remark made in Section 2, can be formulated in the following way.

Theorem $2 \boldsymbol{a}$ (theorem 2 in application to SDiffD). For any Eulerian vector field $\xi=\xi(\mathbf{x}, t) \in \hat{g}(D)$ immobile with respect to the fluid, the quantity

$$
\begin{equation*}
H_{\xi}=(\mathbf{\Omega}, \boldsymbol{\xi}) \doteq(\operatorname{rot} \mathbf{u}, \boldsymbol{\xi}) \tag{3.21}
\end{equation*}
$$

is the first integral for the flow specified by equations (3.15), and

$$
\begin{equation*}
\frac{\partial \xi}{\partial t}=-[\xi, \mathbf{u}]=\{\xi, \mathbf{u}\}_{\mathrm{P}} \tag{3.22}
\end{equation*}
$$

Let the field $\boldsymbol{\Omega} \doteq$ rot $\mathbf{u}$ be tangent to the boundary $\partial D$. In this particular case, $\boldsymbol{\Omega}$ can be regarded both as an element of the Lie coalgebra and as an element of the algebra of the group SDiff $D$, which we take as $\xi$. Then, according to (3.21) and (3.5),

$$
\begin{equation*}
H_{\mathbf{\Omega}}=(\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{u})=\langle\mathbf{u}, \operatorname{rot} \mathbf{u}\rangle=\int_{D} \mathbf{u} \cdot \operatorname{rot} \mathbf{u} \delta \mu(\mathbf{x}) \tag{3.23}
\end{equation*}
$$

Remark. Recall that, for an arbitrary Lie algebra equipped with the scalar product, the value of the cotangent vector $a \in \hat{g}^{*}$ at the tangent vector $\xi \in \hat{g}$ equals the scalar product of the vectors $\eta$ and $\xi$, where $\eta=I^{-1} a$ is the element of the Lie algebra corresponding to $a$, i.e.,

$$
\begin{equation*}
(a, \xi)=\left\langle I^{-1} a, \xi\right\rangle=\langle\eta, \xi\rangle \tag{3.24}
\end{equation*}
$$

for any $a \in \hat{g}^{*}$ and $\xi \in \hat{g}$. Although, in the case under consideration, the definition of the hydrodynamic operator of inertia momentum, $I$, was not given, we know that the Eulerian velocity field $\mathbf{u} \in \hat{g}(D)$ corresponds to the cotangent vector $\operatorname{rot} \mathbf{u} \in \hat{g}^{*}(D)$. This is the basis on which formula (3.24) is applied.

The first integral of motion (3.23) is the well-known topological invariant of helicity $[15,16]$ characterizing the degree of knottiness, or linkage, of the vorticity lines. For example, in the case of only two singly linked vortex rings of intensities $\chi_{1}$ and $\chi_{2}$, we have $H_{\Omega}= \pm \chi_{1} \chi_{2}$, where the sign depends on whether each ring moves in the direction of its 'partner' $(+)$ or in an opposing direction ( - ).

Thus, unlike the case of finite-dimensional GRBs, in the case under consideration, theorems 1 and 2a correspond to different first integrals: the first of them singles out the Lagrangian invariant - the velocity circulation along a closed material contour, whereas the second one selects the integral invariant of helicity. Hence, a proposition follows therefrom that can be formulated as

Theorem 3. Let $G_{N}$ be a finite-dimensional Lie subgroup of SDiffD and be equipped with a right-invariant Riemannian metric induced by the right-invariant Riemannian metric on the SDiff $D$. Then the integral helicity $H_{\Omega}$ of the flow of a homogeneous incompressible fluid described by the equations of motion of the GRB with the group $G_{N}$ as the configuration space either vanishes or does not survive in the development.

Proof. The negation of the above statement would mean that theorems 1 and 2 a are not equivalent to each other for a GRB with a finite-dimensional configuration space. Theorem 3 does not contradict the existence of the invariant (3.23) since the integral helicity must not be conserved if the vorticity field is not tangent to the boundary $\partial D$.

Let us illustrate what has been said with the help of the well-known hydrodynamic interpretation of the mechanical Euler equations of motion (see, e.g., Refs [25, 26]). The group $\mathrm{SO}(3)$ is isomorphic to the group $P(3)$ of affine transformations that map a triaxial ellipsoid surface into itself. The group $P(3)$ is embedded in SDiff $D$ if a three-dimensional domain $D$ is bounded with an ellipsoidal surface

$$
\begin{equation*}
S \doteq \frac{x_{1}^{2}}{a_{1}^{2}}+\frac{x_{2}^{2}}{a_{2}^{2}}+\frac{x_{3}^{2}}{a_{3}^{2}}-1=0 \tag{3.25}
\end{equation*}
$$

where $a_{i}(i=1,2,3)$ are its principal semiaxes. Therefore, exact particular solutions of the hydrodynamic Euler equations corresponding to the group $P(3)$ can be constructed in
the set of the spatially linear velocity fields

$$
\begin{align*}
& \mathbf{W}_{1}=-\frac{a_{2}}{a_{3}} x_{3} \mathbf{j}+\frac{a_{3}}{a_{2}} x_{2} \mathbf{k}, \\
& \mathbf{W}_{2}=-\frac{a_{3}}{a_{1}} x_{1} \mathbf{k}+\frac{a_{1}}{a_{3}} x_{3} \mathbf{i},  \tag{3.26}\\
& \mathbf{W}_{3}=-\frac{a_{1}}{a_{2}} x_{2} \mathbf{i}+\frac{a_{2}}{a_{1}} x_{1} \mathbf{j},
\end{align*}
$$

which satisfy the conditions of orthogonality

$$
\int_{D} \mathbf{W}_{i} \mathbf{W}_{j} \delta \mu=0 \quad(i \neq j)
$$

and boundary tangency $\left(\mathbf{W}_{i} \nabla\right) S=0$, where $i, j=1,2,3$. The velocity field itself has the form

$$
\begin{equation*}
\mathbf{u}(x, t)=\sum_{k=1}^{3} \omega_{k}(t) \mathbf{W}_{k}(\mathbf{x}), \tag{3.27}
\end{equation*}
$$

where quantities $\omega_{k}(t)(k=1,2,3)$, depending only on time and called the Poincaré parameters, are related to $\boldsymbol{\Omega}=\operatorname{rot} \mathbf{u}$ by the formulas

$$
\begin{equation*}
\omega_{k}=\frac{a_{1} a_{2} a_{3}}{a_{k} I_{k}} \Omega_{k} \tag{3.28}
\end{equation*}
$$

and $I_{k}=\sum_{s=1}^{3} a_{s}^{2}-a_{k}^{2}$ are the nonzero elements of the diagonal matrix $I, k=1,2,3$.

By substituting (3.27), (3.28) into the Helmholtz equation (3.15), we find that the Poincare parameters satisfy the following equations:

$$
\begin{equation*}
\dot{\mathbf{m}}=\boldsymbol{\omega} \times \mathbf{m}, \quad \mathbf{m}=I \boldsymbol{\omega}, \tag{3.29}
\end{equation*}
$$

which coincide with the mechanical Euler equations of motion (2.13) up to the substitution $\omega \rightarrow-\omega$. The necessity of this substitution results from the fact that the kinetic energies of the rigid body and fluid define left-invariant and right-invariant metrics, respectively, on their configuration spaces.

Now, let us evaluate the quantities corresponding to the hydrodynamic invariant of energy,

$$
E=\frac{1}{2} \int_{D} \mathbf{u}^{2} \delta \mu
$$

Kelvin's invariant (3.18), and the helicity invariant (3.23) for the solution class under consideration. The substitution of (3.27) and (3.28) into the expression for the energy gives

$$
E=\frac{1}{5} \mu E_{\mathrm{m}}, \quad \mu=\frac{4}{3} \pi a_{1} a_{2} a_{3}, \quad E_{\mathrm{m}}=\frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{m}
$$

where $\mu$ is the total mass of the unit-density fluid in the ellipsoid and $E_{\mathrm{m}}$ formally coincides with the kinetic energy of a mechanical gyroscope.

To evaluate the left-hand side of the equation

$$
\begin{equation*}
\frac{\mathrm{d} K}{\mathrm{~d} t} \equiv \frac{\mathrm{~d} \operatorname{rot} \mathbf{u}}{\mathrm{~d} t} \cdot \delta \boldsymbol{\sigma}+\operatorname{rot} \mathbf{u} \cdot \frac{\mathrm{d} \delta \boldsymbol{\sigma}}{\mathrm{~d} t}=0 \tag{3.30}
\end{equation*}
$$

where

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \equiv \frac{\partial}{\partial t}+(\mathbf{u} \nabla)
$$

in the application to the flow in question, we take as $\delta \boldsymbol{\sigma}$ the element of a plane $P$ passing through the coordinate origin


Figure 3. Material element $\delta \boldsymbol{\sigma}$ of the plane $P$ passing through the origin the ellipsoid center - does not change the location of its center in the motion process.
(ellipsoid center), as shown in Fig. 3. Since we consider the flow with a spatially uniform vorticity, which preserves the immobility of the fluid particle located at the origin, the chosen element (regarded as an element of the liquid surface) will only be deformed and rotated in the space without changing the position of its center. This means that $\delta \boldsymbol{\sigma}=\delta \boldsymbol{\sigma}(t)$ depends only on time and is independent of the spatial coordinates. Then, the substitutions of (3.27) and (3.28) into (3.20) and (3.30) gives

$$
\begin{align*}
& \mathbf{i}=\boldsymbol{\omega} \times \mathbf{l}, \quad l_{i}=\frac{a_{i} \delta \sigma_{i}}{a_{1} a_{2} a_{3}} \quad(i=1,2,3),  \tag{3.31}\\
& \frac{\mathrm{d}(\mathbf{m} \cdot \mathbf{l})}{\mathrm{d} t}=0 . \tag{3.32}
\end{align*}
$$

Thus, for the flow set under consideration, the Kelvin invariant is given by the formula

$$
\begin{equation*}
K_{\mathrm{m}}=\mathbf{m} \cdot \mathbf{l}, \tag{3.33}
\end{equation*}
$$

where I satisfies the Poisson equation (3.31), which reflects the immobility of $\delta \boldsymbol{\sigma}$ with respect to the fluid. Since the immobility of $\boldsymbol{\Omega}=\operatorname{rot} \mathbf{u}$ with respect to the fluid is expressed by equation (3.29) formally identical to (3.31), then, substituting $\mathbf{m}$ for $\mathbf{l}$ in (3.32), we find that the invariance of $\mathbf{m}^{2}$ for the hydrodynamic top is a corollary of Kelvin's theorem, or, eventually, of Theorem 1 for the GRB. Note that, in this case,

$$
\mathbf{m}^{2}=\pi^{-2}\left(\Gamma_{1}^{2}+\Gamma_{2}^{2}+\Gamma_{3}^{2}\right),
$$

where $\Gamma_{i}$ are the velocity circulations along the boundaries of the principal ellipsoid cross sections, $i=1,2,3$.

In the case under consideration, the helicity invariant selected by theorem 2 a vanishes (otherwise the integral helicity would not be preserved because of the invalidity of the boundary conditions), in agreement with Theorem 3. It is important to emphasize that, unlike the case of the hydrodynamic top, the invariant $\mathbf{m}^{2}$ for the mechanical top follows from both theorems.

Note that the hydrodynamic top was first realized in laboratory experiments with solids that had ellipsoidal cavities filled with mercury, which were put in motion by a rotating magnetic field [27].

## 4. Construction of a generalized heavy top

The concept of generalized heavy top (GHT) introduced in Ref. [28] (see also Ref. [29]) is a natural extension of the GRB concept to motion in a potential force field. This concept is
introduced in complete accordance with the notions of the motion of a normal heavy top, taking into account the foliations on an arbitrary Lie group. The definition of the GHT and details of the derivation of the equations of its motion in application to the groups $\mathrm{SO}(n)$ and $\operatorname{SDiff} D$ are given below.

Let $G$ be a Lie group acting in a manifold $N$ [for example, $\mathrm{SO}(3)$ acts in three-dimensional Euclidean space $\left.\mathcal{R}^{3}\right]$, i.e., the $\operatorname{map} f: G \times N \rightarrow N$ is defined so that $f_{g_{1} g_{2}}(x)=f_{g_{1}} f_{g_{2}}(x)$ for any $g_{1}, g_{2} \in G$ and any $x \in N$. The scalar function $\Phi(x)$ and nonnegative function $\rho(x)$ will be called here a potential in the space and the density in the body, respectively. We assume the map $f$ to conserve the density $\rho(x)$ and, therefore, a volume element $\delta \mu \subset N$.

In addition, let $E(\xi)$ be a positive definite quadratic form specified on the Lie algebra $\hat{g}, \xi \in \hat{g}$. Now we will act as if we were dealing with a real rigid body with a fixed point. Let us place the GRB with the density $\rho(x)$, the group $G$ as the configuration space, and the kinetic energy

$$
\begin{equation*}
E(g, \dot{g})=E\left(\mathrm{~L}_{g^{-1} *} \dot{g}\right)=E(\omega)=\frac{1}{2}(I \omega, \omega) \tag{4.1}
\end{equation*}
$$

in an external field with the scalar potential $\Phi(x)$. The function $\varphi_{g}(x)=\Phi\left(f_{g}(x)\right)$ is a potential in the body because the point with a coordinate $x$ in the body has the coordinate $f_{g}(x)$ in the space. Therefore, the potential energy of the dynamical system under consideration,
$U(g)=\int_{V} \rho(x) \varphi_{g}(x) \mathrm{d} \mu(x)=\int_{V} \rho(x) \Phi\left(f_{g}(x)\right) \mathrm{d} \mu(x)$,
is a function specified on the group.
Definition of the GHT. The GHT is a dynamical system whose configuration space is an arbitrary Lie group acting in the manifold $N$ and whose Lagrangian function is specified as

$$
\begin{equation*}
L(g, \dot{g})=E(g, \dot{g})-U(g), \tag{4.3}
\end{equation*}
$$

where $E(g, \dot{g})$ and $U(g)$ are determined by (4.1) and (4.2), respectively.

The equations of motion of the GHT are derived from the least action principle and coincide with the Euler-Lagrange equations for extremals of the Lagrangian function $L(g, \dot{g})$. Let us formulate the functional $L(g, \dot{g})$ only in terms of the Lie algebra $\hat{g}$. A GHT trajectory $g(t)$ on the Lie group $G$ and its image $a(t)$ in the Lie algebra $\hat{g}$ are linked via the relation $g(t)=\exp a(t)$, where the exponential mapping exp: $\hat{g} \rightarrow G$ in the case of matrix groups is specified by the ordinary series: $\exp a=e+a / 1!+a / 2!+\ldots$ ( $e$ is the identity matrix). In the vicinity of zero of the Lie algebra, the following expansion is valid [6]:

$$
\begin{equation*}
\omega=\dot{a}+\frac{1}{2}[\dot{a}, a]+O\left(a^{2}\right), \tag{4.4}
\end{equation*}
$$

because

$$
\begin{aligned}
\omega & =g^{-1} \dot{g}=\exp \{-a(t)\}\left(\dot{a}+\frac{1}{2} \dot{a} a+\frac{1}{2} a \dot{a}+O\left(a^{2}\right)\right) \\
& =\left(1-a+O\left(a^{2}\right)\right)\left(\dot{a}+\frac{1}{2} \dot{a} a+\frac{1}{2} a \dot{a}+O\left(a^{2}\right)\right) \\
& =\dot{a}+\frac{1}{2} \dot{a} a+\frac{1}{2} a \dot{a}-a \dot{a}+O\left(a^{2}\right)=\dot{a}+\frac{1}{2}[\dot{a}, a]+O\left(a^{2}\right) .
\end{aligned}
$$

Substituting $g(t)=\exp a(t)$ into (4.3) and taking into account (4.4) yield the following expression for the Lagrangian function in terms of $a$ and $\dot{a} \in \hat{g}$, which is valid in the vicinity of zero of the Lie algebra:

$$
\begin{equation*}
L(a, \dot{a})=E(a, \dot{a})-U(a) . \tag{4.5}
\end{equation*}
$$

The Euler-Lagrange equation can be written in the form

$$
\begin{equation*}
\frac{\mathrm{d}(\partial L / \partial \dot{a})}{\mathrm{d} t}-\frac{\partial L}{\partial a} \equiv \frac{\mathrm{~d}(\partial E / \partial \dot{a})}{\mathrm{d} t}-\frac{\partial E}{\partial a}+\frac{\partial U}{\partial a}=0 . \tag{4.6}
\end{equation*}
$$

The evaluation of the left-hand side of (4.6) is carried out at $t=0$ assuming that $g(0)=e$. This implies that an arbitrary time is taken as the initial one and the moving CS coincides with a frame of reference instantly immobile with respect to the space. Therefore, the calculation result will be valid for any time:

$$
\begin{aligned}
& \frac{\partial L}{\partial \dot{a}}=\frac{\partial E}{\partial \dot{a}}=m-\frac{1}{2} \operatorname{ad}_{a}^{*} m+O\left(a^{2}\right), \\
& \left.\frac{\mathrm{d}(\partial L / \partial \dot{a})}{\mathrm{d} t}\right|_{t=0}=\left.\frac{\mathrm{d}(\partial E / \partial \dot{a})}{d t}\right|_{t=0} \\
& \quad=\dot{m}-\left.\frac{1}{2} \operatorname{ad}_{\dot{a}}^{*} m\right|_{a=0}=\dot{m}-\frac{1}{2} \operatorname{ad}_{\omega}^{*} m
\end{aligned}
$$

because $\dot{a}=\omega$ at $t=0$. Hence,

$$
\left.\frac{\partial E}{\partial a}\right|_{t=0}=\left.\frac{1}{2} \operatorname{ad}_{\dot{a}}^{*} m\right|_{a=0}=\frac{1}{2} \operatorname{ad}_{\omega}^{*} m .
$$

Before calculating the moment of the external forces $k_{\varphi}=-\partial U / \partial a$, let us recall that the function

$$
\begin{equation*}
\varphi(x, t) \doteq \varphi_{g(t)}(x)=\Phi\left(f_{g(t)}(x)\right)=\Phi\left(f_{\exp a(t)}(x)\right) \tag{4.7}
\end{equation*}
$$

is the potential in the body. Therefore, in accordance with (4.7), the derivative of $\Phi$ in the direction of the vector $\xi \in \hat{g}$ at $a=0$ is equal to

$$
\begin{equation*}
\left(\frac{\partial \Phi}{\partial a}, \xi\right)_{a=0}=\left(\mathrm{d}_{x} \varphi, f_{*} \xi\right) \doteq\left(\mathrm{d}_{x} \varphi, \xi(x)\right), \tag{4.8}
\end{equation*}
$$

where $\mathrm{d}_{x} \varphi$ is the gradient of $\varphi$ at the point $x \in N$ and

$$
\begin{equation*}
f_{*} \xi \doteq \xi(x)=\left.\frac{\mathrm{d} f_{\exp \varepsilon \xi}(x)}{\mathrm{d} \varepsilon}\right|_{\varepsilon=0} \tag{4.9}
\end{equation*}
$$

is the infinitesimal part of the action $f$, i.e., a vector field on the manifold $N$.

