Physics – Uspekhi **48** (10) 999 – 1013 (2005)

METHODOLOGICAL NOTES

PACS numbers: **03.65.–w**, 03.65.Ta, **03.67.–a**

Uncertainty relation and the measurement error – perturbation relation

Yu I Vorontsov

DOI: 10.1070/PU2005v048n10ABEH005794

Contents

Abstract. The origins and physical consequences of the traditionally used relation between the position measurement error and the momentum perturbation, $\Delta_{\mathrm{m}}^2 x\, \Delta_{\mathrm{p}}^2 p \geqslant \hbar^2/4$, are discussed. It is demonstrated that the corresponding increase in the momentum variance for the aposteriori state occurs only in some special cases. The product of $\Delta_{\mathrm{m}}^2 A$ and $\Delta_{\mathrm{p}}^2 B$ is shown to essentially differ from the one given by the uncertainty relation if the commutator $[\hat{A}, \hat{B}]$ is an operator. The error quantum limits for the joint homodyne measurement of quadrature amplitudes for an optical mode are found. It is shown that similar results can be obtained if the quadratures of a harmonic oscillator are estimated by means of continuous position measurement.

1. Introduction

The fundamental relation between the position and momentum variances, known as the Heisenberg uncertainty relation,

$$\Delta^2 x\, \Delta^2 p \geqslant \frac{\hbar^2}{4}\,, \tag{1}$$

is the quantitative formulation of the Heisenberg uncertainty principle [1]. It characterizes the state of an object and directly follows from the fact that the operators $\hat{x}$ and $\hat{p}$ do not commute [2–4]. By definition, the quantities $\Delta^2 x$ and $\Delta^2 p$ depend only on the state of the object and are not related to measurement errors.

Experimental verification of relation (1) implies the following. One prepares an ensemble of many particles. For half of the particles, the positions are measured precisely (with an error much less than Δx), for the other half, the momentum measurement is performed. The obtained sets of numbers are then analyzed and the position and momentum variances are calculated.

There are relations similar in form to relation (1) but with a different meaning. Among them, the best known one is the relation between the error in the position measurement and the momentum perturbation. Traditionally, it is proved by considering a position measurement with the help of a microscope (the Heisenberg microscope) and written using the same notation as in relation (1). We represent it in the form

$$\Delta_{\mathrm{m}}^2 x\, \Delta_{\mathrm{p}}^2 p \geqslant \frac{\hbar^2}{4}\,, \tag{2}$$

where $\Delta_{\mathrm{m}}^2 x$ is the variance of the position measurement error and $\Delta_{\mathrm{p}}^2 p$ is the variance of the momentum perturbation. The error (inaccuracy) of the position measurement is understood as the difference between the measurement result and the true value of the position. The momentum perturbation is understood as the variation of the object momentum as a result of the interaction between the object and the measurement device (meter). According to the definition, neither the measurement error nor the perturbation depends on the state or dynamical parameters of the object.

Yu I Vorontsov Physics Department, Lomonosov Moscow State University, Vorob'evy Gory, 119899 Moscow, Russian Federation
Tel. (7-095) 939 39 03. Fax (7-095) 932 88 20
E-mail: vorontsov@phys.msu.ru

Received 13 May 2005, revised 28 June 2005
Uspekhi Fizicheskikh Nauk **175** (10) 1053 – 1068 (2005)
Translated by M V Chekhova; edited by A M Semikhatov

It is commonly supposed that relation (1) leads logically to the relation between the error variances in the case of a joint (simultaneous) measurement of the position and momentum,

$$\Delta^2_{\mathrm{m.s}} x \, \Delta^2_{\mathrm{m.s}} p \geqslant \frac{\hbar^2}{4} \, . \tag{3}$$

A joint measurement of the position and momentum is understood as a simultaneous interaction of the object with two meters, one of them measuring the position and the other the momentum.

Papers on the simultaneous measurement of the position and momentum deal with the relation that is called the true uncertainty principle for joint measurements in Ref. [5]. We write it as [6]

$$\Delta x_1 \, \Delta x_2 \geqslant \hbar \, , \tag{4}$$

where x_1 and x_2 are observables not of the object but of the meters. The values Δx_1 and Δx_2 are not the measurement errors but the standard deviations of the results of precise measurements of x_1 in one meter and x_2 in the other.

Relations (1)–(3) are as old as quantum mechanics itself. Nevertheless, the connection between them and their consequences are still subject to discussions and arguments.

At the dawn of quantum mechanics, there was a diffused opinion that the uncertainty relation itself follows from the abilities of the measurement devices and a quantum system should be described statistically because of the fundamental nature of the interaction involved in the measurement process [4, 7]. Traces of this viewpoint can still be found in modern textbooks on quantum mechanics, see, e.g., [8–10]. No contradiction is observed between the two different interpretations of the uncertainty relation: as a consequence of the general quantum mechanics principles and as a consequence of the abilities of measurement devices. From this standpoint, the fact that relation (2) is proved using examples (as a rule, involving a microscope) where the device itself (a light beam) is assumed to obey the uncertainty relation is ignored.

Relation (2) is usually understood as follows: measurement of the position of an object with an error $\Delta^2_{\mathrm{m}} x$ always leads to an increase in the variance of the momentum by $\Delta^2_{\mathrm{p}} p \geqslant \hbar^2/(4\Delta^2_{\mathrm{m}} x)$. This interpretation is erroneous. Under certain conditions, an approximate measurement of the position can bring the object into a state in which the momentum variance is less than the initial one [11–13] and less than $\hbar^2/(4\Delta^2_{\mathrm{m}} x)$ [14]. The meter may even leave no trace of its dynamical influence on the object, despite the decrease in the position variance.

Relation (3) is certainly true if it is written for the errors in the measurement of the position and momentum related to the state after the measurement. Otherwise, one could prepare a state contradicting the uncertainty principle. But because the generalized momentum of the object–meter system can be defined in different ways, in some cases relation (3) may not hold for values of the position and generalized momentum during the interaction between the object and the meters.

By analogy with the generalized uncertainty relation [3, 15, 16]

$$\Delta^2 A \, \Delta^2 B \geqslant \frac{\hbar^2}{4} \left|\langle \hat{C} \rangle\right|^2 , \qquad \hat{C} = \frac{[\hat{A}, \hat{B}]}{\mathrm{i}\hbar} \, , \tag{5}$$

one could try to write relations (1)–(3) in the form

$$\Delta^2_{\mathrm{m}} A \, \Delta^2_{\mathrm{p}} B \geqslant \frac{\hbar^2}{4} \left|\langle \hat{C} \rangle\right|^2 , \tag{6}$$

$$\Delta^2_{\mathrm{m.s}} A \, \Delta^2_{\mathrm{m.s}} B \geqslant \frac{\hbar^2}{4} \left|\langle \hat{C} \rangle\right|^2 . \tag{7}$$

But this would be a mistake!

The aim of the present paper is to investigate, in terms of the modern theory of measurements, the origins and the validity domains for relations (2) and (3) as well as to find correct relations instead of erroneous (6) and (7).

In Section 2, it is explained how the state of an object is understood in the quantum theory of measurements after its measurement, and the notions of selective and nonselective measurements are introduced. Using the standard quantum scheme for position measurement, it is proved that relation (2) follows from the uncertainty relation for the observables of the meter. Further, relations $\Delta_{\mathrm{m}} A \, \Delta_{\mathrm{p}} B$ and $\Delta_{\mathrm{m}} B \, \Delta_{\mathrm{p}} A$ are investigated in the case where $[\hat{A}, \hat{B}] = \mathrm{i}\hbar\hat{C}$, the transformation of the object state due to the measurement is described mathematically, and examples are given in which the change in the variance of the object momentum contradicts relation (2).

Next, correlations between the errors in joint measurements for noncommuting observables are investigated. In Section 3, three known measurement models are considered: (1) simultaneous interaction of the object with two meters, one of them being sensitive to the position and the other to the momentum; (2) joint measurement of the position and momentum using an ancillary degree of freedom, such that a joint measurement of noncommuting observables is replaced by a joint measurement of commuting combinations of observables; (3) homodyne measurement of the quadratures and the number of quanta for a single radiation mode. In addition, the origin of relation (4), as well as its connection with relation (3), is pointed out. In Section 4, the errors for the joint estimation of noncommuting observables via continuous position measurement are analyzed, and the stationary state of the object formed by such a procedure is described.

The main results of the work are summarized in Section 5.

2. Measurement error–perturbation relation

A description of the study outlined above inevitably requires using special terms and notions of the quantum theory of measurement. We briefly review the basic ones.

2.1 Some notions of the quantum theory of measurement

Measurement is a fundamentally irreversible process. Irreversibility can arise at the very first stage of the measurement, for instance, when the position of an electron is measured by observing a bright spot on a screen. Such a measurement is called direct. But if the position of an electron is used to measure another observable, for instance, the electric field through which the electron beam is passing, then the second stage of the measurement is irreversible. The first stage, i.e., the interaction of the electron with the field, is in this case reversible. This way of measuring the electric field through the position of an electron is called an indirect measurement.

In this scheme, the electron beam plays the role of a quantum readout system (QRS) [17, 18], i.e., a quantum converter or a microsensor [14]. After an indirect measurement, the object remains in its initial surrounding. An indirect measurement provides information about both the initial (apriori) state of the object and its final (aposteriori) state.

How does the measurement, i.e., interaction of the object with the meter, differ from its interaction with any other physical system? In a meter, the information about the object is converted from the quantum level to the classical one. According to the measurement results, the initial ensemble of objects can be divided into subensembles by means of certain classical operations. The states of these secondary ensembles are called states after a *selective* measurement [14].

Although special quantum properties inhere even in a single quantum system, the only way to verify that an object is prepared in a certain quantum state is to perform experiments on numerous indistinguishable objects or on a single object that is returned to the initial state numerous times. Speaking of the state of a single object always implies that the object is a representative of a certain ensemble. In an experiment on a single object, the selection process is virtual: it only consists of registering the result of the measurement. The state of the object after a selective measurement depends not only on the accuracy but also on the measurement result.

If there is no selection according to the measurement result, then the state of the object after the interaction with the meter can be viewed as a mixture of states that could be obtained through a selective measurement [11, 14]. In such a mixture, the probability distribution for the measured observable is the same as the initial one. Such measurements are called *nonselective* [14].

Because the notion of the state after the measurement is ambiguous, it is important to specify what is meant by the momentum perturbation due to the position measurement. It is useless to define the variance of the momentum perturbation as the variance of the difference $p(\tau) - p(0)$, where $p(\tau)$ and $p(0)$ are the respective momentum values after and before the measurement, because the measurement of $p(0)$ is incompatible with the position measurement. As a value with a physical meaning, one can take the difference of the momentum variances

$$\Delta^2 p(\tau) - \Delta^2 p(0) = \int \big(p - \langle p \rangle\big)^2 w_\tau(p)\,\mathrm{d}p - \int \big(p - \langle p \rangle\big)^2 w_0(p)\,\mathrm{d}p\,, \tag{8}$$

where $w_\tau(p)$ and $w_0(p)$ are the respective probability densities for the momentum in the final and the initial states of the object. In Eqn (8) and in what follows, the integrals are taken from $-\infty$ to ∞.

Expression (8) depends on whether the measurement is selective or nonselective.

2.2 Origins of the traditional measurement error – perturbation relation

We consider the standard quantum scheme for the indirect measurement of the position. For the meter to acquire information about the instantaneous values of the position, the interaction Hamiltonian $\hat{H}_\mathrm{i}$ for the object and the QRS should be linear functions of the position operator. Let $\hat{H}_\mathrm{i} = \alpha \hat{x} \hat{Y}$, where $\hat{Y}$ is some operator related to the QRS (position, momentum, position square, etc.) and α is the coupling coefficient. We represent the Hamiltonian of the object – meter system as

$$\hat{H} = \hat{H}_\mathrm{o} + \hat{H}_\mathrm{i} + \hat{H}_\mathrm{a}\,,$$

where $\hat{H}_\mathrm{o}$ and $\hat{H}_\mathrm{a}$ are the respective Hamiltonians of the object and the meter.

If $\hat{Y}$ is the position operator for the QRS, the equations of motion for the system can be written in the Heisenberg picture as

$$\text{(a)}\quad \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\,[\hat{p}, \hat{H}_\mathrm{o}] + \alpha \hat{Y}\,, \qquad \text{(b)}\quad \frac{\mathrm{d}\hat{P}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\,[\hat{P}, \hat{H}_\mathrm{a}] + \alpha \hat{x}\,, \tag{9}$$

where $\hat{P}$ is the QRS momentum operator, which is conjugate to the position Y. It follows from Eqn (9a) that during the position measurement, the meter acts on the object with a force (the fluctuation back-action force) whose operator is $\hat{F}_\mathrm{b.a} = \alpha \hat{Y}$.

