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Gauge theories as string theories: the first results

A S Gorsky

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Abstract. **The gauge/string theory duality in curved space is discussed mainly using a non-Abelian conformal $N = 4$ supersymmetric gauge theory and the theory of a closed superstring in the $AdS_5 \times S^5$ metric as an example. It is shown that in the supergravity approximation, string duality yields the characteristics of a strong-coupling gauge theory. For a special shape of the contour, a Wilson loop expression is derived in the classical superstring approximation. The role of the hidden integrability in lower-loop calculations in gauge theory and in different approximations of string theory is discussed. It is demonstrated that in the large quantum-number limit, gauge theory operators can be described in terms of the dual string picture. Examples of metrics providing the dual description of gauge theories with broken conformal symmetry are presented, and formulations of the vacuum structure of such theories in terms of gravity are discussed.**

A S Gorsky Russian Federation State Scientific Center 'Alikhanov Institute for Theoretical and Experimental Physics', B. Cheremushkinskaya ul. 25, 119218 Moscow, Russian Federation
Tel. (7-095) 129 94 93. Fax (7-095) 127 08 33
E-mail: gorsky@itep.ru

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1. Introduction

The term 'duality' has a rather long history. It came into being in the theory of strong interactions when Veneziano proposed describing hadron scattering amplitudes by formulas having symmetries with respect to permutation of the s- and t-channels. Such a symmetry implied the possibility of string interpretation, but it took the qualitative picture almost thirty years to acquire a clear quantitative pattern. These years have brought several new types of duality that establish links between varieties of string theory or the theory of strings propagating in different background fields.

In parallel, the idea of duality has been developing in the context of field theory, which makes it possible, for example, to connect theories in the 'electric' and 'magnetic' formulations. Therefore, it needs to be definitively stated what type of duality is being discussed. The subject of the present review is to formulate the duality between non-Abelian gauge theories and the theory of a closed string propagating in a curved space.

The idea of the gauge/string theory duality is most profound in the realm of fundamental interactions. It is intended to formalize the relation between string theory and physically interesting field theories. A qualitative program involving many key ideas was formulated by Polyakov in the 1980s and further elaborated afterwards.

After the duality hypothesis had been quantitatively formulated in Refs [1 – 3], a number of new results, including some unexpected ones, were obtained in field theory and in string theory. In what follows, we try to formulate the ideas underlying the notion of duality, confining ourselves to a minimum of necessary technical details. Our principal objective is to illustrate general ideas by concrete examples of duality amenable to quantitative verification. We also present examples of predictions ensuing from duality for gauge theories in the strong-coupling regime.

During the period since 1997, almost 4,000 publications have appeared concerning various aspects of the gauge/string theory duality. It is therefore impossible to fully reflect all interesting results obtained in recent studies. We hope that the list of references below includes quite enough specialized reviews from which the reader can derive information on purely technical aspects of the problem.

The most optimistically inclined researchers see string theory as the 'theory of everything,' but the duality of gauge and string theories dictates the necessity of a more moderate interpretation. According to the duality hypothesis, the Yang – Mills theory in four dimensions (supersymmetric in the simplest case) is equivalent to the theory of a superstring propagating in a nontrivial geometry in ten dimensions.

The initial idea suggesting such a relationship stems from the duality between open and closed strings that, for example, relates closed string propagation in the tree approximation to the one-loop amplitude in the open string theory. Because the open string has a massless gauge boson and the closed string a massless graviton in their respective spectra, it is natural to expect duality between gauge and gravity theories. However, the closed/open string duality proves insufficient to formulate the duality between gauge theory and string theory and requires the use of additional data obtained during the last decade.

An important part is played by D-branes, which are soliton-like objects of string theory, with a nontrivial field theory defined on their world surface. It is important that an open string can end on a D-brane [4] and its massless gauge mode generates an Abelian gauge field on the brane world surface.

Moreover, the $\mathrm{U}(N_c)$ non-Abelian gauge theory is generated on the world surface of N_c coincident branes [5]. This gives a tool that allows obtaining a gauge field theory at the world surface of the branes with 'predetermined properties.'

In considering the role of D-branes in the closed string theory, it should be borne in mind that any massive object deforms the metric around itself; the D-brane as an object with a nonzero tension is no exception. Therefore, a closed string emanated by a brane propagates in a nontrivial geometry determined by the brane itself. In what follows, we concentrate on the large-N limit of parallel branes and show that the metric is markedly simplified in this limit. Moreover, p-branes are sources of $(p+1)$-form fields, in analogy with the charged point-like particle that serves as a source of a 1-form field, i.e., the vector electromagnetic potential. For this reason, closed strings feel not only metrics but also brane-generated fields of higher forms. It appears that one can, in a sense, forget about branes proper and study closed string dynamics in an external gravitational field corresponding to a large number of branes and in an external p-form field. We try to show that the most difficult part of the task is to find a geometry consistent with the gauge field theory.

Now, an excursion through history is in order. Polyakov suggested the idea of the gauge fields/string theory duality in a higher-dimensional space rather long ago and further developed it over a few decades. An important step was the understanding of the role of one of the 'additional' coordinates (supplementing the four usual ones) as a renormalization-group scale in the four-dimensional field theory [6]. The idea was further promoted by Klebanov [7], who demonstrated the possibility of self-consistently taking into consideration the back reaction of brane-emitted closed strings on the gauge theory at the brane world surface.

These developments culminated in the work of Maldacena [1], who hypothesized that a closed string propagating in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry and in an external 4-form field of constant strength was dual to the gauge theory with $N = 4$ supersymmetries, maximally possible in four dimensions. In Maldacena's original formulation, importance was assigned to massless string modes, that is, string theory was effectively reduced to the theory of supergravity. Also, the holographic principle was implicitly assumed [8] because the gauge theory was formulated at the boundary of AdS_5 and supergravity in the entire ten-dimensional space.

Very soon after the publication of Maldacena's work, it was recognized [2, 3] that the action evaluated on a solution of classical equations of motion in supergravity at fixed values of the supergravity fields is a generating functional for correlators in the gauge theory at the boundary. Fields in the correlators originate at the boundary values of the respective modes from the supergravity Lagrangian. In the course of time, metrics and fields were found whose form determined the string geometry of a dual gauge theory with lower supersymmetry [9 – 11]. Moreover, an example of the geometry was described in which string theory was exactly solvable; its detailed comparison with the corresponding sector of the gauge theory [12] confirmed the validity of the dual description.

This review is organized as follows. First, the necessary notions are introduced and the hypothesis of Maldacena for the $N = 4$ supersymmetric Yang – Mills gauge theory is formulated. In Section 3, duality for the $N = 4$ theory is examined in the supergravity approximation; also, it is shown how correlators in gauge theory can be calculated with the aid of solutions of the classical equations of motion in supergravity. By way of a beautiful illustration of duality in this approximation, it is shown how the universal behavior of viscosity in the strong-coupling region can be predicted in the hydrodynamic regime of gauge theory. In Section 4, we discuss a regime where the classical string approximation holds, and a Wilson loop with the circular geometry is computed in the $N = 4$ theory. In this geometry, computation with the aid of a classical string can be compared with the explicit computation of Feynman diagrams in the case of an arbitrary coupling constant.

Section 5 is concerned with predictions of duality for the matrix of anomalous dimensions of operators with large quantum numbers in a supersymmetric gauge theory. It turns out that eigenvalues of the matrix of the anomalous dimensions of such operators coincide with the classical energy of a string rotating with the corresponding angular momenta in $\mathrm{AdS}_5 \times \mathrm{S}^5$. In this case, the key role is played by the hidden integrability that reflects a high degree of system symmetry.

The integrability can be explicitly demonstrated in the lower orders of the perturbation theory in gauge theory and also for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the classical approximation. The next section presents a brief discussion of the limit where $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry reduces to the so-called *pp*-wave limit, in which it is possible to exactly find a quantum string spectrum coincident with the anomalous dimensions of the respective gauge-theory operators, in the lower orders of the perturbation theory. In this limit, it is possible to exactly identify the Hilbert spaces of the dual theories.

The following two sections are devoted to theories with smaller ($N = 2$ and $N = 1$) supersymmetries, which are more reminiscent of the realistic theory of strong interactions. It is shown that the beta-function of the $N = 2$ theory can be obtained in the framework of the supergravity approximation, and a deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry into a dual geometry of $N = 2$ gauge theory can be explicitly described. Further distortion of the dual geometry for the $N = 1$ supersymmetric theory is studied in Section 8, where it is shown that the main nonperturbative phenomena (a finite number of vacuum states, gluino condensate, exact beta-function) can be described in the dual string theory in the supergravity approximation. Section 9 features some data concerning nonsupersymmetric theories.

The literature on the duality between gauge and string theories is abundant and its full spectrum cannot be covered in a short review. We therefore refer the reader to the comprehensive bibliography and necessary introductory materials contained in the specialized reviews cited below. An introduction to the duality under consideration with a large number of relevant examples can be found in reviews [13, 14]. The calculation of Wilson loops is described in detail in [15] and various aspects of the hidden integrability in Refs [16–18]. Duality in the supergravity approximation for a string in the $N = 2$ theory is discussed at length in Ref. [11] and in the $N = 1$ theory in Refs [19, 20]. The exactly solvable string limit in the geometry of a *pp*-wave and its relation to the sector of special operators in the $N = 4$ gauge theory can be found in Ref. [21].

2. Major elements of duality

Duality was first formulated for the conformal $N = 4$ supersymmetric gauge theory with zero beta-function. The fields of the theory include a gauge boson and six real scalar fields Φ_i and their fermionic superpartners. All these fields are in the adjoint representation of the gauge group $\mathrm{SU}(N_\mathrm{c})$, and the global SO(6) symmetry corresponds to the rotation of scalar fields in a space. The theory incorporates an infinite number of vacuum states, the moduli space of vacua, whose point is parameterized by vacuum values of the scalar fields.

The action of the theory in components has the form

$$S_{N=4} = \frac{1}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4 x \, \mathrm{Tr}\left[F_{\mu\nu}^2 + (D\Phi_i)^2 + [\Phi_i, \Phi_j]^2\right] + \text{fermions} . \tag{1}$$

The duality hypothesis implies that the gauge theory is dual to the type-IIB closed superstring in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric [1]. The metric of the ten-dimensional space in the Poincaré coordinates is given by

$$\mathrm{d}s^2 = \frac{r^2}{R^2}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) + \frac{R^2}{r^2}\,\mathrm{d}r^2 + R^2\,\mathrm{d}\Omega_5^2 , \tag{2}$$

where r is the radial coordinate in AdS_5 and the last term corresponds to the S^5-geometry. This metric is actually a metric generated by a D3-brane in a region close to the horizon.