It is now easy to calculate the moment of the external forces:

$$
\begin{align*}
\left(\frac{\partial U}{\partial a}, \xi\right)_{a=0} & =\int_{V} \rho(x)\left(\frac{\partial \Phi}{\partial a}, \xi\right)_{a=0} \mathrm{~d} \mu \\
& =\int_{V} \rho(x)\left(\mathrm{d}_{x} \varphi, \xi(x)\right) \mathrm{d} \mu=-\left(k_{\varphi}, \xi\right) \tag{4.10}
\end{align*}
$$

Thus, the Euler-Lagrange equation describing the motion of the GHT in the moving frame of reference can be written as

$$
\begin{equation*}
\dot{m}=\operatorname{ad}_{\omega}^{*} m+k_{\varphi}, \quad m=I \omega, \tag{4.11}
\end{equation*}
$$

where the moment of the external forces is given by the formula

$$
\begin{equation*}
\left(k_{\varphi}, \xi\right)=-\int_{V} \rho(x)\left(\mathrm{d}_{x} \varphi, \xi(x)\right) \mathrm{d} \mu \tag{4.12}
\end{equation*}
$$

valid for every $\xi \in \hat{g}$. The systems (4.11) and (4.12) are closed with the evolution equation for the potential $\varphi(x, t)$ :

$$
\begin{equation*}
\dot{\varphi}=\left(\mathrm{d}_{x} \varphi, \omega(x)\right), \tag{4.13}
\end{equation*}
$$

which follows from the definition (4.7) after differentiating it with respect to time at $t=0$.

The motion of the GHT in the immobile frame of reference is described by the equations

$$
\begin{equation*}
\dot{M}=K_{\Phi}, \quad K_{\Phi}=\operatorname{Ad}_{g_{-1}}^{*} k_{\varphi} \tag{4.14}
\end{equation*}
$$

where $K_{\Phi}$ is determined as
$\left(K_{\Phi}, \xi\right)=\left(\operatorname{Ad}_{g^{-1}}^{*} k_{\varphi}, \xi\right)=-\int_{V} R(x, t)\left(\mathrm{d}_{x} \Phi, \xi(x)\right) \mathrm{d} \mu$.
Here, $\xi$ is an arbitrary element of the Lie algebra $\hat{g}$ immobile in the space, $\xi(x)$ is the vector field on the manifold $N$ corresponding to $\xi$, and, by definition, the density of the body in the space $R(x, t)=\rho\left(f_{g^{-1}}(x)\right)$ satisfies the equation

$$
\begin{equation*}
\dot{R}(x, t)=-\left(\mathrm{d}_{x} R, \Omega(t)\right) \tag{4.16}
\end{equation*}
$$

Proof.

$$
\begin{aligned}
& (\dot{M}, \xi)_{t=0}=\left.\frac{\mathrm{d}(M, \xi)}{\mathrm{d} t}\right|_{t=0}=\left.\frac{\mathrm{d}\left(\operatorname{Ad}_{g^{-1}(t)}^{*} m, \xi\right)}{\mathrm{d} t}\right|_{t=0} \\
& \quad=\left.\frac{\mathrm{d}\left(m, \operatorname{Ad}_{g^{-1}(t)} \xi\right)}{\mathrm{d} t}\right|_{t=0} \\
& \quad=\left(\dot{m}, \operatorname{Ad}_{g^{-1}(t)} \xi\right)_{t=0}+\left(m, \frac{\mathrm{~d}\left(\operatorname{Ad}_{g^{-1}(t)} \xi\right)}{\mathrm{d} t}\right)_{t=0} .
\end{aligned}
$$

Further,

$$
\left.\frac{\mathrm{d}\left(\operatorname{Ad}_{g^{-1}(t)} \xi\right)}{\mathrm{d} t}\right|_{t=0}=\left.\frac{\mathrm{d}\left(\operatorname{Ad}_{\exp (-t \omega)}\right)}{\mathrm{d} t}\right|_{t=0} \xi+\dot{\xi}=-\operatorname{ad}_{\omega} \xi
$$

since $\dot{\xi}=0$. Therefore,

$$
(\dot{M}, \xi)=(\dot{m}, \xi)-\left(m, \operatorname{ad}_{\omega} \xi\right),
$$

and, after substituting (4.11), we have

$$
(\dot{M}, \xi)=\left(\operatorname{Ad}_{g-1(t)}^{*} k_{\varphi}, \xi\right) .
$$

Therefrom, equations (4.14) follow. Since $\xi$ is a vector in the space, the density and the potential in the body that appear in (4.12) should be substituted with the corresponding quantities measured in the space:

$$
\begin{aligned}
& \rho(x) \rightarrow R(x, t)=\rho\left(f_{g^{-1}}(t) x\right) \\
& \varphi(x, t)=\Phi\left(f_{g(t)} x\right) \rightarrow \Phi\left(f_{g^{-1}} f_{g} x\right)=\Phi(x) .
\end{aligned}
$$

This yields (4.15).
To pass from the left-invariant to a right-invariant metric, it is enough, as mentioned above, to exchange the places of the moving and immobile frames of references and change the sign of the commutator at all operators linearly depending on it.

Then, the motion of the GHT with an arbitrary Lie group $G$ equipped with a right-invariant metric as the configuration space, is described by the equations:

- In the space,

$$
\begin{equation*}
\dot{M}=-\operatorname{ad}_{\Omega}^{*} M+K_{\Phi}, \quad M=I \Omega, \tag{4.17}
\end{equation*}
$$

where $K_{\Phi}$ and $R(x, t)$ are given by formulas (4.15) and (4.16), respectively. In terms of angular velocity $\Omega$ in the space, (4.17) can be rewritten in the form

$$
\begin{equation*}
\dot{\Omega}=-B(\Omega, \Omega)+K(\Phi), \tag{4.17a}
\end{equation*}
$$

where $B(\Omega, \Omega)$ can be calculated with the help of (2.12) and, by definition,

$$
\begin{equation*}
\langle K(\Phi), \xi\rangle=\left(K_{\Phi}, \xi\right) \tag{4.18}
\end{equation*}
$$

for any $\xi \in \hat{g}$. Equation (4.16) closes the system (4.17)(4.18).

> - In the body,

$$
\begin{equation*}
\dot{m}=k_{\varphi}, \tag{4.19}
\end{equation*}
$$

where $k_{\varphi}$ and $\varphi(x, t)$ are determined according to (4.12) and (4.13).

## 5. Application of the concept of generalized heavy top to $\operatorname{SO}(n)$ and SDiff $D$

### 5.1 A multidimensional heavy top

Let $\Phi(\mathbf{x})$ be the potential of a homogeneous force field $\mathbf{g}=-\nabla \Phi=$ const in which the $n$-dimensional ( $n \mathbf{D}$ ) Euclidean space $\mathcal{R}^{n}$ is situated. A GHT with the configuration space $\mathrm{SO}(n)$ - rotation group of space $\mathcal{R}^{n}$ - moving in this force field will be termed here an $n$-dimensional heavy top ( $n \mathrm{DHT}$ ). The gravity in the body depends on time and is independent of the space coordinate $\mathbf{x}:-\nabla \varphi=\gamma(t)$, where $\varphi=\varphi(\mathbf{x}, t)$ is the potential in the body. The origin of the coordinates is at a fixed point of the top.

The Lie algebra of the group $\mathrm{SO}(n)$ consists of skewsymmetric matrices of the $n$th rank. Therefore, in the case under consideration, $\xi(\mathbf{x})=\xi \mathbf{x}$, where the right-hand side of the equality is the ordinary convolution of tensor $\xi$ with vector $\mathbf{x}$. Then, (4.12) can be written in the form

$$
\begin{align*}
& \left(k_{\varphi}, \xi\right)=\int_{V} \rho(\mathbf{x}) \boldsymbol{\gamma} \cdot \xi \mathbf{x} \mathrm{d} \mu=\boldsymbol{\gamma} \cdot \xi \mathbf{l}_{0}  \tag{5.1}\\
& \mathbf{l}_{0}=\int_{V} \rho(\mathbf{x}) \mathbf{x} \mathrm{d} \mu \tag{5.2}
\end{align*}
$$

where $\mathbf{I}_{0}$ is the position vector of the center of inertia of the top (the total mass of the $n \mathrm{DHT}$ is assumed to be unity) and the dot $\cdot$ denotes the ordinary inner multiplication in $\mathcal{R}^{n}$ defined as follows:

$$
\mathbf{a} \cdot \mathbf{b}=\sum_{i=1}^{n} a_{i} b_{i}
$$

for any orthonormal coordinate basis.
The scalar product in the Lie algebra $\hat{g}$ of $\mathrm{SO}(n)$ for any $\xi, \eta \in \hat{g}$ is expressed by the formula

$$
\begin{equation*}
\langle\xi, \eta\rangle=-\frac{1}{2} \operatorname{tr}(\xi * \eta) \tag{5.3}
\end{equation*}
$$

and identifies $\hat{g}$ and $\hat{g}^{*}$. The asterisk $*$ in (5.3) means the ordinary multiplication of matrices. Therefore, (5.1) can be rewritten as follows:

$$
\begin{equation*}
\left(k_{\varphi}, \xi\right)=\boldsymbol{\gamma} \cdot \xi \mathbf{1}_{0}=\left\langle\boldsymbol{\gamma} \wedge \mathbf{1}_{0}, \xi\right\rangle . \tag{5.4}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
k_{\varphi}=\gamma \wedge \mathbf{l}_{0}, \tag{5.5}
\end{equation*}
$$

where $\gamma \wedge \mathbf{l}_{0} \in \hat{g}$ and $\wedge$ means the Kronecker multiplication of vectors $\gamma$ and $\mathbf{I}_{0} \in \mathcal{R}^{n}$. In an orthonormal CS, the Kronecker product is defined as

$$
\begin{equation*}
\left(\gamma \wedge \mathbf{l}_{0}\right)_{i j}=\gamma_{i} l_{0 j}-\gamma_{j} l_{0 i} . \tag{5.6}
\end{equation*}
$$

Due to the identification of $\hat{g}$ and $\hat{g}^{*}$, we have $\operatorname{ad}_{\xi}^{*} a=-[\xi, a]$. As a result, the equations of motion of the $n$ DHT take the form

$$
\begin{align*}
& \dot{m}=[m, \omega]+\gamma \wedge \mathbf{l}_{0}  \tag{5.7}\\
& \dot{\gamma}=-\omega \gamma, \quad m=I \omega . \tag{5.8}
\end{align*}
$$

In the case of $\mathrm{SO}(3)$, any skew-symmetric third-rank matrix can be identified with a vector in three-dimensional Euclidean space, and vice versa, according to formulas, $\xi_{i}=-\varepsilon_{i j k} \xi_{j k} / 2, \xi_{i j}=-\varepsilon_{i j k} \xi_{k}$. Then, (5.7) and (5.8) can be rewritten as

$$
\begin{align*}
& \dot{\mathbf{m}}=\mathbf{m} \times \boldsymbol{\omega}+\mathbf{l}_{0} \times \boldsymbol{\gamma},  \tag{5.9}\\
& \dot{\boldsymbol{\gamma}}=\boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad \mathbf{m}=I \boldsymbol{\omega}, \tag{5.10}
\end{align*}
$$

which coincides with the classical Euler-Poisson equations of motion of an ordinary heavy top.

### 5.2 A generalized heavy top with the group SDiff $D$ as the configuration space

Let us apply systems (4.17a), (4.18), and (4.16) to SDiffD, where $D$ is a bounded domain of the three-dimensional Euclidean space. Let $R(\mathbf{x}, t)$ be the density of an ideal incompressible stratified fluid filling the domain $D$. By definition of the incompressible fluid, $R(\mathbf{x}, t)$ is a passive scalar, i.e., is conserved over time $\dagger$. In the case under consideration, the role of the angular velocity in the space is played by the Eulerian velocity field $\mathbf{u}=\mathbf{u}(\mathbf{x}, t)$ $(\Omega(t)=\mathbf{u}(\mathbf{x}, t))$. The kinetic energy of the GHT can be defined by the equality

$$
\begin{equation*}
E(\mathbf{u}(\mathbf{x}, t))=\frac{1}{2} \rho_{0} \int_{D} \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t) \mathrm{d} \mu \tag{5.11}
\end{equation*}
$$

which specifies a right-invariant metric on the whole group SDiff $D$. The average density

$$
\rho_{0}=\frac{\int_{D} R(\mathbf{x}, t) \mathrm{d} \mu}{\int_{D} \mathrm{~d} \mu}
$$

is independent of time due to the conservation of the total mass. The scalar product on the Lie algebra $\hat{g}(D)$ corresponding to (5.11) is

$$
\langle\xi(\mathbf{x}), \boldsymbol{\eta}(\mathbf{x})\rangle=\rho_{0} \int_{D} \xi(\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{x}) \mathrm{d} \mu
$$

[^2]for any $\boldsymbol{\xi}(\mathbf{x}), \boldsymbol{\eta}(\mathbf{x}) \in \hat{g}(D)$, i.e., for the divergence-free vector fields tangent to boundary $\partial D$.

According to (4.15) and (4.18),

$$
\begin{aligned}
& \left(K_{\Phi}, \boldsymbol{\xi}(\mathbf{x})\right)=-\int_{V} R(\mathbf{x}, t)\left(\mathrm{d}_{x} \Phi, \boldsymbol{\xi}(\mathbf{x})\right) \mathrm{d} \mu \\
& \quad=-\int_{V} R(\mathbf{x}, t) \mathrm{d}_{x} \Phi \cdot \boldsymbol{\xi}(\mathbf{x}) \mathrm{d} \mu \\
& \quad=-\rho_{0}^{-1}\left\langle R(\mathbf{x}, t) \mathrm{d}_{x} \Phi, \boldsymbol{\xi}(\mathbf{x})\right\rangle=\langle K(\Phi), \boldsymbol{\xi}(\mathbf{x})\rangle .
\end{aligned}
$$

Furthermore, $\left(\mathrm{d}_{x} \Phi, \xi(\mathbf{x})\right)=\mathrm{d}_{x} \Phi \cdot \xi(\mathbf{x})$ because the manifold $N=\mathcal{R}^{3}$ is equipped with the ordinary local scalar product, denoted here by the dot. Hence,

$$
K(\Phi)=-\rho_{0}^{-1} R(\mathbf{x}, t) \nabla \Phi
$$

where the notation $\mathrm{d}_{x}$ for gradient is replaced with the traditional one. Then, equation (4.17a) for the generalized angular velocity can be written, in view of (3.12), as

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}=\mathbf{u} \times \operatorname{rot} \mathbf{u}-\nabla \varphi-\rho_{0}^{-1} R \nabla \Phi \tag{5.12}
\end{equation*}
$$

where $\varphi$ is a gauge function that makes the right-hand side of (5.12) divergence-free. Equation (5.12) is closed with the conditions of zero divergence of $\mathbf{u}$ and the Lagrangian invariance of $R(\mathbf{x}, t)$ [see (4.16)]:

$$
\begin{equation*}
\frac{\partial R}{\partial t}+(\mathbf{u} \nabla) R=0 \tag{5.13}
\end{equation*}
$$

We take into account that
$\mathbf{u} \times \operatorname{rot} \mathbf{u}=-(\mathbf{u} \nabla) \mathbf{u}+\frac{1}{2} \nabla \mathbf{u}^{2}$,
and carry out the formal substitutions $-\nabla \Phi=\mathbf{g}, p / \rho_{0}=$ $\varphi-\mathbf{u}^{2} / 2$, and $R / \rho_{0}=1-\beta T$ to obtain the well-known Oberbeck - Boussinesq equations of the adiabatic convection of an incompressible fluid (see, e.g., Ref. [14]):

$$
\begin{align*}
& \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u} \nabla) \mathbf{u}=-\frac{\nabla p}{\rho_{0}}-\beta T \mathbf{g}  \tag{5.14}\\
& \frac{\partial T}{\partial t}+(\mathbf{u} \nabla) T=0 \tag{5.15}
\end{align*}
$$

where $p$ and $\beta$ should be treated as the pressure and thermalexpansion coefficient of the fluid, respectively, and $T$ is the deviation of temperature from the mean value $T_{0}=$ $\beta^{-1}=$ const.

The equation for the kinetic momentum $M=\operatorname{rot} \mathbf{u} \doteq \boldsymbol{\Omega}$ is nothing but the Friedman equation (see, e.g., Ref. [24]):

$$
\begin{equation*}
\frac{\partial \boldsymbol{\Omega}}{\partial t}+(\mathbf{u} \nabla) \boldsymbol{\Omega}-(\boldsymbol{\Omega} \nabla) \mathbf{u}=\beta \mathbf{g} \times \nabla T \tag{5.16}
\end{equation*}
$$

which results from applying the operator rot to (5.14).