If the measurement is 'instantaneous', i.e., the duration τ of the interaction between the object and the meter is so short that the evolutions of the object and the meter can be ignored ($\tau \to 0$, $\alpha\tau$ is finite), then Eqns (9) imply that

$$\text{(a)}\quad \hat{p}(\tau) = \hat{p}(0) + \alpha\tau \hat{Y}(0)\,, \qquad \text{(b)}\quad \hat{P}(\tau) = \hat{P}(0) + \alpha\tau \hat{x}(0)\,. \tag{10}$$

For the meter, information about the object position is contained in the variation in the QRS momentum. The meter obtains the estimate for the position $x(0)$ of the object by measuring the QRS momentum $P(\tau)$. If the measurement yields a value $\tilde{P}$, then the most probable value is taken as the estimate of the object position,

$$\tilde{x} = \frac{1}{\alpha\tau}\big(\tilde{P} - \big\langle \hat{P}(0) \big\rangle\big)\,, \tag{11}$$

where $\big\langle \hat{P}(0) \big\rangle$ is the QRS momentum mean value in the initial state.

The variance of the measurement error is defined as the conditional (corresponding to a given position) variance of the estimate. In this particular case, it is

$$\Delta_\mathrm{m}^2 x = \frac{1}{(\alpha\tau)^2}\big(\Delta_\mathrm{m}^2 \tilde{P} + \Delta^2 P(0)\big) \geqslant \frac{\Delta^2 P(0)}{(\alpha\tau)^2}\,, \tag{12}$$

where $\Delta_\mathrm{m}^2 \tilde{P}$ is the variance of the QRS momentum measurement error and $\Delta^2 P(0)$ is the QRS momentum variance in the initial state. The distribution of possible estimates for a given state of the object is characterized by the unconditional variance of the estimate,

$$\Delta^2 \tilde{x} = \Delta_\mathrm{m}^2 x + \Delta^2 x(0)\,, \tag{13}$$

where $\Delta^2 x(0)$ is the apriori variance of the object position.

After the interaction with the meter, according to (10), the variance of the object momentum becomes

$$\Delta^2 p(\tau) = \Delta^2 p(0) + (\alpha\tau)^2 \Delta^2 Y(0)\,, \tag{14}$$

which exceeds the variance in the initial state by

$$\Delta_\mathrm{p}^2 p = (\alpha\tau)^2 \Delta^2 Y(0)\,. \tag{15}$$

From (12) and (15), we have

$$\Delta_\mathrm{m}^2 x\, \Delta_\mathrm{p}^2 p \geqslant \Delta^2 P(0)\, \Delta^2 Y(0)\,. \tag{16}$$

Hence, relation (2) is valid as long as the uncertainty relation holds for the QRS state. (In the example with the microscope, the validity of this condition is in fact postulated for the QRS that is a light beam.) Therefore, relation (2) between the position measurement error and the momentum perturbation can only be considered a consequence of the uncertainty relation. In fact, relation (2) is the uncertainty relation for the meter written in terms of the observables of the object.

Note. In some works, the product of relations (13) and (14) is considered [19]. Written in a convenient form, it becomes

$$\Delta^2 \tilde{x}\, \Delta^2 p(\tau) \geqslant \hbar^2 ,$$

which is the relation between the unconditional variance of the position estimate, which depends on the initial states of the object and the meter, and the unconditional variance of the object momentum after the measurement.

2.3 Relation between the measurement error of $\hat{A}$ and the perturbation of $\hat{B}$ in the case where $[\hat{A}, \hat{B}]$ is an operator

The fundamental difference between the uncertainty relation for the object and the measurement error – perturbation relation becomes evident in the case where the commutator of the operator $\hat{A}$ corresponding to the measured observable and the operator $\hat{B}$ corresponding to the perturbed observable is an operator: $[\hat{A}, \hat{B}] = \mathrm{i}\hbar\hat{C}$. For instance, if $\hat{A} = \hat{x}$, $\hat{B} = \hat{p}^2$, then it follows from (5) that

$$\Delta^2 x\, \Delta^2 p^2 \geqslant \hbar^2 \left|\langle \hat{p} \rangle\right|^2 , \tag{17}$$

where $\langle \hat{p} \rangle$ is the mean value of the object momentum.

If a relation similar to (5) were valid for the product of the variances of the position measurement error and the momentum square perturbation, then the variance of the momentum square perturbation would depend not on the initial variance of the momentum but only on its mean value. However, according to (10a), the change in the variance of the momentum square for the object is

$$\begin{aligned} \Delta^2 p^2 &= \Delta^2 p^2(\tau) - \Delta^2 p^2(0) \\ &= 4\left\langle p^2(0)\right\rangle (\alpha\tau)^2 \Delta^2 Y + (\alpha\tau)^4 \Delta^2 Y^2 , \end{aligned} \tag{18}$$

$$\Delta^2 Y^2 = \langle Y^4 \rangle - \langle Y^2 \rangle^2 .$$

From (12) and (18), we obtain

$$\Delta_{\mathrm{m}}^2 x\, \Delta_{\mathrm{p}}^2 p^2 = 4\left\langle p^2(0)\right\rangle \Delta^2 P\, \Delta^2 Y + (\alpha\tau)^2 \Delta^2 P\, \Delta^2 Y^2 . \tag{19}$$

According to relation (5),

$$\Delta^2 P\, \Delta^2 Y^2 \geqslant \hbar^2 \langle Y \rangle^2 .$$

Hence, for $\langle Y \rangle = 0$, the second term in (19) may be equal to zero. The corresponding state was studied in Ref. [20]. In the Hilbert space, a Gaussian state asymptotically tends to this state as $\Delta^2 Y \to 0$.

Because

$$\Delta^2 Y^2 = \langle Y^4 \rangle - \langle Y^2 \rangle^2 = 2(\Delta^2 Y)^2$$

for a Gaussian state of a QRS, the second term in (19) is

$$\frac{\hbar^2 (\alpha\tau)^2}{2}\, \Delta^2 Y \geqslant \frac{\hbar^4 (\alpha\tau)^2}{8\Delta^2 P(0)} = \frac{\hbar^4}{8\Delta_{\mathrm{m}}^2 x} .$$

It follows that the relation between the position measurement error and the momentum square perturbation can be represented as

$$\Delta_{\mathrm{m}}^2 x\, \Delta_{\mathrm{p}}^2 p^2 \geqslant \hbar^2 \left\langle p^2(0)\right\rangle + \frac{\hbar^4}{8\Delta_{\mathrm{m}}^2 x} . \tag{20}$$

This relation is essentially different from uncertainty relation (17), even in the case where the second term can be neglected. The right-hand side of (20) depends on the mean value of the momentum square, while relation (17) involves the square of the momentum mean value.

One can easily prove that the relation between the momentum square measurement error $\Delta_{\mathrm{m}}^2 p^2$ and the position perturbation $\Delta_{\mathrm{p}}^2 x$ differs not only from the corresponding uncertainty relation but also from the position measurement error – momentum square perturbation relation (20). By representing the interaction Hamiltonian in the form $\hat{H}_{\mathrm{i}} = \alpha \hat{p}^2 \hat{Y}$ and solving the corresponding relations, we find that

$$\Delta_{\mathrm{m}}^2 p^2\, \Delta_{\mathrm{p}}^2 x \geqslant \hbar^2 \left\langle p^2(0)\right\rangle . \tag{21}$$

In general, the product of the measurement error and the perturbation $\Delta_{\mathrm{m}}^2 A\, \Delta_{\mathrm{p}}^2 B$ depends on the commutators

$$[\hat{C}, \hat{A}], \quad [[\hat{C}, \hat{A}], \hat{A}], \quad [[[\hat{C}, \hat{A}], \hat{A}], \hat{A}], \quad \ldots .$$

If $[\hat{C}, \hat{A}] = 0$, then

$$\Delta_{\mathrm{m}}^2 A\, \Delta_{\mathrm{p}}^2 B \geqslant \frac{\hbar^2}{4} \left\langle \hat{C}^2(0) \right\rangle . \tag{22}$$

If $[\hat{C}, \hat{A}] = \mathrm{i}\hbar\beta$, where β is a c-number, we have

$$\Delta_{\mathrm{m}}^2 A\, \Delta_{\mathrm{p}}^2 B \geqslant \frac{\hbar^2}{4} \left\langle \hat{C}^2(0) \right\rangle + \beta^2 \frac{\hbar^4}{8\Delta_{\mathrm{m}}^2 A} . \tag{23}$$

Finally, if multiple commutators $[\ldots[\hat{C}, \hat{A}], \hat{A}], \ldots, \hat{A}]$ are nonzero, then the right-hand side of the relation under study contains terms proportional to $\hbar^n / (\Delta_{\mathrm{m}}^{(n-2)} A)$, with $n = 4, 6, 8, \ldots$.

To what state are inequalities (2) and (21) – (23) related? Relations (14) – (23) were derived in the Heisenberg picture, with all averaging performed over the initial states of the object and the meter. In the Schrödinger picture, this corresponds to averaging over the entangled state of the object and the meter, which is formed due to the interaction.

We suppose that before the interaction, the object is in a pure state $|\psi_{\mathrm{o}}\rangle$ and the QRS is in a pure state $|\psi_{\mathrm{a}}\rangle$. Then, after the interaction (at time τ) but before the measurement of $\hat{P}(\tau)$ in the meter, the meter and the object are in the entangled state

$$|\Psi_{\mathrm{ent}}\rangle = \exp\left(-\mathrm{i}\hat{H}_{\mathrm{i}}\tau\right) \left|\psi_{\mathrm{a}}(0)\right\rangle \otimes \left|\psi_{\mathrm{o}}(0)\right\rangle .$$

(Hereafter, we set $\hbar = 1$.) In the x- and Y-representation, the wave function is given by

$$|\Psi_{\mathrm{ent}}\rangle = \iint |Y\rangle\, \psi_{\mathrm{o}}(x)\, \psi_{\mathrm{a}}(Y) \exp\left(-\mathrm{i}\alpha\tau x Y\right) |x\rangle\, \mathrm{d}x\, \mathrm{d}Y . \tag{24}$$

The probability distribution of the object position corresponding to entangled state (24),

$$w_{\text{ent}}(x) = \langle \Psi_{\text{ent}}|x\rangle\,\langle x|\Psi_{\text{ent}}\rangle = \int \left|\psi_{\text{a}}(Y)\right|^2 \left|\psi_{\text{o}}(x)\right|^2 \mathrm{d}Y = w_{\text{o}}(x),$$

is the same as in the initial state $|\psi_{\text{o}}\rangle$. This calculation does not lead to wave packet narrowing due to the interaction. Hence, we are dealing with a nonselective measurement of the position.

For entangled state (24), the probability density of the object momentum is

$$w_{\text{ent}}(p) = \langle \Psi_{\text{ent}}|p\rangle\langle p|\Psi_{\text{ent}}\rangle = \int \left|\varphi_{\text{o}}(\alpha\tau Y + p)\right|^2 \left|\psi_{\text{a}}(Y)\right|^2 \mathrm{d}Y,$$

$$\varphi_{\text{o}}(\alpha\tau Y + p) = \int \exp\left(-\mathrm{i}(p + \alpha\tau Y)x\right)\psi_{\text{o}}(x)\,\mathrm{d}x\,,$$

$$\psi_{\text{o}}(x) = \left\langle x|\psi_{\text{o}}(0)\right\rangle, \qquad \psi_{\text{a}}(Y) = \left\langle Y|\psi_{\text{a}}(0)\right\rangle.$$

The function $w_{\text{ent}}(p)$ is the probability density of a sum of two independent random variables $p + \alpha\tau Y$. This probability distribution for the momentum corresponds to the variance in (14).

2.4 Measurement error and perturbation in the case of a selective measurement

We consider how the state of the object transforms when the initial ensemble is divided into subensembles according to the measurement result.