As mentioned above, the D3-brane is a source of the 4-form field A_4; therefore, the external metric in which the string propagates should be supplemented by a flux of the 4-form field strength, which, in accordance with duality, coincides with the rank of the gauge group:

$$F_5 = dA_4 , \qquad \int_{\mathrm{S}^5} * F_5 = N_\mathrm{c} . \tag{3}$$

The radii of AdS_5 and S^5 coincide and are equal to

$$R^4 = 4\pi g_\mathrm{s} \alpha'^2 N_\mathrm{c} , \tag{4}$$

where g_s is identified with the string coupling constant.

The gauge theory is localized at the boundary of AdS_5, with the conformal group in four dimensions, $\mathrm{SO}(2,4)$, and the R-symmetry group of the $N = 4$ theory, SO(6), being identified with the space isometries of AdS_5 and S^5, respectively. Six additional coordinates in the ten-dimensional space are identified with the vacuum values of the six real scalar fields of the gauge theory.

The key feature that allows duality to be verified at the quantitative level is the coincidence of the dilatation operator in the gauge theory and the string Hamiltonian in radial quantization. Thus, eigenvalues of the dilatation operator that determine the anomalous dimensions of operators in field theory coincide with the string energy calculated in the corresponding solutions of the equations of motion.

The next step must be the identification of parameters in the dual theories. The gauge theory contains the coupling constant g_{YM} and the rank of the gauge group N_c, and string theory involves the string tension T, the string coupling constant g_s, and the radii specifying the curvatures of external geometry.

The effective dimensionless tension of the string is related to the radii by the equation

$$T = \frac{R^2}{2\pi\alpha'} , \tag{5}$$

in other words, the key relations have the form

$$\begin{aligned} &4\pi g_\mathrm{s} = g_{\mathrm{YM}}^2 , \\ &T = \frac{1}{2\pi}(g_{\mathrm{YM}}^2 N_\mathrm{c})^{1/2} = \frac{1}{2\pi}\,\lambda^{1/2} . \end{aligned} \tag{6}$$

Thus, at an arbitrary coupling constant in the gauge theory, the dual description deals with a quantum string propagating in a complex metric and in an external 4-form field. Such a theory is very difficult to analyze, and no acceptable quantum version of it has been proposed thus far. Nevertheless, it is amenable to qualitative analysis in different limits, and this circumstance is extensively used below. In particular, it is possible to consider the limit as $g_\mathrm{s} \to 0$ at a fixed tension, when the classical string approximation is applicable. If we also assume that $T \to \infty$, only massless modes survive in string theory and it is effectively reduced to the zero-mode theory or supergravity.

In what follows, we encounter situations in which one limit or another proves effective. We show that many phenomena in the strong-coupling regime of gauge theory are successfully described in the supergravity approximation. Calculations of the Wilson loop and the anomalous dimensions of operators in gauge theory are convenient to compare with calculations in the classical limit of string theory. We briefly consider the *pp*-wave limit for the metric in which the quantum string spectrum is exactly calculated and compare it with the anomalous dimensions of gauge-theory operators.

3. Supergravity approximation

We consider the limit of the large 't Hooft constant as $T \to \infty$ and all massive modes of the string decouple. It is such a supergravity limit that was considered by Maldacena in his pioneering work. In this limit, the string sigma-model is substituted by the classical IIB supergravity action

$$\begin{aligned} &Z(G_{\mu\nu}, B_2, \Phi, A, C_2, A_4) \\ &\quad = \int d^{10}x\,(-\det G)^{1/2} \exp(-2\Phi) \\ &\quad \times \left[R + 4G^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{12}\,H_3^2\right] \\ &\quad -\frac{1}{2}\int d^{10}x\,(-\det G)^{1/2}\left[G^{\mu\nu}\,\partial_\mu A\,\partial_\nu A + (F_3 - AH_3)^2\right. \\ &\quad \left. + \left(F_5 - \frac{1}{2}\,C_2 \wedge H_3 + \frac{1}{2}\,B_2 \wedge F_3\right)^2\right] \\ &\quad +\frac{1}{2}\int d^{10}x\,A_4 \wedge H_3 \wedge F_3 + \text{fermions}\,, \end{aligned} \tag{7}$$

including the metric $G_{\mu\nu}$, two scalar fields Φ and A, two 2-form fields $C_{\mu\nu} = C_2$ and $B_{\mu\nu} = B_2$ with tensions F_3 and H_3, and the 4-form field $A_{\mu_1,\mu_2,\mu_3,\mu_4} = A_4$ with the tension F_5. In what follows, it is assumed that the self-duality condition is imposed on F_5.

The first qualitative check of duality was undertaken in Refs [2, 3], where correlators in the $N = 4$ gauge theory were compared with solutions of the classical equations of motion in supergravity. It was shown that the supergravity action evaluated on the classical solution with given boundary data is a generating function for correlators in the gauge theory:

$$\begin{aligned} &\left\langle \exp\left(\sum \phi_k O_k\right)\right\rangle_{N=4} \\ &\quad = \exp\left\{-S_{\text{cl}}^{\text{sugra}}\left[\phi_k(x,z) \to \phi_k(x)\right]\right\}. \end{aligned} \tag{8}$$

Here, O_k is the operator in the gauge theory interacting with the supergravity field $\phi_k(x,z)$ and taking the value $\phi_k(x)$ at the boundary of the AdS_5 space.

3.1 $N = 4$ gauge theory in the supergravity approximation; calculation of correlators

We investigate the simplest example of the calculation of a gauge-theory correlator in the dual supergravity theory. We consider the dilaton field Φ in the background metric of the AdS_5 space with the action

$$S(\Phi) = \text{const}\cdot\int d^4x\,dz\,\frac{1}{z^3}\left[(\partial_z\Phi)^2 + (\partial_m\Phi)^2\right], \tag{9}$$

where $m = 1, \ldots, 4$; in the metric

$$ds^2 = \frac{R^2}{z^2}(dz^2 + dx_m^2)\,, \tag{10}$$

the boundary is at $z = 0$. The action evaluated on classical solutions that are regular at the boundary and decrease at large z diverges [2, 3], which suggests the introduction of an infrared cut-off in AdS_5 at $z = \epsilon$.

The normalizable solution of the equations of motion for the dilaton field with the boundary condition

$$\Phi(z = \epsilon, x) = \exp(ikx) = \Phi_0(x)$$

is given by

$$\Phi(x_m, z) = \frac{(kz)^2 K_2(kz)}{(k\epsilon)^2 K_2(k\epsilon)}\exp(ik_m x_m)\,, \qquad k = (k_m^2)^{1/2} \tag{11}$$

(K_2 is the modified Bessel function). It is equally easy to evaluate the action on the solution:

$$S \propto N\int d^4x\int d^4y\;\Phi_0(x)\,\Phi_0(y)\,\frac{1}{\left(\epsilon^2 + |x_m - y_m|^2\right)^4} + O(\epsilon^2)\,. \tag{12}$$

We compare this result with the correlator computed in the gauge theory. Because the dilaton interacts with the operator $\mathrm{Tr}\,F^2$, the generating function is given by

$$Z(\Phi_0) = \left\langle \exp\left(\frac{i}{g_{\text{YM}}^2}\int d^4x\,\Phi_0(x)\left[\mathrm{Tr}\,F^2 + \ldots\right]\right)\right\rangle, \tag{13}$$

where the dots denote the contribution of superpartners and averaging is made with the aid of the standard functional integral of the $N = 4$ theory. In the quadratic approximation in the dilaton field, we have

$$\begin{aligned} &Z(\Phi_0) \\ &\quad \propto \exp\left(-ai\int d^4x\,d^4y\;\Phi_0(x)\,\Phi_0(y)\left\langle \mathrm{Tr}\,F^2(x)\,\mathrm{Tr}\,F^2(y)\right\rangle\right), \end{aligned} \tag{14}$$

where $a = \text{const}$.

Conformal invariance of the theory uniquely fixes the two-point functions

$$\left\langle \mathrm{Tr}\,F^2(x)\,\mathrm{Tr}\,F^2(y)\right\rangle \propto \frac{N_c^2}{|x_m - y_m|^8}\,. \tag{15}$$

It can be seen that calculations in supergravity and gauge theory actually coincide if regularizations in the dual theories are carefully correlated. The introduction of the ultraviolet regularization parameter η_{UV} at $x_m = y_m$ and the assumption of $\eta_{\text{UV}} = \epsilon$ lead to the equality predicted by the duality hypothesis. The above example was generalized for a series of correlators of more sophisticated operators, and the duality hypothesis proved valid in all these cases.

3.2 $N = 4$ gauge theory in the supergravity approximation: viscosity in the hydrodynamic regime

An interesting and somewhat unexpected application of the duality between gauge theories and strings to the calculation of hydrodynamic characteristics of gauge theory in the strong-coupling regime was proposed in Ref. [23]. It turned out that the ratio of the viscosity to the entropy density (a macroscopic characteristic of the hydrodynamic system) can be calculated in the dual supergravity theory. Solutions of the black-hole type in the anti-de-Sitter space were considered as gravitational solutions consistent with field theory at nonzero temperature, with the expression for the sought ratio having a form that depends on universal constants alone.

We find a metric responsible for the dual description of the gauge theory at nonzero temperatures. An important role is played in what follows by the five-dimensional part of the total geometry, which was identified with the black-hole metric in AdS_5 in Ref. [22]:

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\left[-\left(1-\frac{r_0^4}{r^4}\right)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] + \frac{R^2}{1-r_0^4/r^4}\,\mathrm{d}r^2 \,. \tag{16}$$

We note that this metric may be regarded as a metric generated by the brane configuration in the ten-dimensional space.

The temperature in field theory coincides with the Hawking black-hole temperature and the entropy in the field theory coincides with the event horizon area

$$S = \frac{A}{4G}\,, \tag{17}$$

where G is the gravitational constant. It is natural to consider the entropy density, obtained by means of division by an infinite factor corresponding to the volume of a space parallel to the horizon.

