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Gravity and absolute space. The works of Niels Bjern (1865 –1909)

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Abstract. Nearly 20 years before Einstein, Niels Bjern developed a theory of gravity based on what is today known as the equivalence principle — but naturally without invoking the ideas of the special relativity theory. Bjern predicted almost all the effects considered to be tests for general relativity, his calculation formulas for the effects being identical to those of general relativity. An advocate of the absolute space concept, Bjern described the gravitational field in terms of the field of absolute velocities of an inertial space. He only used the ideas of general relativity to describe the precession of the perihelion of Mercury. The reason why Bjern's predictions and those of general relativity are identical is discussed.

1. Introduction

Our ideas about the structure of space and time expressed in the concept of general relativity have been strongly influenced by the sequence of leaps over a number of conceptual barriers: first, with the active participation of Albert Einstein, special relativity, which denounced absolute time, was created, and then Einstein, with the participation of mathematicians Marcel Grossmann and David Hilbert, developed general relativity. Despite the difference between the physical objects in these two theories, the authors' identity led to a strong influence of the ideas of relativity on the theory of gravity, which seemingly emerged as a generalization of special relativity.

A paradox of general relativity is that its experimental base was created more than 300 years before general relativity emerged — one only has to recall the legendary experiments of Galileo Galilei in which he observed the fall of cannonballs and bullets he dropped from the Leaning Tower of Pisa. Comprehension of the equivalence principle did not require the Michelson – Morley experiment nor the ingenious Eötvös experiments. Special relativity was needed only to pose the crucial question: What is an inertial reference frame? Einstein turned to this profound question only in 1911 [1].

And although it is often said that history is incompatible with subjunctive mood, sometimes events, previously unknown, are unearthed, substantiating what had thitherto been pure contemplation. Such an event occurred in the history of the theory of gravity.

2. Niels Bjern (1865 – 1909)

Once, quite unexpectedly, I came across a manuscript file of *Archiv for Naturvidenskab* published in 1888 – 1909 by a group of Norwegian school teachers. It was brought to my attention by Anna Florence, the great-granddaughter of Niels Bjern, who was a teacher of mathematics in a rural school.

Niels Bjern (1865 – 1909) graduated from the University of Christiania (now Oslo), where, among other things, he attended the lectures of Sophus Lie. However, at that time there was no drive for scientific discoveries at the university, and so Niels began teaching mathematics in a rural school. He was friendly with Wilhelm Bjerknes, who got him interested in hydrodynamics. Niels wrote several papers on the flow of viscous liquids but, being strongly opposed to any participa-

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tion in what he called the European scientific race, did not bother sending them to journals. Instead, collaborating with several teachers from his and neighboring schools, he organized a manuscript journal *Archiv for Naturvidenskab*. Pupils of these schools also actively participated in writing papers for the journal. The topics covered by the journal were diverse: butterflies, mists, runic writings, etc. There were also papers on mathematics and physics. And it was in this journal that I came across a series of remarkable papers written by Bjern in which gravity is interpreted in a very unusual way for the 20th-century physics.

2.1 Bjern's work and 20th-century experiments

In his main works, Bjern predicts all the experimental facts that are assumed to verify general relativity.

1. The deflection of a light ray in a gravitational field. This was verified experimentally in 1919.

2. The variation of light frequency as light travels between points with different gravitational potentials. This was verified by Pound and Rebka in 1959.

3. The expansion of the universe (Friedmann, Hubble).

4. A decreasing light frequency in the expansion of the universe.

5. Gravitational energy radiation. This was measured by Hulse and Taylor (1993 Nobel Prize in Physics).

6. Calculation of the angular velocity of rotation of Mercury's perihelion.

Despite the minuteness of these effects, their descriptions in Bjern's theory and general relativity coincide not only in the first order but also in the exact formulas. This suggests that the equivalence principle, which forms the basis of Bjern's theory and general relativity, is a definitive physical principle of the theory of gravity.

However, it must be said that the value of the gravitational red shift measured by Pound and Rebka does not coincide with the value calculated by Bjern, but the correct expression can be derived from his formulas if special relativity is taken into account. The same is true of the calculations of energy losses due to gravitational radiation: Bjern calculated only the plane (nonlinear) gravitational wave; only linearization and taking the relation to matter into account yields Einstein's formula, indirectly verified by Penzias and Wilson.

Briefly, for the results of Bjern's theory to fully coincide with those of general relativity, we add special relativity to the former, and Bjern was able to do this only in his last work, devoted to the precession of Mercury's perihelion.

2.2 Inertial reference frame

In 1891, Niels Bjern published his paper 'An inertial reference frame inside a ball', in which, among other things, he wrote:

"Once I was presented with a small ball. I use it to show my pupils various physical phenomena and, in particular, the motion of a freely thrown ball moving along a parabola. But the ball has a defect: inside it, there is either a pebble or a piece of rubber that found its way there when the ball was made. When you shake the ball, there is a distinctive rattle. My pupils known about this pebble or piece of rubber and call it 'the boss'. Once when I was demonstrating the flight of the ball, one of the pupils asked: 'Does the boss also fly along a parabola?' 'Of course', was my immediate answer. But later I pondered over this question, and for several days I racked my brain wondering about what happens inside the ball during its flight. I did calculations and recalled that in some book of problems I read about the flight of cannonballs. In the solution to that problem, it was recommended to introduce a freely falling reference frame, in which all cannonballs move uniformly and rectilinearly. This meant that from the 'viewpoint' of one of the cannonballs, all the others move uniformly and rectilinearly. Hence, during the ball flight, the 'boss' moves uniformly and rectilinearly (or is at rest) with respect to the walls of the ball.

"This means that an inertial reference frame is implemented inside the ball, while we, standing on the ground, are in a noninertial reference frame. This can easily be verified by letting the ball go — it will not stay at rest, but instead will fall to the ground with a constant acceleration.

"Mathematically, this is simply the consequence of the fact that all bodies, irrespective of their mass, fall to the ground with the same acceleration.

"Physically, this means that only inside a cannonball (or any ball) that is in free flight a truly inertial reference frame is implemented."

Next, Bjern tried to derive some corollaries from this fact, but he arrived at significant results only in 1894.

2.3 Universal inertial reference frame (1894)

Three years later, in 1894, Bjern, enchanted by this idea, published a perfectly developed paper titled 'Universal inertial reference frame', in which he came to the following conclusions concerning the structure of space:

"In the absence of gravity, the absolute inertial reference frame (the absolute Newtonian space) is the immutable Euclidean space. All laws of physics refer to this absolute space. Since in the key law of mechanics, Newton's second law, the central concept is acceleration, the laws of mechanics in a Euclidean space moving uniformly and rectilinearly with respect to the absolute inertial reference frame turn out to be the same as in the absolute one. Thus, we arrive at the concept of many inertial reference frames.

"However, in the presence of gravity, the absolute space ceases to be Euclidean, and a velocity with respect to it proves to be nonuniform, differing from point to point. Gravity forbids the existence of many global equitable inertial reference frames, leaving only one such reference frame to exist."