2.4.1 State of the object after a precise measurement in the QRS. A precise measurement of $\hat{P}$, giving a result $\tilde{P}$, transforms any state into the state $|\tilde{P}\rangle$. This transformation is performed by the projection operator $|\tilde{P}\rangle\langle\tilde{P}|$. Accordingly, the object – QRS system passes from the entangled state $|\Psi_{\text{ent}}\rangle$ into the (nonnormalized) state

$$|\tilde{P}\rangle\langle\tilde{P}|\Psi_{\text{ent}}\rangle = |\tilde{P}\rangle \otimes \left|\psi_{\text{o}}(\tilde{P})\right\rangle,$$

where

$$\left|\psi_{\text{o}}(P)\right\rangle = \int \varphi_{\text{a}}(P + \alpha\tau x)\,\psi_{\text{o}}(x)|x\rangle\,\mathrm{d}x\,,$$

$$\varphi_{\text{a}}(P + \alpha\tau x) = \int \langle P|Y\rangle\,\psi_{\text{a}}(Y)\exp\left(-\mathrm{i}\alpha\tau x Y\right)\mathrm{d}Y.$$

Thus, a precise measurement of the QRS momentum transforms the entangled state into the product of nonentangled states $|\tilde{P}\rangle$ and $\left|\psi_{\text{o}}(\tilde{P})\right\rangle$. The state vector $\left|\psi_{\text{o}}(\tilde{P})\right\rangle$ represents the state of the object after a selective measurement of the position.

Because $\tilde{P}$ sets the estimate $\tilde{x} = \tilde{P}/\alpha\tau$ of the object position, the state vector $\left|\psi_{\text{o}}(\tilde{P})\right\rangle$ can be represented as

$$\left|\psi_{\text{o}}(\tilde{x})\right\rangle = \int \varphi_{\text{a}}(\tilde{x}|x)\,\psi_{\text{o}}(x)\,|x\rangle\,\mathrm{d}x\,, \tag{25}$$

$$\varphi_{\text{a}}(\tilde{x}|x) = (\alpha\tau)^{1/2}\,\varphi_{\text{a}}(\tilde{P} + \alpha\tau x)\,. \tag{26}$$

Transformation of the initial (apriori) state of the object

$$|\psi_{\text{o}}\rangle = \int \psi_{\text{o}}(x_1)\,|x_1\rangle\,\mathrm{d}x_1$$

into aposteriori state (25) can be considered the result of the reduction operator acting on the state $|\psi_{\text{o}}\rangle$ [17, 21],

$$\hat{R}(\tilde{x}) = \int \varphi_{\text{a}}(\tilde{x}|x)\,|x\rangle\langle x|\,\mathrm{d}x = \varphi_{\text{a}}(\tilde{x}|\hat{x})\,. \tag{27}$$

The aposteriori state turned out to be pure because the initial states of both the object and the QRS were pure and the measurement of the QRS momentum was precise.

2.4.2 Aposteriori distributions of the position and momentum of the object. The aposteriori distributions of the object observables can be found from Eqn (25) in accordance with the known rules (considering an idealized measurement).

The normalized wave function corresponding to vector (25) is

$$\psi(x|\tilde{x}) = \frac{\varphi_{\text{a}}(\tilde{x}|x)\,\psi_{\text{o}}(x)}{w^{1/2}(\tilde{x})}\,, \tag{28}$$

where

$$w(\tilde{x}) = \int w(\tilde{x}|x)\,w_{\text{o}}(x)\,\mathrm{d}x$$

is the unconditional distribution density for the estimate $\tilde{x}$.

Transformation (28) of the initial wave function is similar to its transformation by means of a spatial filter with the transmission coefficient $\varphi_{\text{a}}(\tilde{x}|x)$. In other words, a selective measurement of the position changes the state of the object in the same way as a spatial filter whose transmission coefficient depends on the initial state of the object and the result of the position measurement.

According to (28), the aposteriori probability density distribution for the object position,

$$w(x|\tilde{x}) = \frac{w(\tilde{x}|x)\,w_{\text{o}}(x)}{w(\tilde{x})}\,, \tag{29}$$

is connected with the conditional density distribution for the estimate, $w(\tilde{x}|x) = \left|\varphi_{\text{a}}(\tilde{x}|x)\right|^2$, and the apriori density distribution for the position, $w_{\text{o}}(x) = \left|\psi_{\text{o}}(x)\right|^2$, by means of the well-known Bayes relation [22].

By analogy with (29), formula (28), which contains wave functions rather than the corresponding probability density distributions, can be called the *quantum analog of the Bayes relation*.

The aposteriori probability density distribution for the object momentum is

$$w(p|\tilde{x}) = \frac{1}{2\pi}\left|\int \varphi_{\text{a}}(\tilde{x} + x)\,\psi_{\text{o}}(x)\exp\left(-\mathrm{i}px\right)\mathrm{d}x\right|^2.$$

The aposteriori mean value of the momentum square is

$$\langle p^2\rangle = \int \left|\psi'(\tilde{x}|x)\right|^2 \mathrm{d}x = \int \left|\varphi'_{\text{a}}(\tilde{x}|x)\,\psi_{\text{o}}(x) + \varphi_{\text{a}}(\tilde{x}|x)\,\psi'_{\text{o}}(x)\right|^2 \mathrm{d}x\,. \tag{30}$$

2.4.3 Examples of the object state transformation due to selective measurement. *Example* 1. If the initial states of the object and the QRS are purely Gaussian, with minimal uncertainties, the aposteriori state of the object is also

Gaussian, the inverse variance of the position being given by

$$\frac{1}{\Delta^2 x(\tilde{x})} = \frac{1}{\Delta_0^2 x} + \frac{1}{\Delta_\mathrm{m}^2 x}\,.$$

The aposteriori distribution of the momentum is also Gaussian, with the variance

$$\Delta^2 p(\tilde{x}) = \frac{\hbar^2}{4\Delta^2 x(\tilde{x})}\,.$$

Hence, the momentum variance is increased by

$$\Delta_\mathrm{p}^2 p = \frac{\hbar^2}{4\Delta_\mathrm{m}^2 x}\,. \tag{31}$$

Relation (31), describing a selective measurement of the position, is the equivalent of relation (2). This example was traditionally considered a proof of the validity of relation (2) in the case of selective measurement of the position. At first sight, it confirms the widespread opinion that a selective measurement of the position inevitably increases the momentum variance in accordance with relation (2). However, this statement is not true. After a selective measurement of the position, the momentum variance may become smaller than its initial value. To show this, one could again consider Gaussian states, but with the correlation between the position and momentum of the object [11]. This reasoning can be made even more evident by considering other examples [12, 13].

Example 2. Let the function $\psi_\mathrm{o}(x)$ have the shape of a rectangular pulse:

$$\psi_\mathrm{o}(x) = \begin{cases} \dfrac{1}{(2l)^{1/2}} & \text{for} \quad -l < x < l\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{32}$$

The corresponding momentum variance is $\Delta_\mathrm{o}^2 p = \infty$. Let

$$\varphi_\mathrm{a}(\tilde{x}|x) = \begin{cases} \left(\dfrac{k}{3\pi}\right)^{1/2} \big(1 + \cos k(x - \tilde{x})\big) & \text{for} \quad -\dfrac{\pi}{k} \leqslant x - \tilde{x} \leqslant \dfrac{\pi}{k}\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{33}$$

where $\pi/k < l$.

We suppose that the measurement gives the value $\tilde{x}$ such that $|\tilde{x}| \leqslant l - \pi/k$. Functions (32) and (33) corresponding to this case are plotted in Fig. 1. According to Eqn (28), the a posteriori wave function coincides in shape with the meter function $\varphi_\mathrm{a}(\tilde{x}|x)$.

The function $\varphi_\mathrm{a}(\tilde{x}|x)$ in (33) and its first derivative are continuous. Hence, the aposteriori variance of the momentum is finite. Thus, the position measurement can transform the initial state with an infinite momentum variance into a state with a finite momentum variance.

In this example, it is not necessary to assume the apriori momentum variance to be infinite. There are many functions $\psi_\mathrm{o}(x)$ and $\varphi_\mathrm{a}(\tilde{x}|x)$ for which, in the case of certain results of the position measurement, the aposteriori momentum variance becomes less than the apriori one.

The aposteriori variance of the momentum is finite in this example only at $|\tilde{x}| < l - \pi/k$. Otherwise, the aposteriori momentum variance is infinite, similarly to the variance in the initial state. Moreover, the aposteriori probability distribution function of the momentum is lower than the apriori distribution.

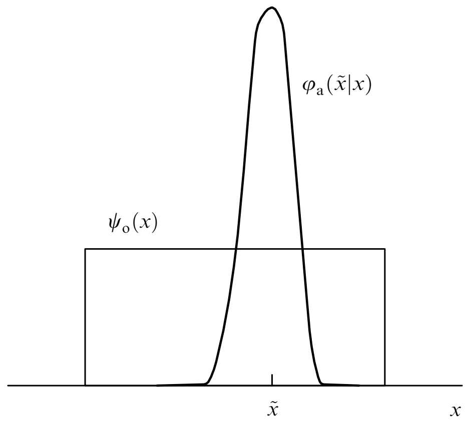

Figure 1. Example 2 illustration.

We note the following nontrivial fact. The aposteriori variance of the object momentum corresponding to Fig. 1,

$$\Delta^2 p(\tilde{x}) = (\alpha\tau)^2\,\Delta^2 Y\,,$$

coincides with (15), i.e., is equal to the unconditional variance of the momentum perturbation. Thus, in this example, the apriori uncertainty of the object momentum has no influence on the aposteriori state.

In the case shown in Fig. 1, the measurement acts as a spatial filter and selects only particles whose positions are in the domain where $\psi_\mathrm{o}'(x) = 0$. This looks as if all particles with such positions had the same momentum before the measurement. This statement does not contradict the uncertainty relation because it describes the past, while the uncertainty relation considers the present and the future.

Similarly to the previous example, this situation could be erroneously considered as proving the statement that the meter perturbs the momentum in accordance with relation (2). The variance decrease in the aposteriori state could be viewed as a consequence of the special initial state of the object. However, under certain measurement conditions, the variance of the object momentum can be equal to its initial value. In other words, the meter can leave no trace of its random dynamical influence.

Example 3. This time, we suppose that the function $\varphi_\mathrm{a}(\tilde{x}|x)$ is rectangular-shaped while the function $\psi_\mathrm{o}(x)$ is bell-shaped:

$$\varphi_\mathrm{a}(\tilde{x}|x) = \begin{cases} \dfrac{1}{(2l)^{1/2}}\,, & -l < x - \tilde{x} < l\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{34}$$

$$\psi_\mathrm{o}(x) = \begin{cases} \left(\dfrac{k}{3\pi}\right)^{1/2} \big(1 + \cos k(x - \tilde{x})\big)\,, & -\dfrac{\pi}{k} \leqslant x - \tilde{x} \leqslant \dfrac{\pi}{k}\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{35}$$

where $\pi/k < l$. For $|\tilde{x}| < l - \pi/k$, the aposteriori state is identical to the apriori one. Despite the interaction with the meter, which was in a state with $\Delta^2 Y = \infty$, corresponding to $\Delta_\mathrm{p}^2 p = \infty$, the variance of the object momentum has not

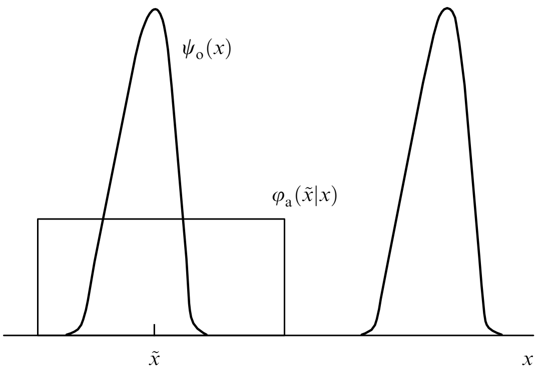

Figure 2. Example 3 illustration.

changed. True, the uncertainty of the position has not changed, either.

But we next suppose that the initial wave function of the object is a sequence of nonoverlapping identical bell-shaped functions,

$$\psi_N(x) = \frac{1}{\left\|\psi_N(x)\right\|^{1/2}} \sum_{m=1}^{N} \psi_{\rm o}(x + mL)\,,$$

with L being the period and only a single 'bell' being covered by the meter function $\varphi_{\rm a}(\tilde{x}|x)$ (Fig. 2). The aposteriori wave function then consists of a single 'bell', as in the previous example.

The apriori uncertainty of the position is mainly determined by the distance between the first and the last 'bells' of the sequence, while the aposteriori one is given by the width of a single 'bell'. In this case, the measurement leads to a decrease in the position variance. But what happens with the momentum variance? It does not change!