Viscosity is computed using the Kubo formula in terms of the equilibrium correlation function of the energy–momentum tensor components T_{xy} in the supersymmetric gauge theory:

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega}\int \mathrm{d}t\,\mathrm{d}\mathbf{x}\,\big\langle T_{xy}(t,\mathbf{x}),\ T_{xy}(0,0)\big\rangle\,\exp(\mathrm{i}\omega t)\,. \tag{18}$$

The optical theorem permits us to relate the correlator to the cross section of absorption of a graviton propagating normally to the brane on which the four-dimensional field theory is defined and polarized in the xy plane [7]:

$$\eta = \frac{\sigma_{\mathrm{abs}}(\omega=0)}{16\pi G}\,. \tag{19}$$

On the other hand, it can be shown that the graviton absorption cross section in the given problem coincides with the scalar absorption cross section and, in the low-energy limit, depends on the geometric area of the horizon. Thus, for the ratio of interest, the following equality holds [23]:

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_{\mathrm{B}}}\,. \tag{20}$$

Surprisingly, the answer is universal. Moreover, it was demonstrated that it is independent of the metric characteristics in the dual theory provided the event horizon is present.

The universality of the answer made it possible to hypothesize that the result obtained in the supersymmetric theory actually represents the lower boundary for the ratio in any relativistic field theory at a finite temperature and zero chemical potential. Corrections to this value obtained in nonconformal theories confirm the hypothesis [24], but its status in the general case remains to be elucidated. It is worthwhile to mention an optimistic view expressed by certain authors that this characteristic may prove very helpful in the studies of collisions between heavy ions, where its universality can be verified.

4. $N=4$ gauge theory and classical string: calculation of the Wilson loop

In the limit where explicit quantitative calculations are feasible, the string tension remains finite and the string constant is small; in other words, the classical approximation for strings is valid. Therefore, in solving the classical equations in the sigma-model on $AdS_5\times S^5$ with the chosen boundary conditions on the string world surface, it is possible to find quantities of interest in the dual gauge theory in the strong-coupling regime.

As an example illustrating the efficiency of this approximation, we consider the computation of the Wilson loop in the $N=4$ gauge theory with a circular geometry. The Wilson loop is naturally associated with the world line of the W-boson. It is convenient to examine the case where the $SU(N+1)$ gauge symmetry is broken to $SU(N)\times U(1)$ and the vacuum average of the scalar field that determines the scale of this violation is sufficiently large. In this case, the W-boson is a heavy object.

By virtue of supersymmetry, the phase factor of the heavy boson includes contributions by the vector and scalar fields:

$$W(C) = \frac{1}{N}\,\mathrm{Tr}\,P\exp\left[\oint \mathrm{d}\tau\,\big(\mathrm{i}A_\mu(x)\,\dot{x}^\mu + \Phi_i(x)\,\theta^i|\dot{x}|\big)\right]. \tag{21}$$

Here, C is the closed contour parameterized by $x^\mu(\tau)$ and θ^i is the unit vector in the 'internal' space in the direction of the symmetry breaking. We show that the vacuum average of the circular Wilson loop in the strong-coupling regime coincides with the result in the dual classical sigma-model.

In dual theories,

$$S = \exp\big[-ML(C)\big]\big\langle W(C)\big\rangle \tag{22}$$

can be calculated in the limit of the large W-boson mass M. The string theory considers the sigma-model action calculated on the string world surface whose boundary coincides with the Wilson loop at the boundary of AdS_5. The path integral is represented in the form

$$S = \int \mathrm{D}X^\mu\,\mathrm{D}Y^i\,\mathrm{D}h_{ab}\,\mathrm{D}\Theta^\alpha \times \exp\left(-\frac{\lambda}{4\pi}\int_D \mathrm{d}^2\sigma\,h^{1/2}\,\frac{h^{ab}}{Y^2}\,(\partial_a X^\mu\,\partial_b X^\mu + \partial_a Y^i\,\partial_b Y^i)\right) + \text{fermions}\,, \tag{23}$$

where X and Y are coordinates in the ten-dimensional space in which the sigma-model is defined.

At a large string tension, the functional integral is computed semiclassically and the saddle-point value is [26]

$$-\ln\big\langle W(C)\big\rangle = \frac{\lambda^{1/2}}{2\pi}\,A(C) - ML(C)\,. \tag{24}$$

The world-surface area of a string in AdS_5 with the boundary contour C diverges; nonetheless, it can be shown that the divergence is absorbed by the renormalization of the W-boson mass.

Thus, the finite part of the area implies the following prediction for the average of the Wilson loop in the strong-coupling regime:

$$\big\langle W(C)\big\rangle = \exp\left(c\lambda^{1/2}\right), \tag{25}$$

where c is a contour-dependent positive number. Taking zero modes in the classical solution into account, the prediction for the Wilson loop on the string-theory side can be written as

$$\langle W(C)\rangle = \lambda^{-3/4}\exp\left[\frac{\lambda^{1/2}}{2\pi}A(C)\right]\sum_{n=0}^{\infty}c_n\lambda^{-n/2}\,. \tag{26}$$

For an arbitrary contour, the check of duality is a difficult task because it is impossible to obtain an exact result in the strong-coupling regime on the gauge-theory side. But such a comparison is possible for a circular loop. The leading term

$$\langle W(C)\rangle = \left(\frac{2}{\pi}\right)^{1/2}\lambda^{-3/4}\exp\lambda^{1/2} \tag{27}$$

obtained in the classical sigma-model for the circular contour can be compared with the answer obtained by the summation of loop corrections in gauge theory.

High supersymmetry makes it possible to sum the loop corrections for a circular loop in the $N=4$ theory [27]. The central point in Ref. [27] is the discovery of the cancellation of diagrams with internal vertices for a circular contour in high orders of the perturbation theory. In other words, the problem is effectively reduced to the summation of ladder diagrams. The explicit summation of planar ladder diagrams in the strong-coupling regime yields a result fully coincident with relation (27). Thus, calculation of the Wilson loop provides an accurate test for duality in the classical string approximation.

5. $N = 4$ theory in the classical string approximation: integrability and anomalous dimensions of operators

5.1 Integrability in gauge theories

In this section, we try to demonstrate that a hidden integrability allows verifying duality in lower orders of the perturbation theory in the coupling constant. Of primary importance is the identification of the anomalous dimensions of operators in the gauge theory with the energies of classical string configurations. We show below that in certain cases, the integrability allows predicting anomalous dimensions in the gauge theory.

We start the discussion with the explanation of the role of integrability in the four-dimensional gauge theory. At first sight, the appearance of integrable systems with a finite number of the degrees of freedom in gauge field theory may seem unexpected. Indeed, an integrable system with a finite number of the degrees of freedom has canonical variables defined in terms of phase space, while the Hamiltonian depending on the canonical variables determines evolution in 'physical' time. On the other hand, there is a field system in four dimensions with an infinite number of the degrees of freedom that is not integrable *per se*.

It turns out that integrability emerges in string theory in different limits [31 – 34] at a certain effective description. The meaning of the degrees of freedom in such integrable systems, and the identification of a time variable and the corresponding Hamiltonian are not self-evident; they are brought out in each concrete limit being considered. We discuss integrable systems that describe renormalization-group evolution of the local operators in QCD and supersymmetric theories.

The renormalization-group equation can be represented in the Hamiltonian form if the scaling logarithm is interpreted as a time variable in a dynamical system. It turns out that the corresponding Hamiltonian in the one-loop approximation coincides, in the simplest case, with the Hamiltonian of the Heisenberg spin chain

$$H_{s=1/2} = -\sum_{n=1}^{L}\left(S_nS_{n+1}-\frac{1}{4}\right), \tag{28}$$

where $S_n = (S_n^x, S_n^y, S_n^z)$ is the operator of spin 1/2 in the nth site of a chain L, with the assumed periodic boundary conditions $S_{L+1} = S_1$.

Heisenberg model (28) is integrable and the spectrum can be found using the Bethe ansatz. Generalization to an arbitrary-spin magnet has been performed in Refs [28, 29], and the corresponding Hamiltonian of a spin-s magnet has the form [29]

$$H_s = \sum_{n=1}^{L}H(J_{n,n+1})\,,\qquad J_{n,n+1}(J_{n,n+1}+1) = (S_n+S_{n+1})^2\,. \tag{29}$$

The operator $J_{n,n+1}$ is connected with the sum of spins on the neighboring sites, $S_n^2 = s(s+1)$, and the function $H(x)$ is expressed through the harmonic sum:

$$H(x) = \sum_{l=x}^{2s-1}\frac{1}{l+1} = \psi(2s+1)-\psi(x+1)\,, \tag{30}$$

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)\,.$$

For $s = 1/2$, the two-particle spin can take the values $J_{n,n+1} = 0$ and $J_{n,n+1} = 1$. In this case, $H(0) = 1$ and $H(1) = 0$; therefore, in accordance with (28), the Hamiltonian $H(J_{n,n+1})$ is given by the projector on the subspace $J_{n,n+1} = 0$:

$$H(J_{n,n+1}) = \frac{1}{4} - S_nS_{n+1}\,.$$

Calculations of the anomalous dimensions and asymptotic forms of the scattering cross sections in the Regge regime [30] in QCD showed that the evolution operator is $[\psi(J)-\psi(1)]$, where J has the meaning of the Lorentz SL(2) spin in the computation of anomalous dimensions or conformal spin in the analysis of the cross section asymptotic forms. Also, it was explicitly demonstrated that the appearance of the ψ-function is a universal feature of gauge theories associated with the presence of a massless vector particle. Comparison of explicit calculations in QCD and spin-chain Hamiltonians revealed the hidden integrability of the evolution equations in QCD [31 – 34].

In Bjorken's kinematic limit of scattering processes at high energies in QCD, the short-distance dynamics is separated from the nonperturbative infrared dynamics and is described in terms of the renormalization-group logarithmic evolution of local composite operators constructed from the fundamental fields and covariant derivatives. In general, operators of the same canonical dimension mix during the evolution described by the Callan – Symanzik renormalization group equations

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}O_n(x) = \sum_k \gamma_{n,k}(g)\,O_k(x)\,, \tag{31}$$

where $\gamma_{n,k}$ is the mixing matrix calculated in the form of a perturbation series in powers of the coupling constant $g = g(\mu^2)$.

The size of the mixing matrix is determined by the symmetries of the operators being considered, and the matrix itself may be interpreted as a Hamiltonian acting in the operator space [35]. The logarithm of the renormalization-group scale $\tau = \ln\mu$ plays the role of a time variable, and Eqn (31) takes the form of the Schrödinger equation. It turns out that the one-loop evolution of a broad class of operators is described by the Heisenberg magnets of different spins, each being realized in terms of the $\mathrm{SL}(2,R)$ generators [34, 36, 37]. The number of sites in the spin chain coincides with the number of fundamental fields of the theory from which a given composite operator is constructed. If the operator contains quark fields in the fundamental representation, its evolution is typically described by an open spin chain; if the operator contains only fields in the adjoint representation, the spin chain is closed.