Bjern 'takes' the sun and 'builds' around a field of velocities with respect to an inertial reference frame. He writes:

"...A freely falling hollow ball constitutes an inertial reference frame. But if we take the sun's gravitational field into account, the parts of the ball that are farthest and closest to the sun have different accelerations and also the accelerations of the lateral points, pointing to the sun, are slightly nonparallel. In a nonuniform gravitational field, the accelerations are the same only within infinitely small regions. Hence, Euclidean systems exist only in infinitesimal regions. The universal inertial reference frame is not Euclidean and consists of many balls concurrently flying by inertia."

To describe extended physical systems, one is forced to build a continuous collection of infinitely small reference frames. The mathematical tool for building such a collection is the Hamilton – Jacobi equation. For a set of balls freely falling in a gravitational potential $\phi(\mathbf{r})$, the equation for the action S is

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + m\,\phi(\mathbf{r}) = 0\,. \tag{1}$$

This yields the formula for the velocity of an inertial ball

$$\mathbf{V} = \frac{\nabla S}{m}.$$

In the static case, $\partial S/\partial t = 0$, and Eqn (1), after introduction of the function $s = S/m$, becomes

$$\frac{(\nabla s)^2}{2} + \phi(\mathbf{r}) = 0. \tag{2}$$

Bjern called

$$\mathbf{V}(\mathbf{r}) = \nabla s, \quad V = \sqrt{-2\phi} \tag{3}$$

the field of absolute velocities. He writes:

"A gravitational field makes an inertial reference frame unique. The unique field of absolute velocities found from equations (2) and (3) with a nonuniform potential $\phi(\mathbf{r})$ does not allow Galilean transformations from one inertial reference frame to another. Such a transformation can be carried out only within an infinitely small region, where the absolute velocity may be assumed constant, and from the standpoint of mechanics, the reference frames moving uniformly (in a small region) are of equal status. But in the entire space, the field of absolute velocities proves to be unique.

"My main conclusion was that previously, physics considered reference frames moving uniformly and rectilinearly with respect to an absolute inertial reference frame; in the presence of gravity, such reference frames simply do not exist.

"Only in the absence of the gravitational potential is the absolute velocity the same over the entire space, and all uniformly moving reference frames prove to be of equal status in relation to mechanical motion."

Next Bjern models the sun as a spherical body of mass M, assuming that s depends only on its radius, and obtains

$$\phi = -\frac{kM}{r},$$
$$V = V_r = \sqrt{-2\phi} = \sqrt{\frac{2kM}{r}}. \tag{4}$$

At infinity, the field of absolute velocities is zero.

The space outside the sun with the above field of absolute velocities Bjern calls the sun's inertial reference frame.

2.3.1 Motion of bodies in the sun's inertial reference frame. The fact that the introduced inertial reference frame is an effective physical instrument was demonstrated by Bjern with an example describing the motion of bodies in the sun's inertial reference frame as the motion of free particles. He uses Lagrangian language to describe the motion. For a free body of unit mass (because mass has no effect on the law of motion), in a spherical coordinate system, the Lagrangian is

$$L = \frac{1}{2}\left[(\dot{r} - V)^2 + r^2\dot{\varphi}^2\right]. \tag{5}$$

The momenta

$$p_r \equiv p = \dot{r} - V, \quad p_\varphi \equiv l = r^2\dot{\varphi} = \text{const} \tag{6}$$

determine the Hamiltonian

$$H = \dot{r}p + \dot{\varphi}l - L = \frac{\dot{r}^2 - V^2}{2} + \frac{l^2}{2r^2} = \\ = \frac{1}{2}\left(\dot{r}^2 + \frac{l^2}{r^2}\right) - \frac{kM}{r} = E. \tag{7}$$

This expression coincides with the one for the Hamiltonian of a particle in a spherical gravitational field and determines the motion along conic sections.

2.3.2 Conclusions. At the end of his work Bjern concludes that:

(1) classical Euclidean inertial reference frames have infinitely small dimensions;

(2) the universal inertial reference frame is not Euclidean and is not related to any bulk solid object;

(3) the general world Euclidean reference frame is noninertial, which is determined by the field of absolute velocities associated with the gravitational potential in this reference frame; and

(4) specifying the field of absolute velocities is equivalent to specifying the gravitational potential.

2.4 A more general case of motion in a gravitational field

In the next paper, also written in 1894, Bjern examines a more general case of a gravitational potential. If there is an arbitrary absolute velocity field $\mathbf{V}(\mathbf{r})$ related to the gravitational potential through Eqn (2), the motion of a free particle in such a field is described by the Lagrangian

$$L = \frac{(\dot{\mathbf{r}} - \mathbf{V})^2}{2}; \quad \mathbf{p} = \dot{\mathbf{r}} - \mathbf{V};$$
$$H = \mathbf{p}\dot{\mathbf{r}} - L = \frac{\dot{\mathbf{r}}^2}{2} - \frac{V(r)^2}{2} = \frac{\dot{\mathbf{r}}^2}{2} + \phi(r). \tag{8}$$

In this general case, too, the description of the absolute velocity by a field is equivalent to the description by the gravitational potential — the Hamiltonian is the same.

2.5 Propagation of light (1896)

However, Bjern came to realize that the concept of an inertial reference frame is much broader than that of a gravitational potential, which affects only mechanical systems. He began to study the propagation of light in moving reference frames, adhering to the classical Newtonian concepts of absolute space and absolute motion. He assumes that in the absolute inertial reference frame, the eikonal equation has the well-known form

$$\frac{\omega_0^2}{c^2} - \mathbf{k}^2 = 0; \quad \omega_0 = \frac{\partial\psi}{\partial t}; \quad \mathbf{k} = \nabla\psi. \tag{9}$$

In the transition from the absolute reference frame at rest to an inertial reference frame moving with respect to the former with a velocity $\mathbf{V}$, the derivatives with respect to spatial variables do not change, while the derivatives (of a scalar) with respect to the time variable acquire a 'convective' addition:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \mathbf{V}\nabla. \tag{10}$$

Hence, in transition from the absolute reference frame to the moving reference frame, the wave vector of the light ray, $\mathbf{k}$, does not change, but the frequency is transformed according to the law

$$\omega_0 = \omega + (\mathbf{V}\cdot\mathbf{k}). \tag{11}$$

Bjern studies the dispersion equation in a reference frame moving with the absolute velocity $\mathbf{V}(\mathbf{r})$:

$$\frac{1}{c^2}\left[\omega + (\mathbf{V}\cdot\mathbf{k})\right]^2 - \mathbf{k}^2 \stackrel{\text{def}}{\equiv} 2\,h(\omega, \mathbf{k}) = 0\,. \tag{12}$$

The theory of first-order partial differential equations states that for rays (or characteristics), there is a parameter τ in terms of which the characteristics are expressed via the equations

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} = \frac{\partial h}{\partial \mathbf{k}}\,, \quad \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}\tau} = -\frac{\partial h}{\partial \mathbf{r}}\,. \tag{13}$$

In the spherically symmetric case with field (14),

$$h = \frac{1}{2}\left(k_r^2 + \frac{k_\varphi^2}{r^2} - \frac{\left(\omega + V(r)\,k_r\right)^2}{c^2}\right) = 0\,, \tag{14}$$

whence

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\partial h}{\partial k_r} = k_r\left(1 - \frac{V^2}{c^2}\right) - \frac{V\omega}{c^2}\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{k_\varphi}{r^2}\,, \tag{15}$$

where ω and k_φ are constants.