## 6. Fundamental invariants of a generalized heavy top

### 6.1 First integrals of the multidimensional heavy top

First, let us consider the first integrals of the $n$ DHT associated with the invariance of its Lagrangian function with respect to the transformations of $\mathcal{R}^{n}$ that conserve the gravity $\mathbf{g}=-\nabla \Phi$ (see Section 5.1). The first integrals, which, according to the

Noether theorem, correspond to one-parameter subgroups $p(s) \subset \mathrm{SO}(n)(p(s) \mathbf{g}=\mathbf{g})$ preserving the Lagrangian function, can be represented in an explicit form.

The equations of motion of the $n \mathrm{DHT}$ in the space can be written as

$$
\begin{equation*}
\dot{M}=\mathbf{g} \wedge \mathbf{L}(t), \quad \dot{\mathbf{L}}(t)=\Omega(t) \mathbf{L}(t) \tag{6.1}
\end{equation*}
$$

where $\mathbf{L}(t)=g(t) \mathbf{I}_{0}$ is the position vector of the center of inertia of the top in the space and $\Omega(t)=\operatorname{Ad}_{g(t)} \omega(t)=g \Omega g^{-1}$ $(g(t)$ is a trajectory of $n$ DHT on $\mathrm{SO}(n))$.

Let $\mathbf{A}$ and $\mathbf{B} \in \mathcal{R}^{n}$ be vectors immobile with respect to the space and orthogonal to $\mathbf{g}$. Then,

$$
\begin{equation*}
\Pi_{\mathrm{m}}=\langle M, \mathbf{A} \wedge \mathbf{B}\rangle \tag{6.2}
\end{equation*}
$$

is the first integral of the $n \mathrm{DHT}$ corresponding to the subgroup of rotations of $\mathcal{R}^{n}$ that do not affect a subspace of $\mathcal{R}^{n}$ orthogonal to $\mathbf{A}$ and $\mathbf{B}$. Indeed, according to the formula (see Section 5.1) A $\cdot \xi \mathbf{B}=\langle\mathbf{A} \wedge \mathbf{B}, \xi\rangle$, valid for any skewsymmetric matrix $\xi \in \hat{g}$, we have

$$
\begin{aligned}
\frac{\mathrm{d} \Pi_{\mathrm{m}}}{\mathrm{~d} t} & =\langle\dot{M}, \mathbf{A} \wedge \mathbf{B}\rangle=\langle\mathbf{g} \wedge \mathbf{L}(t), \mathbf{A} \wedge \mathbf{B}\rangle \\
& =-\langle\mathbf{L}(t) \wedge \mathbf{g}, \mathbf{A} \wedge \mathbf{B}\rangle=-\mathbf{L}(t) \cdot(\mathbf{A} \wedge \mathbf{B}) \mathbf{g}=0
\end{aligned}
$$

since $\mathbf{A} \cdot \mathbf{g}=0$ and $\mathbf{B} \cdot \mathbf{g}=0$ imply $(\mathbf{A} \wedge \mathbf{B}) \mathbf{g}=0$.
It is clear that the number of independent invariants of this kind is $C_{n-1}^{2}=(n-1)(n-2) / 2$, whereas the dimension of the configuration space of the $n$ DHT equals the dimension of $\mathrm{SO}(n)$ plus $n$, i.e., $n(n-1) / 2+n=(n+1) n / 2$. In the frame of reference frozen in the body, $\Pi_{\mathrm{m}}=\langle m, \mathbf{a} \wedge \mathbf{b}\rangle$ provided that $\dot{\mathbf{a}}=-\omega \mathbf{a}$ and $\dot{\mathbf{b}}=-\omega \mathbf{b}$. In addition, the equations of motion (5.7) and (5.8) in the body possess the energy integral $E=\langle m, \omega\rangle / 2-\mathbf{l}_{0} \cdot \gamma$ (by construction), and $\gamma^{2}=\gamma \cdot \gamma$ because of the immobility of the gravity in the space. At $n=3$, there is a unique first integral of the kind of (6.2), which corresponds to the projection of the angular momentum of the classical heavy top onto the direction of gravity.

### 6.2 What is the potential vorticity?

Let us now compare the first integrals

$$
\begin{equation*}
E_{\mathrm{m}}=\frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega}-\boldsymbol{\gamma} \cdot \mathbf{l}_{0}, \quad \Pi_{\mathrm{m}}=\mathbf{m} \cdot \boldsymbol{\gamma}, \quad \gamma^{2}=\gamma \cdot \gamma \tag{6.3}
\end{equation*}
$$

of the Euler-Poisson equations of motion (5.9), (5.10) for a classical heavy top with the invariants of the hydrodynamic equations (5.12) and (5.13),

$$
\begin{equation*}
E_{\mathrm{h}}=\frac{1}{2} \rho_{0} \int_{D} \mathbf{u}^{2} \delta \mu-\int_{D} \mathbf{g} \cdot \mathbf{x} R \delta \mu, \quad \Pi_{\mathrm{h}}=\boldsymbol{\Omega} \cdot \nabla R \tag{6.4}
\end{equation*}
$$

where $\quad R=R(\mathbf{x}, t), \quad \mathbf{g}=-\nabla \Phi, \quad \mathbf{\Omega} \doteq \operatorname{rot} \mathbf{u}, \quad \mathrm{d} \Pi_{\mathrm{h}} / \mathrm{d} t=0$, $\mathrm{d} R / \mathrm{d} t=0$, and $\mathrm{d} / \mathrm{d} t=\mathrm{\partial} / \mathrm{\partial} t+\mathbf{u} \nabla$.

The total energy equal to the sum of the kinetic and potential energies is conserved as the Hamiltonian function of the GHT independent explicitly of time. Therefore, $E_{\mathrm{m}}$ is a mechanical prototype of $E_{\mathrm{h}}$ by the construction of the GHT. The invariance of $\gamma^{2}$ is a corollary of the immobility of the gravity in space, whereas the Lagrangian invariance of $R=R(\mathbf{x}, t)$ means the immobility of the density with respect to the 'body' (fluid). Since the passage from the left-invariant to a right-invariant metric means the exchange of roles
between the mobile and immobile coordinate systems, $\gamma^{2}$ can be regarded as a mechanical analog of the Lagrangian invariant $R(\mathbf{x}, t)$.

It is well known that, from the hydrodynamic viewpoint, the invariance of the quantity $\Pi_{\mathrm{h}}=\boldsymbol{\Omega} \cdot \nabla R$ (called the potential vorticity) means the applicability of the Kelvin circulation theorem to closed material contours entirely lying on a surface of constant density, and vice versa. Indeed, the evolution of an element $\delta \sigma_{0}$ of an oriented fluid surface is governed by (3.20), which follows from the conservation of the volume element $\delta \mu=\delta \mathbf{l} \cdot \delta \boldsymbol{\sigma}_{0}$ (see Section 3). By differentiating (5.13), it is easy to show that $\nabla R(\mathbf{x}, t)$ also obeys equation (3.20). Therefore, the Lagrangian invariance of $\Pi_{\mathrm{h}}=\boldsymbol{\Omega} \cdot \nabla R$ implies the Lagrangian invariance of

$$
\begin{equation*}
K_{0}=\boldsymbol{\Omega} \cdot \delta \boldsymbol{\sigma}_{0}=\oint_{C_{0}} \mathbf{u} \cdot \delta \mathbf{l} \tag{6.5}
\end{equation*}
$$

(and vice versa), where $C_{0}$ is the closed material contour bounding $\delta \boldsymbol{\sigma}_{0}$. Generally speaking, without any connection to the PV, the invariance of $K_{0}$ follows from the fact that motion along a surface of constant density is the flow of an incompressible homogeneous fluid for which Kelvin's theorem is valid.

The invariance of the PV is associated with the invariance of the potential and kinetic energies of the fluid with respect to the following transformations of configuration space SDiff $D$ [30]. The right translation $R_{h}: g \rightarrow g h, g, h \in \operatorname{SDiff} D$ is a transformation of the Lagrangian variables, which are attached to fluid particles. The density $R(\mathbf{x}, t)$ is a Lagrangian invariant, i.e., it is also moving with the fluid. If $h \in \operatorname{SDiff} D$ preserves the density distribution, $R(\mathbf{x}, t)=$ $R(h \mathbf{x}, t)$, then the right translation $R_{\mathrm{h}}$ changes neither the kinetic energy (by virtue of its right invariance) nor the potential energy of the fluid:

$$
\begin{aligned}
U(g) & =\int_{D} R(\mathbf{x}) \Phi(g \mathbf{x}) \mathrm{d} \mu(\mathbf{x})=\int_{D} R(h \mathbf{x}) \Phi(g h \mathbf{x}) \mathrm{d} \mu(h \mathbf{x}) \\
& =\int_{D} R(\mathbf{x}) \Phi(g h \mathbf{x}) \mathrm{d} \mu(\mathbf{x})=U(g h) .
\end{aligned}
$$

The PV corresponds to the indicated group of symmetries of the Lagrangian function $L(\mathbf{u}, g)=E(\mathbf{u}, g)-U(g)$. In the dynamics of the stratified fluid, the surface of constant density plays the role of an equipotential surface. Therefore, $\Pi_{\mathrm{h}}=\boldsymbol{\Omega} \cdot \nabla R$ - the projection of the hydrodynamic kinetic momentum $\mathbf{M}=\boldsymbol{\Omega} \doteq \operatorname{rot} \mathbf{u}$ onto the density gradient - is conserved by analogy with the conservation of $\Pi_{\mathrm{m}}=\mathbf{m} \cdot \gamma=-\mathbf{m} \cdot \nabla \varphi$, the projection of the mechanical kinetic momentum onto the gravity, i.e., the gradient of the potential. Thus, from the viewpoint of the GRB theory, both invariants, $\Pi_{\mathrm{h}}$ and $\Pi_{\mathrm{m}}$, have the same origin.

It is particularly reasonable to note that, in a sense, the revealed analogy extends to the equations of gas dynamics. By analogy with an ideal inhomogeneous incompressible fluid, which stratifies into nonintersecting surfaces of constant density, an ideal compressible fluid stratifies into nonintersecting isentropic surfaces because a material particle that belongs to an isentropic surface at the initial time will always remain on it due to the Lagrangian invariance of the specific entropy $S=S(\mathbf{x}, t)$ (according to the definition of an ideal compressible fluid). Since Kelvin's theorem is valid for isentropic flows, the following proposition holds.


Figure 4. The material cylinder end-walls situated on the isentropic surfaces $S(\mathbf{x}, t)=S_{0}$ and $S(\mathbf{x}, t)=S_{0}+\delta S$ at the initial moment will remain on them all the time.

In application to an infinitesimal closed material contour $C_{0}$ that belongs entirely to an isentropic surface $S(p, \rho)=$ $S_{0}=\mathrm{const}$, where $p=p(\mathbf{x}, t)$ is the pressure and $\rho=\rho(\mathbf{x}, t)$ is the density of the fluid, the quantity $K_{0}$ given by (6.5) is a Lagrangian invariant.

Along with surface $S(\mathbf{x}, t)=S_{0}$, let us consider an infinitesimally close isentropic surface $S(\mathbf{x}, t)=S_{0}+\delta S$ and draw, through the contour $C_{0}$, a cylindrical surface whose intersection with the additional isentropic surface is also a closed contour (Fig. 4). The end walls of the selected material particle will reside on the corresponding isentropic surfaces. Then, the mass of the material particle is

$$
\begin{equation*}
\delta \mu=\rho h \mathbf{n} \cdot \delta \boldsymbol{\sigma}_{0} \tag{6.6}
\end{equation*}
$$

where $\mathbf{n}$ is the normal to the surface $S(\mathbf{x}, t)=S_{0}$ directed along $\operatorname{grad} S$ and $h$ is the height of the cylinder.

Since the isentropic surfaces are infinitesimally close,

$$
\begin{equation*}
\delta S=\operatorname{grad} S \cdot \mathbf{n} h \tag{6.7}
\end{equation*}
$$

From formulas (6.6) and (6.7), it follows that

$$
\begin{equation*}
\delta \boldsymbol{\sigma}_{0}=\frac{\delta \mu}{\delta S} \frac{\operatorname{grad} S}{\rho} \tag{6.8}
\end{equation*}
$$

By substituting (6.8) into (6.5) and taking into account the conservation of $\delta S$ and of the mass of the chosen volume, which consists of the same material particles, we find that the invariance of $K_{0}$ implies the invariance of the gasdynamic PV

$$
\begin{equation*}
\Pi_{\mathrm{E}}=\frac{\operatorname{rot} \mathbf{u} \cdot \operatorname{grad} S}{\rho} \tag{6.9}
\end{equation*}
$$

called the Ertel invariant. Ertel [12] was the first to prove in 1942 the invariance of this quantity directly from the equations of motion.

This highly elegant and physically transparent proof of the invariance of the gas-dynamical PV, which has been included in some textbooks on geophysical hydrodynamics (see, e.g., Ref. [31]), was implemented for the first time by Moran [32] also in 1942 and then, apparently independently, reconstructed by Charney [33] in 1948 in his celebrated article dedicated to the dynamics of large-scale atmospheric motions. This proof can also be found in later papers, e.g., Refs [34, 35]. A Hamiltonian treatment of Ertel's invariant was proposed by Salmon [36-38].

The concept of PV and its interpretation in terms of Kelvin's theorem are successfully used in processing atmospheric observational data and describing the atmospheric motions in the natural coordinate system defined by the surfaces of constant PV and entropy [39-43].

### 6.3 The direct hydrodynamic interpretation of the Euler - Poisson equations and their invariants

Let us remember that the group $P(3)$ of the affine mappings of an ellipsoid into itself is isomorphic to $\mathrm{SO}(3)$ and is a subgroup of $\operatorname{SDiff} D$, where $D$ is the domain bounded by the given ellipsoid (3.25). Therefore, the Euler-Poisson equations describe exact particular solutions of the Oberbeck-Boussinesq equations [26, 28]. Indeed, consider the motion of an ideal, weakly stratified incompressible fluid inside the ellipsoid (3.25) with the principal semiaxes $a_{1} \neq a_{2} \neq a_{3} \neq a_{1}$ in the class of spatially linear velocity fields (3.27) and temperature fields

$$
\begin{equation*}
T(\mathbf{x}, t)=\mathbf{x} \cdot \nabla T=\frac{\partial T}{\partial x_{1}} x_{1}+\frac{\partial T}{\partial x_{2}} x_{2}+\frac{\partial T}{\partial x_{3}} x_{3} \tag{6.10}
\end{equation*}
$$

where $\nabla T$ is independent of $\mathbf{x}$.
In terms of $\boldsymbol{\Omega} \doteq \operatorname{rot} \mathbf{u}$ and $\mathbf{q}=\beta \nabla T$, equations (5.16) and (5.15) can be rewritten as follows:

$$
\begin{align*}
& \frac{\partial \boldsymbol{\Omega}}{\partial t}+\{\mathbf{u}, \boldsymbol{\Omega}\}_{\mathbf{P}}=\mathbf{g} \times \mathbf{q}  \tag{6.11}\\
& \frac{\partial \mathbf{q}}{\partial t}+(\mathbf{u} \nabla) \mathbf{q}=-\mathbf{q} \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \tag{6.12}
\end{align*}
$$

where $\{\mathbf{A}, \mathbf{B}\}_{\mathrm{P}}=(\mathbf{A} \nabla) \mathbf{B}-(\mathbf{B} \nabla) \mathbf{A}$ denotes the Poisson brackets of the vector fields $\mathbf{A}$ and $\mathbf{B}$, and the right-hand side of (6.12) in the tensor notation has the form $-q_{\mathrm{s}} \partial u_{\mathrm{s}} / \partial x_{i}$.

By substituting (3.27) and (6.10) into (6.11) and (6.12) and taking into account (3.26), it is easy to show that the vector $\boldsymbol{\omega}=\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ determined by the Poincaré parameters and the vector

$$
\begin{equation*}
\boldsymbol{\sigma}=\beta\left(a_{1} \frac{\partial T}{\partial x_{1}} \mathbf{i}+a_{2} \frac{\partial T}{\partial x_{2}} \mathbf{j}+a_{3} \frac{\partial T}{\partial x_{3}} \mathbf{k}\right) \tag{6.13}
\end{equation*}
$$

satisfy the equations

$$
\begin{align*}
& \dot{\mathbf{M}}=\boldsymbol{\omega} \times \mathbf{M}+g \mathbf{l}_{0} \times \boldsymbol{\sigma},  \tag{6.14}\\
& \dot{\boldsymbol{\sigma}}=\boldsymbol{\omega} \times \boldsymbol{\sigma} . \tag{6.15}
\end{align*}
$$

Here, $\mathbf{M}=I \boldsymbol{\omega}$, and $I$ is a diagonal matrix with the nonzero elements $I_{1}=a_{2}^{2}+a_{3}^{2}, I_{2}=a_{3}^{2}+a_{1}^{2}, I_{3}=a_{1}^{2}+a_{2}^{2}$ and

$$
\mathbf{1}_{0}=a_{1} \cos \alpha_{1} \mathbf{i}+a_{2} \cos \alpha_{2} \mathbf{j}+a_{3} \cos \alpha_{3} \mathbf{k}
$$

where $\cos \alpha_{i}$ are the direction cosines of the gravity $\mathbf{g}$ with respect to the ellipsoid principal axes, $i=1,2,3$.

The equations (6.14) and (6.15) coincide, up to the formal substitutions $\boldsymbol{\omega} \rightarrow-\boldsymbol{\omega}, \mathbf{M} \rightarrow-\mathbf{m}$, and $g \boldsymbol{\sigma} \rightarrow-\gamma$, with the Euler-Poisson equations (5.9) and (5.10). As was mentioned above, this substitution is dictated by the contradistinction of the mechanical and hydrodynamic metrics specified on the corresponding configuration spaces.