The appearance of the $\mathrm{SL}(2,R)$ group as a spin-chain structure group is not accidental and results from the fact that it is a reduction of the total conformal group $\mathrm{SO}(2,4)$ in four dimensions to the light-cone kinematics, relevant in the Bjorken limit. As shown in Ref. [38], operators that realize conformal group representations are multiplicatively renormalized in the one-loop approximation.

In Ref. [40], the integrability was used for an in-depth study of the anomalous dimensions of the twist-3 operators important for the description of power corrections in QCD. We note that the simplest integrable structure for higher-twist operators arises in the large-N_c limit. For arbitrary N_c, the situation is much more complicated because it is necessary to take into account the interaction between the degrees of freedom at all sites of the lattice, and not only between nearest neighbors, as in the large-N_c limit.

Although the one-loop integrability of the evolution equations was first described in QCD, it also aroused great interest in the context of supersymmetric gauge theories. Integrability was rediscovered for a class of scalar operators in the $N = 4$ theory in Ref. [43] and formulated for the generic operators in the $N = 4$ theory in Ref. [41], where it was shown that the full structure group in the supersymmetric case was the supergroup $\mathrm{SU}(2,2|4)$. Consecutive simplification of the structure group from the $N = 4$ theory to a nonsupersymmetic theory ($N = 0$) was evident in the light-cone gauge as described in Ref. [42]. We note that in the supersymmetric case, the simplest description is that of the integrable structure for the mixing matrix of scalar operators of the type $\mathrm{Tr}\left\{\Phi_1^{J_1}(0)\,\Phi_2^{J_2}(0)\,\Phi_3^{J_3}(0)\right\}$; this mixing matrix is identified with the Heisenberg $\mathrm{SO}(6)$-magnet of length $J_1 + J_2 + J_3$ [43, 44].

Our purpose is to study the role of integrability in the context of duality between gauge theories and string theory. The string excitation spectrum to be compared with the spectrum of anomalous dimensions in gauge theory is generally unknown, with the exception of the limiting *pp*-wave geometry [12]. Therefore, the anomalous dimensions of operators with large quantum numbers have to be studied in order to compare string energy with perturbative calculations in field theory; this allows using the classical approximation for the string. We show that the classical string described as a sigma-model can be obtained from a magnet in the thermodynamic limit. There is good agreement between the spectrum of anomalous dimensions of a broad class of operators and classical energies of the corresponding string configurations. We also discuss an example of inverse relations when for special solutions the string equations of motion are reduced to the equations of motion of a finite-dimensional integrable system.

5.2 String as the thermodynamic limit of a spin chain

The dilatation operator in the Yang–Mills theory in the one-loop approximation coincides with the Hamiltonian of a spin chain in which the number of sites coincides with the number of fields involved in the composite operator. For example, the operator $\mathrm{Tr}\,\Phi^J$ (where Φ is a certain field in the theory) corresponds to a chain of length J; in other words, at large J, it is natural to consider the thermodynamic limit of the spin chain. We show that the thermodynamic limit of the spin chain may be identified with the Hamiltonian of a string propagating over a certain submanifold of $\mathrm{AdS}_5\times\mathrm{S}^5$. Thus, the spin chain may actually be regarded as a string discretization in the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry.

As an example illustrating such an interpretation, we show that the XXX spin chain with the structure group $\mathrm{SU}(2)$ determining the one-loop evolution of the operators composed of products of powers of two complex scalar fields Φ_1 and Φ_2 describes, in the thermodynamic limit, a classical string propagating in the S^3-submanifold of $\mathrm{AdS}_5\times\mathrm{S}^5$. The sigma-model corresponding to such a string arises from the spin chain in the long-wave approximation. Corrections to the classical sigma-model behave as $1/J$ and are suppressed in the thermodynamic limit.

Technically, the transition from a chain to a classical string is realized using the coherent-state formalism [47]. Let $|ss\rangle$ be the state with the total spin s and projection on the z axis $S_z = s$. The coherent state corresponding to the representation $-s$ of the $\mathrm{SU}(2)$ group is defined as

$$|\mathbf{n}\rangle = \exp(\mathrm{i}S_x\phi)\exp(\mathrm{i}S_y\theta)\,|ss\rangle\,, \tag{32}$$

where $\mathbf{n}$ is the unit vector, $\mathbf{n}^2 = 1$,

$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \tag{33}$$

and θ and ϕ are the spherical angles.

The Hamiltonian of the spin chain

$$H = \frac{\lambda}{4\pi^2}\sum_{k=1}^{J}\left(\frac{1}{4} - \mathbf{S}_k\mathbf{S}_{k+1}\right) \tag{34}$$

can be expanded in coherent states. The partition function $\mathrm{Tr}\exp(-Ht)$ can be represented as a functional integral over the variables $\mathbf{S}_k = s\,\mathbf{n}_k$ with the action

$$S(\mathbf{n}) = s\sum_{k=1}^{J}\int \mathrm{d}t\int_0^1 \mathrm{d}\tau\,\mathbf{n}_k[\partial_t\mathbf{n}_k\,\partial_\tau\mathbf{n}_k] - \frac{\lambda}{8\pi^2}\,s^2\int\mathrm{d}t\sum_{k=1}^{J}(\mathbf{n}_k - \mathbf{n}_{k+1})^2\,, \tag{35}$$

with the condition $\mathbf{n}_{J+1} = \mathbf{n}_1$.

In the long-wave approximation, the vectors $\mathbf{n}_k(t)$ vary only insignificantly along the spin chain. This allows introducing the function $\mathbf{n}(\sigma,t)$, continuously dependent on the variable σ, that takes values from zero to the chain length J

with the action

$$S = -s\int \mathrm{d}t\,\mathrm{d}\sigma\,\partial_t\phi\cos\theta - \frac{\lambda}{8\pi^2}\,s^2\int \mathrm{d}t\,\mathrm{d}\sigma\left[(\partial_\sigma\theta)^2 + (\partial_\sigma\phi)^2\sin^2\theta\right]. \tag{36}$$

As shown in Ref. [47], the action for spin $s = 1/2$ coincides with action

$$S_{\mathrm{str}} = \frac{R^2}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\left[G_{\mu\nu}\,\partial_\tau X^\mu\,\partial_\tau X^\nu - G_{\mu\nu}\,\partial_\sigma X^\mu\,\partial_\sigma X^\nu\right] \tag{37}$$

for a classical string propagating in the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\psi^2 + \mathrm{d}\varphi_1^2 + \mathrm{d}\varphi_2^2 + 2\cos(2\psi)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,. \tag{38}$$

To obtain the string action, it is convenient to fix the gauge $t = \chi\tau$, consider the limit as $\partial_\tau X^i \to 0$ and $\chi \to \infty$ at $\chi\partial_\tau X^i = \mathrm{const}$, and identify the variables

$$\varphi_2 = -\frac{1}{2}\,\phi\,, \qquad \psi = \frac{1}{2}\,\theta\,. \tag{39}$$

Elimination of the variable φ_1 with the help of the classical equations of motion immediately yields (36).

In a similar way, it is possible to find the classical field action from a more general spin chain with the structure group $\mathrm{SU}(3)$ [48] and also for the string configuration having both the Lorentz spin S and the R-charge J [49]. Corrections to sigma-model action (36) containing higher field derivatives can also be taken into account.

5.3 Classical string configurations and integrable systems

We have shown that string configurations arise in the consideration of the thermodynamic limit of integrable spin chains. In this section, we demonstrate that the inverse relation is equally valid; in other words, special solutions of the string equations of motion in $\mathrm{AdS}_5 \times \mathrm{S}^5$ reduce to finite-dimensional integrable systems. For this, it is convenient to consider the string action

$$S = \frac{\lambda^{1/2}}{4\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau[G_{mn}^{\mathrm{AdS}}\,\partial y_m\,\partial y_n + G_{kl}^{\mathrm{S}^5}\,\partial x^k\,\partial x^l] \tag{40}$$

with the string tension determined by the coupling constant of the gauge theory. By introducing constraints with Lagrange multipliers, the above action can be rewritten in a somewhat different form:

$$S = \frac{\lambda^{1/2}}{4\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau\left[\partial X_m\,\partial X_m + \Lambda_x(X^2 - 1) + \partial Y^k\partial Y^k + \Lambda_y(Y^2 + 1)\right], \tag{41}$$

where X_n $(n = 1, \ldots, 6)$ and Y_k $(k = 0, \ldots, 5)$ are two sets of coordinates describing the embedding of our geometry into R^6 with the respective signatures $(6, 0)$ and $(4, 2)$. In addition to analyzing the action, we must impose Virasoro constraints on the dynamical variables. The Virasoro constraints reflect the vanishing of the two-dimensional energy – momentum tensor on the string world surface:

$$\dot{Y}_k\dot{Y}_l + Y'_kY'_l + \dot{X}_n\dot{X}_n + X'_nX'_n = \dot{Y}_kY'_k + \dot{X}_nX'_n = 0\,, \tag{42}$$

We also impose the periodic boundary conditions

$$Y_k(\sigma + 2\pi) = Y_k(\sigma)\,, \qquad X_n(\sigma + 2\pi) = X_n(\sigma)\,. \tag{43}$$

The global symmetries $\mathrm{SO}(2,4)$ and $\mathrm{SO}(6)$ allow defining a set of conserved charges:

$$\begin{aligned} S_{kl} &= \lambda^{1/2}\int \mathrm{d}\sigma\,(Y_k\dot{Y}_l - Y_l\dot{Y}_k)\,, \\ J_{nm} &= \lambda^{1/2}\int \mathrm{d}\sigma\,(X_n\dot{X}_m - X_m\dot{X}_n)\,. \end{aligned} \tag{44}$$

Among them, six symmetry algebra generators are naturally distinguished: the energy $E = S_{05}$, the Lorentz spins S_{12} and S_{34}, and the angular momenta J_{12}, J_{34}, and J_{56} corresponding to rotations in S^5. The set of conserved charges (44) parameterizes the general solutions of the classical equations of motion in the sigma-model [50].

To characterize the gauge theory operator in the dual description of the string, it is necessary to find an appropriate solution of the string equations of motion satisfying constraints (42) and boundary conditions (43). A simplest example is

$$\begin{aligned} &Y_5 + \mathrm{i}\,Y_0 = \exp(\mathrm{i}t)\,, \\ &X_{2i-1} + \mathrm{i}X_{2i} = r_i(\sigma)\exp\left[\mathrm{i}\omega_i\tau + \mathrm{i}\alpha_i(\sigma)\right], \end{aligned} \tag{45}$$

where $i = 1, 2, 3$ and Y-coordinates are taken to be zero. Substitution of this ansatz into the sigma-model action leads to [51]

$$L = \sum_{i=1}^{3}\left(r_i'^2 + r_i^2\alpha_i'^2 - \omega_i^2 r_i^2\right) - \Lambda_x\sum_{i=1}^{3}\left(r_i^2 - 1\right). \tag{46}$$

The solution of the equation of motion for α_i yields $\alpha_i' = v_i/r_i^2$, where v_i are integration constants.