The quantity k_r can be expressed via Eqn (14) as

$$k_r^2\left(1 - \frac{V^2}{c^2}\right) - 2\,\frac{\omega}{c}\,\frac{V}{c}\,k_r + \frac{k_\varphi^2}{r^2} - \frac{\omega^2}{c^2} = 0\,,$$

$$k_r = \frac{V\omega/c^2 - \sqrt{1 - (k_\varphi^2/r^2)\,(1 - V^2/c^2)}}{1 - V^2/c^2}\,,$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = k_r\left(1 - \frac{V^2}{c^2}\right) - \frac{V\omega}{c^2} = \sqrt{1 - \frac{k_\varphi^2}{r^2}\left(1 - \frac{V^2}{c^2}\right)}$$

$$= \sqrt{1 - \frac{k_\varphi^2}{r^2}\left(1 - \frac{2\,k\,M}{r\,c^2}\right)}\,.$$

These expressions determine the differential equation for the path of the ray:

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{\dot r}{\dot\varphi} = \frac{r^2}{k_\varphi}\sqrt{1 - \frac{k_\varphi^2}{r^2}\left(1 - \frac{2\,k\,M}{r\,c^2}\right)}\,. \tag{16}$$

We see that Bjern predicts the deflection of a light ray by gravity. The deviation of the ray from a straight line is determined by the maximum value of $2\,k\,M/(r\,c^2)$ with r equal to the sun's radius. After calculating this quantity, Bjern writes that the effect in the sun's field is negligible and has practically no effect on the observed celestial map.

We note, however, that this formula coincides exactly with the formula in general relativity (e.g., see Ref. [2]).

2.6 Variation of light frequency

In Eqn (11), ω is a constant, but the physical frequency in an inertial reference frame is ω_0, which changes from point to point.

Mentioning this fact, Bjern arrives at an effect that is today known as the gravitational red shift (or simply the red shift).

It must also be noted that Bjern, not being familiar with the concept of proper time (before special relativity came into being), assumed that the change in ω_0 manifests itself in experiments, and hence with the correct (from the general relativity standpoint) dispersion relation (12), the corrections calculated by him are not quite right.

2.7 Cosmology (1887)

Bjern had already arrived at the conclusion that absolute space is non-Euclidean, but elementary reasoning, the presence of a velocity field, led him to the idea that the space is not static.

Bjern was in search of a simple problem in which absolute space would be described explicitly. In 1897, he took up the problem of dustlike (stellar) matter distributed with a uniform density ρ. Due to the gravitational attraction, this system cannot be static, but all the dust particles move by inertia (in the gravitational field). To calculate their motion, we select a certain particle, or point, which we call the 'center of the world'. Another point at a distance r from this center moves because of the attractive force of the mass inside the sphere of radius r,

$$M = \frac{4}{3}\pi\rho r^3\,. \tag{17}$$

The energy conservation law implies, as in the problem of a spherical body, that

$$V^2 = \frac{2kM}{r} = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2.$$

Because the mass inside the sphere does not change during such motion, the above expression can be considered a differential equation, whose solution is

$$r^3 = \frac{9kM}{2}\,t^2\,. \tag{18}$$

The recession velocity of a star from the 'center' at each instant is proportional to their separation and is directed away from the center:

$$v = \frac{2}{3t}\,r\,, \quad \mathbf{v} = \frac{2}{3t}\,\mathbf{r}\,. \tag{19}$$

For example, if the sun's velocity is $\mathbf{v}_0$, the star's velocity with respect to the sun is

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_0 = \frac{2}{3t}\,(\mathbf{r} - \mathbf{r}_0)\,, \tag{20}$$

as if the sun were the chosen center.

Thus, any star may act as the 'center of the world': any motion with respect to it is such as if this center were at rest. In particular, at $\mathbf{r} = 0$, the velocity is zero — a freely dropped body is at rest with respect to a given point — and Newton's first law is obeyed. This multitude of stars with distances measured by law (18) constitute a nonstatic universal inertial reference frame.

To explain the variation of the distance between stars, one is forced to introduce a time-dependent scale

$$r(t) = m(t)\,\bar r\,, \quad m(t) = \left(\frac{t}{t_0}\right)^{2/3}\,, \tag{21}$$

where $\bar r$ is the constant (angular) distance between stars.

Thus, the inertial reference frame in this problem has been constructed explicitly, and its geometric properties prove to be dynamical.

Having mastered the technique of describing the motion of light, Bjern demonstrated the variation of the frequency of freely propagating light as the world scale changes. The dispersion relation has the form

$$\frac{\omega^2}{c^2} = \frac{\mathbf{k}^2}{m^2(t)}\,.$$

Because the reference frame is inertial, ω is the absolute frequency. Due to the uniformity of space, $\mathbf{k}$ is constant, and the frequency proves to be inversely proportional to the scale.

2.8 Sophus Lie

The year 1899 saw the paper by Sophus Lie and Niels Bjern, 'The dynamics of space'. This paper was written by Bjern alone and was published in the same manuscript journal *Archiv for Naturvidenskab*. In the introduction, Bjern describes in detail the role that Sophus Lie played in writing the paper.

"Karl Bjerknes, with whom I often socialized thanks to my friendship with his brother Wilhelm and who showed great interest in my works, often said that I must show them to Sophus Lie. When in the autumn of 1898 Lie returned to Norway, Bjerknes was able to arrange a meeting with Lie. However, he warned me that because of Lie's bad health, the professor could spare me only one hour.

"We arrived at the professor's residence on a dark autumn morning. The professor met us in a sullen mood; he had a puffy face with a bluish tint. However, he was hospitable and even put on a semblance of recalling me as a student. We settled in armchairs, and he prepared to listen to me. I began telling him about my idea of replacing the concept of a gravitational field by that of a field of absolute velocities and about the deflection of light rays by the sun's field. His interest grew with every passing minute. When I told him about the cosmological problem, he jumped out of his armchair, paced the room for several minutes, and then exclaimed; 'I always told Felix [probably, Klein] that our world cannot be everywhere flat as Euclidean triangles! You also began with the Euclidean world, only your overall scale varies. But the stars are not distributed evenly over the sky, which means that scale is different in different parts of the World. This simply means that all components of the quadratic Riemannian element of space change. This requires a new theory, which I am now working out.'

"The one hour had long since passed. Lie very emphatically dismissed the hints made by his family that it was time to end the meeting. We were invited to dinner. The professor ate almost nothing, was very quiet, all in his thoughts, and it seemed that he emanated an internal glow. His face gained color, and even the sickly blue tint disappeared.

"After dinner, he very calmly, as if speaking not to me but to himself, began formulating what had to be done.

1. The space is Riemannian and is described by a Riemannian quadratic form.
2. The time dependence of the elements of the quadratic form must be determined from the least-action principle, which must include time derivatives, in contrast to the Laplace theory.
3. But this means that these equations must have wave solutions and the Lagrangian must contain a constant determining the velocity of the gravitational wave.
4. The transformation to the absolute inertial reference system where the absolute velocity field is everywhere zero requires that the new coordinates be time-dependent.
5. Transformation formulas for tensors from the absolute inertial frame to a noninertial reference frame must derived.