Indeed, the mean momentum square calculated for the sequence $\psi_N(x)$ is

$$\begin{aligned}\langle p_N^2\rangle &= \frac{1}{\|\psi_N\|}\int_{-\infty}^{\infty} \left|\psi_N'(x)\right|^2 {\rm d}x = \frac{N}{\|\psi_N\|}\int_{-\pi/k}^{\pi/k} \left|\psi_{\rm o}'(x)\right|^2 {\rm d}x \\ &= \frac{1}{\|\psi_{\rm o}\|}\int_{-\pi/k}^{\pi/k} \left|\psi_{\rm o}'(x)\right|^2 {\rm d}x = \langle p_{\rm o}^2\rangle\,,\end{aligned}$$

where $\langle p_{\rm o}^2\rangle$ is the mean momentum square corresponding to a single 'bell' $\psi_{\rm o}(x)$.

In this example, the fact that the momentum variance is constant does not mean that the momentum distribution is constant. The function $w_{\rm o}(p)$ for a single bell is smooth, while the function $w_N(p)$ has a periodic modulation. The measurement of the position has destroyed the quantum correlations in the initial pure state. If the initial state were a mixture of bell-shaped states, the aposteriori momentum distribution would be the same as the initial one.

The absence of the fluctuation influence of the meter on the aposteriori state of the object does not mean the absence of any dynamical effect. In the state of the meter described by (34), the mean value $\alpha\tau\langle Y\rangle$ of the back-action force is zero. If one changes $\langle Y\rangle$ without changing the absolute value of the function $\varphi_{\rm a}(\tilde{x}|x)$, the mean momentum of the object in the aposteriori state changes by $\alpha\tau\langle Y\rangle$. (This is the difference between the measurement considered here and the so-called interaction-free measurement [23].)

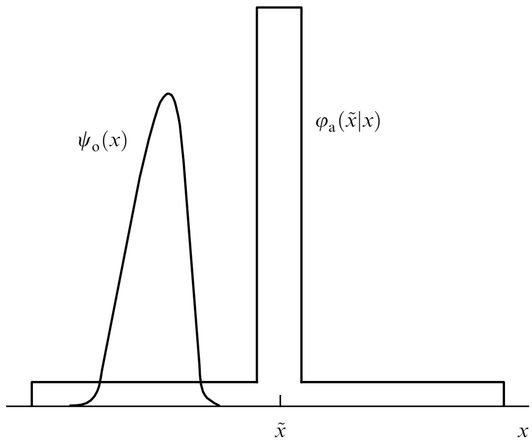

Figure 3. An example demonstrating that the aposteriori variance of the object momentum can be less than $\hbar^2/(4\Delta_{\rm m}^2 x)$.

The situation considered in this example can occur if the position of a body is measured through the arrival time of a reflected photon, the departure time of the photon having a rectangular probability distribution $w(\tau)$. The picture looks as if photons for which $w(\tau)$ is flat had the same momenta.

A similar situation can occur if the phase of an oscillator is measured by means of a position null detector accompanied by the interaction of the oscillator with the meter in the vicinity of zero position [24].

It is also worth noting that the variance fully characterizes the distribution of an observable only for Gaussian states. In other cases, using the momentum variance as a characteristic of a state is justified by the fact that this variance determines the minimal value of the mean kinetic energy of the object.

2.4.4 Relation between the aposteriori momentum variance and the position measurement error. The relation between $\Delta^2 p(\tilde{x})$ and $\hbar^2/(4\Delta_{\rm m}^2 x)$ has been studied in detail in Ref. [14]. As we have seen, the aposteriori momentum variance $\Delta^2 p(\tilde{x})$ can be less than the apriori one and can be equal to the momentum perturbation variance. But can it be less than $\hbar^2/(4\Delta_{\rm m}^2 x)$? Mensky has shown that such a situation is possible (Fig. 3). After the position measurement, the momentum variance remains the same as in the initial state. As regards the value

$$\Delta_{\rm m}^2 x = \int (\tilde{x} - x)^2 \left|\varphi_{\rm a}(\tilde{x}|x)\right|^2 {\rm d}x\,,$$

for relatively low wings of the function $\varphi_{\rm a}(\tilde{x}|x)$, it depends only on the width of its central part, which can be arbitrarily small. Hence, the value of $\hbar/(\Delta_{\rm m} x)$ can be larger than $\Delta p(\tilde{x})$. However, the momentum variance for entangled state (24) is equal to (14) and always exceeds $\hbar^2/(4\Delta_{\rm m}^2 x)$.

3. Errors in the joint measurement of position and momentum

A joint (simultaneous) measurement of two observables is usually understood as a measurement in which two meters simultaneously interact with the object. There are two versions of the mathematical model describing such a measurement for the position and momentum. In the first version, one of the QRS's is directly sensitive to the position and the other to the momentum [5, 6]. The Hamiltonian of the interaction between the object and the meter is given by a sum of two terms,

$$\hat{H}_{\mathrm{i}} = -\alpha\hat{x}\hat{Y} + \beta\hat{p}\hat{Z}\,.$$

In the other version, the measurement scheme involves another degree of freedom $(y,\, p_y)$ in the minimal-uncertainty state $(\Delta y\,\Delta p_y = \hbar/2)$.

The interaction Hamiltonian is taken in the form

$$\hat{H}_{\mathrm{i}} = -\alpha(\hat{x}-\hat{y})\hat{Y} + \beta(\hat{p}+\hat{p}_y)\hat{Z}\,. \tag{36}$$

This enables converting the joint measurement of noncommuting variables $\hat{x}$ and $\hat{p}$ into the joint measurement of commuting variables

$$\hat{Q}_{-} = \hat{x}-\hat{y}\,, \qquad \hat{P}_{+} = \hat{p}+\hat{p}_y\,.$$

(A measurement of this kind is considered optimal in Ref. [25].) Formally, the measurement accuracy for the observables $\hat{Q}_-$ and $\hat{P}_+$ is not limited. According to this, the error limits of estimating x and p are equal to the uncertainties of y and p_y.

In reality, joint measurements of noncommuting observables are performed in completely different schemes, which do not correspond to the interactions introduced above. In what follows, we consider some examples of such as a homodyne measurement of quadrature amplitudes for a radiation mode and a scheme of continuous position measurement.

We first consider the features of standard schemes using the example of a nonselective joint measurement of position and momentum. (The state after the selective measurement in the first scheme was studied in Refs [5, 6].)

3.1 Direct joint measurement of position and momentum

The total Hamiltonian of the object and the QRS in this scheme is

$$\hat{H} = \hat{H}_{\mathrm{o}} - \alpha\hat{x}\hat{Y} + \beta\hat{p}\hat{Z} + \hat{H}_{\mathrm{a},x} + \hat{H}_{\mathrm{a},p}\,, \tag{37}$$

where $\hat{H}_{\mathrm{a},x}$, $\hat{H}_{\mathrm{a},p}$ and $\hat{Y}$, $\hat{Z}$ are the Hamiltonians and the position operators of the QRS that are sensitive to x and p, respectively.

We assume that $\hat{H}_{\mathrm{o}} = \hat{p}^2/2m$. Hamiltonian (37) leads to the equations

$$\begin{aligned} &\text{(a)}\quad \frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{\hat{p}}{m} + \beta\hat{Z}\,,\\ &\text{(b)}\quad \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \alpha\hat{Y}\,, \end{aligned} \tag{38}$$

$$\begin{aligned} &\text{(a)}\quad \frac{\mathrm{d}\hat{P}_y}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\hat{P}_y, \hat{H}_{\mathrm{a},x}] + \alpha\hat{x}\,,\\ &\text{(b)}\quad \frac{\mathrm{d}\hat{P}_z}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\hat{P}_z, \hat{H}_{\mathrm{a},p}] - \beta\hat{p}\,. \end{aligned} \tag{39}$$

In the instantaneous-measurement approximation ($\alpha\tau^2 \to 0$, $\beta\tau^2 \to 0$), their solutions are

$$\begin{aligned} &\text{(a)}\quad \hat{x}(\tau) = \hat{x}(0) + \beta\tau\hat{Z}\,,\\ &\text{(b)}\quad \hat{p}(\tau) = \hat{p}(0) + \alpha\tau\hat{Y}\,, \end{aligned} \tag{40}$$

$$\begin{aligned} &\text{(a)}\quad \hat{P}_y(\tau) = \hat{P}_y(0) + \alpha\tau\,\hat{x}(0) + \frac{1}{2}\,\alpha\beta\tau^2\hat{Z}\,,\\ &\text{(b)}\quad \hat{P}_z(\tau) = \hat{P}_z(0) - \beta\tau\,\hat{p}(0) - \frac{1}{2}\,\alpha\beta\tau^2\hat{Y}\,. \end{aligned} \tag{41}$$

After $P_y(\tau)$ and $P_z(\tau)$ have been measured, one can estimate the position and momentum of the object at any time instant. In particular, one can obtain their estimates before the measurement $(x(0),\, p(0))$, at the final moment of interaction $(x(\tau),\, p(\tau))$, and at the moment $\tau/2$ $(x(\tau/2),\, p(\tau/2))$. If $\langle P_y(0)\rangle = \langle P_z(0)\rangle = \langle Y\rangle = \langle Z\rangle = 0$, then in all three cases, the estimate of the position, according to the maximum probability method, is $\tilde{P}_y/\alpha\tau$ and the estimate of the momentum is $\tilde{P}_z/\beta\tau$, where $\tilde{P}_y$ and $\tilde{P}_z$ are the results of the QRS momentum measurement. The estimation errors are different.

Estimation errors for $x(0)$ and $p(0)$. It follows from (41) that the estimation error variances for $x(0)$ and $p(0)$ in the approximation of the precise measurement for $P_y(\tau)$ and $P_z(\tau)$ are

$$\begin{aligned} &\text{(a)}\quad \Delta^2_{\mathrm{m.s}}\,x(0) = \Delta^2_{\mathrm{m}}x + \frac{1}{4}\,\Delta^2_{\mathrm{p}}x\,,\\ &\text{(b)}\quad \Delta^2_{\mathrm{m.s}}\,p(0) = \Delta^2_{\mathrm{m}}\,p + \frac{1}{4}\,\Delta^2_{\mathrm{p}}\,p\,, \end{aligned} \tag{42}$$

where

$$\Delta^2_{\mathrm{m}}x = \frac{\Delta^2 P_y(0)}{(\alpha\tau)^2}\,, \qquad \Delta^2_{\mathrm{p}}x = (\beta\tau)^2\,\Delta^2 Z\,,$$

$$\Delta^2_{\mathrm{m}}\,p = \frac{\Delta^2 P_z(0)}{(\beta\tau)^2}\,, \qquad \Delta^2_{\mathrm{p}}\,p = (\alpha\tau)^2\,\Delta^2 Y$$

are the variances of the measurement errors and the perturbation variances for separate measurements of position and momentum. The second terms on the right-hand sides of relations (42) appear due to the influence of the cross-perturbation of the position by the momentum meter and the cross-perturbation of the momentum by the position meter on the measurement errors. After simple transformations of the product $\Delta^2_{\mathrm{m.s}}\,x(0)\,\Delta^2_{\mathrm{m.s}}\,p(0)$, taking into account that

$$\Delta^2 P_y\,\Delta^2 Y \geqslant \frac{\hbar^2}{4}\,, \qquad \Delta^2 P_z\,\Delta^2 Z \geqslant \frac{\hbar^2}{4}\,,$$

$$\Delta^2 P_y\,\Delta^2 P_z + \left(\frac{\alpha\beta\tau^2}{2}\right)^4 \Delta^2 Y\,\Delta^2 Z \geqslant \frac{1}{8}\,(\alpha\beta\tau^2)^2\hbar^2\,,$$

we obtain

$$\Delta^2_{\mathrm{m.s}}\,x(0)\,\Delta^2_{\mathrm{m.s}}\,p(0) \geqslant \frac{\hbar^2}{4}\,. \tag{43}$$

The minimum is achieved at

$$\Delta^2 Y\,\Delta^2 Z = \frac{\hbar^2}{(\alpha\beta\tau^2)^2}\,. \tag{44}$$

In principle, the second terms in (42) can be eliminated by measuring in the QRS not $P_y(\tau)$ and $P_z(\tau)$ but the combinations

$$\hat{Q}_x = \hat{P}_y - \frac{1}{2}\,\alpha\beta\tau^2\hat{Z}\,,$$

$$\hat{Q}_{\mathrm{p}} = \hat{P}_z + \frac{1}{2}\,\alpha\beta\tau^2\hat{Y}\,.$$

However, this cannot change relation (43). A joint precise measurement of these combinations is impossible because they do not commute. On the right-hand sides of (42), $\Delta_{\mathrm{p}}^2 x$ and $\Delta_{\mathrm{p}}^2 p$ then appear instead of $\Delta_{\mathrm{m}}^2 Q_x$ and $\Delta_{\mathrm{m}}^2 Q_{\mathrm{p}}$.