The emerging Lagrangian describes the finite-dimensional integrable Neuman – Rosochatius system with five integrals of motion. Three of them are identified with v_1, v_2, and v_3 and the remaining two are

$$I_i = r_i^2 + \sum_{j\neq i}^{3}\frac{1}{\omega_i^2 - \omega_j^2}\left[(r_i r_j' - r_i r_j')^2 + \frac{v_i^2 r_j^2}{r_i^2} + \frac{v_j^2 r_i^2}{r_j^2}\right], \tag{47}$$

where

$$\sum_{i=1}^{3} I_i = 0\,.$$

As a result, the energy of the system depends on the frequencies ω_i and five integers interrelated by the Virasoro constraint.

Thus, by computing the classical energy of a finite-dimensional dynamical system as a function of all integrals of motion, we automatically calculate the anomalous dimensions of the operators in the gauge theory with the same set of quantum numbers relative to global symmetries [52, 53]. Today, there are many solutions of the sigma-model equations of motion for which the anomalous dimensions of the corresponding operators have been obtained [50, 16]. Whenever a comparison with the data of explicit calculations in the field theory is feasible, the results of the two computations coincide.

We note that a system with a finite number of degrees of freedom can at first sight describe only a very narrow class of solutions of the equations of motion and, accordingly, of the gauge theory operators. However, because the given sigma-model on $AdS_5 \times S^5$ is integrable at the classical level, it is possible to use powerful tools known in the theory of integrable systems. Specifically, the so-called Bäcklund transformation may be applied to the simplest solutions corresponding to the Neuman–Rosochatius system in order to generate more sophisticated solutions in the sigma-model corresponding to the general operators in the field theory [53].

5.4 General structure of the relation between spin chains and strings

We have considered examples of the relation between the spin chains determining the evolution of the operators in gauge theory and the solutions of the equations of motion in a string sigma-model. We now discuss the general structure of this relation. In the first place, it is necessary to elucidate the relation between the structure group of the chain and the type of solution of the string equations of motion. We recall that the general string configuration in $AdS_5 \times S^5$ is determined by five quantum numbers S_1, S_2, J_1, J_2, J_3, where S_1 and S_2 define charges in the Lorentz group and the J_i correspond to charges in the R-symmetry group in the gauge theory. Because the quantum numbers of the operators must coincide with the quantum numbers of the string solution, it is easy to see that renormalization of the operators with two nonzero R-charges, J_1 and J_2, is described by SU(2) spin chains and that of operators with one nonzero Lorentz spin S by the $SL(2, R)$ chain. The most general operators in the supersymmetric theory are described by a chain with the structure supergroup $SO(4, 2|2) \times SO(6)$.

Another obvious problem is the coupling constant dependence. The energy evaluated on the classical solution in string theory shows a complex dependence on the coupling constant. For the comparison with perturbative calculations, it is necessary to expand the exact string answer in a series. Generally speaking, analyticity of the exact answer with respect to the coupling constant cannot be warranted, but it does hold for a broad class of operators with large quantum numbers. As an example, we give the first term in the expansion of the string energy in the coupling constant for a state with large quantum numbers (J_1, J_2) [16],

$$E_{\mathrm{str}} = \frac{2}{\pi^2} K(x)\left[E(x) - (1-x)K(x)\right], \qquad \frac{J_2}{J_1+J_2} = 1 - \frac{E(x)}{K(x)}, \tag{48}$$

where $K(x)$ and $E(x)$ are the standard elliptic integrals of the 1st and 2nd kinds, respectively. Expression (48) appears to be rather complicated, but it exactly coincides with the anomalous dimension of the $\mathrm{Tr}\, \Phi_1^{J_1} \Phi_2^{J_2}$-type operators calculated as the energy of state in the SU(2) spin chain with $(J_1 + J_2)$ sites; this demonstrates the validity of duality in the one-loop approximation for the given class of operators.

The agreement between the results of calculations in the spin chain and the string raises the problem of integrability of the next terms of expansion of the dilatation operators in the coupling constant. It turns out that explicit computation leads to an integrable Hamiltonian that describes evolution of the operators in the scalar sector in two loops:

$$H^{2\,\mathrm{loop}} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}(1 - P_{k,k+1}) + \frac{\lambda^2}{128\pi^4}\sum_{k=1}^{J}(-4 + 6P_{k,k+1} - P_{k,k+1}P_{k+1,k+2} - P_{k+1,k+2}P_{k,k+1}) + O(\lambda^3), \tag{49}$$

where $P_{i,j}$ is the operator of permutation of the ith and jth sites. For $S = 1/2$, it can be represented in the more traditional form $(\mathbf{S}_i\mathbf{S}_j - \mathrm{const})$. Calculations of the spectrum of the two-loop Hamiltonian exactly reproduce the string solution energies expanded to the second order in the coupling constant.

Despite success in the first two loops of the $N = 4$ theory, the situation in the next loops cannot be considered fully satisfactory. A few candidate integrable systems responsible for the higher-loop dilatation operators have been proposed [67, 68]. However, starting from three loops, discrepancies arise between anomalous dimensions of the operators with a large R-charge J in the nonleading term in $1/J$ computed in the perturbation theory and in the classical string approximation [69]. A new candidate integrable structure has recently been proposed that takes all loops into account [45] and reproduces the highly nontrivial three-loop result for the anomalous dimensions of operators with a large Lorentz spin [70]. However, its status as an exact answer remains to be clarified.

We note that it is possible to compare integrable structures in the spin chain and the classical string in terms of geometric objects, i.e., higher-genus Riemann surfaces. The fact is, the Jacobians of such Riemann surfaces are complex Liouville tori along which the classical evolution develops. Parameters of these surfaces are given by a complete set of the integrals of motion of a dynamical system (in this case, the spin chain). On the other hand, classical solutions of the string sigma-model are also parameterized by moduli of Riemann surfaces. Reference [66] demonstrated coincidence of Riemann surfaces occurring in the gauge theory through spin chains and in the description of classical solutions of sigma-models. This gives hope that the hidden integrability will make it possible to more exactly formulate the dual description in higher orders of the perturbation theory.

6. $N = 4$ gauge theory and quantum string; the *pp*-wave limit

At present, there is no explicit quantum answer for the spectrum of a string propagating in $AdS_5 \times S^5$, which makes a comprehensive comparison of Hilbert spaces of gauge and string theories impossible. However, there is a special degeneration of the $AdS_5 \times S^5$ geometry to the Penrose limit where the exact answer is nevertheless possible to obtain [54]. The Penrose limit describes a region near the special null-geodesic, with the string effectively reducing to a point-like particle. The string spectrum in the *pp*-wave limit was found and studied in Refs [12, 55, 56].

For the explicit description of the metric in the *pp*-wave limit, it is convenient to introduce the variables

$$x^+ = \frac{t+\chi}{2\mu}, \qquad x^- = \mu R^2(t-\chi), \tag{50}$$

where χ is the angular variable on S^5 and μ is an auxiliary scale. Considering the limit as $R \to \infty$, we obtain the *pp*-wave metric in the form

$$ds^2 = -4\,dx^+\,dx^- - z^2\,dx^{+2} + \sum_{i=1}^{8} dz_i^2 . \tag{51}$$

Eight flat coordinates z_i correspond to a part of the coordinates from $AdS_5 \times S^5$, and the string behaves as a particle rotating with a large angular momentum J about the angular coordinate χ in S^5.

It is easy to see that the light-cone string Hamiltonian is

$$H = 2p^- = i(\partial_t + \partial_\chi) = \Delta - J , \tag{52}$$

and its spectrum is exactly calculable in the limit

$$R \to \infty , \qquad \Delta \sim J \to \infty , \qquad \frac{J^2}{R^4} = \text{const} . \tag{53}$$

String quantization in the *pp*-wave metric is reduced to the quantization of a system of oscillators; as a result, the string spectrum takes the form

$$\Delta - J = \sum_k N_k \left(1 + \frac{\lambda k^2}{J^2}\right)^{1/2} , \tag{54}$$

where k corresponds to the number of the Fourier harmonic, N_k is the total occupation number of the oscillatory mode, and the Virasoro constraint imposes additional restrictions on the quantum numbers:

$$P = \sum_k k N_k = 0 .$$

Our objective is to identify the string spectrum with the spectrum of anomalous dimensions of a class of operators in the $N = 4$ gauge theory. In the first place, it is necessary to identify operators dual to the string states in the Penrose limit. We recall that the effective string length J must be identified with the number of fields contained in the composite operator. The string ground state can be identified with an operator composed of scalars $Z = \Phi_1 + i\Phi_2$:

$$|0, J\rangle \leftrightarrow \operatorname{Tr} Z^J . \tag{55}$$

This operator has the charge J with respect to the rotation plane in the *pp*-wave.

Oscillatory excitations of the string correspond to the inclusion of other scalar fields from the Lagrangian of the $N = 4$ theory to the composite operator. The best known are the so-called BMN-operators, which are identified with the string excitation modes as

$$\begin{gathered} a_0^{i+}\,|0, J\rangle \leftrightarrow \operatorname{Tr} \Phi_i Z^J , \\ a_n^{i+} a_{-n}^{j+}\,|0, J\rangle \leftrightarrow \sum_l \exp\left(2\pi i\,\frac{nl}{J}\right) \operatorname{Tr} \Phi_i Z^l \Phi_j Z^{J-l} . \end{gathered} \tag{56}$$

Using this correspondence, it is possible to compare eigenvalues of the anomalous dimension matrix of the operators that mix among themselves and the string energy spectrum.

The string energy is an exact function of the ratio λ/J^2 of the coupling constant to the angular momentum. Thus, we have the first example of a prediction of the anomalous dimension of operators in string theory in an arbitrary order of the coupling constant. For comparison with the known loop computations, it is necessary to expand the exact spectrum in the perturbative region. The first terms of the expansion in the coupling constant exactly reproduce calculations in the framework of the supersymmetric gauge theory; this provides an explicit example of duality verification in a situation where the string is regarded as a quantum object.