"He not only formulated these questions but also in almost all aspects worked out plans for finding the answers. It was already evening. When we were saying good-bye, he told me: 'This is a one-week assignment. In exactly one week, come back to me, we shall see what we have. I also need that week. Now I know what I must aim my theory at. Felix will die with envy.'

"To my deep regret, I did not see him in a week or ever after. Then next time I saw him was at his funeral.

"In this work and in those that followed I tried to realize the ideas expressed by professor Lie at that one meeting we had. I considered my research only a hobby and didn't expect such enthusiasm from a merited professor."

2.9 Absolute inertial reference frame

The first topic that the professor touched on was the concept of the absolute inertial reference frame. He said:

" 'It is customary to say that an inertial reference frame is an unalterable Euclidean space. I don't know whether you realized that you showed that space has quite other properties. First, it is dynamic. Its Riemannian quadratic form is time-dependent, and although in your work only the scale changes, which is due to the homogeneity of the problem, in the general inhomogeneous case, all components should change. That is, a general inertial reference frame must have a nonzero Christoffel tensor.'

"To my question as to how this could be possible, the professor replied:

" 'Well, for example, our world may turn out to be a three-dimensional Riemannian space. All directions are of equal status, but no matter what direction you choose, after traveling the same distance, you will end up at the same point on the other side. If the radius of this sphere is very large, the world would seem flat to you, just as we think of Earth as being flat. However, the radius of the world must be enormous compared to the Earth's radius. What is more important, we showed that the world radius must be time-dependent. You and I still don't know how to calculate this dependence, but today we must work out the principles from which this dependence can be derived.

" 'The time dependence of the Riemannian quadratic form is needed in order that at each point of an inertial reference frame, Newton's first law be valid, but in a much more rigorous form: a resting body should remain at rest. There is no sense of speaking of uniform motion in the general case, unless in infinitely small regions, where all spaces are flat. It is inside your ball that Newton's first law is fully valid.

" 'You and I are calmly sitting here in our armchairs, but our reference frame is not inertial. I drop a pencil, but it does not remain in a state of rest. From the viewpoint of your absolute velocity field, this field is zero at all points in an inertial reference frame. Hence, the problem consists of two parts: building the theory in a general inertial reference frame and recalculating all the results for an arbitrary reference frame. Mathematically, the second part is purely technical (mainly thanks to my works).' "

2.10 Principle of least action

Next the two discussed the equations of the dynamics of space in the universal inertial reference frame.

"The professor stressed the fact that the dynamics of the Riemannian quadratic form must be determined from Hamilton's principle of least action. It was clear that he was especially fond of this principle. He said: 'The Lagrangian function for various fields is represented by the difference of the densities of the kinetic and potential energies.' "

The two began the 'construction' of the kinetic energy by studying the dimensions of various quantities. The dimension of the gravitational constant is [$kg^{-1}\ m^3\ s^{-2}$]. The dimension of the Lagrangian is the same as that of energy, [kg $m^2\,s^{-2}$], and the dimension of the Lagrangian density for a field is [kg $m^{-1}\,s^{-2}$]. The Lagrangian density times the gravitational constant has the dimension [$m^2\,s^{-4}$], i.e., the dimension of the square of the time derivative of the scale, [s^{-2}], multiplied by the square of a certain velocity.

"The professor said:

"This, undoubtedly, is the speed of propagation of gravity! I heard that it is even much higher than the speed of light but not infinite. Gravity can also propagate in the form of waves. Let us call this speed U. Now we can construct an expression for the kinetic energy. My new work that I began only recently is called "Variation of invariant integrals'. It is devoted to building integrals that are invariant under coordinate transformations. Invariance under infinitely small transformations leads to interesting differential identities. But so far my new theory was purely abstract. Now you have supplied me with a marvelous area for its application.' "

As a result of this discussion, the authors of the paper 'The dynamics of space' arrived at an expression for the kinetic energy density (I use the modern notation with upper and lower indices):

$$T = \frac{U^2}{2k}\left(g^{ij}g^{kl} - g^{ik}g^{jl}\right)\dot{g}_{ik}\,\dot{g}_{jl}\sqrt{\det(g)}\,. \tag{22}$$

Lie insisted that the sign be a minus, because a plus would produce only oscillatory solutions rather than infinite expansion, as was the case in Bjern's cosmological problem:

"What is left is to correctly write the potential energy density. Here my new work gives an almost unique answer. The potential energy density must be proportional to the scalar of spatial curvature."

Thus, the authors arrive at an expression for the action of the dynamics of space in an inertial reference frame,

$$S_g = \frac{U^2}{2k}\int\left[(g^{ij}g^{kl} - g^{ik}g^{jl})\,\dot{g}_{ik}\,\dot{g}_{jl} + q\,U^2 R\right]\sqrt{g}\,d^3x\,dt\,, \tag{23}$$

where q is a constant that has yet to be determined.

"The professor said:

" 'I am giving you a rather complicated problem. You have a week to solve it. You must build a solution, as in electrodynamics, for a plane wave where all the components of the Riemannian quadratic form depend on time and only one coordinate, as if the space were two-dimensional. This should be easy for you. I recollect how easily you calculated the second Gaussian quadratic form knowing the first. [Sophus Lie, indeed, remembered me as a student — I was very proficient in differential geometry.] You must take $dl^2 = m^2(x,t)\,dx^2 + A^2(x,t)\,dy^2 + B^2(x,t)\,dz^2$ and calculate the Gaussian curvature of this form. If you set $A = 0$, you will obtain a two-dimensional surface, for which you will calculate everything very easily. Similarly, set $B = 0$. Your final answer must include these two variants as particular cases. You also need to calculate the relation between A and B, but even without this you have enough work for a week. Meanwhile I will see how to simplify the calculations. The idea is that at any moment in time, we can select the coordinate x such that the coefficient m becomes equal to unity. However, to establish $m(x,t) = 1$, we first need to perform a variation.'

"This task I have not fulfilled to this day, but after this work I will certainly turn to gravitational waves."

2.11 Coordinate transformations

Then the two discussed reexpressing the results in a non-inertial reference frame of a general type.