The first integrals of system (6.14), (6.15)

$$
E=\frac{1}{2} \mathbf{M} \cdot \boldsymbol{\omega}+g \mathbf{l}_{0} \cdot \boldsymbol{\sigma}, \quad \Pi=\mathbf{M} \cdot \boldsymbol{\sigma}, \quad \Theta=\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}
$$

express the conservation of the total hydrodynamic energy $E_{\mathrm{h}}$ and the Lagrangian invariance of the $\mathrm{PV}, \Pi_{\mathrm{h}}$, and temperature [for the density, see (6.4)] for the class of exact solutions under consideration.

It is interesting and important to note that the hydrodynamic heavy top has also been realized in extremely sophisticated laboratory experiments with convection inside an ellipsoidal cavity [44].

Remark. The first integrals of the $n \mathrm{DHT}$ (at $n>3$ ) should be compared with the invariants of the equations of motion for an ideal $n$-dimensional stratified incompressible fluid, i.e., the GHT with the configuration space $\operatorname{SDiff} D$, where $D$ is a bounded domain of the $n$-dimensional Euclidean space $\mathcal{R}^{n}$. The dynamics of the ideal $n$-dimensional homogeneous incompressible fluid is discussed in Ref. [8] and is more likely of mathematical interest.

## 7. Generalized magnetohydrodynamic system. What are Woltjer's invariants?

### 7.1 Equations of motion and their invariants

Let us consider a generalized rigid body whose kinetic energy

$$
\begin{equation*}
E(\Omega)=\frac{1}{2}(M, \Omega)=\frac{1}{2}(I \Omega, \Omega)=\frac{1}{2}\langle\Omega, \Omega\rangle \tag{7.1}
\end{equation*}
$$

is a positive definite quadratic function of the angular velocity $\Omega=R_{g^{-1} *} \dot{g}$ in the space. The kinetic energy $E(\Omega)$ specifies a right- invariant metric on the entire configuration space $G$ of the GRB and the symmetric inertia operator $I$ identifies $\hat{g}$ and $\hat{g}^{*}$. The element $h \in \hat{g}$, immobile in the body, will be called here the magnetic-field strength in the body. Then, $H=\operatorname{Ad}_{g} h$ is the magnetic-field strength in the space. The quantity $J=I H \in \hat{g}^{*}$ is, by definition, the electric-current density in the space and $j=\operatorname{Ad}_{g}^{*} J$ is the electric current in the body.

Definition of the GMHDS. A dynamical system with an arbitrary Lie group as the configuration space and with the Lagrangian function

$$
\begin{equation*}
L(g, \dot{g})=E(\Omega)-\frac{1}{2}(J, H)=\frac{1}{2}(M, \Omega)-\frac{1}{2}(J, H) \tag{7.2}
\end{equation*}
$$

is called the generalized magnetohydrodynamic system (GMHDS).

In accordance with the least action principle, the motion of GMHDS is described by the equations [29]:

- in the space,

$$
\begin{align*}
& \dot{M}=-\operatorname{ad}_{\Omega}^{*} M+\operatorname{ad}_{H}^{*} J,  \tag{7.3}\\
& \dot{H}=[\Omega, H] \tag{7.4}
\end{align*}
$$

- in the body

$$
\begin{equation*}
\dot{m}=\operatorname{ad}_{h}^{*} j, \quad \dot{h}=0 \tag{7.5}
\end{equation*}
$$

In terms of the angular velocity, equation (7.3) can be written in the form

$$
\begin{equation*}
\dot{\Omega}=-B(\Omega, \Omega)+B(H, H), \tag{7.6}
\end{equation*}
$$

where the bilinear operator $B(\xi, \eta)$ is given by formula (2.12).
These equations can be derived using the approach similar to that used in Section 4 to obtain the GHT equations of motion. A detailed discussion of the GMHDS and its applications is contained in Ref. [8]. In almost exact
correspondence to Section 3, the following propositions analogous to Theorems 1 and 2 a are valid due to the immobility of the magnetic field with respect to the body.

Theorem 4. The orbits of the adjoint representation of the Lie group $G$ in its Lie algebra $\hat{g}$ are invariant manifolds for the flow given by equation (7.4) in the algebra.

Proof. The magnetic-field strength $H(t)$ in the space can be obtained from $h$ via the action of the adjoint representation and $h$ is immobile in the body, Q.E.D.

Theorem 5. For any element $a \in \hat{g}^{*}$, immobile with respect to the body $(\dot{a}=0)$, the quantity

$$
\begin{equation*}
H_{A}=(A, H) \tag{7.7}
\end{equation*}
$$

at $A=\operatorname{Ad}_{g-1}^{*} a$ is the first integral of the system given by (7.3), (7.4), and the equation

$$
\begin{equation*}
\dot{A}=-\mathrm{ad}_{\Omega}^{*} A \tag{7.8}
\end{equation*}
$$

The last proposition follows from the immobility of the element $a \in \hat{g}^{*}$ in the body (cf. theorem 2a).

Theorem 6. The quantity

$$
\begin{equation*}
H_{\mathrm{c}}=(M, H) \tag{7.9}
\end{equation*}
$$

is the first integral for the flow given by equations (7.3), (7.4) on $\hat{g} \times \hat{g}$.

Proof.

$$
\begin{aligned}
& \frac{\mathrm{d}(M, H)}{\mathrm{d} t}=(\dot{M}, H)+(M, \dot{H}) \\
& \quad=-\left(\operatorname{ad}_{\Omega}^{*} M, H\right)+\left(\operatorname{ad}_{H}^{*} J, H\right)+(M,[\Omega, H]) \\
& \quad=-(M,[\Omega, H])+(J,[H, H])+(M,[\Omega, H])=0 .
\end{aligned}
$$

Except the above-mentioned invariants, system (7.3), (7.4) preserves the total energy

$$
\begin{equation*}
E=\frac{1}{2}(M, \Omega)+\frac{1}{2}(J, H) \tag{7.10}
\end{equation*}
$$

by construction.

### 7.2 Application to SDiff $D$ and $\operatorname{SO}(3)$

The equations of motion (7.6) and (7.4) for the GMHDS with the configuration space SDiff $D$ can be written, using formula (3.12), in the form of the magnetohydrodynamic equations for an ideal, perfectly conducting, incompressible homogeneous fluid of unit density:

$$
\begin{align*}
& \frac{\partial \mathbf{u}}{\partial t}+\operatorname{rot} \mathbf{u} \times \mathbf{u}=\operatorname{rot} \mathbf{H} \times \mathbf{H}-\nabla \varphi,  \tag{7.11}\\
& \frac{\partial \mathbf{H}}{\partial t}=-\{\mathbf{u}, \mathbf{H}\}_{\mathrm{P}} . \tag{7.12}
\end{align*}
$$

They are closed by the zero-divergence conditions for the Eulerian fields of velocity $\mathbf{u}=\mathbf{u}(\mathbf{x}, t)$ and magnetic strength $\mathbf{H}=\mathbf{H}(\mathbf{x}, t)$

$$
\begin{equation*}
\operatorname{div} \mathbf{u}=0, \quad \operatorname{div} \mathbf{H}=0 . \tag{7.13}
\end{equation*}
$$

Here, $\varphi$ is the above-mentioned Bernoulli function $\varphi=p+\mathbf{u}^{2} / 2$ (see Section 3), $p$ is the pressure, and $\{,\}_{\mathrm{P}}$ denotes the Poisson brackets of the vector fields [see (3.6)].

Equation (7.12) is nothing but the Helmholtz equation for $\mathbf{H}$ and means the immobility of the magnetic field in the 'body' (the 'frozenness' in the fluid). The first integrals (7.10)
and (7.9) (Theorem 6) correspond to the total MHD energy

$$
\begin{equation*}
E_{\mathrm{h}}=\frac{1}{2}\langle\mathbf{u}, \mathbf{u}\rangle+\frac{1}{2}\langle\mathbf{H}, \mathbf{H}\rangle=\frac{1}{2} \int_{D}\left(\mathbf{u}^{2}+\mathbf{H}^{2}\right) \delta \mu \tag{7.14}
\end{equation*}
$$

and to the cross-helicity invariant [17]

$$
\begin{equation*}
H_{\mathrm{c}}=\langle\mathbf{u}, \mathbf{H}\rangle=\int_{D} \mathbf{u} \cdot \mathbf{H} \delta \mu=\int_{D} \mathbf{u} \cdot \operatorname{rot} \mathbf{A} \delta \mu, \tag{7.15}
\end{equation*}
$$

$\mathbf{A}=\mathbf{A}(\mathbf{x}, t)$ being the vector potential of the magnetic field.
It follows from theorem 4, i.e., from the immobility of $\mathbf{H}$ with respect to the fluid and the conservation of the volume element, that the orbits of the adjoint representation of the flow (7.12) are isovortical fields of the magnetic vector potential. In other words, the quantity

$$
\begin{equation*}
K_{H}=\mathbf{H} \cdot \delta \boldsymbol{\sigma}=\oint_{C} \operatorname{rot}^{-1} \mathbf{H} \cdot \delta \mathbf{l} \tag{7.16}
\end{equation*}
$$

where $\delta \boldsymbol{\sigma}$ is the area element of an oriented surface bounded by the material contour $C$, is the Lagrangian invariant (Kelvin's circulation theorem for the magnetic field).

If the vector magnetic potential $\mathbf{A}=\operatorname{rot}^{-1} \mathbf{H}$ belongs to the Lie algebra of the group SDiff $D$ (which can be achieved using the gauge invariance), then $\mathbf{H}$ can be regarded as an element of the Lie coalgebra immobile in the fluid, to which theorem 5 is applicable. As a result, we obtain the invariant of the magnetic-field helicity
$H_{\mathrm{W}}=(\mathbf{H}, \mathbf{H})=\left\langle\operatorname{rot}^{-1} \mathbf{H}, \mathbf{H}\right\rangle=\int_{D} \operatorname{rot}^{-1} \mathbf{H} \cdot \mathbf{H} \delta \mu$,
which is also associated with the name of Woltjer [17].
Remark. Generally speaking, what has been said of the invariant (7.16) is a heuristic consideration rather than an exact proof because the definition of the inertia tensor has not been given. Therefore, the fact that $\mathbf{A}$ is an element of the Lie algebra of the group SDiff $D$, i.e., a divergence-free vector field tangent to the boundary, strictly speaking, does not imply that $\operatorname{rot} \mathbf{A}$ is an element of the Lie coalgebra $\hat{g}^{*}(D)$. The rigorous approach described in Ref. [8] is based on the construction of the GMHDS not simply on an arbitrary Lie group but on the semidirect product $G \times \hat{g}^{*}$.

The equations of motion of the GMHDS (7.3) and (7.4) as applied to $\mathrm{SO}(3)$ can be written in the form

$$
\begin{align*}
& \dot{\mathbf{M}}=\boldsymbol{\Omega} \times \mathbf{M}-\mathbf{H} \times \mathbf{J}, \quad \mathbf{M}=I \boldsymbol{\Omega},  \tag{7.18}\\
& \dot{\mathbf{H}}=\boldsymbol{\Omega} \times \mathbf{H}, \quad \mathbf{J}=I \mathbf{H}, \tag{7.19}
\end{align*}
$$

and their first integrals, corresponding to the hydrodynamic invariants (7.14)-(7.16), are expressed by the formulas

$$
\begin{align*}
& E_{\mathrm{m}}=\frac{1}{2} \mathbf{M} \cdot \mathbf{\Omega}+\frac{1}{2} \mathbf{J} \cdot \mathbf{H}  \tag{7.20}\\
& H_{\mathrm{cm}}=\mathbf{M} \cdot \mathbf{H}, \quad H_{\mathrm{Wm}}=\mathbf{H} \cdot \mathbf{H} \tag{7.21}
\end{align*}
$$

Equations (7.18) and (7.19) coincide up to the substitution $\boldsymbol{\Omega} \rightarrow-\boldsymbol{\Omega}$, with the Kirchhoff equations of motion [9] (see also Ref. [8]) of a rigid body in a potential flow of an ideal incompressible fluid immobile at infinity. The equations are written in the CS immobile with respect to the body. According to such mechanical interpretation, $\mathbf{M}$ and $\mathbf{H}$ mean the total angular momentum and the total momentum of the body-fluid system, respectively. The first integrals
represent the total energy, the projection of the total angular momentum onto the direction of the total momentum, and the immobility of the total momentum in the space, respectively. The correspondence of the second of invariants (7.21) to the invariant (7.16) becomes obvious if we recall that the latter is a corollary of the immobility of the magnetic field in the fluid.

It is reasonable to note that the Kirchhoff equations describe exact particular solutions of the MHD equations (see Refs $[10,26]$ ) as the mechanical Euler and Euler - Poisson equations describe exact particular solutions of the equations of motion of ideal homogeneous and inhomogeneous fluid, respectively. The system of equations

$$
\begin{align*}
& \dot{\mathbf{M}}=\boldsymbol{\Omega} \times \mathbf{M}+\mathbf{g l}_{0} \times \boldsymbol{\sigma}-\mathbf{H} \times \mathbf{J}, \quad \mathbf{M}=I \boldsymbol{\Omega},  \tag{7.22}\\
& \dot{\boldsymbol{\sigma}}=\boldsymbol{\Omega} \times \boldsymbol{\sigma},  \tag{7.23}\\
& \dot{\mathbf{H}}=\boldsymbol{\Omega} \times \mathbf{H}, \quad \mathbf{J}=I \mathbf{H} \tag{7.24}
\end{align*}
$$

describe exact particular solutions of the equations of convection of ideal incompressible, stratified, electrically conducting fluid in a magnetic field.

## 8. A generalized rigid body in the Coriolis force field:

 mechanical and hydrodynamic treatmentsFirst, let us formulate the equations of motion of an ordinary rigid body with a fixed point in a CS rotating with a constant angular velocity $\omega_{0}$. It should, however, be specified more exactly what kind of rotating coordinate system is meant. From a group-theoretic viewpoint, as was shown in Section 2, a substantial difference between the mechanical and hydrodynamic systems is that their kinetic energies define different invariant metrics on the corresponding configuration spaces. In practice, this particularly means the following. When hydrodynamically interpreting the solutions of mechanical equations, and vice versa, we should take into account that the corresponding mobile and immobile frames of reference exchange roles. For example, a Lagrangian description of fluid motion conforms in mechanics to the description in a CS immobile in space, and an Eulerian description to the description in a frame of reference 'frozen' in the body. Therefore, in view of the application of the mechanical equations for the Eulerian description of fluid motion in the Coriolis force field, we are interested in the motion of a rigid body in the frame of reference that rotates with a constant angular velocity $\boldsymbol{\omega}_{0}$ around the axis immobile with respect not to the space but to the body. Otherwise, the equations of motion of the mechanical and hydrodynamic gyroscopes will not be equivalent to each other, at least because the angular velocity of rotation of the CS in the body depends on time and obeys the Poisson equation.

Let us exploit the known formula that relates derivatives with respect to time for an arbitrary vector $\mathbf{A}$ in an immobile and rotating frames of reference,

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{A}}{\mathrm{~d} t}=\left(\frac{\mathrm{d} \mathbf{A}}{\mathrm{~d} t}\right)_{\mathrm{r}}+\boldsymbol{\omega}_{0} \times \mathbf{A} \tag{8.1}
\end{equation*}
$$

where the subscript r marks the derivative with respect to time in the rotating frame of reference.

Let $\boldsymbol{\Omega}$ and $\mathbf{M}$ be the angular velocity and kinetic momentum with respect to the space, respectively, and $\boldsymbol{\omega}_{\mathrm{r}}$
and $\mathbf{m}_{\mathrm{r}}$ be the angular velocity and kinetic momentum with respect to the new frame of reference, so that $\boldsymbol{\Omega}=\boldsymbol{\omega}_{\mathrm{r}}+\boldsymbol{\omega}_{0}$ and $\mathbf{M}=\mathbf{m}_{\mathrm{r}}+\mathbf{m}_{0}\left(\mathbf{m}_{0}=I \boldsymbol{\omega}_{0}\right)$. We apply formula (8.1) and take into account the fact that $\mathrm{d} \mathbf{M} / \mathrm{d} t=0$ (the conservation of angular momentum), thus having

$$
\begin{equation*}
\left(\frac{\mathrm{d}\left(\mathbf{m}_{\mathrm{r}}+\mathbf{m}_{0}\right)}{\mathrm{d} t}\right)_{\mathrm{r}}+\left(\boldsymbol{\omega}_{\mathrm{r}}+\boldsymbol{\omega}_{0}\right) \times\left(\mathbf{m}_{\mathrm{r}}+\mathbf{m}_{0}\right)=0 . \tag{8.2}
\end{equation*}
$$

Now let us pass to the frame of reference immobile with respect to the body, which rotates relative to the original frame of reference with the angular velocity $-\boldsymbol{\omega}_{0}$. Then, according to (8.1),

$$
\left(\frac{\mathrm{d}\left(\mathbf{m}_{\mathrm{r}}+\mathbf{m}_{0}\right)}{\mathrm{d} t}\right)_{\mathrm{r}}=\left(\frac{\mathrm{d}\left(\mathbf{m}_{\mathrm{r}}+\mathbf{m}_{0}\right)}{\mathrm{d} t}\right)_{\mathrm{c}}-\boldsymbol{\omega}_{0} \times\left(\mathbf{m}_{\mathrm{r}}+\mathbf{m}_{0}\right)
$$

where the subscript c marks the derivative with respect to time in the coordinate system immobile in the body. Substituting the last formula into (8.2) and taking into account that $\dot{\mathbf{m}}_{0}=0$ in the chosen coordinate system, represent the equations of motion of a rigid body with a fixed point in the form

$$
\begin{equation*}
\dot{\mathbf{m}}=\left(\mathbf{m}+\mathbf{m}_{0}\right) \times \boldsymbol{\omega}, \quad \mathbf{m}=I \boldsymbol{\omega}, \quad \mathbf{m}_{0}=I \boldsymbol{\omega}_{0} \tag{8.3}
\end{equation*}
$$

(where the subscripts $r$ and $c$ are omitted), the tensor of inertia momenta $I$ being independent of time.