Estimation errors for $x(\tau)$ and $p(\tau)$. Relations (41) can be represented in the form

$$\begin{aligned}
&\text{(a)}\quad \hat{P}_y(\tau) = \hat{P}_y(0) + \alpha\tau\hat{x}(\tau) - \frac{1}{2}\,\alpha\beta\tau^2\hat{Z}\,,\\
&\text{(b)}\quad \hat{P}_z(\tau) = \hat{P}_z(0) - \beta\tau\hat{p}(\tau) + \frac{1}{2}\,\alpha\beta\tau^2\hat{Y}\,.
\end{aligned} \tag{45}$$

For the estimates that we consider, the variances are the same as in (42). Their product,

$$\Delta_{\mathrm{m.s}}^2\, x(\tau)\,\Delta_{\mathrm{m.s}}^2\, p(\tau) \geqslant \frac{\hbar^2}{4}\,, \tag{46}$$

as expected, confirms the rule that one cannot prepare a state contradicting the uncertainty relation.

The origin and physical meaning of relation (4). Setting $x_1 = P_y(\tau)/\alpha\tau$ and $x_2 = P_z(\tau)/\beta\tau$, we obtain from Eqns (41) that

$$\Delta^2 x_1 = \Delta_{\mathrm{m}}^2 x + \Delta^2 x(0)\,, \qquad \Delta^2 x_2 = \Delta_{\mathrm{m}}^2\, p + \Delta^2 p(0)\,.$$

After simple algebra, we find that

$$\Delta^2 x_1\,\Delta^2 x_2 \geqslant \hbar^2\,.$$

This is the relation between the unconditional variances of the measurement results in two QRS's.

Estimation errors for $x(\tau/2)$ and $p(\tau/2)$. From (40) and (41), it follows that

$$\begin{aligned}
&\text{(a)}\quad \hat{P}_y(\tau) = \hat{P}_y(0) + \alpha\tau\hat{x}\left(\frac{\tau}{2}\right),\\
&\text{(b)}\quad \hat{P}_z(\tau) = \hat{P}_z(0) - \beta\tau\hat{p}\left(\frac{\tau}{2}\right).
\end{aligned} \tag{47}$$

The variances of the estimates

$$\Delta_{\mathrm{m.s}}^2\, x\left(\frac{\tau}{2}\right) = \frac{\Delta^2 P_y}{(\alpha\tau)^2}\,, \qquad \Delta_{\mathrm{m.s}}^2\, p\left(\frac{\tau}{2}\right) = \frac{\Delta^2 P_z}{(\beta\tau)^2}$$

are not related to each other and can be arbitrarily small simultaneously. This can be written as

$$\Delta_{\mathrm{m.s}}^2\, x\left(\frac{\tau}{2}\right)\Delta_{\mathrm{m.s}}^2\, p\left(\frac{\tau}{2}\right) > 0\,. \tag{48}$$

It turns out that a joint measurement of the position and momentum can yield their exact values at the time instant corresponding to the middle of the interaction interval. This does not mean that one can prepare a state contradicting the uncertainty principle. (In the framework of quantum mechanics, one cannot obtain a correct result contradicting its foundations.)

We note that at time $\tau/2$, the object is interacting with the QRS, and according to (38), the operator $\hat{p} = m\hat{\dot{x}} - m\beta\hat{Z}$ is not equal to the momentum operator for the free object ($\hat{p}$ is the generalized momentum operator in a coupled system). In theoretical mechanics, the generalized momentum is defined as the derivative of the Lagrangian with respect to the velocity. Because the Lagrangian, in its turn, is defined up to a full time derivative of an arbitrary function of the coordinates, it is possible to redefine the generalized momentum of the coupled system conjugate to the position x such that it becomes equal to the kinematic momentum $m\dot{x}$. However, the Hamiltonian and the corresponding equations are then changed. For this redefined generalized momentum, it is also impossible to precisely measure its value at the time moment $\tau/2$.

We consider the measurement error for the value $m\dot{x}(\tau/2)$. According to (38a),

$$m\dot{x}\left(\frac{\tau}{2}\right) = p\left(\frac{\tau}{2}\right) - m\beta Z\,.$$

Accordingly, the measurement error variance is

$$\begin{aligned}
\Delta_{\mathrm{m.s}}^2(m\dot{x}) &= \Delta_{\mathrm{m.s}}^2\, p\left(\frac{\tau}{2}\right) + (m\beta)^2\,\Delta^2 Z\\
&= \frac{\Delta^2 P_z}{(\beta\tau)^2} + (m\beta)^2\Delta^2 Z \geqslant \frac{\hbar m}{\tau}\,.
\end{aligned}$$

The expression $\sqrt{\hbar m/\tau}$ is the so-called standard quantum limit for the accuracy of momentum measurement [26, 27]. As $\tau \to 0$, we obtain $\Delta_{\mathrm{m}}^2(m\dot{x}) \to \infty$. Hence, precise measurement of the kinematic momentum at a time instant during the interaction with the meter is impossible.

3.2 Joint measurement of position and momentum by means of an ancillary degree of freedom

In the measurement scheme described by interaction Hamiltonian (36), the total Hamiltonian of the object, the ancillary system, and the QRS interacting with them takes the form

$$\hat{H} = \hat{H}_{\mathrm{o}} + \hat{H}_y - \alpha\hat{Q}_-\hat{Y} + \beta\hat{P}_+\hat{Z} + \hat{H}_{\mathrm{a},x} + \hat{H}_{\mathrm{a},p}\,. \tag{49}$$

Without the loss of generality, we can assume that $\hat{H}_{\mathrm{o}} = \hat{P}^2/2m$ and $\hat{H}_y = \hat{p}_y^2/2m_y$.

In the instantaneous-measurement approximation ($\alpha\tau^2 \to 0,\ \beta\tau^2 \to 0$), the corresponding Heisenberg equations have the solutions

$$\begin{aligned}
&\hat{P}_y(\tau) = \alpha\tau\hat{Q}_- + \hat{P}_y(0)\,, \qquad \hat{P}_z(\tau) = \beta\tau\hat{Q}_+ + \hat{P}_z(0)\,,\\
&\hat{Q}_-(\tau) = \hat{Q}_-(0)\,, \qquad \hat{P}_+(\tau) = \hat{P}_+(0)\,,\\
&\hat{y}(\tau) = \hat{y}(0) + \beta\tau\hat{Z}\,, \qquad \hat{p}_y(\tau) = \hat{p}_y(0) - \alpha\tau\hat{Y}\,.
\end{aligned}$$

Measurement of $\hat{P}_y(\tau)$ and $\hat{P}_z(\tau)$ yields the estimates

$$\tilde{Q}_- = \frac{\tilde{P}_y}{\alpha\tau}\,, \qquad \tilde{P}_+ = \frac{\tilde{P}_z}{\beta\tau}\,.$$

Their conditional variances are

$$\Delta_{\mathrm{m}}^2 Q_- = \frac{\Delta^2 P_y(0)}{(\alpha\tau)^2}\,, \qquad \Delta_{\mathrm{m}}^2 P_+ = \frac{\Delta^2 P_z(0)}{(\beta\tau)^2}\,.$$

From the estimates $\tilde{Q}_-$ and $\tilde{P}_+$, we obtain the estimates

$$\begin{aligned}
&\tilde{x}(0) = \tilde{Q}_- + \langle y(0)\rangle\,, \qquad \tilde{p}(0) = \tilde{P}_+ - \langle p_y(0)\rangle\,,\\
&\tilde{x}(\tau) = \tilde{Q}_- + \langle y(\tau)\rangle\,, \qquad \tilde{p}(\tau) = \tilde{P}_+ - \langle p_y(\tau)\rangle\,,
\end{aligned}$$

whose conditional variances are

$$\begin{aligned}
&\Delta_{\mathrm{m.s}}^2\, x(0) = \Delta_{\mathrm{m}}^2 Q_- + \Delta^2 y(0)\,,\\
&\Delta_{\mathrm{m.s}}^2\, p(0) = \Delta_{\mathrm{m}}^2 P_+ + \Delta^2 p_y(0)\,,\\
&\Delta_{\mathrm{m.s}}^2\, x(\tau) = \Delta_{\mathrm{m}}^2 Q_- + \Delta^2 y(\tau)\,,\\
&\Delta_{\mathrm{m.s}}^2\, p(\tau) = \Delta_{\mathrm{m}}^2 P_+ + \Delta^2 p_y(\tau)\,.
\end{aligned}$$

The values $\Delta^2_{\text{m}} Q_-$ and $\Delta^2_{\text{m}} P_+$ can be arbitrarily small simultaneously. In this approximation, the product

$$\Delta^2_{\text{m.s}}\, x(0)\, \Delta^2_{\text{m.s}}\, p(0) = \Delta^2 y(0)\, \Delta^2 p_y(0) \geqslant \frac{\hbar^2}{4}$$

is the same as in (43) in the scheme of the direct joint measurement of position and momentum.

The situation is much worse with the variances of the estimates $\tilde{x}(\tau)$ and $\tilde{p}(\tau)$. The product

$$\Delta^2_{\text{m.s}}\, x(\tau)\, \Delta^2_{\text{m.s}}\, p(\tau) \geqslant \frac{9\hbar^2}{4} \tag{50}$$

is nine times larger than its optimal value. This is because after the precise measurement of P_y and P_z, the object remains entangled with the ancillary degree of freedom. This entanglement can be eliminated by either precisely measuring one of the observables $y(\tau)$ and $p_y(\tau)$ or performing their joint measurement. After a precise measurement of $y(\tau)$, we have $\Delta^2_{\text{m.s}}\, x(\tau) = \Delta^2_{\text{m}} Q_-$. Further, as $\Delta^2 P_y(0)/(\alpha\tau)^2 \to 0$, we have

$$\Delta^2_{\text{m.s}}\, x(\tau)\, \Delta^2_{\text{m.s}}\, p(\tau) \geqslant \frac{\hbar^2}{4}\,.$$

The same result is obtained in the case of the joint measurement of $y(\tau)$ and $p_y(\tau)$ with the errors satisfying the relation $\Delta^2_{\text{m.s}}\, y(\tau)\, \Delta^2_{\text{m.s}}\, p_y(\tau) = \hbar^2/4$.

Thus, a joint measurement of the position and momentum of an object using an ancillary degree of freedom provides the optimal result, similarly to the direct measurement of the position and momentum in the initial state $(x(0), p(0))$. However, as a procedure for preparing the state of the object [measuring $x(\tau)$ and $p(\tau)$], this scheme is far from optimal if it is not accomplished by a measurement of the ancillary degree of freedom.

In some papers, the product of apriori variances of the observables $\hat{Q}_-(0)$ and $\hat{P}_+(0)$ is considered:

$$\Delta^2 Q_-(0)\, \Delta^2 P_+(0) = \big(\Delta^2 x(0) + \Delta^2 y(0)\big)\big(\Delta^2 p(0) + \Delta^2 p_y(0)\big) \geqslant \hbar^2\,. \tag{51}$$

This relation is the analog of (4). However, it should be stressed that relation (51) does not result from the noncommutativity of the operators $\hat{Q}_-$ and $\hat{P}_+$. These operators do commute. Relation (51) results from the independence of the states of the object and the ancillary degree of freedom.

There have been no experimental realizations of joint indirect measurements corresponding to Hamiltonians (37) and (49). No realistic method of performing such a measurement has been proposed, either. It is therefore interesting to discuss the known practical schemes that do not correspond to the standard joint measurement scheme but actually provide simultaneous estimation of several noncommuting observables. These are the scheme of homodyne measurement of observables for a single radiation mode and the scheme of continuous position measurement.