We note that in the one-loop approximation, the dilatation operator in the $N = 4$ theory in the sector of scalar operators coincides with the Hamiltonian of a spin chain having the structure group SO(6) [43]. It also permits us to reflect string states in the spin-chain states. Specifically, if the consideration is restricted to operators composed of only two complex scalar fields, the structure group reduces to SU(2). The string ground state corresponds to all spins aligned in one direction and string excitations to the flip of part of the spins.

7. Dual descriptions of nonconformal theories; $N = 2$ supersymmetric gauge theory

In this section, we discuss the dual description of the $N = 2$ gauge theory in the supergravity approximation. Metrics in the dual description and higher-form fields have a more complicated structure compared with the $N = 4$ case; nonetheless, they can be directly represented. We demonstrate how the simplest facts known in the gauge theory can be reproduced in the dual description. We start by recalling the main facts about the $N = 2$ supersymmetric theory.

The gauge theory without additional matter fields is described by a supermultiplet of fields in the adjoint representation that includes the gauge vector field, two Majorana fermions, and a complex scalar Φ. The theory is asymptotically free and the β-function arises only in one loop. The classical theory has the global $SU(2) \times U(1)$ R-symmetry group, but the U(1)-part is broken to Z_{4N_c} at the quantum level. An infinite number of the vacuum states of the theory are parameterized by vacuum values of the complex scalar. The nonperturbative energy action taking instanton effects into account was found in Ref. [57].

In the dual description in the supersymmetric theory, the metric is [11]

$$\begin{aligned} ds^2 = {} & H^{-1/2}\eta_{\mu\nu}\,dx^\mu\,dx^\nu \\ & + H^{1/2}\left[d\rho^2 + \rho^2\,d\theta^2 + \delta_{mn}\,dx^n\,dx^m\right] , \end{aligned} \tag{57}$$

and the higher-form field corresponding to the $N = 2$ gauge theory is given by

$$\tilde{F}_5 = d(H^{-1}\,dx^0 \wedge \ldots dx^3) + {*}d(H^{-1}\,dx^0 \wedge \ldots dx^3) , \tag{58}$$

$$c + ib = 4\pi\alpha' g_s N_c \ln \frac{z}{\rho_0} , \tag{59}$$

where $\rho^2 = (x_4^2 + x_5^2)$, $z = \rho \exp(i\theta)$, $r^2 = x_6^2 + \ldots + x_9^2$, and H is the known function of radial coordinates. We note that the solution involves a complex scalar field $c + ib$ and a set of various-degree form fields from the NS–NS and R–R sectors:

$$\tilde{F}_5 = F_5 - C_2 \wedge H_3 , \qquad H_3 = dB_2 . \tag{60}$$

In brane terms, the solution can be described as a set of fields induced by the bound state of D3-branes that are defined on the orbifold C^2/Z_2. Following the general logic, the gauge theory is identified as a theory on the branes' world surface.

To describe a few quantitative characteristics of the theory, it is necessary to find parameters of the gauge theory in the low-energy limit from the Born–Infeld action at the brane world surface. Rather simple calculations for the coupling constant and $\theta_{\rm YM}$ lead to

$$\frac{1}{g_{\rm YM}^2} = \frac{1}{16\pi^2\alpha' g_{\rm s}}\int B_2 = \frac{N_{\rm c}}{4\pi^2}\ln\frac{\rho}{\rho_0}\,, \tag{61}$$

$$\theta_{\rm YM} = \frac{1}{2\pi^2\alpha' g_{\rm s}}\int C_2 = -2N_{\rm c}\theta\,. \tag{62}$$

The solution of the equations of motion in supergravity has a symmetry of the form

$$\theta \to \theta + \frac{\pi k}{2N_{\rm c}}\,. \tag{63}$$

Using the relation with $\theta_{\rm YM}$, we note that it corresponds to the known $Z_{4N_{\rm c}}$-symmetry in the $N=2$ supersymmetric gauge theory.

Another essential characteristic of gauge theory is its β-function. In order to find the β-function in the dual theory, it is necessary to carefully relate the scale μ in the gauge theory to coordinate ρ in the gravitational solution.

We consider an operator with simple renormalization-group properties. The most convenient for this purpose is a scalar field whose vacuum expectation value can be related to the coordinate z as $\phi = (2\pi\alpha')^{-1}z$. Then, it is easy to obtain the relation

$$\rho = 2\pi\alpha'\mu\,. \tag{64}$$

Substitution of this relation into gravitational solution (61) gives the behavior of the running coupling constant

$$\frac{1}{g_{\rm YM}^2} = \frac{N_{\rm c}}{4\pi^2}\ln\frac{\mu}{\Lambda}\,, \tag{65}$$

which exactly reproduces the β-function of the $N=2$ theory.

We note that the perturbative behavior of the $N=2$ theory is fairly well reproduced in the framework of the dual theory, while the complete nonperturbative low-energy action remains to be found, the reason being the singular behavior of the function H that determines the solution in the infrared region [11]. The singularity is believed to disappear if string modes are taken into consideration, but this has never been demonstrated explicitly.

8. Dual descriptions of nonconformal theories; $N=1$ supersymmetric gauge theory

The $N=2$ supersymmetric theory is essentially different from realistic models by virtue of its having an infinite number of vacuum states. It would therefore be interesting to find a dual description for the more realistic $N=1$ supersymmetric gauge theory. We recall its main features that differentiate it from $N=2$ gauge theories. The $N=1$ theory includes a vector gauge field and the Majorana gluino field in the adjoint representation. The action of the theory

$$L_{N=1} = \frac{1}{g_{\rm YM}^2}\int {\rm d}^4x\,({\rm Tr}\,F^2 + {\rm i}\bar{\lambda}D\lambda) \tag{66}$$

has much in common with QCD. Specifically, the theory is asymptotically free and has a mass gap.

Unlike the $N=2$ theory, the $N=1$ theory has the finite number $N_{\rm c}$ of vacua for the gauge group ${\rm SU}(N_{\rm c})$. The anomaly multiplet includes anomalies in the dilatation current, the supercurrent, and the current corresponding to the R-symmetry. The respective symmetries are already broken in the one-loop approximation. In contrast to the one-loop β-function in the $N=2$ theory, that in the $N=1$ theory is constructed from the contributions of all loops and the exact answer can be obtained as [58]

$$\beta = -\frac{N_{\rm c}g_{\rm YM}^2}{16\pi^2}\left(1-\frac{N_{\rm c}g_{\rm YM}^2}{8\pi^2}\right)^{-1}. \tag{67}$$

We note that all loop contributions, starting from the second loop, are of the infrared origin. Therefore, generally speaking, it is possible to determine the β-function in the Wilson sense, as containing only the one-loop contribution.

In what follows, we discuss the dual description of the R-symmetry broken to $Z_{2N_{\rm c}}$ by the one-loop anomaly. Moreover, nonperturbative effects generate a gluino condensate in the theory, which leads to further spontaneous symmetry breaking to Z_2. The expression for the condensate is

$$\langle{\rm Tr}\,\lambda^2\rangle = \Lambda^3\exp\left(2\pi{\rm i}\,\frac{k}{N_{\rm c}}\right), \qquad k=0,1,\ldots,N_{\rm c}-1\,, \tag{68}$$

where Λ is the infrared scale of the theory. The gluino condensate is an order parameter of the theory and the number k in (68) labels the vacuum state.

8.1 Solution of the equations of motion in supergravity

Currently, there are two solutions of the supergravity equations that provide for the dual description of the $N=1$ gauge theory: the Maldacena–Nuñez [9] and Klebanov–Strassler [10] solutions interrelated through a sequence of transformations. This section deals with the Maldacena–Nuñez solution generated by branes wrapped around compact submanifolds. We recall that in the previous solutions dual to the $N=4$ and $N=2$ gauge theories, we had to consider D3-branes embedded in ten-dimensional space. However, in the $N=1$ case, we must consider $N_{\rm c}$ coincident D5-branes wrapped around a compact two-dimensional cycle.

The gauge theory is defined at the world surface of the $N_{\rm c}$ D5-branes in the background metric

$$\begin{aligned}{\rm d}s^2 = \exp\Phi\,{\rm d}x^2 + g_{\rm s}N_{\rm c}\exp\Phi\Big[&\exp(2h)\,({\rm d}\theta_1^2+\sin^2\theta_1\,{\rm d}\phi_1^2)\\ &+{\rm d}\rho^2+\sum_{a=1}^{3}(\omega^a-A^a)^2\Big]\,,\end{aligned} \tag{69}$$

$$\exp(2\Phi) = \frac{\sinh\rho}{2\exp h}\,, \tag{70}$$

$$F_3 = 2g_{\rm s}N_{\rm c}\prod_{a=1}^{3}(\omega^a-A^a) - g_{\rm s}N_{\rm c}\sum_{a=1}^{3}F^a\wedge\omega^a\,, \tag{71}$$

where

$$A^1 = -\frac{1}{2a(\rho)}\,\mathrm{d}\theta_1\,,$$
$$A^2 = \frac{1}{2a(\rho)}\sin\theta_1\,\mathrm{d}\phi_1\,, \tag{72}$$
$$A^3 = -\frac{1}{2}\cos\theta_1\,\mathrm{d}\phi_1\,,$$

$$\exp h = \rho\coth(2\rho) - \frac{\rho^2}{\sinh^2(2\rho)} - \frac{1}{4}\,,$$
$$a(\rho) = \frac{2\rho}{\sinh\rho}\,. \tag{73}$$

The left-invariant forms are given by

$$2\omega^1 = \cos\psi\,\mathrm{d}\theta_1 + \sin\psi\sin\theta_2\,\mathrm{d}\phi_2\,,$$
$$2\omega^2 = \sin\psi\,\mathrm{d}\theta_2 - \cos\psi\sin\theta_2\,\mathrm{d}\phi_2\,, \tag{74}$$
$$2\omega^3 = \mathrm{d}\psi + \cos\theta_2\,\mathrm{d}\phi_3\,,$$

and $F^a = \nabla A^a$.

Unlike the $N=2$ geometry, the solution defines a nonsingular metric and depends on five angular variables and a radial variable ρ, which we again connect with the energy scale in the gauge theory. The parameters of the gauge theory, as in the $N=2$ case, can be obtained by the substitution of the solution of the supergravity equations of motion into the low-energy expansion of the Born – Infeld action:

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{1}{16\pi^3 g_{\mathrm{s}}}\int_{S^2}\exp(-\Phi)\,(\det G)^{1/2} = \frac{N_{\mathrm{c}}}{4\pi^2}\,\rho\tanh\rho\,, \tag{75}$$

$$\theta_{\mathrm{YM}} = -\frac{1}{2\pi g_{\mathrm{s}}}\int_{S^2} C_2 = -N_{\mathrm{c}}\psi\,. \tag{76}$$

It easy to see from the solution that the limit of large values of the radial coordinate ρ corresponds to the ultraviolet region of the field theory, where the coupling constant is small, and U(1)-rotations associated with the R-symmetry correspond to shifts in the angular variable ψ.