Let the spatial coordinates of the inertial reference frame be denoted by $\bar{x}^i$ and those of an arbitrary reference frame by $x^j(\bar{x},t)$. The transformations of the spatial variables x^i may be time-dependent. The absolute velocity vector appears here, and the quadratic Riemannian element is transformed as

$$V^i = \frac{\partial x^i}{\partial t}\,; \quad g^{ij} = \frac{\partial x^i}{\partial \bar{x}^k}\,\frac{\partial x^j}{\partial \bar{x}^l}\,\bar{g}^{kl}\,. \tag{24}$$

If a function depends on the coordinates, its time derivatives are expressed differently in different reference frames. The authors of 'The dynamics of space' call the time derivative in an inertial reference frame the total time derivative. According to the rule of differentiating a composite function,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x^i}\,\frac{\partial x^i}{\partial t} = \frac{\partial F}{\partial t} + V^i\frac{\partial F}{\partial x^i}\,. \tag{25}$$

For a tensor function, the expression is somewhat more complicated, because there is also a transformation involving the indices, but these were the transformations introduced and studied by Lie (today, we call them Lie variations). Under an infinitesimal transformation of the coordinates of an inertial reference frame, $x^i = \bar{x}^i + \xi^i(x,t)$, the tensor Q^i_{jk} of rank 3, for instance, undergoes the Lie variation

$$\delta_\xi Q^i_{jk} = \xi^i_{,s}\,Q^s_{jk} - \xi^s_{,j}\,Q^i_{sk} - \xi^s_{,k}\,Q^i_{js} - \xi^s\,Q^i_{jk,s}\,.$$

If we go back to the inertial reference frame, $\xi^s = -V^s\,dt$, and the transformation is expressed in terms of the field of absolute velocities V^s and its spatial derivatives, which by adding and subtracting the appropriate connections are reduced to covariant derivatives (the standard notation includes a semicolon):

$$\frac{d}{dt}\,Q^i_{jk} = \frac{\partial}{\partial t}\,Q^i_{jk} - V^i_{;s}Q^s_{jk} + V^s_{;j}Q^i_{sk} + V^s_{;k}Q^i_{js} + V^sQ^i_{jk;s}\,. \tag{26}$$

For instance, for a contravariant vector field A^i, we have

$$\frac{d}{dt}\,A^i = \frac{\partial A^i}{\partial t} - A^j\,\frac{\partial V^i}{\partial x^j} + V^j\,\frac{\partial A^i}{\partial x^j} = \frac{\partial A^i}{\partial t} - V^i_{;j}A^j + V^jA^i_{;j}\,. \tag{27}$$

Similarly, for a covariant vector field,

$$\frac{dB_i}{dt} = \frac{\partial B_i}{\partial t} + V^j_{;i}B_j + V^jB_{i;j}\,. \tag{28}$$

The expression for the covariant time derivative of the metric tensor components is especially important. Because the spatial covariant derivatives of the tensor are zero, we have

$$\frac{dg_{ij}}{dt} = \frac{\partial g_{ij}}{\partial t} + V_{i;j} + V_{j;i}\,. \tag{29}$$

Now all the relations obtained in an inertial reference frame can be carried over to an arbitrary reference frame by replacing the partial time derivative with the total time derivative. This, in particular, determines the additional term from the absolute velocity field in the kinetic energy in the Lagrangian.

A similar substitution of time derivatives must be performed when we write the equations or Lagrangians for other fields.

2.12 The gravitational wave (1901)

Only after more than two years following the memorable discussion with Sophus Lie was Niels Bjern finally able to finish his work on the problem of a plane gravitational wave, which made it possible to refine the coefficients in action (23). He writes:

"With the quadratic form

$$\mathrm{d}l^2 = m(x,t)^2\,\mathrm{d}x^2 + A(x,t)^2\,\mathrm{d}y^2 + B(x,t)^2\,\mathrm{d}z^2\,,$$

in an inertial reference frame, the kinetic energy is expressed as

$$T = 2\left[\left(\frac{\dot m}{m}\right)^2 + \left(\frac{\dot A}{A}\right)^2 + \left(\frac{\dot B}{B}\right)^2 - \left(\frac{\dot m}{m}+\frac{\dot A}{A}+\frac{\dot B}{B}\right)^2\right] \times mAB = -4\big[\dot m\,(\dot A B + A\dot B) + m\dot A\dot B\big]\,.$$

Calculating the scalar curvature is a more complicated problem. After many attempts I finally understood what Professor Lie had in mind when he mentioned the possibility of setting $m(x,t)$ equal to unity at some moment in time. This can be done, but because this presupposes a time-dependent transformation of the coordinate x of the inertial reference frame, it is necessary to introduce an absolute velocity field $V_x(x,t) \equiv V$, and hence the total time derivatives (at $m=1$) become

$$\frac{\mathrm{d}m}{\mathrm{d}t} = V'\,,\quad \frac{\mathrm{d}A}{\mathrm{d}t} = \dot A + VA'\,,\quad \frac{\mathrm{d}B}{\mathrm{d}t} = \dot B + VB'\,.$$

But this is compensated by the simple way in which the Gaussian curvature can be calculated:

$$R\sqrt{g} = -2(A''B + AB'' + A'B')\,.$$

For such a wave, the Lagrangian described by formula (23) assumed the form

$$L = -4\Big\{V'\big[(\dot A + VA')\,B + A\,(\dot B + VB')\big] + (\dot A + VA')(\dot B + VB')\Big\} - q(A''B + AB'' + A'B')\,.$$

Three variations (with respect to A, B, and V) yield three differential equations. We are interested, however, in an inertial reference frame, where $V = 0$, and assuming that this relation is valid after the variation, we thus arrive at three equations for the two functions A and B (at $q = 4$):

$$\ddot A - A'' = 0\,;\quad \ddot B - B'' = 0\,;\quad \dot A'B + A\dot B' = 0\,.$$

The first two equations yield the functions A and B as wave solutions, arbitrary functions of $x - Ut$.

The third equation establishes the relation between these two functions (A and B), which depend on a single variable:

$$A''B + AB'' = 0\,.$$

This equation can be solved rather simply if we introduce a parameter as $A = F\exp(\psi)$ and $B = F\exp(-\psi)$, which leads to the equation

$$F'' + (\psi')^2 F = 0\,.$$

Professor Lie said that we determine the action up to a dimensionless factor, and therefore the expression derived earlier can be split into four expressions, which can be conveniently done by introducing total time derivatives of the elements of the Riemannian quadratic form as follows:

$$w^i_j = \frac{g^{ik}}{2U}\,(\dot g_{kj} + V_{k;j} + V_{j;k})\,.$$

The action is then given by the formula

$$S = \frac{U^4}{2k}\int\big(w^i_j w^j_i - (w^j_j)^2 + R\big)\sqrt{g}\,\mathrm{d}^3x\,\mathrm{d}t\,. \tag{30}$$

(We remind the reader that we use the modern notation for covariant derivatives and summation over repeated indices — *D B.*)

"Professor Lie said: 'The numerical factor in front of this expression must be found by establishing the relation to matter fields'. This relation is just to be established by the method of his new theory called 'Variation of invariant integrals', which he hoped to finish soon."

A similar wave solution in general relativity can be found in Ref. [3].

2.13 The dynamics of a spherical world (1903)

In this work, Bjern arrived at the solutions for the dynamics of a spherical world, which were found in general relativity by A A Friedmann almost 20 years later.