3.3 Joint measurement of quadrature amplitudes for a radiation mode

The classical quadrature amplitudes (QAs)

$$X_1 = A\cos\theta\,, \qquad X_2 = A\sin\theta\,,$$

defined as the real and imaginary parts of the complex amplitude $\tilde{A} = X_1 + \mathrm{i}X_2$, correspond in quantum optics to the following operators of an oscillation mode:

$$\hat{x}_\theta = \hat{x}\cos\theta - \hat{p}\sin\theta = \left(\frac{\hbar}{2}\right)^{1/2} \big(\hat{a}^\dagger \exp(-\mathrm{i}\theta) + \hat{a}\exp(\mathrm{i}\theta)\big)\,,$$

$$\hat{p}_\theta = \hat{p}\cos\theta + \hat{x}\sin\theta = \mathrm{i}\left(\frac{\hbar}{2}\right)^{1/2} \big(\hat{a}^\dagger \exp(-\mathrm{i}\theta) - \hat{a}\exp(\mathrm{i}\theta)\big)\,,$$

$$\hat{x} = \left(\frac{\hbar}{2}\right)^{1/2} (\hat{a}^\dagger + \hat{a})\,, \qquad \hat{p} = \mathrm{i}\left(\frac{\hbar}{2}\right)^{1/2} (\hat{a}^\dagger - \hat{a})\,,$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators. The commutator $[\hat{x}_\theta, \hat{p}_\theta] = \mathrm{i}\hbar$ corresponds to the commutator of the position and momentum operators. The phase θ is a classical value depending on the choice of the time origin (in experiments, it is defined with respect to the phase of the reference oscillator).

To measure QAs, one can use the heterodyne method [29] or the homodyne method [30–32]. In both cases, the object under measurement is a wave incident on the detector during the measurement time.

A joint measurement of x_θ and p_θ was performed in Ref. [30] using an eight-port (with four inputs and four outputs) homodyne scheme. A simple homodyne scheme consists of a (50 : 50)-beamsplitter and two photodetectors (Fig. 4) and hence has two input ports and two output ports. The signal is fed to one of the input ports of the beamsplitter and the reference wave having the same frequency is fed to the other port.

The output modes of an ideal beamsplitter are related to the input ones as

$$\hat{b}_1 = \frac{1}{\sqrt{2}}\big(\hat{a}_1 \exp(\mathrm{i}\theta) + \mathrm{i}\hat{a}_2\big)\,, \qquad \hat{b}_2 = \frac{1}{\sqrt{2}}\big(\hat{a}_2 + \mathrm{i}\hat{a}_1 \exp(\mathrm{i}\theta)\big)\,.$$

The photon number difference for the output beams is

$$\hat{N}_{21} = \hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2 = \mathrm{i}\big(\hat{a}_1^\dagger \exp(-\mathrm{i}\theta)\hat{a}_2 - \hat{a}_1 \exp(\mathrm{i}\theta)\hat{a}_2^\dagger\big)\,. \tag{52}$$

Averaging the right-hand side of (52) over the coherent state $|\alpha\rangle$ of the reference wave (having zero phase), we

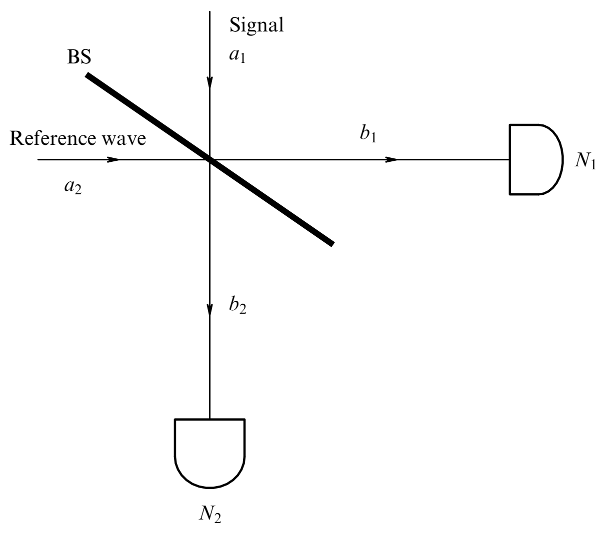

Figure 4. Homodyne measurement of quadrature amplitudes.

obtain

$$\mathrm{i}|\alpha|\big(\hat{a}_1^\dagger \exp(-\mathrm{i}\theta) - \hat{a}_1 \exp(\mathrm{i}\theta)\big) = \left(\frac{2}{\hbar}\right)^{1/2} |\alpha|\,\hat{p}_\theta = \left(\frac{2}{\hbar}\right)^{1/2} \langle n_{\mathrm{LO}}\rangle^{1/2}\,\hat{p}_\theta\,,$$

where $\langle n_{\mathrm{LO}}\rangle$ is the mean number of the probe wave quanta hitting the detector during the measurement time. Measuring N_{21}, we find the maximum-probability estimate:

$$\tilde{p}_\theta = \left(\frac{\hbar}{2}\right)^{1/2} \frac{\tilde{N}_{21}}{\langle n_{\mathrm{LO}}\rangle^{1/2}}\,.$$

The estimate variance depends on the amplitude and phase variances of the reference wave. The conditional variance of the difference number of quanta,

$$\hat{D}N_{21}^{\mathrm{LO}} = \langle\alpha|\hat{N}_{21}^2|\alpha\rangle - \langle\alpha|\hat{N}_{21}|\alpha\rangle^2 = \hat{a}^\dagger\hat{a}\,, \tag{53}$$

is related to the amplitude and phase uncertainties of the reference wave and depends on the state of the signal wave. Correspondingly, the variance of the QA estimate is also an operator:

$$\hat{D}_{\mathrm{m}}\,p_\theta = \frac{\hbar\hat{D}N_{21}^{\mathrm{LO}}}{2\langle n_{\mathrm{LO}}\rangle} = \frac{\hbar\hat{a}^\dagger\hat{a}}{2\langle n_{\mathrm{LO}}\rangle}\,. \tag{54}$$

Practical optical homodyne schemes use laser radiation as a reference. The laser radiation is called coherent, but this does not mean that a single mode is in a pure coherent state $|\alpha\rangle$. For laser radiation, the phase variance is not constant; it grows in accordance with the diffusion law. Phase diffusion can be neglected compared with the phase uncertainty of the coherent state only within a time interval t_{c} satisfying the inequality $(Dt_{\mathrm{c}})^{1/2} \ll 1/\langle n\rangle$, where D is the phase diffusion coefficient. For $D = 10^3$, $t_{\mathrm{c}} \ll 10^{-3}/\langle n\rangle^2$. Therefore, for instance, in experiments on squeezed light generation, the signal wave and the reference wave are obtained from the same beam by splitting it [33]. In this case, both waves have a random phase, but the influence of diffusion on the phase difference of the signal and reference waves is substantially reduced. (In real experiments, the measurement error is also largely influenced by the quantum efficiencies of the detectors.)

We note that even the usual homodyne scheme is in fact a scheme for the joint measurement of noncommuting observables: a QA and the number of photons. The number of photons for the signal wave can be estimated by subtracting the mean number of photons in the reference wave from the total number of registered photons. For such an estimation, the error variance is equal to the photon number variance of the reference wave $\langle n_{\mathrm{LO}}\rangle$. Hence, the product of the QA measurement error variance $\Delta^2_{\mathrm{m.s}}\,p_\theta = \langle D_{\mathrm{m}}\,p_\theta\rangle$ averaged over the state of the signal field and the variance $\Delta^2_{\mathrm{m.s}}\,n$ of the photon number measurement error is

$$\Delta^2_{\mathrm{m.s}}\,p_\theta\,\Delta^2_{\mathrm{m.s}}\,n = \frac{\hbar}{2}\,\langle\hat{a}^\dagger\hat{a}\rangle\,. \tag{55}$$

This product essentially differs from the one entering the uncertainty relation

$$\Delta^2 p_\theta\,\Delta^2 n \geqslant \frac{1}{4}\,\big|\langle x_\theta\rangle\big|^2 \tag{56}$$

corresponding to the commutator $[\hat{p}_\theta, \hat{n}] = -\mathrm{i}\hat{x}_\theta$.

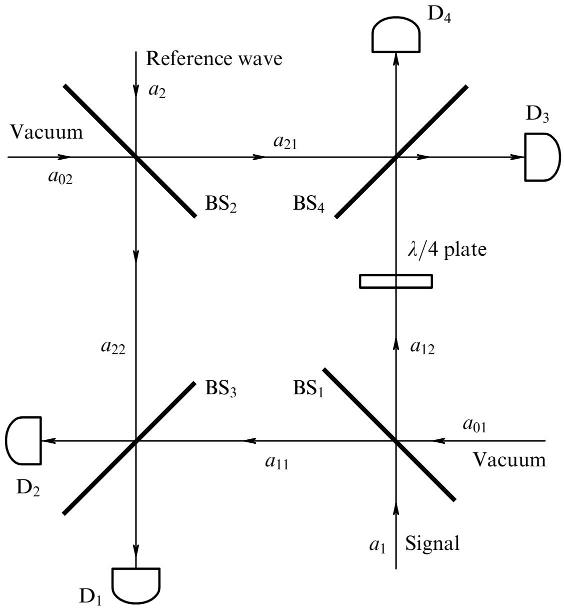

Figure 5. Eight-port homodyne scheme for the joint measurement of quadrature amplitudes.

A scheme for the joint measurement of two QAs is shown in Fig. 5. First, both the signal and the reference waves are split into two beams by two beamsplitters. Then one of the secondary signal beams and one of the secondary reference beams are sent to a homodyne detector. The other homodyne detector registers the second signal beam and the second reference beam, into which a phase shift of $\pi/2$ is introduced. As a result, the first homodyne detector measures one QA and the other measures the other QA. The scheme performs a joint measurement of two QAs for a single radiation mode. But in contrast to the schemes described by Eqns (37) and (49), the object (a wave) is here absorbed at the end of the measurement.

This joint measurement of noncommuting variables has a payoff: there is a fundamental limitation of the measurement accuracy. The measurement has inevitable errors that are caused by the vacuum fields entering the scheme through the free ports of the signal and reference beamsplitters. As a result, the signal and reference waves arrive to the homodyne detectors together with the vacuum fields a_{01} and a_{02}:

$$\hat{a}_{11} = \frac{1}{\sqrt{2}}\,\big(\hat{a}_1 \exp(\mathrm{i}\theta) + \mathrm{i}\hat{a}_{01}\big)\,, \qquad \hat{a}_{22} = \frac{1}{\sqrt{2}}\,(\hat{a}_2 + \mathrm{i}\hat{a}_{02})\,.$$

The difference in the photon numbers on the photodetectors is given by

$$\hat{N}_{21} = \frac{\mathrm{i}}{\sqrt{2}}\,(\hat{a}_{11}^\dagger\hat{a}_{22} - \hat{a}_{11}\hat{a}_{22}^\dagger)\,.$$

Averaging the right-hand side of this relation over the coherent state for the reference wave and over the vacuum state for $\hat{a}_{02}$, we obtain

$$\frac{1}{2\hbar^{1/2}}\,|\alpha|(\,\hat{p}_\theta + \hat{x}_{01})\,,$$

where $\hat{x}_{01} = (\hbar/2)^{1/2}(\hat{a}_{01}^\dagger + \hat{a}_{01})$ is the quadrature of the vacuum field a_{01}.

After measuring N_{21}, one obtains the estimate for the sum $p_\theta + x_{01}$. The error of this estimate, similarly to (54), tends to zero as the energy of the reference wave increases. The variance of the p_θ estimation error tends in this case to

$$\Delta^2_{\rm m.s}\, p_\theta = \langle \hat{x}^2_{01} \rangle \geqslant \frac{\hbar}{2}\,.$$

Similarly, the other half of the scheme provides the estimate for the sum $(x_\theta + p_{02})$. The limit of the estimation error of x_θ is

$$\Delta^2_{\rm m.s} x_\theta = \langle p^2_{02} \rangle \geqslant \frac{\hbar}{2}\,.$$

The product

$$\Delta^2_{\rm m.s}\, p_\theta\, \Delta^2_{\rm m.s} x_\theta \geqslant \frac{\hbar^2}{4} \tag{57}$$

is the same as for schemes (37) and (49).

From the viewpoint of accuracy and simplicity, the eight-port setup has no advantage over the usual homodyne scheme. The eight-port setup has been developed not for the measurement of QAs but as a scheme for the operational definition of the sine and cosine phase difference operators [28, 30]. A joint estimate for both QAs of a single mode of optical radiation can also be obtained by means of the heterodyne technique, with the same accuracy. The same limiting accuracy can be achieved for a joint estimate of both QAs of a harmonic oscillator by means of a continuous position measurement.

4. Measurement errors and perturbation for the continuous position measurement

A measurement involving a single QRS is only interesting from the methodological standpoint. In real schemes, multiple quantum readout systems are used, such as beams of electrons or photons.