8.2 The physics of $N=1$ gauge theory in the dual description

We now move to the dual description of the main characteristics of the $N=1$ theory; we first discuss the geometry of R-symmetry breaking. As mentioned earlier, U(1) may be expected to break down to $Z_{2N_{\mathrm{c}}}$ in the ultraviolet region; it is natural to reproduce this symmetry breaking in the dual description. We recall that the ultraviolet behavior corresponds to large ρ, when $a(\rho)\to 0$. In this limit, rotation about the angular coordinate ψ is a metric isometry and the symmetry breaking is due only to the C_2-form field flux. It is easy to see that the shift

$$\psi \to \psi + \frac{2\pi k}{N_{\mathrm{c}}} \tag{77}$$

is a symmetry of the solution of the equations of motion in supergravity, in excellent agreement with the expected $Z_{2N_{\mathrm{c}}}$-symmetry in the dual description.

For the analysis of further symmetry breaking to Z_2, it is necessary to consider arbitrary values of the radial coordinate ρ and study how the solution-giving functions depend on it. The most essential fact is that the function $a(\rho)$ is multiplied by $\cos\phi$ and $\sin\psi$; in other words, at arbitrary ρ, only the shift

$$\psi \to \psi + 2\pi N_{\mathrm{c}} \tag{78}$$

remains a symmetry of the solution, which corresponds to the Z_2-symmetry remaining in the infrared region.

In order to determine the β-function, it is necessary to find the relation between the radial coordinate ρ of the solution and the energy scale μ in gauge theory. For this, it is convenient to use the dual identification of the operator in the gauge theory that is not deformed at the quantum level,

$$\langle\lambda^2\rangle \leftrightarrow a(\rho)\,, \tag{79}$$

which implies the relation

$$\frac{\Lambda^3}{\mu^3} = \frac{2\rho}{\sinh(2\rho)}\,. \tag{80}$$

Now, it is possible to find the β-function of the theory as

$$\beta_{\mathrm{YM}} = \frac{\partial g_{\mathrm{YM}}}{\partial\rho}\,\frac{\partial\rho}{\partial\big(\ln(\mu/\Lambda)\big)}\,. \tag{81}$$

At large ρ,

$$\frac{\partial g_{\mathrm{YM}}}{\partial\rho} = -\frac{N_{\mathrm{c}} g^2_{\mathrm{YM}}}{8\pi^2}\,, \tag{82}$$

$$\frac{\partial\rho}{\partial\big(\ln(\mu/\Lambda)\big)} = \frac{3}{2}\left(1 - \frac{N_{\mathrm{c}} g^2_{\mathrm{YM}}}{8\pi^2}\right)^{-1}. \tag{83}$$

The combination of these two equations immediately leads to an answer for the β-function that exactly coincides with the perturbative result obtained in the gauge theory [59]. We note that the gravitational computation predicts the existence of nonperturbative corrections to the 'exact' β-function, although the origin of possible corrections in the gauge theory remains to be clarified.

9. Duality and anomalous dimensions of operators in the nonsupersymmetric Yang – Mills theory

We now discuss examples of duality for a nonsupersymmetric gauge field theory. For all the difficulty of analysis of nonsupersymmetric theories, we consider two well-established results concerning anomalous dimensions of the operators, leaving aside other, less rigorous assertions encountered in the literature. First, we discuss the integrable structure of the one-loop dilatation operator in the sector of self-dual gluonic operators and its string realization. Second, we demonstrate how a universal prediction for the anomalous dimensions of certain gauge-theory operators in the strong-coupling regime can be obtained.

9.1 Classical string and gluonic operators

We consider a class of local operators of high canonical dimension composed of gluon fields

$$\prod_{j=1}^{L} F_{\mu_j\nu_j}(0)\,. \tag{84}$$

We show that renormalization of these operators in one loop is described by an integrable Heisenberg spin chain with unit spins at each site [60]. For the description in what follows, it is convenient to pass to Euclidean space and decompose the tensor into irreducible components using the 't-Hooft symbols:

$$F_{\mu\nu} = \eta^A_{\mu\nu} F^A_+ + \bar{\eta}^A_{\mu\nu} F^A_- \,. \tag{85}$$

The self-dual and anti-selfdual components are transformed under the Lorentz group as tensors of the respective types $(1,0)$ and $(0,1)$. Straightforward calculations indicate that the Hamiltonian describing renormalization-group equations in one loop preserves the number of fields in a composite operator (or, equivalently, the number of sites in the spin chain). The total Hamiltonian of the interaction between nearest neighbors can be expanded in the projectors $P^P_{(j_1,j_2)}$ for the spin components j and parity P [60]:

$$\begin{aligned} H_{12} = {} & 7\big(P_{(2,0)} + P_{(0,2)}\big) + P_{(1,0)} + P_{(0,1)} \\ & - 11\big(P^+_{(0,0)} + P^-_{(0,0)}\big) + 3P^-_{(1,1)} \,. \end{aligned} \tag{86}$$

If the operators composed only of self-dual operators F^A_+ are considered, the Hamiltonian reduces to

$$H^{\mathrm{sd}}_{12} = 7P_{(2,0)} + P_{(1,0)} - 11P_{(0,0)} \,. \tag{87}$$

The respective projectors in (87) are given by

$$P_{(2,0)} F^A_+ F^B_+ = \frac{1}{2}\left(F^A_+ F^B_+ + F^B_+ F^A_+ - \frac{2}{3}\,\delta^{AB} F^C_+ F^C_+\right),$$

$$P_{(1,0)} F^A_+ F^B_+ = \frac{1}{2}\left(F^A_+ F^B_+ - F^B_+ F^A_+\right),$$

$$P_{(2,0)} F^A_+ F^B_+ = \frac{1}{3}\,\delta^{AB} F^C_+ F^C_+ \,.$$

These projectors can be reduced to the operators of permutation $PF^A_+F^B_+ = F^B_+F^A_+$ and of taking the trace $KF^A_+F^B_+ = \delta^{AB}F^C_+F^C_+$ and to the identity operators $IF^A_+F^B_+ = F^A_+F^B_+$:

$$P_{(2,0)} = \frac{1}{2}(I+P) - \frac{1}{3}\,K \,,$$

$$P_{(1,0)} = \frac{1}{2}(I-P) \,,$$

$$P_{(0,0)} = \frac{1}{3}\,K \,.$$

Therefore, the Hamiltonian of two-particle interactions

$$H^{\mathrm{sd}}_{12} = 4I_{12} + 3P_{12} - 6K_{12} = 7 + 3\mathbf{s}_1\mathbf{s}_2(1 - \mathbf{s}_1\mathbf{s}_2) \tag{88}$$

coincides with the Hamiltonian of a spin chain with unit spin that was studied earlier and can be diagonalized by the Bethe-ansatz method [28, 29].

The main peculiarity of the given class of operators is the proportionality of their anomalous dimension to the total spin operator S or, accordingly, to the total length of the spin chain. For the dual description of such operators, it is necessary to find an appropriate solution of the equations of motion of the classical string. It turned out that the adequate solution describes a string rotating in two independent planes of AdS_5 with quantum numbers $(\mathrm{S}, \mathrm{S}, 0, 0, 0)$.

At small S, the energy of the classical string is given by [61]

$$E = 2(mS)^{1/2} + O(S^{2/3}) \,, \tag{89}$$

where m is the string winding number, in agreement with the results in the flat case. However, if operators with a large quantum number ($S \gg 1$) are considered (which justifies the classical string approximation), the energy behaves as

$$E = 2S + \frac{3}{4}(4m^2S)^{1/3} + \ldots \,, \tag{90}$$

in accordance with the one-loop answer. We note that in the region of very large S, the classical string solution becomes unstable.

In the foregoing, we discussed integrable structures in the $N=4$ gauge theory and in the nonsupersymmetric theory. Similar integrable structures were found in other cases. Suffice it to mention that the one-loop renormalization of the scalar operators in the $N=2$ theory is described by the XXZ spin chain [62].

9.2 Anomalous dimensions in the strong-coupling regime

It seems appropriate to briefly mention certain general properties of the operators in the nonsupersymmetric theory that can be obtained with the use of the dual string description. Operators with a large Lorentz spin S correspond to the string rotating with a large angular momentum in AdS_5. In the strong-coupling regime, the anomalous dimensions of the twist-2 operator $F_{+\perp}(D_+)^S F_{+\perp}$ coincide with the energy of a doubly folded classical string rotating in AdS_5 [25]:

$$\gamma^{(\mathrm{tw}=2)}_S = \frac{\lambda^{1/2}}{2\pi}\ln S^2 \,. \tag{91}$$

To obtain (91), it is convenient to consider a string with its center at $\rho = 0$ in the global coordinates with the Nambu–Goto action. In the gauge $\sigma_1 = \tau$, $\sigma_2 = \rho$, the induced metric entering the action takes the form

$$G_{MN}\,\partial_a X^M\,\partial_b X^N = \begin{pmatrix} -\cosh^2\rho + \dot{\phi}^2\sinh^2\rho & 0 \\ 0 & 1 \end{pmatrix}, \tag{92}$$

where $\phi = \phi(\tau)$ is the azimuthal angle of a point of the string, τ plays the role of time in AdS, ρ is the radial coordinate, and $\dot{\phi} \equiv \partial\phi/\partial\tau$ is the corresponding angular velocity.