It is evident that, in the invariant form, i.e., in the form applicable to an arbitrary Lie group equipped with a left- or right-invariant metric, equations (8.3) and the equation for the generalized angular velocity in the notation of Section 2 take the form

$$
\begin{align*}
& \dot{m}= \pm\left\{\omega, m+m_{0}\right\}, \quad m=I \omega, \quad m_{0}=I \omega_{0}  \tag{8.4}\\
& \dot{\omega}= \pm B\left(\omega, \omega+\omega_{0}\right) \tag{8.5}
\end{align*}
$$

where the + and - signs refer to the left- and right-invariant metrics, respectively.

Now, let us apply the concept of the GRB in the Coriolis force field to the group SDiff $D$ with a right-invariant metric. In this case, $\omega=\mathbf{u}(\mathbf{x}, t)$ is the Eulerian velocity field, $m=\operatorname{rot} \mathbf{u}=\boldsymbol{\Omega}(\mathbf{x}, t)$ is the vorticity, $\omega_{0}=\mathbf{u}_{0}(\mathbf{x}, t)$ is the Eulerian velocity field corresponding to the rotation of the fluid with a constant angular velocity $\boldsymbol{\Omega}_{0}$, and $\{\}=,\{,\}_{\mathrm{P}}$ are the Poisson brackets of the vector fields, i.e., $\{\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x})\}_{\mathrm{P}}=(\mathbf{a} \nabla) \mathbf{b}-(\mathbf{b} \nabla) \mathbf{a}$. Then,

$$
m_{0}=\operatorname{rot} \mathbf{u}_{0}=\operatorname{rot}\left(\boldsymbol{\Omega}_{0} \times \mathbf{x}\right)=2 \boldsymbol{\Omega}_{0}
$$

and, in view of (3.12), equations (8.4) and (8.5), whose righthand sides are taken with the minus sign, can be written in the form of the equations of motion for a rotating ideal, incompressible homogeneous fluid:

$$
\begin{equation*}
\frac{\partial \boldsymbol{\Omega}}{\partial t}+\left\{\boldsymbol{\Omega}+2 \boldsymbol{\Omega}_{0}, \mathbf{u}\right\}_{\mathrm{P}} \equiv \frac{\partial \boldsymbol{\Omega}}{\partial t}+(\mathbf{u} \nabla) \boldsymbol{\Omega}-\left[\left(\boldsymbol{\Omega}+2 \boldsymbol{\Omega}_{0}\right) \nabla\right] \mathbf{u}=0 \tag{8.6}
\end{equation*}
$$

$\frac{\partial \mathbf{u}}{\partial t}+2 \boldsymbol{\Omega}_{0} \times \mathbf{u}=\mathbf{u} \times \operatorname{rot} \mathbf{u}-\operatorname{grad} \varphi$,
where $\varphi$ is the Bernoulli function that includes the potential of the centrifugal forces.

Thus, the coefficient 2 at the Coriolis term in the hydrodynamic equations appears because the vorticity, in terms of which the equation for the generalized kinetic momentum is written, is equal to the doubled angular velocity of local fluid rotation.

## 9. The motions of Euler and Euler - Poisson tops in the Coriolis force field as mechanical prototypes of global geophysical flows

The existence of mechanical prototypes of the fundamental hydrodynamic invariants suggests that there are other mechanical characteristics reflecting less obvious but important properties of fluid motion. Most interesting is predicting, on the basis of mechanical equations, such qualitative features of fluid behavior that are difficult (if possible at all) to find by means of the numerical integration of the hydrodynamic equations. We should obviously begin with the comparison of mechanical motions with global-scale flows, since in this case we can expect most similarity. The quasigeostrophic approximation for the equations of motion of a rotating fluid is a natural filter eliminating small-scale components. In this context, let us consider the behavior of the classical gyroscope and a heavy top in the Coriolis force field at various Rossby numbers; by comparing the quasigeostrophic and exact solutions of their equations of motion, we will try to assess the possibility of extracting certain conclusions about the global geophysical flows from such a comparison [45].

The motion of an ideal, weakly stratified, incompressible rotating fluid is described by the equations

$$
\begin{align*}
& \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u} \nabla) \mathbf{u}+2 \mathbf{\Omega}_{0} \times \mathbf{u}=-\frac{\nabla p}{\rho_{0}}-\mathbf{g} \beta T  \tag{9.1}\\
& \frac{\partial T}{\partial t}+(\mathbf{u} \nabla) T=0, \quad \operatorname{div} \mathbf{u}=0 \tag{9.2}
\end{align*}
$$

[cf. (5.14) and (5.15)]. The respective equations for the vorticity $\boldsymbol{\Omega}=\operatorname{rot} \mathbf{u}$ and $\mathbf{q}=\beta \nabla T$ can be written in the form [cf. (6.11), (6.12), and (8.6)]:

$$
\begin{align*}
& \frac{\partial \boldsymbol{\Omega}}{\partial t}+\left\{\boldsymbol{\Omega}+2 \boldsymbol{\Omega}_{0}, \mathbf{u}\right\}_{\mathrm{P}}=\mathbf{g} \times \mathbf{q}  \tag{9.3}\\
& \frac{\partial \mathbf{q}}{\partial t}+(\mathbf{u} \nabla) \mathbf{q}=-\mathbf{q} \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \tag{9.4}
\end{align*}
$$

Substituting (3.27) and (6.10) into (9.3) and (9.4) and taking into account (3.26) yield the equations [cf. (6.14), (6.15)]

$$
\begin{equation*}
\dot{\mathbf{M}}=\boldsymbol{\omega} \times\left(\mathbf{M}+2 \mathbf{M}_{0}\right)+g \mathbf{l}_{0} \times \boldsymbol{\sigma}, \quad \dot{\boldsymbol{\sigma}}=\boldsymbol{\omega} \times \boldsymbol{\sigma}, \tag{9.5}
\end{equation*}
$$

which describe the fluid motion inside the ellipsoid (3.25) that rotates with a constant angular velocity $\boldsymbol{\Omega}_{0}$ about the axis that passes through the center of the ellipsoid. Here, $\mathbf{M}=I \omega$, and $\mathbf{M}_{0}=I \omega_{0}$, the components of $\omega_{0}$ being expressed through the components of $\boldsymbol{\Omega}_{0}$ via relations analogous to (3.28). Other notation is the same as in Section 6.3. We will also use the additional relations

$$
\begin{equation*}
\omega_{1}=-\frac{a_{3}}{a_{2}} \frac{\partial v}{\partial z}, \quad \omega_{2}=\frac{a_{3}}{a_{1}} \frac{\partial u}{\partial z}, \tag{9.6}
\end{equation*}
$$

which follow from (3.26) and (3.27).

Up to the substitutions $\boldsymbol{\omega} \rightarrow-\boldsymbol{\omega}, 2 \boldsymbol{\omega}_{0} \rightarrow-\boldsymbol{\omega}_{0}$, and $\boldsymbol{\sigma} \rightarrow-\boldsymbol{\gamma}$, system (9.5) coincides with the equations of motion of a heavy top in the Coriolis force field [cf. (5.9), (5.10) with (8.3) taken into account]:

$$
\begin{aligned}
& \dot{\mathbf{m}}=\left(\mathbf{m}+\mathbf{m}_{0}\right) \times \boldsymbol{\omega}+g \mathbf{l}_{0} \times \gamma \\
& \dot{\gamma}=\gamma \times \boldsymbol{\omega}, \quad \mathbf{m}=I \boldsymbol{\omega}, \quad \mathbf{m}_{0}=I \boldsymbol{\omega}_{0}
\end{aligned}
$$

In the absence of stratification $(\mathbf{q}=\beta \nabla T=0)$, the fluid motion is governed by the equation

$$
\begin{equation*}
\dot{\mathbf{M}}=\boldsymbol{\omega} \times\left(\mathbf{M}+2 \mathbf{M}_{0}\right), \tag{9.7}
\end{equation*}
$$

which we will call the equation of motion of a barotropic top, and system (9.5), the equations of motion of a baroclinic top.

### 9.1 The motion of a barotropic top

In terms of $\mathbf{M}_{\mathrm{a}}=\mathbf{M}+2 \mathbf{M}_{0}$, equation (9.7) can be rewritten as

$$
\begin{equation*}
\dot{\mathbf{M}}_{\mathrm{a}}=\boldsymbol{\omega} \times \mathbf{M}_{\mathrm{a}} . \tag{9.7a}
\end{equation*}
$$

Multiplying (9.7) by $\boldsymbol{\omega}$ and (9.7a) by $\mathbf{M}_{\mathrm{a}}$, we obtain two first integrals,

$$
\begin{align*}
& 2 E=\boldsymbol{\omega} \cdot \mathbf{M}=I_{1} \omega_{1}^{2}+I_{2} \omega_{2}^{2}+I_{3} \omega_{3}^{2},  \tag{9.8}\\
& \mathbf{M}_{\mathbf{a}}^{2}=\left(M_{1}+2 M_{01}\right)^{2}+\left(M_{2}+2 M_{02}\right)^{2}+\left(M_{3}+2 M_{03}\right)^{2}, \tag{9.9}
\end{align*}
$$

which mean the conservation of kinetic energy and the validity of Kelvin's circulation theorem (see Section 3), respectively, for the flows under consideration.

By using invariants (9.8) and (9.9), as in the case $\boldsymbol{\Omega}_{0}=0$ (see, e.g., Ref. [23]), we can form an idea of the behavior of the barotropic top without integrating its equations of motion. In the kinetic momentum space, a trajectory of the top is the intersection of the 'energy' ellipsoid

$$
\frac{M_{1}}{2 E I_{1}}+\frac{M_{2}}{2 E I_{2}}+\frac{M_{3}}{2 E I_{3}}=1
$$

with the 'circulation' sphere

$$
\frac{\left(M_{1}+2 M_{01}\right)^{2}}{M_{\mathrm{a}}^{2}}+\frac{\left(M_{2}+2 M_{02}\right)^{2}}{M_{\mathrm{a}}^{2}}+\frac{\left(M_{3}+2 M_{03}\right)^{2}}{M_{\mathrm{a}}^{2}}=1
$$

of radius $M_{\mathrm{a}}=\left|\mathbf{M}+2 \mathbf{M}_{0}\right|$ centered at $\mathbf{M}=-2 \mathbf{M}_{0}$.
Typical phase portraits of the dynamical system (9.7) are shown in Fig. 5 for various values of the Rossby number $\varepsilon=|\mathbf{M}| /\left|2 \mathbf{M}_{0}\right|$. These portraits are of a certain hydrodynamic interest since they illustrate the process of gradual disappearance of the complex elements of motion with the enhancement of the Coriolis force influence. It can be seen that the hyperbolic points successively vanish as the Rossby number diminishes from $\varepsilon=\infty$. The global geophysical flows correspond to small Rossby numbers at which the trajectories of the barotropic top are formed in essence by the intersections of the energy ellipsoid with the family of planes orthogonal to the vector $\mathbf{M}_{0}$. Hence, two conclusions follow: (a) the phase portrait of the geophysical motions of a barotropic top consists of closed elliptical trajectories and has no hyperbolic point; (b) at small Rossby numbers, the projection of the kinetic momentum onto the direction $\mathbf{M}_{0}$ is virtually [up to $O\left(\varepsilon^{2}\right)$ ] conserved in time.

$$
a_{1}=3, \quad a_{2}=1, \quad a_{3}=2, \quad \varepsilon=10
$$

$a_{1}=3, \quad a_{2}=1, \quad a_{3}=2, \quad \varepsilon=0.1$
b




Figure 5. The phase portraits of a barotropic top in the angular momentum space at various Rossby numbers.

The description of motion along a closed trajectory is most simple if $\mathbf{M}_{0}$ coincides in its direction with one of the principal axes of the ellipsoid. Let $\mathbf{M}_{0}$ be directed along the $x_{3}$, or $z$, axis. Then the coordinate form of (9.7) is

$$
\begin{align*}
& I_{1} \dot{\omega}_{1}=\left(I_{3}-I_{2}\right) \omega_{2} \omega_{3}+2 I_{3} \omega_{0} \omega_{2}, \\
& I_{2} \dot{\omega}_{2}=\left(I_{1}-I_{3}\right) \omega_{1} \omega_{3}-2 I_{3} \omega_{0} \omega_{1},  \tag{9.10}\\
& I_{3} \dot{\omega}_{3}=\left(I_{2}-I_{1}\right) \omega_{1} \omega_{2} .
\end{align*}
$$

At a small Rossby number, $\varepsilon \ll 1$, we have $\dot{M}_{3}=O\left(\varepsilon^{2}\right)$, $M_{3}=M_{30}+O\left(\varepsilon^{2}\right)$, and $M_{30}=$ const $=O(\varepsilon)$. In terms of the variables $X=\sqrt{I_{2}} M_{1}$ and $Y=\sqrt{I_{1}} M_{2}$, the equations of motion can be rewritten, up to $O\left(\varepsilon^{2}\right)$, in the form

$$
\begin{equation*}
\dot{X}=2 \frac{I_{3}}{\sqrt{I_{1} I_{2}}} \omega_{0} Y, \quad \dot{Y}=-2 \frac{I_{3}}{\sqrt{I_{1} I_{2}}} \omega_{0} X . \tag{9.11}
\end{equation*}
$$

Hence, it follows that the endpoint of the vector $\mathbf{M}$, or $\mathbf{M a}_{\mathrm{a}}$ (which is the same), moves along the elliptical orbit $M_{1}^{2} / I_{1}+M_{2}^{2} / I_{2}=$ const with the angular velocity

$$
\begin{equation*}
\sigma=-2 \frac{I_{3}}{\sqrt{I_{1} I_{2}}} \omega_{0}=-2 \frac{a_{1} a_{2}}{\sqrt{I_{1} I_{2}}} \Omega_{0} \tag{9.12}
\end{equation*}
$$

i.e., in the direction opposite to that of the revolution of the coordinate system.

In view of the dual treatment of the equations of motion of a rigid body with a fixed point, such precessions of a barotropic top, as well as the approximate invariance of the projection of its kinetic momentum onto the $\mathbf{M}_{0}$ direction, could be regarded, to a certain degree of scepticism, as mechanical prototypes of the Rossby waves transporting the atmospheric kinetic momentum in a direction opposite to the Earth's rotation, and of the approximate Lagrangian invariance of the vertical vorticity (the Charney-Obukhov equation) of global geophysical flows, respectively.

### 9.2 The quasigeostrophic approximation for the equations of motion of a baroclinic top

The three first integrals of system (9.5),

$$
\begin{align*}
& E=\frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{M}+g \mathbf{l}_{0} \cdot \boldsymbol{\sigma},  \tag{9.13}\\
& \Pi=\left(\mathbf{M}+2 \mathbf{M}_{0}\right) \cdot \boldsymbol{\sigma}, \quad \Theta=\boldsymbol{\sigma} \cdot \boldsymbol{\sigma},
\end{align*}
$$

express the conservation of the total energy

$$
E_{\mathrm{h}}=\frac{1}{2} \rho_{0} \int_{D} \mathbf{u}^{2} \delta \mu+\rho_{0} \beta \int_{D} \mathbf{g} \cdot \mathbf{x} T \delta \mu
$$

and the Lagrangian invariance of the PV,

$$
\Pi_{\mathrm{h}}=\left(\boldsymbol{\Omega}+2 \mathbf{\Omega}_{0}\right) \cdot \nabla T
$$

and of the temperature, $T$, for the considered class of solutions of the original hydrodynamic equations (9.1), (9.2) [cf. (6.3), (6.4)]. Thus, we have a complete collection of necessary tools to construct the desired approximation. Let us recall in this context the general scheme that is employed in geophysical hydrodynamics to derive the quasigeostrophic approximation for the equations of motion of a baroclinic atmosphere. This scheme is as follows (see, e.g., Ref. [31]).
I. The Rossby number $\varepsilon=U / f_{0} L=O\left(\omega / f_{0}\right)$ and the dimensionless quantities

$$
\begin{equation*}
\xi=\frac{f_{0}^{2} L^{2}}{g H_{0}}=O(\varepsilon), \quad \eta=\frac{N^{2} H_{0}}{g}=O(\varepsilon) \tag{9.14}
\end{equation*}
$$

are assumed to be small parameters, the same order of their smallness being not necessary but chosen to simplify the calculations. Here, $f_{0}$ is the Coriolis parameter (the doubled latitudinal average of the vertical component of the angular velocity of the Earth's rotation), $L$ and $H_{0}$ are the characteristic horizontal and vertical scales of the global flows, $U$ and $\omega$ are their characteristic velocity and vertical vorticity, and $N$ is the Brunt-Väisälä frequency; in application to equations (9.1) and (9.2), it is given by the relationship $N^{2}=g \beta \partial T / \partial z$, provided that $\partial T / \partial z>0$.
II. The motion is assumed to be quasistatic and quasigeostrophic, i.e., to satisfy the thermal wind equations up to $O(\varepsilon)$ (from the mechanical viewpoint, they mean an approximate equilibrium between the Coriolis force and pressure, expressed in terms of vorticity - see below).
III. The derivation is based on the conservation equations for the PV and potential temperature (in the case under consideration, this role of the latter is played by $T$ ), which are expanded in powers of the small parameter $\varepsilon$ up to the terms $O\left(\varepsilon^{2}\right)$.

Let $\mathbf{g}$ coincide with the principal axis $O z$ of the ellipsoid (semiaxis $a_{3}$ ) about which the ellipsoid rotates with the angular velocity $\Omega_{0}$. Let us apply the above-described scheme to obtaining the quasigeostrophic approximation for equations (9.5).