4.1 Classical characteristics of a continuous measurement

A real measurement of the position results in a sequence of position estimates $\tilde{x}(t_j)$ taken at different times. Under certain conditions, a discrete sequence can be represented by a function $\tilde{x}(t)$. (One can use another representation of continuous measurements [14].) The fluctuation back-action force, which in the case of a single measurement corresponds to the operator $\alpha(t - t_0)\hat{Y}$, in this case corresponds to the operator

$$\hat{F}_{\rm b.a} = \sum_j \alpha(t - t_j)\, \hat{Y}_j\,,$$

where Y_j is the position and t_j is the time at which the jth QRS starts interacting with the object. Each interaction has a finite duration τ.

For a single measurement, the approximate estimate $\tilde{x}$ of the position can be represented as the result of a precise measurement of the observable $\hat{\tilde{x}} = \hat{x} + \hat{x}_{\rm er}$, where $\hat{x}_{\rm er}$ is the operator of error for a single measurement and $\hat{x}_{\rm er} = \hat{P}(0)/\alpha\tau$ in accordance with (11). Similarly, for a continuous measurement, the function $\tilde{x}(t)$ can be regarded as a result of a precise measurement of the observable

$$\hat{\tilde{x}}(t) = \hat{x}(t) + \hat{x}_{\rm er}(t)\,, \tag{58}$$

where $\hat{x}(t)$ is the operator of the object position and $\hat{x}_{\rm er}(t)$ is the operator of error for a continuous measurement of the position.

In the continuous measurement regime, the object position differs from its unperturbed value $\hat{x}_0(t)$. In a linear system,

$$\hat{x}(t) = \hat{x}_0(t) + \int_0^t \hat{F}_{\rm b.a}(t')\, K(t - t')\, {\rm d}t'\,, \tag{59}$$

where

$$K(t - t') = \frac{1}{{\rm i}\hbar}\big[\hat{x}(t), \hat{x}(t')\big]$$

is the Green's function of the object. (Other approaches to the continuous measurement problem are considered in Ref. [14].)

To quantitatively characterize the measurement error and the back-action force, one uses the covariance functions

$$B_x(t_1, t_2) = \frac{1}{2}\left\langle \big\{\hat{x}_{\rm er}(t_1), \hat{x}_{\rm er}(t_2)\big\} \right\rangle,$$

$$B_F(t_1, t_2) = \frac{1}{2}\left\langle \big\{\hat{F}_{\rm b.a}(t_1), \hat{F}_{\rm b.a}(t_2)\big\} \right\rangle,$$

$$B_{xF}(t_1, t_2) = \frac{1}{2}\left\langle \big\{\hat{x}_{\rm er}(t_1), \hat{F}(t_2)\big\} \right\rangle.$$

In the case of stationary measurements, they correspond to the spectral densities $S_x(\omega)$, $S_F(\omega)$, and $S_{xF}(\omega)$ that satisfy the relation equivalent to the uncertainty relation [17, 21, 34]:

$$S_x(\omega)\, S_F(\omega) = \Delta^2 Y(0)\, \Delta^2 P(0) \geqslant \frac{\hbar^2}{4}\,, \tag{60}$$

and, taking the correlation into account [17, 21],

$$S_F(\omega)\, S_x(\omega) - |S_{xF}|^2 \geqslant \frac{\hbar^2}{4} + \hbar\omega\big|{\rm Im}\, S_{xF}(\omega)\big|^2\,. \tag{61}$$

In the approximation of instantaneous interaction between separate quantum readout systems and the object, correlation functions are δ-shaped and the spectral densities are frequency-independent. In this case, the error variance for the measurement of the position averaged over time $\bar{\tau}$ is

$$\Delta^2_{\rm m} \bar{x} = \Delta^2 \left(\frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} \hat{x}_{\rm er}(t)\, {\rm d}t \right) = \frac{S_x}{\bar{\tau}}\,.$$

The variance of the momentum perturbation during the same time is

$$\Delta^2_{\rm p}\, p = \Delta^2 \left(\int_0^{\bar{\tau}} \hat{F}_{\rm b.a}(t)\, {\rm d}t \right) = S_F\, \bar{\tau}\,.$$

The relation

$$\Delta^2_{\rm m} \bar{x}\, \Delta^2_{\rm p}\, p = S_x S_F \geqslant \frac{\hbar^2}{4}$$

is the analog of relation (16).

The result $\tilde{x}(t)$ of a continuous position measurement is a classical object. To obtain the necessary information, one can apply any number of classical transformations tor this result. In the classical calculation, the only quantum limit is given by relation (61).

4.2 Errors in the joint estimation of quadrature amplitudes for a harmonic oscillator

As an example, we consider the errors in the joint estimation of quadrature amplitudes for a harmonic oscillator,

$$\hat{X}_1 = \hat{x}(0) = \frac{\hat{x}_0}{(m\omega)^{1/2}}\,, \qquad \hat{X}_2 = \frac{\hat{p}(0)}{m\omega} = \frac{\hat{p}_0}{(m\omega)^{1/2}}\,,$$

where m and ω are the mass and the frequency of the oscillator. In this case, the function $\tilde{x}(t)$ is a realization of the random process

$$\hat{\tilde{x}}(t) = \hat{X}_1 \cos\omega t + \hat{X}_2 \sin\omega t + \frac{1}{m\omega}\int_0^t \hat{F}_{\mathrm{b.a}}(t')\sin\omega(t-t')\,\mathrm{d}t' + \hat{x}_{\mathrm{er}}(t)\,. \qquad (62)$$

It is not only the randomness of $x_{\mathrm{er}}(t)$ that hinders the estimation of X_1 and X_2, but also the random perturbation of the oscillator by the force $F_{\mathrm{b.a}}(t)$.

The estimates for the quadrature amplitudes (QAs) have been obtained in Ref. [17] using the classical method of optimal estimation. The conditional variances for these estimates are

$$\Delta^2_{\mathrm{m.s}} X_1 = \Delta^2_{\mathrm{m.s}} X_2 = \frac{(S_x S_F)^{1/2}}{m\omega} \geqslant \frac{\hbar}{2m\omega}\,. \qquad (63)$$

Their product

$$\Delta^2_{\mathrm{m.s}} X_1\, \Delta^2_{\mathrm{m.s}} X_2 \geqslant \left(\frac{\hbar}{2m\omega}\right)^2 = \frac{1}{4}\left|\langle[\hat{X}_1, \hat{X}_2]\rangle\right|^2 \qquad (64)$$

is similar to (57). (The estimates were calculated under the condition that $S_{xF} = 0$ and in the approximation $S_F/S_x m\omega^2 \ll 1$, which means that the perturbation of the oscillator position during a single oscillation period is relatively small.)

Knowing the estimates for $\tilde{X}_1$ and $\tilde{X}_2$, one can estimate the initial energy and phase of the oscillator:

$$\tilde{W} = \frac{m\omega^2}{2}(\tilde{X}_1^2 + \tilde{X}_2^2)\,, \qquad \tilde{\varphi} = -\arctan\frac{\tilde{X}_1}{\tilde{X}_2}\,.$$

For $\Delta_{\mathrm{m.s}} X_{1,2} \ll \langle X_{1,2}\rangle$, the variances of the estimation errors are

$$\Delta^2_{\mathrm{m.s}} W = 2\omega\langle W\rangle (S_x S_F)^{1/2}\,, \qquad \Delta^2_{\mathrm{m.s}}\varphi = \frac{\omega}{2\langle W\rangle}(S_x S_F)^{1/2}\,. \qquad (65)$$

In the quantum limit,

$$\Delta^2_{\mathrm{m}} W = \hbar\omega\langle W\rangle = (\hbar\omega)^2\langle n\rangle\,, \qquad \Delta^2_{\mathrm{m}}\varphi = \frac{\hbar\omega}{4\langle W\rangle} = \frac{1}{4\langle n\rangle}\,. \qquad (66)$$

The product

$$\frac{1}{(\hbar\omega)^2}\,\Delta^2_{\mathrm{m.s}} X_1\, \Delta^2_{\mathrm{m.s}} W = \frac{\hbar\langle n\rangle}{2m\omega}$$

is similar to (55).

Square roots of the limit measurement errors in (63) and (66) are called the standard quantum limits (SQLs) of the measurement errors for the corresponding observables of the oscillator. (By definition, an SQL of a measurement error is the limit mean square value of the estimate for some observable calculated from the results of the position measurement.)

We note the following. The time moment to which the estimates of observables are related is not connected with the beginning of the position measurement. Estimation errors are the same for the values of observables related to the unperturbed state (before the measurement) and the current state formed by the measurement.

4.3 The state produced via continuous selective measurement

In practice, a continuous measurement is a rather dense sequence of discrete measurements. Each measurement reduces the state of the object. Between the measurements, the object evolves without any interaction. In the instantaneous-interaction approximation, the change of the state is described by means of reduction and free-evolution operators, which in turn act on the object starting from its initial state. If separate measurements have little influence on the state and the intervals between them are small, the discrete variation of the state can be regarded as a continuous one. The rate of the free evolution and the rate at which the state changes due to reduction contribute to the rate of the state variation with time [35]. The equation describing the time variation of the state depends on the states of the quantum readout systems and further measurements in them.

For a pure Gaussian state of the QRS, the reduction operator corresponding to a single position measurement has the form

$$\hat{R}(\tilde{x}_j) = G\exp\left[-\frac{(x-\tilde{x}_j)^2}{4\Delta^2_{\mathrm{m}} x}\right],$$

where G is a normalization factor. After n measurements taken within a time interval δt, the state is (without an account for the evolution)

$$\begin{aligned}
\left|\psi(t)\right\rangle &= \prod_{j=1}^{n}\hat{R}(\tilde{x}_j)|\psi_0\rangle \\
&= G^n\exp\left[-\sum_j\frac{(\hat{x}-\tilde{x}_j)^2}{4\Delta^2_{\mathrm{m}} x}\right]|\psi_0\rangle \\
&= B\exp\left[-\frac{\big(\hat{x}-\tilde{x}(t)\big)^2 n}{4\Delta^2_{\mathrm{m}} x}\right]|\psi_0\rangle \\
&= B\exp\left[-\frac{\big(\hat{x}-\tilde{x}(t)\big)^2 v\,\delta t}{4\Delta^2_{\mathrm{m}} x}\right]|\psi_0\rangle\,,
\end{aligned} \qquad (67)$$

where $\tilde{x}(t)$ is the average number of counts in time δt, v is the measurement rate, and B is a normalization factor.

The error variance of the position multiple measurement during a time δt is related to the spectral density S_x of the measurement error as

$$\frac{\Delta^2_{\mathrm{m}} x}{v\,\delta t} = \frac{S_x}{\delta t}\,.$$

It follows from (67) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = -k\big(\hat{x}-\tilde{x}(t)\big)^2\left|\psi(t)\right\rangle\,, \qquad k = \frac{1}{S_x}\,. \qquad (68)$$

Adding a term corresponding to the free evolution to the right-hand side of (68), we obtain the equation for the variation of the state in the course of a continuous position measurement. Its generalization to the case of an arbitrary initial state of the object has the form

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\hat{H}_0, \hat{\rho}] - k\left\{\left(\hat{x} - \tilde{x}(t)\right)^2, \hat{\rho}\right\}, \tag{69}$$

where $\hat{\rho}$ is the density operator of the object.

Equation (69) was first obtained by Mensky [14] using a different approach to the problem of continuous measurements, namely, the method of restricted path integrals. The approach discussed here was used earlier by Rembovsky [35], who derived an equation taking the errors of measurement in the QRS, correlations in the QRS state, and the degeneracy of the QRS observables into account.

We note that Eqn (69) could be obtained with the position operator replaced by the operator of any other observable. Hence, the symbol x in (69) can denote any observable. The state $\hat{\rho}(t)$ is a state after a selective measurement and it corresponds to the record $\tilde{x}(t)$ obtained during the measurement. The function $\tilde{x}(t)$ is a realization of random process (62).

Solutions of Eqn (69), as well as of a more general equation, have been studied in Ref. [36]. It was found that regardless of the initial state, the state of the object tends to a Gaussian one, with the position and momentum variances being constant but the time dependence of the mean values given by the realization functional $\tilde{x}(t)$. A special feature of the state formed this way is the correlation between the position and momentum.