Hence, the action is given by

$$\begin{aligned} S_{\mathrm{cl}} &= 4\,\frac{R^2}{2\pi\alpha'}\int \mathrm{d}\tau \int_0^{\rho_0} \mathrm{d}\rho\,\big(\cosh^2\rho - \dot{\phi}^2(\tau)\sinh^2\rho\big)^{1/2} \\ &\equiv \int \mathrm{d}\tau\, L[\phi] \,. \end{aligned} \tag{93}$$

The factor 4 accounts for the number of segments of the folded string rotating about $\rho = 0$, and the maximum value of the radial coordinate $\rho \leqslant \rho_0$ is derived from the condition

$$\coth^2\rho - \dot{\phi}^2(\tau) \geqslant 0 \,. \tag{94}$$

Equation (93) describes a classical mechanical model of a rotating bar with the Lagrangian $\mathcal{L}[\phi]$, the energy

$$\begin{aligned} E &= \dot{\phi}\,\frac{\partial}{\partial\dot{\phi}}\,L[\phi] - L \\ &= -4\left(\frac{\alpha_{\mathrm{s}} N_{\mathrm{c}}}{\pi}\right)^{1/2} \int_0^{\rho_0} \mathrm{d}\rho\,\frac{\cosh^2\rho}{\big(\cosh^2\rho - \dot{\phi}^2\sinh^2\rho\big)^{1/2}} \,, \end{aligned} \tag{95}$$

and the spin

$$S = \frac{\partial}{\partial\dot\phi}\, L[\phi] = -4\left(\frac{\alpha_s N_c}{\pi}\right)^{1/2} \int_0^{\rho_0} d\rho\, \frac{\dot\phi \sinh^2\rho}{(\cosh^2\rho - \dot\phi^2 \sinh^2\rho)^{1/2}}\,. \quad (96)$$

Integrals of motion on the classical trajectory take the values E and S, and action (93) on the classical trajectory is

$$S_{\rm cl} = \int d\tau\,(J\dot\phi - E) = 2\gamma_S(\alpha_s)\ln\frac{r_{\max}}{r_{\min}}\,, \quad (97)$$

where

$$\tau_{\max,\min} = \ln r_{\max,\min}\,,$$

$$\gamma_S(\alpha_s) = \frac{1}{2}\int_0^{2\pi}\frac{d\tau}{2\pi}\,(S\dot\phi - E) = \frac{1}{2}\,(-E + S\omega)\,, \quad (98)$$

and $\omega = \dot\phi$ is the angular velocity of the bar.

The anomalous dimension is defined as the coefficient in the action in front of time in the AdS space. In the long-wave limit,

$$\rho_0 = \frac{1}{2}\ln\frac{1}{\eta} \gg 1\,, \qquad \omega = 1 + 2\eta\,; \quad (99)$$

as $\eta \to 0$, it is possible to find the string energy and angular momentum:

$$E = 2\left(\frac{\alpha_s N_c}{\pi}\right)^{1/2}(\eta^{-1} - \ln\eta)\,,$$
$$S = 2\left(\frac{\alpha_s N_c}{\pi}\right)^{1/2}(\eta^{-1} + \ln\eta)\,. \quad (100)$$

Substitution of these relations in (98) yields

$$\gamma_S(\alpha_s) = 2\left(\frac{\alpha_s N_c}{\pi}\right)^{1/2}\ln S\,, \quad (101)$$

which gives the anomalous dimension of the operator in the strong-coupling regime.

The generalization to the case of higher-twist operators $F_{+\perp}D_+^{S_1}F_{+\perp}\ldots D_+^{S_{L-1}}F_{+\perp}$ is possible. In this case, the string effectively splits into L components, each reaching the boundary of AdS_5. The energy of the corresponding string configuration is [63]

$$\gamma^{(\rm tw=L)}_{S_1,\ldots,S_{L-1}} = \frac{\lambda^{1/2}}{2\pi}\ln q_L(S_1,\ldots,S_{L-1})\,. \quad (102)$$

In this expression, q_L may be identified with an integral of motion for classical string. For $S_k \sim S \gg 1$ with $k = 1,\ldots,L-1$, $q_L \sim S^L$. It should be borne in mind that the logarithmic behavior of the anomalous dimensions of operators with large quantum numbers is universal for all gauge theories [64, 65].

9.3 Calculation of anomalous dimensions in the open string theory

We demonstrate that the logarithmic growth of the anomalous dimensions of operators with large Lorentz spins (91) and (102) can be obtained in terms of Wilson lines in the gauge theory or, equivalently, with the use of an open string in the AdS_5 metric. The key factor is the relation between the anomalous dimensions of the operators with a large number of covariant derivatives along the light cone and the so-called anomalous renormalization of the Wilson line for a contour with a cusp [64, 65].

It was found rather long ago [71] that the Wilson loop

$$W[C] = \mathrm{Tr}\left\{P\exp\left(ig\int_C dx^\mu A_\mu(x)\right)\right\}$$

acquires a nontrivial anomalous dimension $\Gamma_{\rm cusp}(\lambda,\theta)$ if the integration contour contains a cusp,

$$\langle W[C]\rangle \sim \mu^{\Gamma_{\rm cusp}(\lambda,\theta)}\,, \quad (103)$$

where μ is the ultraviolet cut-off. The relation between the anomalous dimensions of the twist-3 operators with Lorentz spin S and the anomaly is given by [64, 65]

$$\gamma_S^{(\rm tw=2)}(\lambda) = 2\Gamma_{\rm cusp}(\lambda,\theta=\ln S) \quad (104)$$

and holds at all values of the coupling constant λ. In the weak-coupling region at $\theta \gg 1$, we have

$$\Gamma_{\rm cusp}(\lambda,\theta) = \theta\left[\frac{\lambda}{4\pi^2} + O(\lambda^2)\right]\,, \quad (105)$$

with a few of the following terms of the perturbative expansion being known.

The dual string description allows $\Gamma_{\rm cusp}(\lambda,\theta)$ to be computed in the strong-coupling region using an open string. In this limit, it is possible to consider the Wilson contour with a cusp that bounds the world surface of an open string propagating in the AdS_5 metric. The answer for the vacuum value of the Wilson contour at $\theta \gg 1$ reduces to the computation of the minimal surface [72, 73]:

$$\Gamma_{\rm cusp}(\lambda,\theta) = \theta\left[\left(\frac{\lambda}{4\pi^2}\right)^{1/2} + O(\lambda^0)\right]\,. \quad (106)$$

Using Eqns (104) and (106), it is possible to reproduce the result in the strong-coupling region (91) obtained with the help of a closed string [25].

The correspondence in (104) can be extended to higher-twist operators. If an operator contains L fundamental fields and the total number of covariant derivatives is $S \gg L$, its anomalous dimension can be obtained from a contour composed of L Wilson contours in the fundamental representation of the gauge group, with the number of cusps varying from 4 to $2L$ [63]. At large N_c, the vacuum value of Wilson loops is factored into the product of vacuum averages; therefore, the minimal surface of L contours with cusps calculated in the strong-coupling region and corresponding to the product of $k = 2,\ldots,L$ cusped contours is given by the sum of k 'elementary areas':

$$2\,\Gamma_{\rm cusp}(\lambda,\theta=\ln S) \leqslant \gamma_S^{(\rm tw=L)}(\lambda) \leqslant L\,\Gamma_{\rm cusp}(\lambda,\theta=\ln S)\,. \quad (107)$$

We note that the anomalous dimensions of the higher-twist operators are not determined by their Lorentz spin S alone. In fact, a band structure emerges for the anomalous dimensions parameterized by the additional hidden quantum numbers [63]. An explicit example of such a band structure can be studied in the weak-coupling regime, where the internal band structure is parameterized by the higher integrals of motion in the SL(2) spin chain.

10. Conclusion

In this brief review, we have tried to cover the most promising (in our view) lines of research on the gauge/string duality and the results obtained thus far in this field. Evidently, only the very first steps have been made on this path; nevertheless, even some modest progress achieved during recent years emphasizes the essential advantages of the approach being considered. For a few decades, the concept of duality discussed in the present review seemed to be a rather academic area of research until the results of the studies demonstrated their applicability to the analysis of the most intricate problems concerning the behavior of gauge theory in the strong-coupling regime.

We emphasize that the dual string description yielded concrete quantitative predictions for gauge theories, part of which have subsequently been confirmed by explicit calculations in field theory. Also, it is worthy noting that duality has and will have a counter-influence on string theory. Even now, it is clear that the transitional behavior between the perturbative and nonperturbative regimes in field theory requires knowledge of quantum gravity; therefore, well-known phenomena of field theory may shed light on a number of deep-lying problems in the gravity sector.

Of course, it would be most interesting to have the dual string description of the Standard Model. However, neither the metric in the dual description nor the corresponding higher-form fields has been found, despite extensive studies to this effect and the indisputable applicability of the general scheme to this case. According to the most optimistic point of view, the dual description will also give a powerful impetus to the solution of the confinement problem.

As regards the weak-coupling regime, the main hopes are linked with the summation of the perturbation theory series in the dual theory. Although numerous cancelations occur only in the $N=4$ theory, some examples considered in the foregoing text indicate that the string description permits us to fix universal properties of the perturbation theory series. Specifically, it would be extremely interesting to find the string realization of the Regge regime in QCD along with the corresponding effective degrees of freedom. The first steps in this direction have been made in Refs [74–76].

We tried to substantiate the extremely important part played by the hidden integrability of dual theories, at least in certain sectors or regimes. Indeed, the integrability reflects the existence of additional symmetries that were not found in preceding studies; it most strongly manifests itself in the reduction to the appropriate kinematic sectors. In particular, its role was explicitly observed in renormalization-group dynamics of the operators on a light cone in QCD and for generic operators in supersymmetric gauge theories.

Notwithstanding indubitable progress in this field, the most essential question, "what kind of hidden symmetry is responsible for the integrability in gauge theory?", remains open. The first attempts to consistently elucidate this symmetry showed [77–79] that it must be associated with the so-called nonlocal conservation laws known from the theory of integrable systems.

Apart from general questions pertaining to integrability, a few specific problems are worth mentioning. They include, among others, verification of the integrability of the dilatation operators in higher loops in the gauge theory and of the dual sigma-model at the quantum level. In any case, the methods of integrable systems have already demonstrated their efficiency in the studies of duality between gauge theories and strings; there is little doubt that they will find further application in the class of problems being considered.

In conclusion, we mention certain results obtained quite recently. For example, it was shown how the U(1)-problem is solved in the dual gravity theory [80]; also, a metric for nonconformal supersymmetric gauge theories with fundamental matter was found [81]. Some authors undertook to obtain the physical characteristics of mesons in the standard QCD in the dual theory [82].

We barely touched on the problem of the dual string description of gauge theory from first principles. Only minimal progress has been made thus far toward resolving this issue. Nevertheless, a few seemingly promising results deserve to be mentioned. A new approach to the summation of instanton effects has been developed [83], which has provided a basis for the hypothesis that gauge theory actually plays the role of an effective theory of microscopic gravitational degrees of freedom [84].

On the other hand, it has been noticed [85] that loop calculations in four-dimensional field theory may be reformulated as tree diagrams in five-dimensional space in the AdS_5 metric. Finally, a new mechanism of generating an effective gravity theory from the 'condensation' of special states in gauge theory was proposed in Ref. [86]. At the same time, the key problem of physical mechanisms underlying the generation of the metric condensate in quantum gravity remains to be solved despite some positive trends and advancements.

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