"In explaining the meaning of 'nonflat' to me in relation to space, Sophus Lie gave the example of a three-dimensional sphere. I have already mastered the calculation of the Gaussian curvature of three-dimensional spaces. In particular, for a three-dimensional sphere of radius r, the curvature is $6/r^2$. The professor chided me for being so attached to the idea of flat space. So I decided to examine the dynamics of a three-dimensional spherical space with a time-dependent radius. The kinetic energy can also be easily calculated. As in the flat case,

$$\dot g_{ij} = \frac{2\dot r}{r}\,g_{ij}\,,$$

whence, with the measure of volume given by $\sqrt{g} = 2\pi^2 r^3$, we have the Lagrangian density

$$L = \frac{6\pi^2 U^2}{k}\,(r\dot r^2 - r)$$

and the Hamiltonian density

$$H = \frac{6\pi^2 U^2}{k}\,(r\dot r^2 + r)\,.$$

Because the system has only one degree of freedom, the entire dynamics can be derived from the energy conservation law

$$r\dot r^{\,2} + r = r_{\mathrm{m}}\,, \tag{31}$$

where r_{m} is the integration constant, with the meaning of the maximum possible radius.

"The above differential equation can easily be solved. Its solution is well known to be a cycloid with the radius $r_{\mathrm{m}}/2$. In such a world, the radius (and all scales) increases from zero to r_{m} and then decrease to zero.

"It is doubtful that this solution is of any practical importance, but it has educational value and can be used to study the possible configurations of our world, which, as I understood almost a decade ago, is sure to be nonstatic."

2.14 Precession of Mercury's perihelion

At the end of 1908, Niels Bjern wrote a paper in which he studied the ways in which the nascent theory of special relativity can contribute to his theory. First of all, he believed that the speed of a gravitational wave, which enters his equations as an undefined constant and, as Lie believed, was much higher than the speed of light, is equal to the latter.

However, the central part of this paper, which was published in the 1909 spring issue of *Archiv for Naturvidenskab*, deals with aspects of allowing for the modifications that special relativity introduces into the dynamics of a particle. In this connection, Bjern believed that a particle moves according to the principle of least proper time, but he developed this principle in a nonstandard way. He looked for the parametric dependence of the coordinates $x^i(\tau)$ and time $t(\tau)$ on the proper time:

$$\tau = \int_A^B \frac{c^2\,\mathrm{d}\tau^2}{c^2\,\mathrm{d}\tau^2}\,\mathrm{d}\tau = \int_A^B \frac{c^2\,\mathrm{d}t^2 - (\mathrm{d}\mathbf{r} - \mathbf{V}(r)\,\mathrm{d}t)^2}{c^2\,\mathrm{d}\tau^2}\,\mathrm{d}\tau\,. \qquad (32)$$

Instead of finding extrema of the proper time, Bjern looked for the extrema of the action $S = -c^2\tau/2$, proportional to this time, with the Lagrangian

$$L = \frac{1}{2}\left[\left(\dot{\mathbf{r}} - \mathbf{V}(r)\dot{t}\right)^2 - c^2\,\dot{t}^2\right] = -\frac{c^2}{2}\,, \qquad (33)$$

where the dot denotes the derivative with respect to τ.

We take the field of absolute velocities of a spherical mass M, i.e., $V_r \equiv V = \sqrt{2kM/r}$, and write the Lagrangian in spherical coordinates:

$$L = \frac{1}{2}\left[(\dot{r} - V\dot{t})^2 + r^2\dot{\varphi}^2 - c^2\dot{t}^2\right]. \qquad (34)$$

Because the coefficients is this expression depend neither on t nor on φ, the time and the angle are cyclic coordinates and the conjugate momenta (ε and l) are constant along the trajectories:

$$p_\varphi = r^2\,\dot{\varphi}\,, \quad \dot{\varphi} = \frac{l}{r^2}\,;$$
$$p_t = V\dot{r} + (c^2 - V^2)\,\dot{t}\,, \quad \dot{t} = \frac{c^2\,\varepsilon - V\dot{r}}{c^2 - V^2}\,.$$

Substituting these expressions in the Lagrangian (which for a free particle coincides with the Hamiltonian), we obtain

$$2L = \frac{\dot{r}^2 - c^2\,\varepsilon^2}{1 - V^2/c^2} + \frac{l^2}{r^2} = -c^2\,,$$

which implies that

$$\left(\frac{\mathrm{d}r}{\mathrm{d}c\,\tau}\right)^2 + \left(1 + \frac{l^2}{c^2\,r^2}\right)\left(1 - \frac{2\,k\,M}{r\,c^2}\right) = \varepsilon^2\,. \qquad (35)$$

Expressing $\mathrm{d}\tau$ in terms of $\mathrm{d}\varphi$ via the momentum conservation law,

$$\mathrm{d}\tau = \frac{r^2\,\mathrm{d}\varphi}{l}\,,$$

and introducing the variable $x = l/(c\,r)$ and the notation $kM/(lc) \equiv \alpha$, we reduce Eqn (35) to

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right)^2 + W(x) = \varepsilon^2 - 1\,, \quad W(x) \equiv x^2 - 2\alpha x - 2\alpha x^3\,. \qquad (36)$$

This equation has the form of the energy conservation law for the one-dimensional motion of a particle in a potential field $W(x)$. The angle φ plays the role of time here.

We emphasize that Eqn (36) coincides with the expression for the motion of a particle in the Schwarzschild field.

Circular motion corresponds to the point x_0 at the minimum of the function $W(x)$ (time, i.e., the angle, increases, while the radius does not change):

$$\frac{\mathrm{d}W}{\mathrm{d}x} = 2x_0 - 2\alpha - 6\alpha x_0^2 = 0\,, \quad x_0 = \frac{1 - \sqrt{1 - 12\alpha^2}}{6\alpha}\,.$$

Examining trajectories close to circles, Bjern studied small oscillations of a trajectory near a circular orbit. Their frequencies (in angle φ) are given by

$$\Omega^2 = \frac{1}{2}\,\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = 1 - 6\alpha x_0 = \sqrt{1 - 12\alpha^2}\,. \qquad (37)$$

The classical approach is to ignore the additional term $12\alpha^2$. Then $\Omega = 1$ and $T = 2\pi$. The period of these oscillations coincides with the revolution period, and the trajectory is closed (an ellipse).

However, the theory of relativity introduces corrections. The frequency decreases, the period of oscillations proves to be larger than 2π, and the particle traces an ellipse that rotates with the angular velocity $\Omega \approx 1 - 3\alpha^2$ along the direction of rotation. In each full revolution, the angle increases by

$$\delta\varphi = 6\pi\alpha^2\,.$$

This explains the precession of Mercury's perihelion (or the secular rotation of the orbit of Mercury), first discovered by Le Verrier, who analyzed the results of observations over three centuries, beginning with the time of the Danish astronomer Tycho Brahe — the major axis of Mercury's elliptic orbit rotates every hundred years by an angle of about 40 seconds of arc.

2.15 What followed

On May 28, 1909, Niels Bjern lost his life tragically in the mountains in the presence of his pupils. The manuscript journals were kept in the families of some teachers and pupils. The spectacular development of the theory of relativity in the 20th century led to a situation in which when Bjern's work, published in a provincial journal, found its way to a physicist, he/she (as Anna Florence said) put it aside with a smile, and finally all Bjern's work was forgotten.

Anna Florence was surprised by my interest in Bjern's work that she showed me. But the explanation is simple.

During the last decade, I found the key issue in general relativity that had been lost in the process of creating the theory under the paradigm of relativity, and that is global time, which fits perfectly into the mathematical structure of general relativity.