As the parameters $\varepsilon, L^{2}$, and $H_{0}$, it is natural to take the quantities

$$
\begin{align*}
& \varepsilon=\frac{\omega}{2 \omega_{0}} \quad\left(\omega=\sqrt{\omega_{1}^{2}+\omega_{2}^{2}+\omega_{3}^{2}}\right),  \tag{9.15}\\
& 2 L^{2}=I_{3}=a_{1}^{2}+a_{2}^{2}, \quad H_{0}=a_{3} .
\end{align*}
$$

Then,

$$
\begin{align*}
& \xi=\frac{4 \omega_{0}^{2}\left(a_{1}^{2}+a_{2}^{2}\right)}{2 g a_{3}}=\frac{2 \omega_{0}^{2} I_{3}}{g a_{3}}=O(\varepsilon),  \tag{9.16}\\
& N^{2}=g \beta \frac{\partial T}{\partial z}=\frac{g \sigma_{3}}{a_{3}}, \quad \eta=\frac{N^{2} a_{3}}{g}=\sigma_{3}=O(\varepsilon) . \tag{9.17}
\end{align*}
$$

For the original hydrodynamic equations (9.1) and (9.2), the thermal wind equation is

$$
\begin{equation*}
-2\left(\boldsymbol{\Omega}_{0} \nabla\right) \mathbf{u}=\beta \mathbf{g} \times \nabla T+O(\varepsilon), \tag{9.18}
\end{equation*}
$$

meaning that the ratio of the omitted to the retained terms is $O(\varepsilon)$. In the coordinate form, this equation can be rewritten as follows:

$$
\begin{equation*}
\frac{\partial u}{\partial z}=-\frac{g \beta}{2 \Omega_{0}} \frac{\partial T}{\partial y}+O(\varepsilon), \quad \frac{\partial v}{\partial z}=\frac{g \beta}{2 \Omega_{0}} \frac{\partial T}{\partial x}+O(\varepsilon) . \tag{9.18a}
\end{equation*}
$$

For the model equations (9.5), the thermal wind equation takes the form

$$
\begin{equation*}
\boldsymbol{\omega} \times 2 \mathbf{M}_{0}+g \mathbf{1}_{0} \times \boldsymbol{\sigma}=O(\varepsilon), \tag{9.19}
\end{equation*}
$$

or, in the coordinate form $\left[\mathbf{I}_{0}=\left(0,0,-a_{3}\right)\right]$,

$$
\begin{equation*}
\omega_{2}=-\frac{a_{3} g \sigma_{2}}{2 I_{3} \omega_{0}}+O(\varepsilon), \quad \omega_{1}=-\frac{a_{3} g \sigma_{1}}{2 I_{3} \omega_{0}}+O(\varepsilon) \tag{9.19a}
\end{equation*}
$$

By taking into account (9.6) and (9.19a), we find that $-\omega_{2}$ and $-\omega_{1}$ can be treated as the components of the thermal wind up to contraction-expansion transformations:

$$
-\omega_{1} \propto \frac{\partial v}{\partial z} \propto \frac{\partial T}{\partial x}, \quad-\omega_{2} \propto-\frac{\partial u}{\partial z} \propto \frac{\partial T}{\partial y} .
$$

Further, for the sake of convenience, the quantities $\omega_{1}$ and $\omega_{2}$ will be called the thermal wind components, which is correct, however, up to their sign.

According to (19.19a) and (19.16),

$$
\frac{\omega_{2}}{\omega_{0}} \propto \frac{\omega_{1}}{\omega_{0}} \propto O(\varepsilon) \propto \frac{\sigma_{2}}{O(\varepsilon)} \propto \frac{\sigma_{1}}{O(\varepsilon)}
$$

hence,

$$
\begin{equation*}
\sigma_{1} \propto \sigma_{2}=O\left(\varepsilon^{2}\right) \tag{9.20}
\end{equation*}
$$

Let us write the model equations (9.5) in the coordinate form:

$$
\begin{align*}
& I_{1} \dot{\omega}_{1}=\left(I_{3}-I_{2}\right) \omega_{2} \omega_{3}+2 I_{3} \omega_{0} \omega_{2}+g a_{3} \sigma_{2}  \tag{9.21}\\
& I_{2} \dot{\omega}_{2}=\left(I_{1}-I_{3}\right) \omega_{1} \omega_{3}-2 I_{3} \omega_{0} \omega_{1}-g a_{3} \sigma_{1} \\
& I_{3} \dot{\omega}_{3}=\left(I_{2}-I_{1}\right) \omega_{2} \omega_{3}, \\
& \dot{\sigma}_{1}=\omega_{2} \sigma_{3}-\omega_{3} \sigma_{2}  \tag{9.22}\\
& \dot{\sigma}_{2}=\omega_{3} \sigma_{1}-\omega_{1} \sigma_{3} \\
& \dot{\sigma}_{3}=\omega_{1} \sigma_{2}-\omega_{2} \sigma_{1} \tag{9.23}
\end{align*}
$$

The system (9.21)-(9.23) has, in particular, the following families of fixed points describing stationary regimes of rotations about the principal axes:
(A) $\omega_{1}=\omega_{2}=0, \quad \sigma_{1}=\sigma_{2}=0, \quad \omega_{3}=\omega_{30}, \quad \sigma_{3}=\sigma_{30}$;
(B) $\omega_{1}=\omega_{3}=0, \quad \sigma_{1}=\sigma_{3}=0, \quad \omega_{2}=\omega_{20}, \quad \sigma_{2}=\sigma_{20}$, $2 I_{3} \omega_{0} \omega_{20}+g a_{3} \sigma_{20}=0 ;$
(C) $\omega_{2}=\omega_{3}=0, \quad \sigma_{2}=\sigma_{3}=0, \quad \omega_{1}=\omega_{10}, \quad \sigma_{1}=\sigma_{10}$, $2 I_{3} \omega_{0} \omega_{10}+g a_{3} \sigma_{10}=0$.

The variables with the subscript 0 can assume arbitrary real values (do not confuse the variables with the external parameter $\omega_{0}$ ). It is easy to see that any representative of family (B) or (C) is a nontrivial, strictly geostrophic stationary regime at any $\omega_{0} \neq 0$. In accordance with the above estimates (9.20) and the thermal wind equations (9.19a), in view of (9.16), the equation (9.23) gives $\dot{\sigma}_{3}=o\left(\varepsilon^{3}\right)$. Therefore, $\sigma_{3}=\sigma_{30}$ is a constant to a high accuracy, and the last two equations of system (9.22) can be rewritten in the form

$$
\begin{equation*}
\dot{\sigma}_{1}=\omega_{2} \sigma_{30}-\omega_{3} \sigma_{2}, \quad \dot{\sigma}_{2}=\omega_{3} \sigma_{1}-\omega_{1} \sigma_{30} \tag{9.24}
\end{equation*}
$$

Now, by eliminating $\sigma_{1}$ and $\sigma_{2}$ from (9.24) with the use of (9.19a), we obtain the system

$$
\begin{align*}
& \dot{\omega}_{1}=-\left(\frac{g a_{3} \sigma_{30}}{2 I_{3} \omega_{0}}+\omega_{3}\right) \omega_{2},  \tag{9.25}\\
& \dot{\omega}_{2}=\left(\frac{g a_{3} \sigma_{30}}{2 I_{3} \omega_{0}}+\omega_{3}\right) \omega_{1},
\end{align*}
$$

which can be interpreted as an analog of the equation for the 'potential' temperature (more precisely, the equations for the components of the potential temperature gradient) reduced by means of the expansion in powers of $\varepsilon$ and written in terms of the thermal wind components.

Now, it remains to clarify what the quasigeostrophic PV is. By virtue of the above estimates, the expression for

$$
\Pi=\left(\mathbf{M}+2 \mathbf{M}_{0}\right) \cdot \boldsymbol{\sigma}=I_{1} \omega_{1} \sigma_{1}+I_{2} \omega_{2} \sigma_{2}+I_{3} \omega_{3} \sigma_{3}+2 I_{3} \omega_{0} \sigma_{3}
$$

[see (9.13)] can be rewritten in the form

$$
\Pi=I_{3}\left(2 \omega_{0}+\omega_{3}\right) \sigma_{30}+O\left(\varepsilon^{3}\right) .
$$

Therefore, the quasigeostrophic PV is

$$
\begin{equation*}
\Pi_{\mathrm{G}}=I_{3}\left(2 \omega_{0}+\omega_{3}\right) \sigma_{30}, \quad \dot{\Pi}_{\mathrm{G}}=2 I_{3} \sigma_{30} \dot{\omega}_{3} \tag{9.26}
\end{equation*}
$$

and its evolution is described by the first equation of system (9.22).

Thus, the quasigeostrophic approximation of the sixthorder system (9.21)-(9.23), describing the motion of a baroclinic top, reduces to the three-component dynamical system

$$
\begin{align*}
& I_{3} \dot{\omega}_{3}=\left(I_{2}-I_{1}\right) \omega_{1} \omega_{2}, \\
& \dot{\omega}_{1}=-\left(\frac{g a_{3} \sigma_{30}}{2 I_{3} \omega_{0}}+\omega_{3}\right) \omega_{2},  \tag{9.27}\\
& \dot{\omega}_{2}=\left(\frac{g a_{3} \sigma_{30}}{2 I_{3} \omega_{0}}+\omega_{3}\right) \omega_{1} .
\end{align*}
$$

System (9.27) corresponds to the equations for the slow variables in the theory of relaxation oscillations (see, e.g., Ref. [46]) and, in the case under consideration, describes the
slow evolution of the main components of the global geophysical flows, i.e., the vertical vorticity and thermal wind.

Let us divide the left- and right-hand sides of equations (9.27) by $\omega_{0}^{2}$. Then, in view of (9.15), (9.17), and the relationship $4 \omega_{0}^{2}=f_{0}^{2}$, the additive constant that appears in the parentheses in the last two equations takes the form

$$
\begin{equation*}
S=\frac{g a_{3} \sigma_{30}}{2 I_{3} \omega_{0}^{2}}=\frac{N^{2} H_{0}^{2}}{f_{0}^{2} L^{2}} \tag{9.28}
\end{equation*}
$$

Here, $S$ is a similarity criterion, well known in geophysical fluid dynamics and appearing in the quasigeostrophic equation of the PV for a baroclinic atmosphere as a given parameter called the stratification parameter (see, e.g., Ref. [31]).

In the dimensionless variables $X=-\omega_{1} / \omega_{0}, \quad Y=$ $-\omega_{2} / \omega_{0}, Z=S+\omega_{3} / \omega_{0}$, and $\tau=\omega_{0} t$ (slow time), system (9.27) can be rewritten in the following, extremely simple form:

$$
\begin{equation*}
\dot{X}=-Y Z, \quad \dot{Y}=Z X, \quad \dot{Z}=\Gamma X Y \tag{9.29}
\end{equation*}
$$

where $\Gamma=\left(I_{2}-I_{1}\right) / I_{3}=\left(a_{1}^{2}-a_{2}^{2}\right) /\left(a_{1}^{2}+a_{2}^{2}\right)$.
Further, we can assume that $a_{1}>a_{2}$ without a loss of generality, so that $0<\Gamma<1$. The system (9.29) has two positive definite invariants,

$$
\begin{equation*}
2 E_{\mathrm{G}}=\Gamma X^{2}+Z^{2}, \quad \Theta_{\mathrm{G}}=X^{2}+Y^{2} \tag{9.30}
\end{equation*}
$$

According to Obukhov's theorem (see Ref. [26]), this means that the quasigeostrophic approximation for the equations of motion of the baroclinic top is equivalent to the Euler equations of motion of a classical gyroscope whose dependent variables are the vorticity and the components of the thermal wind. It is worth noting in this context that the quasigeostrophic approximation for the reduced equations of motion for rotating shallow water also coincides with the mechanical Euler equations and describes a slow evolution of the Rossby waves [19].

The families $(A)-(C)$ of stationary solutions of the original system (9.21)-(9.23) exhaust the set of fixed points of the reduced equations (9.29); in terms of the new variables they can be written as
(A) $X=Y=0, Z=Z_{0}$,
(B) $X=Z=0, \quad Y=Y_{0}$,
(C) $\quad Y=Z=0, \quad X=X_{0}$.

Here, the quantities marked with the subscript 0 can assume arbitrary real values; the zero solution $X=Y=Z=0$ at $S \neq 0$ describes circulation around the vertical axis and is in essence also a nontrivial representative of family (A).

The above-mentioned first integrals should be treated as the total energy and an analog of the Lagrangian invariance of potential temperature. At $S=0$, the second term in the energy expression is the kinetic energy of vertical vorticity, whereas the first term defined by one of the thermal wind components should in fact be interpreted as a measure of the available potential energy. The reason for this is as follows. The kinetic energy of the horizontal vorticity, neglected in the quasigeostrophic approximation, is generated by the horizontal inhomogeneity of the potential temperature and, therefore, is attributed to the potential energy of the quasigeostrophic system. (This is a general statement valid
for any global geophysical flow.) In the dimensional variables, the stationary solution of system (9.29)

$$
\begin{equation*}
X=X_{0} \neq 0, \quad Y=Z=0 \tag{9.31}
\end{equation*}
$$

with a nonzero $X$ component of the thermal wind has a horizontal vorticity $\omega_{10} \propto-\partial T / \partial x$ to which the kinetic energy

$$
E_{X}=\frac{1}{2} I_{1} \omega_{10}^{2}=\frac{1}{2}\left(a_{2}^{2}+a_{3}^{2}\right) \omega_{10}^{2}
$$

corresponds. The same value of the vorticity directed in the $Y$ direction corresponds to the stationary solution

$$
\begin{equation*}
X=0, \quad Y=X_{0} \neq 0, \quad Z=0 \tag{9.32}
\end{equation*}
$$

and the corresponding kinetic energy is

$$
E_{Y}=\frac{1}{2} I_{2} \omega_{10}^{2}=\frac{1}{2}\left(a_{1}^{2}+a_{3}^{2}\right) \omega_{10}^{2}
$$

Note that the first integrals do not prohibit a transition from the state (9.31) to the unsteady state

$$
X=0, \quad Y=X_{0} \neq 0, \quad Z=\sqrt{\Gamma} X_{0}
$$

in which the components $X$ and $Y$ exchange their roles but forbid an inverse transition because of the violation of the law of conservation of energy. Therefore, the difference between the kinetic energies corresponding to the horizontal vorticities of the states (9.31) and (9.32),

$$
\Delta E=E_{Y}-E_{X}=\frac{1}{2}\left(a_{1}^{2}-a_{2}^{2}\right) \omega_{10}^{2}
$$

is the excess potential energy of the state (9.31) with respect to the state (9.32), i.e., the available potential energy that can generate the vertical vorticity.

It follows from the phase portrait of system (9.29), shown in Fig. 6, that the fixed point (B) is stable, whereas (C) as a hyperbolic point is unstable. In essence, Fig. 6 illustrates the Eady's results [47], who was the first to find the baroclinic instability in the case of a horizontally uniform zonal flow with a vertical velocity shear due to the pole-equator temperature difference. In other words, Eady found that a


Figure 6. The phase portrait of the quasigeostrophic motion of the baroclinic top in the space of dimensionless components of thermal wind, $X$ and $Y$, and the vertical vorticity $Z$, which illustrates the baroclinic instability mechanism.
flow with zero vertical vorticity and a nonzero thermal wind $\partial u / \partial z=-\left(g \beta / 2 \Omega_{0}\right) \partial T / \partial y$ turns out to be unstable because of the excess of available potential energy, which is transformed into the kinetic energy of vertical vorticity, giving rise to the atmospheric cyclogenesis. It is this mechanism that describes model (9.29).

If $S \neq 0$, we cannot treat quantity $Z^{2}$ as a measure of the kinetic energy. Therefore, it is not an accident that the exact and quasigeostrophic solutions come to best agreement at $S=0$, as we will see in Section 9.3: the larger the departure of $S$ from zero, the larger the divergence between the exact and quasigeostrophic trajectories. Therefore, the stratification parameter can in a sense be regarded as a measure of the deviation of the original model trajectories from the slow manifold described by system (9.29).

Remark. The following question may arise: why is $S=O(1)$, rather than $S=0$, the best value for the atmosphere? The point is that the equations of atmospheric motion are formulated for the deviations from the static equilibrium state, with a stable vertical profile of potential temperature to which a positive $N^{2}$ corresponds. However, in our case, we started with an equilibrium state with $N^{2}=0$, which is explained by the choice of the Oberbeck-Boussinesq approximation for the deviations from the static equilibrium state with a uniform profile of the average temperature, $T_{0}=\beta^{-1}=$ const.

### 9.3 Comparison of the quasigeostrophic and exact motions of a baroclinic top depending on stratification parameter at small initial Rossby numbers

For a spherical ( $a_{1}=a_{2}=a_{3}$ ) or cylindrical symmetry, the analytical solutions of the quasigeostrophic triplet can be compared with the known analytical solutions of the original model equations. Such a comparison was done for a spherically symmetric top in Ref. [48]. However, these are the least interesting examples, in which the nonlinear interaction of the $\omega$ components responsible for the generation of the ageostrophic component of motion are partly or entirely excluded [see (9.21)]. To avoid such simplifications, we preferred to integrate the approximate and primary model equations numerically, using methods that allow checking the calculation errors to a high accuracy. (The nonintegrability of the equations of motion of an asymmetric heavy top was already proven by S Kovalevskaya. A modern proof is presented in Ref. [49].) Specifically, the question is the use of the following two criteria of integration accuracy: (a) the degree to which the invariants (9.13) and (9.30) are really conserved and (b) the precision to which the trajectory returns to the starting point if the backward integration is carried out.