Under ideal measurement conditions, which correspond to Eqn (69), the quasi-stationary state of a harmonic oscillator is a pure Gaussian state with the parameters

$$\Delta^2 x = \frac{\hbar}{2m\omega},$$

$$\Delta^2 p = \frac{\hbar m\omega}{2}\left[1 + \left(\frac{\hbar k}{m\omega^2}\right)^2\right],$$

$$\Delta^2(xp) = \frac{\hbar^2 k}{2m\omega},$$

which satisfy the uncertainty relation

$$\Delta^2 x\,\Delta^2 p - \Delta^2(xp) = \frac{\hbar^2}{4},$$

where m and ω are the mass and the frequency of the oscillator.

Continuous measurement of the position of a free particle produces a pure Gaussian state with the parameters

$$\Delta^2 x = \left(\frac{\hbar}{4km}\right)^{1/2}, \qquad \Delta^2 p = (\hbar^3 km)^{1/2}, \qquad \Delta(xp) = \frac{\hbar}{2}.$$

From the recorded results of the continuous measurement of some observable, one can calculate the state of the object at any time moment. As in the case of a single measurement, the aposteriori state of the object depends on its initial state, the state of the QRS, and the accuracy of the $P(\tau)$ measurement in the QRS.

5. Conclusion

The analysis presented above was focused on the measurement error — perturbation relation and the relation between the measurement errors of noncommuting variables.

The study of the standard scheme for the position measurement has shown that relation (2) between the position measurement error and the perturbation of the object momentum results from the uncertainty relation for the observables of the measurement device. In other words, relation (2) is secondary with respect to the uncertainty relation. This is why interpretation of the uncertainty relation as a consequence of both the general principles of quantum mechanics and the capabilities of measurement devices is contradictory.

It is erroneous to understand relation (2) as stating that a position measurement with an error $\Delta_{\mathrm{m}}^2 x$ would always lead to a $\Delta_{\mathrm{p}}^2 p \geqslant \hbar^2/(4\Delta_{\mathrm{m}}^2 x)$ increase in the variance of the object momentum compared to its initial value. An increase in the momentum variance in accordance with (2) occurs only during a nonselective measurement and in some special cases of selective measurements.

For a selective measurement, the aposteriori momentum variance is determined not by its apriori value and the accuracy of the position measurement but by the aposteriori state of the object, which depends on the apriori states of the object and the meter and the measurement result $\tilde{x}$. For pure initial states and a precise measurement in the meter, the aposteriori wave function $\psi(x|\tilde{x})$ is equal to the product of the initial wave function of the object $\psi_{\mathrm{o}}(x)$ and the meter function $\varphi_{\mathrm{a}}(\tilde{x}|x)$ [see (28)]. A selective measurement of the position changes the state of the object as a spatial filter with the transmission function depending on the state of the meter and the measurement result.

With other conditions being permanent, the aposteriori momentum variance depends on the measurement result and can be larger or smaller than the initial one. In some cases, the momentum variance of the object in the aposteriori state (a) is independent of its initial value and the fluctuation back-action of the meter; (b) can be less than $\hbar^2/(4\Delta_{\mathrm{m}}^2 x)$.

It is only the unconditional momentum variance, i.e., the momentum variance in the aposteriori state $\psi(x|\tilde{x})$ averaged over all $\tilde{x}$, that increases in accordance with relation (2).

In general, where an observable $\hat{A}$ is measured with an error $\Delta_{\mathrm{m}}A$, an observable $\hat{B}$ is perturbed by $\Delta_{\mathrm{p}}B$, and the commutator of $\hat{A}$ and $[\hat{A}, \hat{B}] = \mathrm{i}\hbar\hat{C}$ is an operator, the product $\Delta_{\mathrm{m}}^2 A\,\Delta_{\mathrm{p}}^2 B$ is essentially different from the corresponding uncertainty relation and depends on the secondary commutators $[\ldots[\hat{C}, \hat{A}], \hat{A}]\ldots, \hat{A}]$ [see (22), (23)]. If $[\hat{C}, \hat{B}] \neq [\hat{C}, \hat{A}]$, then the product of the measurement error and the perturbation is not symmetric: $\Delta_{\mathrm{m}}^2 A\,\Delta_{\mathrm{p}}^2 B \neq \Delta_{\mathrm{m}}^2 B\,\Delta_{\mathrm{p}}^2 A$.

The relation between the errors of a joint measurement of noncommuting observables has been investigated for both commonly used models of position and momentum joint measurement and other examples, such as the homodyne measurement of quadrature amplitudes for a radiation mode and the continuous position measurement.

In a standard joint measurement of position and momentum, directly [described by Hamiltonian (37)] or with the help of an ancillary degree of freedom [described by Hamiltonian (36)], the variances of the measurement errors for the position and momentum values before the interaction with the meter $(x(0), p(0))$ and after the interaction $(x(\tau), p(\tau))$ satisfy relation (3). The product of joint measurement errors for the position and momentum values taken in the middle of the interaction with the meter depends on the definition of the generalized momentum of the object. For Hamiltonian (37), the estimate variances $\Delta^2_{\mathrm{m.s}} x(\tau/2)$ and $\Delta^2_{\mathrm{m.s}} p(\tau/2)$ can be arbitrarily small simultaneously. However, the generalized momentum is then not equal to the kinematic one: $p(\tau/2) \neq m\dot{x}(\tau/2)$.

For a joint measurement of observables $\hat{A}$ and $\hat{B}$ whose commutator is an operator, the product of error variances differs from the value given by the corresponding uncertainty relation, similarly to the product $\Delta^2_{\mathrm{m}} A \, \Delta^2_{\mathrm{p}} B$.

The eight-port homodyne scheme for the measurement of QAs of a radiation mode, considered in Section 3.3 and shown in Fig. 5, is an example of a measurement with an ancillary degree of freedom. In this scheme, the errors of the joint QA measurement for the initial mode satisfy relation (3).

A simple (four-port) homodyne scheme, which is usually considered as measuring only QAs, is in fact a scheme for the joint measurement of a quadrature amplitude and the photon number. In the case of a coherent reference wave, the variance in the QA estimation error has been shown to essentially depend on the uncertainty of not only the amplitude of the reference wave but also its phase. Therefore, the estimation error for a single QA depends on the values of both QAs. As a result, the variance of a QA estimation error is proportional to the number of quanta in the signal wave [see (55)]. The relation between the mean error variances for the joint measurement of a QA and the number of quanta,

$$\Delta^2_{\mathrm{m.s}} p_\theta \, \Delta^2_{\mathrm{m.s}} n \geqslant \frac{\hbar \langle n \rangle}{2} \,, \tag{70}$$

is considerably different from the uncertainty relation for the same observables.

Joint estimation of observables can be alternatively performed by means of a continuous position measurement. The estimates can be obtained from the realization $\tilde{x}(t)$ in the framework of the classical estimation theory. Their quantum limits are determined by the quantum restrictions on the fluctuation spectral densities in the meter. The relations between the error variances for a joint estimation of QAs of a harmonic oscillator and the number of quanta are the same as in the homodyne scheme. The estimation error variances correspond to the fact that any initial state of the object is transformed into a Gaussian one with constant variances.

The author is much grateful to V B Braginsky and F Ya Khalili, who took it upon themselves to read the paper and made several valuable remarks.

This work was supported by the Russian Foundation for Basic Research (project 03-02-16975a).

References

1. Heisenberg W *Die physikalischen Prinzipien der Quantentheorie* (Leipzig: S. Hirzel, 1930) [Translated into English: *The Physical Principles of the Quantum Theory* (Chicago, Ill.: The Univ. of Chicago Press, 1930); Translated into Russian (Moscow: Gostekh-teorizdat, 1932)]
2. Elyutin P V, Krivchenkov V D *Kvantovaya Mekhanika s Zadachami* (Quantum Mechanics with Tasks) (Moscow: Nauka, 1976)
3. Dodonov V V, Man'ko V I *Tr. Fiz. Inst. Akad. Nauk SSSR* **183** 200 (1987)
4. Louisell W H *Radiation and Noise in Quantum Electronics* (New York: McGraw-Hill, 1964) [Translated into Russian (Moscow: Nauka, 1972)]
5. Arthurs E, Kelly J L *Bell Syst. Tech. J.* **44** 725 (1965)
6. Stenholm S *Ann. Phys.* (New York) **218** 233 (1992)
7. Mandelstam L I *Lektsii po Optike, Teorii Otnositel'nosti i Kvantovoi Mekhanike* (Lectures in Optics, Relativity Theory, and Quantum Mechanics) (Moscow: Nauka, 1972) p. 325
8. Wichmann E H *Quantum Physics* (Berkeley Physics Course, Vol. 4) (New York: McGraw-Hill, 1967) [Translated into Russian (Moscow: Nauka, 1974)]
9. Blokhintsev D I *Osnovy Kvantovoi Mekhaniki* (Foundations of Quantum Mechanics) (Moscow: Nauka, 1983) p. 75
10. Wigner E P *Symmetries and Reflections*: *Scientific Essays* (Bloomington – London: Indiana Univ. Press, 1970) [Translated into Russian (Moscow: Mir, 1971), p.143]
11. Vorontsov Yu I *Vestn. Mosk. Univ. Ser. 3. Fiz. Astron.* (6) 7 (1998)
12. Vorontsov Yu I *Vestn. Mosk. Univ. Ser. 3. Fiz. Astron.* (1) 24 (1999)
13. Vorontsov Yu I *Opt. Spektrosk.* **96** 747 (2004) [*Opt. Spectrosc.* **96** 679 (2004)]
14. Mensky M B *Kvantovye Izmereniya i Dekogerentsiya. Modeli i Fenomenologiya* (Quantum Measurements and Decoherence. Models and Phenomenology) (Moscow: Fizmatlit, 2001); *Quantum Measurements and Decoherence. Models and Phenomenology* (Dordrecht: Kluwer Acad. Publ., 2000)
15. Robertson H P *Phys. Rev.* **35** 667 (1930)
16. Schrödinger E *Izbrannye Trudy po Kvantovoi Mekhanike* (Selected Works in Quantum Mechanics) (Moscow: Nauka, 1976)
17. Vorontsov Yu I *Teoriya i Metody Makroskopicheskikh Izmerenii* (Theory and Methods of Macroscopic Measurements) (Moscow: Nauka, 1989)
18. Braginsky V B, Vorontsov Y I, Thorne K S *Science* **209** 547 (1980)
19. Braginsky V B et al. *Phys. Rev. D* **67** 082001 (2003)
20. Vorontsov Yu I *Vestn. Mosk. Univ. Ser. 3. Fiz. Astron.* (4) 12 (1996)
21. Braginsky V B, Khalili F Ya *Quantum Measurement* (Cambridge: Cambridge Univ. Press, 1992)
22. Korn G A, Korn T M *Mathematical Handbook for Scientists and Engineers* 2nd ed. (New York: McGraw-Hill, 1968) [Translated into Russian (Moscow: Nauka, 1978)]
23. Vyatchanin S P, Khalili F Ya *Usp. Fiz. Nauk* **174** 765 (2004) [*Phys. Usp.* **47** 705 (2004)]
24. Braginsky V B, Khalili F Ya, Kulaga A A *Phys. Lett. A* **202** 1 (1995)
25. Holevo A S *Veroyatnostnye i Statisticheskie Aspekty Kvantovoi Teorii* (Probabilistic and Statistical Aspects of Quantum Theory) (Moscow: Nauka, 1980) [Translated into English (Amsterdam: North-Holland Publ. Co., 1982)]
26. Braginskiĭ V B, Vorontsov Yu I *Usp. Fiz. Nauk* **114** 41 (1974) [*Sov. Phys. Usp.* **17** 644 (1975)]
27. Vorontsov Yu I *Usp. Fiz. Nauk* **164** 89 (1994) [*Phys. Usp.* **37** 81 (1994)]
28. Vorontsov Yu I *Usp. Fiz. Nauk* **172** 907 (2002) [*Phys. Usp.* **45** 847 (2002)]
29. Shapiro J H, Wagner S S *IEEE J. Quantum Electron.* **QE-20** 803 (1984)
30. Noh J W, Fougères A, Mandel L *Phys. Rev. A* **45** 424 (1992)
31. Mandel L *Opt. Commun.* **42** 437 (1982)
32. Smithey D T et al. *Phys. Scripta* **T48** 35 (1993)
33. Bowen W P et al. *J. Opt. B*: *Quantum Semiclass. Opt.* **4** 421 (2002)
34. Giffard R P *Phys. Rev. D* **14** 2478 (1976)
35. Rembovsky Yu A *Phys. Lett. A* **297** 300 (2002)
36. Rembovsky Yu A *Phys. Lett. A* **305** 1 (2002)