3. Bjern's dynamical space and general relativity

In most cases, Bjern's calculation formulas coincide with similar expressions in general relativity, although he did not even think about time transformations. Only in the problem of the precession of Mercury's perihelion did he use the concept of proper time of special relativity, but even then, to describe the dynamical space, he employed his old model of an inertial reference frame around a gravitating mass and arrived at an expression for the trajectory of motion that is identical to the expression in general relativity.

Moreover, to describe the gravitational red shift, Bjern derived an expression for the variation in frequency valid only in an inertial reference frame and was unable to correctly recalculate the result if a (noninertial) observer is at rest, because there was still no concept of proper time. However, his eikonal equation, from which all the observed effects can be derived, coincides with the eikonal equation in general relativity.

Thus, if we locally add special relativity to Bjern's theory, the resulting theory coincides with general relativity in its main aspects. To see where the two theories coincide, we must try to find Newton's global time and Bjern's dynamical space in general relativity.

3.1 Extended inertial reference frame in general relativity

Many researchers (e.g., see Ref. [2]) noticed the remarkable simplicity of the equations of gravity in a synchronous reference frame:

$$\mathrm{d}s^2 = dt^2 - \gamma_{ij}(t,x)\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{38}$$

In this reference frame, $g^{00} = 1$, the time is global and flows uniformly everywhere. Only the components of the three-dimensional metric change in time. In this reference frame, which Bjern's theory interprets as being inertial, general relativity and Bjern's theory coincide.

Indeed, by slightly changing the definitions adopted in Ref. [2],

$$\mu_{ij} = \frac{1}{2}\,\dot{\gamma}_{ij}\,; \quad \mu_i^i = \frac{\partial \ln\sqrt{\gamma}}{\partial t}\,,$$

we obtain Hilbert's action for the gravitational field, which coincides with Bjern–Lie's action up to a term outside the integral,

$$-\frac{1}{2}\,R\sqrt{g} = \frac{\partial}{\partial t}\,(\mu_i^i\sqrt{\gamma}) + \frac{\sqrt{\gamma}}{2}\,(\mu_j^i\mu_i^j - \mu_i^i\mu_j^j) + \frac{\sqrt{\gamma}}{2}\,R_3\,, \tag{39}$$

where R_3 is the scalar curvature of the three-dimensional manifold $\tau = \mathrm{const}$ (see Ref. [3]).

When we go over to noninertial reference frames in general relativity, we must recalculate the proper time of the moving observer, while there is no such process in Bjern's theory. In general relativity, transforming only the (time-dependent) spatial coordinates $x^i = x^i(t,\bar{x})$, where $\bar{x}$ are coordinates in the synchronous reference frame, gives rise to an off-diagonal component of the metric,

$$g^{0i} = \frac{\partial x^i}{\partial ct} \equiv \frac{V^i}{c}\,, \tag{40}$$

and the metric becomes (see Ref. [4])

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \gamma_{ij}(\,\mathrm{d}x^i - V^i)(\,\mathrm{d}x^j - V^j)\,. \tag{41}$$

Obviously, the off-diagonal entries of the metric, $g^{0i}c \equiv V^i$, appear to be velocities of some sort. The coincidence is even more apparent when we turn to the inverse metric tensor. The general relativity expression that enters the Hamilton–Jacobi equation and the eikonal equation,

$$g^{\alpha\beta}\,\frac{\partial f}{\partial x^\alpha}\,\frac{\partial f}{\partial x^\beta} = \left(\frac{\partial f}{\partial t} + V^i\,\frac{\partial f}{\partial x^i}\right)^2 - \gamma^{ij}\,\frac{\partial f}{\partial x^i}\,\frac{\partial f}{\partial x^j}\,, \tag{42}$$

is transformed in Bjern's theory in exactly the same manner due to an additional 'transfer' term,

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + V^i\,\frac{\partial}{\partial x^i}\,.$$

Hence, the results of all of Bjern's calculations related to the Hamilton–Jacobi and eikonal equations coincide with the corresponding expressions of general relativity.

The concept of global time is used rather often in general relativity. For instance, all cosmological problems are formulated and solved in global time in a synchronous reference frame.

3.2 The external Schwarzschild problem

The effects of light ray deflection in a gravitational field and the rotation of Mercury's perihelion, which were predicted by Bjern and coincide with the corresponding expressions in general relativity, are related to an inertial reference frame in the vicinity of the gravitating mass. Because the equivalence principle applies in both theories, we can use it to establish the reasons for this coincidence.

In 1911, in Ref. [1], Einstein formulated the local equivalence principle, namely, that an inertial reference frame is realized in a freely falling reference frame (an elevator). He predicted remarkable effects (gravitational red shift and deflection of light rays in a gravitational field). However, he examined only uniform (homogeneous) gravitational fields. An important point is that all the reasoning in the above-cited paper pertained to global classical time.

We now take the nonuniformity of a gravitational field into account. We take a spherically symmetric body of mass M. To describe physical phenomena in the vicinity of such a body, we introduce infinitely small inertial reference frames that are implemented in freely falling (along the radius) laboratories (dust particles) and introduce the Minkowski metric in each of these laboratories,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \bar{\mathrm{d}}x^2 - \bar{\mathrm{d}}y^2 - \bar{\mathrm{d}}z^2\,, \tag{43}$$

where $\bar{\mathrm{d}}$ means that there is no respective global coordinate — its variation is possible only within an infinitely small region. We note that this restriction is imposed in (43) only on the spatial coordinates. With respect to time, a laboratory (a local inertial reference frame) may have finite (but, possibly, limited) dimensions.

Now, if we relate the local coordinates in the laboratory to the global spherical coordinates in the vicinity of the gravitating mass,

$$\bar{\mathrm{d}}x = r\,\mathrm{d}\theta\,, \quad \bar{\mathrm{d}}y = r\sin\theta\,\mathrm{d}\varphi\,, \quad \bar{\mathrm{d}}z = \mathrm{d}r - V\,\mathrm{d}t\,, \tag{44}$$

and substitute these formulas in (43), we arrive at the following expression for the metric:

$$\mathrm{d}s^2 = (c^2 - V^2)\,\mathrm{d}t^2 + 2V\,\mathrm{d}r\,\mathrm{d}t - \left[\mathrm{d}r^2 + r^2(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right]. \tag{45}$$

For the variable t to also be global, we must synchronize the clocks in different laboratories. This can be done by specifying their motion such that at infinity, where space–time is flat, their velocities are zero — then this variable measures the common time for observers in freely falling laboratories who were at rest at infinity. The energy conservation law yields

$$\frac{mV^2}{2} - \frac{kmM}{r} = 0\,; \quad V = \sqrt{\frac{2kM}{r}}\,. \tag{46}$$

The global reference frame thus constructed is characterized by the same (unique) physical time:

$$\mathrm{d}s^2 = \left(1 - \frac{2kM}{c^2 r}\right)c^2\,\mathrm{d}t^2 + 2\sqrt{\frac{2kM}{c^2 r}}\,c\,\mathrm{d}t\,\mathrm{d}r - \mathrm{d}r^2 - r^2(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,; \tag{47}$$

here, the components of the inverse tensor are

$$g^{00} = 1\,; \quad g^{0r} = \frac{V}{c}\,. \tag{48}$$

The sections $t = \mathrm{const}$ form the absolute space. In this metric, the absolute space is the flat Euclidean space. The only aspect in which this metric differs from the Minkowski metric is the radial field of Bjern's absolute velocities.