In all examples given below, the errors in the conservation of the above-mentioned invariants and in the return to the initial point did not exceed $10^{-6}$ and $10^{-4} \%$, respectively. All numerical experiments aimed at the comparison of the quasigeostrophic and 'exact' trajectories were implemented for an ellipsoid with principal semiaxes of $a_{1}=3, a_{2}=1$, and $a_{3}=2$. The unstable stationary point of the quasigeostrophic triplet model, $\boldsymbol{\omega} / \omega_{0}=(0.1,0,0)$, initially perturbed to $\boldsymbol{\omega}^{\prime} / \omega_{0}=\left(0,0,10^{-5}\right)$, was taken as the initial state, the corresponding values of $\sigma_{1}$ and $\sigma_{2}$ were calculated using the thermal wind formulas (9.19a). The stratification parameter $S$ was the only quantity varied from run to run.

The calculation results are presented in Figs 7 and 8 in the form of the projections of the approximate and 'exact' phase trajectories onto the two-dimensional $(X, Y)$ and three-
dimensional $(X, Y, Z)$ subspaces for different positive and negative initial $S$ values (recall that $S$ is an invariant only for the quasigeostrophic model). The parameter $S$ was varied within the range $0 \leqslant|S| \leqslant 1$.

First of all, we note that, at $|S| \ll 1$, the phase portraits of the approximate and primary models virtually coincide. This suggests the very existence of a slow manifold. At small and moderate $S$ values, the 'exact' trajectories are reflected inside or outside from the slow manifold, depending on the sign of $S$, as if the manifold were a curved mirror. The higher the value of $|S|$, the larger the deviation amplitude. We emphasize that the above-noted property persists even when the ageostrophic amplitude becomes comparable in its magnitude with the geostrophic component (or even exceeds it), at positive changes in the stratification parameter up to $S=1$ and, at its negative changes, up to $|S| \approx 0.64$. The transpiercing of the mirror occurs from outside at $S \approx-0.65$, and the trajectory fills the previously unavailable domain within a finite time interval $\left(\Delta \tau \sim 10^{2}\right)$, which seems to be accompanied with the origin of chaos. This is illustrated in Figs 9 and 10 , which show the Poincaré maps and frequency spectra for subcritical and supercritical $S$ values. The hierarchy of models occupying intermediate positions between the quasigeostrophic and exact equations of motion of a baroclinic top were constructed in Ref. [50]. These models permit analytically describing the regular oscillations of a baroclinic top at moderate values of the parameter $S$.

It is interesting to note that the described situation is reminiscent of the results of Lorenz [19], who compared the exact solutions of the truncated (according to the Galerkin method) equations of atmospheric motion with the solutions of their quasigeostrophic approximation. In Lorenz's model, the roles of slow and fast motions are played by the planetary and inertial gravitational waves, respectively. In our case, at $S \neq 0$, the slow evolution of the vertical vorticity and thermal wind is accompanied by high-frequency inertial gravitational oscillations, which periodically move the phase trajectories away from the slow manifold (see Figs 7 and 8).

## 10. The motion of a baroclinic top under the action of external heating, friction, and the beta effect

### 10.1 Taking into account friction and external heating

As is well known, kinetic energy dissipates in geophysical hydrodynamic systems mainly in the planetary boundary layer, which slows down the motion of the free atmosphere according to a nearly linear (in velocity) friction law. The external heating, as is not infrequently done in theoretical studies, can be taken into account using Newton's formula. According to it, the heat influx is proportional to the temperature deviations from the background value. The temperature field that is established in the motionless fluid due to the nonuniform external heating and the thermal conduction of the fluid is called the background temperature distribution. Then, under these assumptions, the viscous motion of a baroclinic top is described by the equations

$$
\begin{align*}
& \dot{\mathbf{M}}=\boldsymbol{\omega} \times\left(\mathbf{M}+2 \mathbf{M}_{0}\right)+g \mathbf{l}_{0} \times \boldsymbol{\sigma}-\lambda \mathbf{M},  \tag{10.1}\\
& \dot{\boldsymbol{\sigma}}=\boldsymbol{\omega} \times \boldsymbol{\sigma}+\mu\left(\boldsymbol{\sigma}_{B}-\boldsymbol{\sigma}\right) .
\end{align*}
$$

Here, the quantities $\lambda$ and $\mu$ having the dimension of inverse time can be treated as some effective coefficients of friction


Figure 7. The phase portraits of quasigeostrophic and 'exact' motions of the baroclinic top in the plane $(X, Y)$ for various values of the parameter $S$ : (a) $S=0.2$, (b) $S=-0.2$, (c) $S=0.6$, (d) $S=-0.6$, (e) $S=0.65$, (f) $S=-0.65$. The heavy curves correspond to quasigeostrophic trajectories; the light curves, to 'exact' trajectories. At $S=0$, the approximate and exact trajectories virtually coincide.
and heat conductivity, respectively, and $\boldsymbol{\sigma}_{\mathrm{B}}$ corresponds to the spatially linear distribution of the background temperature. Other notation is as before.

## 10.2 'Toy' Hadley and Rossby circulations

Let us consider the typical geophysical situation in which a viscous baroclinic top moves under the action of horizon-


Figure 8. The phase portraits of quasigeostrophic and 'exact' motions of the baroclinic top in space $(X, Y, Z)$ for subcritical values of the parameter $S$, (a) $S=0.6$, (b) $S=-0.2$, (c) $S=-0.6$, and the supercritical value (d) $S=-0.65$. The heavy curves correspond to quasigeostrophic trajectories, the light curves, to 'exact' trajectories.


Figure 9. The Poincare maps in planes $(X, Z)$ and $(Y, Z)$ at (a) subcritical value $S=-0.6$ and (b) supercritical value $S=-0.65$.
tally nonuniform external heating. As before, we assume that $a_{1}>a_{2}$ and vectors $\mathbf{g}$ and $\boldsymbol{\Omega}_{0}$ have opposite directions and are parallel to the $x_{3}$ axis. To switch on the mechanism of baroclinic instability, we direct the gradient of background temperature along the $x_{1}$ axis. In this case, $\boldsymbol{\sigma}_{\mathrm{B}}=\left(\sigma_{\mathrm{B} 1}, 0,0\right)$.

The quasigeostrophic approximation for system (10.1) can be obtained using the procedure described in Section 9.2:

$$
\begin{align*}
& I_{3} \dot{\omega}_{3}=\left(I_{2}-I_{1}\right) \omega_{1} \omega_{2}-\lambda I_{3} \omega_{3}, \\
& \dot{\omega}_{1}=-\omega_{2} \omega_{3}-\mu \omega_{1}-\frac{\mu g a_{3} \sigma_{\mathrm{B} 1}}{2 I_{3} \omega_{0}},  \tag{10.2}\\
& \dot{\omega}_{2}=\omega_{3} \omega_{1}-\mu \omega_{2} .
\end{align*}
$$

Note that the quantity $\sigma_{3}$ appears in the system (10.2) neither as a parameter [cf. (9.27)] nor as a variable with its own equation, since $\sigma_{3}$ finally vanishes because of the homogene-


Figure 10. The frequency spectra corresponding to the cases represented by (a) Fig. 7c, (b) Fig. 7d, (c) Fig. 7e, and (d) Fig. 7f.
ity of the background temperature distribution (the result of numerical integration).

In the dimensionless dependent variables

$$
X=\frac{\omega_{1}}{\mu}, \quad Y=\frac{\omega_{2}}{\mu}, \quad Z=\frac{\omega_{3}}{\mu}, \quad \tau=\mu t
$$

system (10.2) takes the form

$$
\begin{equation*}
\dot{X}=-Y Z-X-D, \quad \dot{Y}=Z X-Y, \quad \dot{Z}=\Gamma X Y-\zeta Z . \tag{10.3}
\end{equation*}
$$

Such a choice of the slow time is dictated by the fact that, for geophysical systems, as a rule, $\mu^{-1} \gtrsim \lambda^{-1} \gg \Omega_{0}^{-1}$ (for example, the cooling time of the Earth's atmosphere is about 10 days). The quantity $\zeta=\lambda / \mu$ can be interpreted as an effective Prandtl number and $D=g a_{3} \sigma_{\mathrm{B} 1} / 2 I_{3} \omega_{0} \mu$ is the dimensionless thermal drive.

Let the vector $\boldsymbol{\sigma}_{\mathrm{B}}$ point in the negative direction of axis $x_{1}$, i.e., $D=-|D|$. In this case, the natural convection due to external heating can be described by positive values of $\omega_{2}$. System (10.3) has two types of stationary solutions. One of them is the Hadley regime $(\mathrm{H})$ and the other corresponds to the Rossby regimes $\left(\mathrm{R}_{ \pm}\right)$:

$$
\begin{align*}
(\mathrm{H}) \quad X & =|D|, \quad Y=Z=0  \tag{10.4}\\
\left(\mathrm{R}_{ \pm}\right) \quad X & =D_{0} \equiv\left(\frac{\zeta}{\Gamma}\right)^{1 / 2}, \\
Y & = \pm D_{0}^{1 / 2}\left(|D|-D_{0}\right)^{1 / 2},  \tag{10.5}\\
Z & = \pm D_{0}^{-1 / 2}\left(|D|-D_{0}\right)^{1 / 2} .
\end{align*}
$$

The meaning of these solutions becomes clear if we compare their energetic characteristics with the energetic characteristics of the real global geophysical flows. We begin with the fact that, according to the analysis of the solutions of the primary model (see Refs [26,51,52]), in the H regime both the quantities $Y$ and $Z$ are negligibly small but strictly positive for any $D \neq 0$ if the quasigeostrophic equilibrium is valid. The smallness and positiveness of $Y$ mean that the natural convection, arising in the cross sections orthogonal to the $x_{2}$ axis, is extremely inefficient from the viewpoint of heat transport from the heater to refrigerator. The intense circulation about the $x_{1}$ axis does not increase the efficiency of this regime. As a result, the temperature field established in the fluid virtually coincides with the background distribution (according to the thermal wind relations, $X=-D$ and $Y=0$ mean that $\nabla T=\nabla T_{\mathrm{B}}$ ). From the energetic viewpoint, the Hadley regime observed in laboratory and numerical experiments on modeling the global atmospheric circulation (GAC) is characterized by similar peculiarities, i.e., by a powerful but inefficient zonal flow orthogonal to the pole-equator direction and very weak natural convection in the meridional (radial) plane (see Fig. 11a). The intensity of the meridional circulation is two orders of magnitude less than that of the zonal flow [53].

In $\mathrm{R}_{ \pm}$regimes, the situation is radically different. The intensity of circulation about the $x_{1}$ axis and, according to the thermal wind equations, temperature difference along the same axis are independent of $D$, i.e., of the power of external heating. If $\Delta T$ is the above-mentioned temperature difference, then $\Delta T / \Delta T_{\mathrm{B}}=\left|D_{0} / D\right|<1$, i.e., in $\mathrm{R}_{ \pm}$regimes the heat engine under consideration becomes substantially more


Figure 11. Schematic representation of the Hadley (a) and Rossby (b) regimes observed in experiments with rotating annular channels filled with a fluid that is heated in the periphery and cooled in the center. The above view and side view are shown. The compensation cells in the radial cross sections, which provide an oncoming bottom zonal flow to maintain the conservation of angular momentum, are not shown. (c) The circulation over Antarctica [57].
efficient. The intensities of fluid rotations about the axes orthogonal to $\nabla T_{\mathrm{B}}$ increase according to the law $\sqrt{|D|-D_{0}}$ with the growth of $|D|$. In the $\mathrm{R}_{+}$regime, both these rotations favor the heat transport from the heater to the refrigerator, whereas, in the $\mathrm{R}_{-}$regime, the circulation about the $x_{2}$ axis occurs in the direction opposite to that of the natural convection. Previously, researchers associated this phenomenon, observed under natural and laboratory conditions, with the so-called negative viscosity effect [54]. Again, from the energetic viewpoint, the above-mentioned situation is similar to atmospheric and laboratory Rossby regimes. Indeed, Fig. 11b, in which the results of laboratory experiments on modeling the GAC are schematically presented (see, for example, the monograph [53] and reviews [55, 56]), shows that the negative influence of 'contranatural' convection in the radial (meridional) plane is compensated by a powerful horizontal jet stream, which transfers heat in the required


Figure 12. The critical curve for the $\mathrm{R}_{-}$regime in the plane of the external similarity criteria $\left(\mathrm{Ta}, \mathrm{Ro}_{\mathrm{T}}\right)$; the behavior of the lower branch is consistent with the quasigeostrophic theory of the baroclinic top.
direction due to alternating contacts with the heat and cold sources. This effect is also stimulated by large-scale vortices circumflexed by the jet stream, and the role of these vortices in the 'toy' Rossby regimes is played by the vertical vorticity $\omega_{3}$. A similar pattern is also observed in the atmosphere (see Fig. 11c [57] based on observational data from the USSR Hydrometeorological Center).

In the context of the above-mentioned properties of the H and $R_{ \pm}$regimes, it seems to be interesting to investigate their domains of existence and stability, to compare them with the domains of the corresponding regimes of global geophysical flows, natural and simulated in laboratory. According to (10.5), $|D|=D_{0}$ is the lower boundary of the region of existence of the $\mathrm{R}_{ \pm}$regimes. In geophysical hydrodynamics, the convection of a rotating fluid is described in terms of the thermal Rossby number $\mathrm{Ro}_{\mathrm{T}}$ and the Taylor number Ta , which, as applied to the model in question, can be defined as follows:

$$
\begin{equation*}
\mathrm{Ro}_{\mathrm{T}}=\frac{g a_{3}\left|\sigma_{\mathrm{BI}}\right|}{2 I_{3} \omega_{0}^{2}}, \quad \mathrm{Ta}=\frac{\omega_{0}^{2}}{\lambda^{2}} . \tag{10.6}
\end{equation*}
$$

In terms of these parameters,

$$
\begin{equation*}
D=\operatorname{sign}\left(\sigma_{\mathrm{B} 1}\right) \mathrm{Ro}_{\mathrm{T}} \mathrm{Ta}^{1 / 2} \zeta . \tag{10.7}
\end{equation*}
$$

Then, in the plane of external parameters $\left(\mathrm{Ta}, \mathrm{Ro}_{\mathrm{T}}\right)$, the above-mentioned lower boundary is the curve

$$
\begin{equation*}
\mathrm{Ro}_{\mathrm{T}}=(\zeta \Gamma \mathrm{Ta})^{-1 / 2}, \tag{10.8}
\end{equation*}
$$

which, in the quasigeostrophic approximation, coincides with the lower boundary of the stability region for the regimes $\mathrm{R}_{ \pm}$. This is not surprising. Another thing is surprising: this curve coincides with the asymptotic for the lower boundary of the existence and stability region of the laboratory Rossby regimes observed in annular channels, as determined theoretically by Lorenz [58] on the basis of a truncated two-layer model of the baroclinic flow.

A detailed investigation carried out in $\operatorname{Refs}[26,52]$ on the basis of the primary model equations (10.1) and corrected by our additional calculations shows that the existence and stability domains of Rossby regimes have shapes similar to that presented in Fig. 12, the top branch asymptotically


Figure 13. The experimental stability diagram for various convection regimes that are observed in annular channels with a fluid subjected to horizontally inhomogeneous heating [55]. The squared angular velocity of the general rotation, $\Omega^{2}$, is proportional to the Taylor number, and the parameter $\theta$ - the ratio of the horizontal density difference to $\Omega^{2}$ - is proportional to the thermal Rossby number.
behaving as $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{1 / 2}$. This does not mean that the existence and stability domains for the $\mathrm{R}_{+}$and $\mathrm{R}_{-}$regimes coincide. In particular, the solid curve in Fig. 12 reproduces the stability boundary for regime $\mathrm{R}_{-}$, whereas regime $\mathrm{R}_{+}$is stable not only inside the domain bounded by this curve but also in its outer vicinity.

The stability curve of the Rossby regimes in annular channels found by Lorenz in his above-cited work differs from the curve shown in Fig. 12 by the behavior of the top branch, which, according to Lorenz, asymptotically approaches a constant. It would be too bold to give a strict explanation to this distinction because the models are different. We only note that, unlike Lorenz, we did not use the quasigeostrophic approximation in constructing the top branch.

Now, comparing the theoretical stability boundaries of the Rossby regimes with the experimental critical curve presented in Fig. 13, we can see that, although all the curves are anvil-shaped, the lower branch of the experimental curve behaves as $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{-1}$ rather than as $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{-1 / 2}$. For the top branch, the experimental data are insufficient. The reason for this disagreement remained obscure until recently. We will return to this question in Section 10.3.

To conclude this section, let us note two points. First, the critical curve for regime $R_{+}$was not constructed because of the weak difference between regimes $\mathrm{R}_{+}$and H near the upper branch. Second, although regimes $R_{ \pm}$are equivalent in the quasigeostrophic approximation from the stability viewpoint, the transition $H \rightarrow R_{+}$dominates over $H \rightarrow R_{-}$in the framework of the original model.

### 10.3 The influence of the slope of the general rotation axis with respect to gravity

One can see from the experimental diagram in Fig. 13 that, in the domain bounded by the critical curve of the Rossby


Figure 14. Two orientations of the ellipsoid with respect to the nonparallel directions of the gravity and general rotation, which produce the same beta effect in the quasigeostrophic approximation, if $\varphi$ is small.
regimes, both strictly periodic and irregular auto-oscillations are observed along with stationary regimes. However, in the framework of the problem considered in Section 9, we failed to find self-oscillations in system (10.1). We overlooked the important factor that a global geophysical flow is nothing but the oblique convection of a rotating fluid developing in conditions where the axis of general rotation is not parallel to the gravity.

Let $\omega_{0}$ and $\mathbf{g}$ form an angle $\varphi$, as shown in Fig. 14 for two orientations of the ellipsoid with respect to the gravity and general rotation. The angle $\varphi$ is assumed to be so small that it does not affect the thermal wind relations (9.19a). To a certain extent, this imitates the situation related to the presence of the so-called beta effect on the Earth - the latitudinal dependence of the vertical projection of the Earth's angular velocity. The point is that the equations of atmospheric motion are formulated precisely in terms of this projection, but the derivation of the quasigeostrophic approximation neglects the dependence of the projection on the latitude in the thermal wind relations, being based on the latitudinal average.

Since $\varphi$ is small, the quasigeostrophic approximation of system (10.1) for both orientations can be written in terms of $X, Y$, and $Z$ as

$$
\begin{align*}
\dot{X} & =-Y Z-X-D \\
\dot{Y} & =X Z-Y  \tag{10.9}\\
\dot{Z} & =\Gamma X Y-\beta Y-\zeta Z
\end{align*}
$$

where $\beta=\beta_{0} \mathrm{Ta}^{1 / 2} \zeta$ and $\beta_{0}=2\left(a_{1} / a_{3}\right) \varphi$ or $\beta_{0}=2\left(I_{1} / I_{3}\right) \varphi$, depending on whether the orientation corresponds to Fig. 14a or 14 b .

$$
\text { At } \Gamma=0\left(a_{1}=a_{2}\right) \text {, the change of variables }
$$

$$
X=\frac{\zeta}{\beta} z-D, \quad Y=-\frac{\zeta}{\beta} y, \quad Z=x
$$

reduces system (10.9) to Lorenz's deterministic stochastic equations [22]:

$$
\begin{equation*}
\dot{x}=\zeta(y-x), \quad \dot{y}=-x z-y+r x, \quad \dot{z}=y x-b z \tag{10.10}
\end{equation*}
$$

where $b=1$ and $r=(\beta / \zeta) D=\operatorname{sign}\left(\sigma_{\mathrm{BI}}\right) \beta_{0} \zeta \mathrm{Ro}_{\mathrm{T}} \mathrm{Ta}$.
Therefore, under specific conditions discussed below, system (10.9) describes the stochastic regimes of 'toy' geophysical flows. System (10.9) and its irregular solutions were first obtained in 1980 [48], but, at that time, the investigation was restricted to studying these solutions without comparisons with the corresponding solutions of system
(10.1) and without a geophysical treatment of the results. In the same year Lorenz found that equations (10.10) describe not only Rayleigh convection but also a slow manifold in the reduced system of interacting Rossby and inertial gravitational waves.

Let us note that equations (10.9) and (10.10) are invariant with respect to the substitution $D \rightarrow-D, \quad \beta \rightarrow-\beta$ $\left(\sigma_{\mathrm{B} 1} \rightarrow-\sigma_{\mathrm{B} 1}, \varphi \rightarrow-\varphi\right)$. Thus, we define the beta effect to be positive (negative) if $D \beta>0$ or $\sigma_{\mathrm{B} 1} \varphi>0$ ( $D \beta<0$ or $\sigma_{\mathrm{B} 1} \varphi<0$ ). Since the beta effect violates the symmetry of the original force configuration, there are two different types of Rossby regime, depending on its sign. As before, we assume that $\sigma_{\mathrm{B} 1}<0$. Therefore, for a positive beta-effect, i.e., at $D<0$ and $\beta<0$, the Hadley and Rossby stationary regimes are described by the formulas $\dagger$

$$
\begin{aligned}
(\mathrm{H}) \quad X & =|D|, \quad Y=Z=0, \\
\left(\mathrm{R}_{ \pm}\right) \quad X & =D_{1} \equiv \sqrt{\frac{\beta^{2}}{4 \Gamma^{2}}+\frac{\zeta}{\Gamma}-\frac{|\beta|}{2 \Gamma},} \\
Y & = \pm D_{1}^{1 / 2}\left(|D|-D_{1}\right)^{1 / 2}, \\
Z & = \pm D_{1}^{-1 / 2}\left(|D|-D_{1}\right)^{1 / 2} .
\end{aligned}
$$

For a negative beta effect, i.e., at $D<0$ and $\beta>0$,

$$
\begin{aligned}
(\mathrm{H}) \quad X & =|D|, \quad Y=Z=0, \\
\left(\mathrm{R}_{ \pm}\right) \quad X & =D_{2} \equiv \sqrt{\frac{\beta^{2}}{4 \Gamma^{2}}+\frac{\zeta}{\Gamma}}+\frac{|\beta|}{2 \Gamma}, \\
Y & = \pm D_{2}^{1 / 2}\left(|D|-D_{2}\right)^{1 / 2}, \\
Z & = \pm D_{2}^{-1 / 2}\left(|D|-D_{2}\right)^{1 / 2} .
\end{aligned}
$$

Therefore, the equalities $|D|=D_{1,2}$ specify the lower boundaries of the existence domains of the Rossby regimes. In the plane of external parameters $\left(\mathrm{Ta}, \mathrm{Ro}_{\mathrm{T}}\right)$, these boundaries are described by the curves [according to (10.6) and to the corresponding expression for $\beta$ ]

$$
\begin{equation*}
\mathrm{Ro}_{\mathrm{T}}=\sqrt{\frac{\beta_{0}^{2}}{4 \Gamma^{2}}+\frac{1}{\zeta \Gamma \mathrm{Ta}}} \mp \frac{\left|\beta_{0}\right|}{2 \Gamma}, \tag{10.11}
\end{equation*}
$$

which behave asymptotically, at $\mathrm{Ta} \rightarrow \infty$, as $\mathrm{Ro}_{\mathrm{T}}=$ $\left(\zeta\left|\beta_{0}\right| \mathrm{Ta}\right)^{-1}$ and $\mathrm{Ro}_{\mathrm{T}}=\beta_{0} / \Gamma=$ const for positive and negative beta effects, respectively.

It is easy to show that, in the framework of the reduced equations (10.9), formulas (10.11) also describe the critical curves for regimes H and R . It can be seen from a comparison between (10.8) and (10.11) that the beta effect destabilizes or stabilizes the H regime depending on the sign of the beta effect.

The results of our numerical investigation of the stability of regimes H and $\mathrm{R}_{-}$on the basis of the unreduced equations of motion (10.1) are shown in Fig. 15 for $\zeta=1$ and both positive and negative beta effects; Fig. 16 for $\zeta=3$ and $\sigma_{\mathrm{B} 1} \varphi>0$; and Fig. 17 for $\zeta=0.4$ and $\sigma_{\mathrm{B} 1} \varphi<0$. The calculations were carried out at principal semiaxes of $a_{1}=3$, $a_{2}=2$, and $a_{3}=1$ and at an angle of $|\varphi|=1^{\circ}$ between $-\mathbf{g}$ and $\boldsymbol{\Omega}_{0}$. For each of these examples, the asymptotics of the lower
$\dagger$ In the case under consideration, the term 'beta effect' should be regarded as a conventional term not corresponding to the generally used term. It should be replaced with 'sloping effect'. (Translator's note.)


Figure 15. The critical curves of the $\mathrm{R}_{-}$regime for a positive $(2,3)$ and negative $(1,4)$ beta effect at $|\varphi|=1^{\circ}$ and $\zeta=1$. The dashed lines refer to the corresponding quasigeostrophic critical curves.


Figure 16. The critical curves of the $\mathrm{R}_{-}$regime, periodic (C), and stochastic (L) regime at $\zeta=3$ for a positive beta effect ( $\sigma_{\mathrm{B} 1} \varphi>0,|\varphi|=1^{\circ}$ ). Symbol L means that the chaos is similar to that of the Lorenz attractor.
critical branch agrees with the corresponding 'quasigeostrophic' stability curve. Contrary to our expectations, the quasigeostrophic equations of motion satisfactorily describe or, at least, reflect the principal features of the exact phase trajectories even at 'ageostrophic' (insufficiently small) $\mathrm{Ro}_{\mathrm{T}}$ provided that $\mathrm{Ta} \gtrsim 50$.

In the domains bounded by the critical curves of the regime R_shown in Fig. $15(\zeta=1)$, we have found no selfoscillations. The figure illustrates the influence of the beta effect on the stability of stationary Rossby regimes. The critical curve displaces down or up, and the decay rate of the lower branch increases or decreases depending on whether $\sigma_{\mathrm{B} 1} \varphi>0$ of $\sigma_{\mathrm{B} 1} \varphi<0$. However, the growth of the upper branch slows down noticeably under the action of any nonzero beta effect. It is worth noting that the beta effect favors the transition $\mathrm{H} \rightarrow \mathrm{R}_{-}$, which dominates under certain conditions.

Curve $1-1^{\prime}$ in Fig. 16 separates the stability domains of stationary H and $\mathrm{R}_{-}$regimes. A chaotic regime related to


Figure 17. The critical curves of the $\mathrm{R}_{-}$regime, periodic (C), and stochastic (X) regimes at $\zeta=0.4$ for a negative beta effect ( $\sigma_{\mathrm{B} 1} \varphi<0,|\varphi|=1^{\circ}$ ). The symbol X means that the origin of chaos is not known.

Lorenz's attractor is observed in domain L bounded with curves 2-2' and 3-3'. Note that the dependence $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{-1}$ corresponding to branch 2-2' was found numerically and has no theoretical explanation. At $\zeta>1$, the critical curves move into the region of small Rossby numbers, which favors the mutual approach of exact and quasigeostrophic phase trajectories. In particular, Fig. 18a illustrates the proximity of the exact and quasigeostrophic stochastic attractors, which is typical of the overwhelming part of domain $L$.

The following observations were made in the numerical simulation of chaos in domain $L$.
(1) The quasigeostrophic model (10.9) describes well the established regimes, but the exact and quasigeostrophic transients can be substantially different. Furthermore, the well-established exact and quasigeostrophic trajectories differ substantially in phase.
(2) As a rule, the 'exact' chaos is accompanied by a quasigeostrophic chaos, whereas the inverse is not true. In particular, the quasigeostrophic chaos persists in the outer vicinity of branch $3-3^{\prime}$.
(3) Sometimes the $\mathrm{R}_{-} \rightarrow \mathrm{L}$ transition means a passage into a metastable state. The Lorenz attractor persists for a long time, after which it slowly degenerates into a $\mathrm{R}_{+}$regime, i.e., a double transition $\mathrm{R}_{-} \rightarrow \mathrm{L} \rightarrow \mathrm{R}_{+}$is realized. The situation is typical of the neighborhood of curve 2-2', which makes its construction difficult. If the period of general rotation is assumed to be a day, the metastable state can last from several months to several decades. It is worth emphasizing that $\mathrm{R}_{+}$and L regimes coexist, i.e., both are stable in domain L or, at least, one of them is metastable.

No self-oscillations are observed in the numerical experiments at a negative beta effect and $\zeta>1$. This is not too surprising, since, at $\zeta>1$, the quasigeostrophic model satisfactorily describes the motion of a baroclinic top in the overwhelming part of the stability domain of Rossby regimes and the sign reversal of the beta effect implies the sign reversal of the coefficient $r$ in Lorenz's system (10.10). According to the original Lorenz's treatment, this system describes the Rayleigh convection of fluid heated from below and $r$ is the Rayleigh number. A sign-reversed $r$ means heating from above. In this case, the onset of self-oscillations can hardly be expected.


Figure 18. (a) The phase portrait of 'quasigeostrophic' and exact Lorenz chaos in the space ( $X, Y, Z$ ) for a positive beta effect, $\zeta=3, \mathrm{Ta}=450$, and $\mathrm{Ro}_{\mathrm{T}}=0.2$. (b) The phase portrait of the self-oscillations in the outer vicinity of curve $3^{\prime}-3$ (see Fig. 16) at $\zeta=3, \mathrm{Ta}=450, \mathrm{Ro}_{\mathrm{T}}=1$. (c) The spectral diagrams for the case presented in Fig. 18b. The heavy curves correspond to the quasigeostrophic approximation and the light curves to the exact solution.

The situation changes if $\zeta<1$ with the beta-effect remaining negative. This is illustrated in Fig. 19, which corresponds to $\zeta=0.4$ and $\sigma_{\mathrm{B} 1} \varphi<0\left(|\varphi|=1^{\circ}\right)$. In this case, the critical curve shown in Fig. 17 is shifted into the region of 'ageostrophic' Rossby numbers. Therefore, the quasigeostrophic equations (10.9) work only in the vicinity of the lower critical branch. Regular self-oscillations and chaos of an unknown origin emerge in domains C and X (see Fig. 17), respectively, and they are described only by the unreduced equations.

To summarize this section, it is worth noting two points. First, which is most important, under the influence of the beta effect, the asymptotics of the lower branch of the critical curve $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{-1 / 2}$ is replaced with the asymptotics $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{-1}$ or $\mathrm{Ro}_{\mathrm{T}}=$ const, depending on the positiveness or negativeness of the beta effect, respectively. These changes emerge at $|\varphi|=1^{\circ}$ and at quite realistic values of the Taylor number, typical of the majority of the laboratory experiments discussed in Refs [53, 55] (see also the diagram in Fig. 13). The experiments were carried out with rotating annular containers filled with a fluid with a free surface. The deformation of the free surface under the influence of the centrifugal force produces a positive beta effect corresponding to small but finite $\varphi$ values. The smallness of this angle seemingly prompted Lorenz [58, 59] to neglect the beta effect due to the centrifugal force in constructing his 8 - and 12 -component models. This led to the asymptotics $\mathrm{Ro}_{\mathrm{T}} \sim \mathrm{Ta}^{-1 / 2}$. Our investigation shows that taking into


Figure 19. The phase portrait of stochastic regime $X$ (a) and its spectral diagram (b) at $\zeta=0.4, \mathrm{Ta}=3200, \mathrm{Ro}_{\mathrm{T}}=4.5$.
account the small slope of $\boldsymbol{\Omega}_{0}$ with respect to $-\mathbf{g}$ gives a result consistent with experiment, which was noted by A E Gledzer [60] (see Fig. 13).

Second, it is the beta-effect that induces regular and chaotic self-oscillations in the model under consideration, which are really observed in laboratory experiments (see Fig. 13). To discover and describe them, Lorenz constructed a 12 -component system instead of the 8 -component one. The degree of consistency between the self-oscillations found here and reality remains to be evaluated, which requires, in particular, a more accurate determination of the coefficients of friction, $\lambda$, and heat conductivity $\mu$, corresponding to the experiments. However, the very fact of the onset of low-order turbulence in global geophysical flows due to the beta effect seems to be important for understanding the nature of their unpredictability. A detailed and more dedicated discussion of these questions is given in Ref. [60].

## 11. Conclusion

The revealed analogies between the fundamental mechanical and hydrodynamic invariants indicative of the generality of their origin will, I hope, be fairly satisfying to hydrodynamicists and theoretical physicists, irrespective of practical applications. This refers to both the quasigeostrophic approximations for a baroclinic top, which coincide (in the specific cases mentioned in Sections 9 and 10) with the mechanical Euler equations, and to the well-known Lorenz system that describes low-order turbulence. They are formulated in terms of the principal characteristics of the global geophysical flows and, therefore, can be regarded as 'toy' models of the general circulation of the inviscid and viscous atmospheres, respectively, that share common symmetries with the real atmosphere.

The observations made can be conceptually useful for the interpretation of hydrodynamic phenomena using the fundamental invariants of motion and for the construction of
simplified hydrodynamic models retaining the principal symmetry of the primary hydrodynamic equations. In this respect, Theorem 3 is of some interest. It has recently been recognized that the helicity of the velocity field plays an essential, if not decisive, part in the development of tornados and tropical cyclones. In view of the complexity of these meteorological objects and of the difficulty of their description, a temptation arises as to construct a finite-dimensional model of the hydrodynamic Euler equations describing the nonstationary motion of an ideal homogeneous fluid with a nonzero helicity. According to Theorem 3, there are no timedependent solutions of the equations of motion of an ideal homogeneous fluid strictly described by a finite-dimensional dynamical system that belongs to the GRB class and has a nonzero helicity invariant. In this context, Professor V I Yudovich expressed in a private discussion "a very strong suspicion that the group $\operatorname{SDiff} D$ does not have totally geodesic subgroups other than those generated by spatially linear fields." To a certain extent, I share this suspicion. However, as long as this is nothing more than a suspicion, Theorem 3 is not useless.

In this review, we tried to show that global flows really 'inherit' fundamental features of mechanical motions. From this viewpoint, a top in gravitational and Coriolis force fields can be regarded as a mechanical prototype of rotating planetary atmospheres and their laboratory analogs: it reproduces motions analogous to Rossby waves, the approximate invariance of the vertical vorticity of a homogeneous incompressible rotating fluid, Eady's mechanism of baroclinic instability, the energetic cycle and stability domain of the fundamental Hadley and Rossby regimes, and the low-order turbulence and unpredictability of global geophysical flows. In essence, this means that these properties of geophysical flows could be predicted based on the analysis of motions of the barotropic and baroclinic tops. A sceptical reader would say that this forecast has been done post factum. This is correct or almost correct. Nevertheless, we succeeded in explaining an experimental result related to the lower branch of the critical curve of the Rossby regime and disagreeing with theory. In addition, we hope that the results of Section 10 could be of interest to climatologists. The point is the possibility of the coexistence of essentially different regular and stochastic - regimes of motion and the nonzero probability of spontaneous transitions from one metastable regime to another.

What has been said also gives the hope that system (7.22)-(7.24) with a general rotation taken into account could be used to predict fundamental features of global motions in the Earth's inner liquid core and on the Sun and other stars, whose hydrodynamics has not yet been adequately studied. The hydrodynamic helicity can be taken into account by combining spatially linear fields with the fields describing Hill's vortex [61] (see also more easily accessible publications [62, 63]).

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[^1]:    $\dagger$ In the book [8], Theorem 1 is formulated as follows: Each solution $m(t)$ of the Euler equation belongs to the same coadjoint orbit for all $t$. In other words, the group coadjoint orbits are invariant submanifolds for the flow of the Euler equation in the dual space $\hat{g}^{*}$ to the Lie algebra. (Translator's note.)

[^2]:    $\dagger$ The density of a fluid element is conserved. (Translator's note.)