At the same time, this metric transforms into the diagonal Schwarzschild metric if we replace the time t with the formal time $\tilde{t}$:

$$\mathrm{d}t = \,\mathrm{d}\tilde{t} - \frac{V\,\mathrm{d}r}{c^2 - V^2}\,. \tag{49}$$

After this substitution, we obtain

$$\mathrm{d}s^2 = \left(1 - \frac{2kM}{c^2 r}\right)c^2\,\mathrm{d}\tilde{t}^2 - \frac{\mathrm{d}r^2}{1 - (2kM/c^2 r)} - r^2(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,. \tag{50}$$

Thus, Bjern was actually operating in a Schwarzschild metric reduced to global time.

Expression (47) for the external metric of a gravitating mass was derived in 1927 by Painlevé [5] from the Schwarzschild metric by a transformation that is the inverse of (49).

There is a great difference between these two expressions. In metric (47), the variable i is the global time, while in (50), $\tilde{t}$ is simply a formal time variable endowed with physical meaning only asymptotically, as $r \to \infty$, but on the other hand, the metric is diagonal. This difference manifests itself in the geometry of the three-dimensional sections $t = \mathrm{const}$. In metric (47), this is a flat Euclidean space, while in metric (50), such a section has a singularity on the sphere $r = r_g$, inside which the section becomes pseudo-Euclidean.

3.3 Global time in general relativity

All four coordinates in general relativity are of the same formal nature. If we want to go over to the time of a synchronous reference frame with $g^{00} = 1$ (but, possibly, in noninertial spatial coordinates with $V^i \neq 0$), we must find a global variable $t(x^\alpha)$ such that $g^{00} = 1$. This standard transformation of the metric component

$$\bar{g}^{00} = g^{\alpha\beta}\frac{\partial t}{\partial x^\alpha}\,\frac{\partial t}{\partial x^\beta} = 1 \tag{51}$$

proves to be the Hamilton–Jacobi equation for the trajectories of freely falling dust particles (laboratories).

As an instructive example, we construct the global time system for the Kerr metric.

Equation (51) can be written in terms of the Boer–Lindqvist coordinates as [2]

$$\frac{1}{\rho^2}\left(\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right)\left(\frac{\partial t}{\partial\tau}\right)^2 - \frac{\Delta}{\rho^2}\left(\frac{\partial t}{\partial r}\right)^2 - \frac{1}{\rho^2}\left(\frac{\partial t}{\partial\theta}\right)^2 - \frac{1}{\Delta\sin^2\theta}\left(1 - \frac{2Mr}{\rho^2}\right)\left(\frac{\partial t}{\partial\varphi}\right)^2 + \frac{4Mar}{\rho^2\Delta}\,\frac{\partial t}{\partial\tau}\,\frac{\partial t}{\partial\varphi} = 1, \tag{52}$$

where $\Delta = r^2 + a^2 - 2Mr$ and $\rho^2 = r^2 + a^2\cos^2\theta$.

Because the coefficients in this equation depend neither on t nor on φ, the conjugate momenta are constants:

$$\tau = \varepsilon t + l\varphi + f(\theta, r)\,.$$

The conditions at $r = 0$ and infinity yield

$$\tau = t \pm u(r)\,, \quad u(r) = \frac{\sqrt{2Mr(r^2+a^2)}}{\Delta}\,. \tag{53}$$

The substitution $\mathrm{d}t = \,\mathrm{d}\tau + u\,\mathrm{d}r$ changes the metric components as

$$\bar{g}^{00} = 1\,, \quad g^{0r} = V^r = \frac{\sqrt{2Mr(r^2+a^2)}}{\rho^2}\,,$$

after which the spatial section $\tau = \mathrm{const}$ acquires the metric

$$\gamma_{rr} = \frac{\rho^2}{\Delta} + \frac{2Mr(r^2+a^2)(2Mr-\rho^2)}{\rho^2\Delta^2}\,,$$

$$\gamma_{r\varphi} = \frac{\sqrt{2Mr(r^2+a^2)}}{\rho^2}\,\frac{2Mar}{\Delta}\sin^2\theta\,,$$

$$\gamma_{\theta\theta} = \rho^2\,, \quad \gamma_{\varphi\varphi} = \left(r^2 + a^2 + \frac{2Mr}{\rho^2}\,a^2\sin^2\theta\right)\sin^2\theta$$

with an everywhere positive determinant

$$\det(\gamma_{ij}) = \rho^4\sin^2\theta\,.$$

At $a = 0$, this metric becomes the Schwarzschild metric in the global time t with a flat spatial section.

3.4 Uniqueness of the inertial reference frame

From the standpoint of general relativity, there are two questions about a synchronous reference frame:

1. Does there exist a global synchronous reference frame, i.e., are there coordinates in which the synchronization conditions ($g^{00} = 1$ and $g^{0i} = 0$) are satisfied in the entire space and at all times?

2. If such a reference frame exists, is it unique?

It might be possible to construct a four-dimensional metric $g^{ij}(x)$ such that there is no unique global solution of Hamilton–Jacobi equation (51) and such a four-dimensional world cannot be split into global time and space. However, Bjern believed that the first question is meaningless because the equations describing the dynamics of the components of the spatial metric are already formulated in terms the absolute (global) time.

But the second question, of the uniqueness of space (of a synchronous reference frame), is meaningful in both theories.

We suppose that we have a synchronous reference frame with metric (38). Time-dependent coordinate transformations $\tilde{x}^i = f^i(t, x)$ lead to a noninertial reference frame with the absolute velocity field $V^i = \dot{x}^i$, but time remains the same under such transformations. However, we can try to pass to the time τ of a moving system of particles, with the synchronism preserved. Locally, such a transformation exists (see Ref. [2]):

$$\tau = \tau(t, x), \quad \dot{\tau}^2 - \gamma^{ij}\tau_{,i}\tau_{,j} = 1, \quad \dot{\tau}\dot{x}^j - \gamma^{jk}\frac{\partial\tau}{\partial x^k} = 0\,.$$

There are four-dimensional spaces where such transformations can be performed globally, for instance, the Minkowski space in Cartesian coordinates, where synchronism is preserved under any Lorentz transformations. The situation is here the same as with groups of motion: most spaces do not allow such global transformations.

We examine the Schwarzschild–Painlevé space with metric (47). At infinity, synchronized laboratories in such a metric are at rest. We then pass to a system of clocks that are in uniform motion (with a small velocity) at infinity and had a plane front f_1 before they passed the attractive center; at this front, all clocks showed the same time. The clocks that move past the attractive center ($r \gg r_g = 2kM/c^2$) move along hyperbolas turned toward the gravitating mass. The angle of rotation of the asymptote after passing the center increases as the impact parameter decreases (for the paths s_1, s_2, s_3, and s_4), which leads to a twisting of the clock flow front after the attractive center O has been passed and, as a consequence, measuring the time by the clocks moving in this way (the synchronism front f_3) produces ambiguous results.

This is probably what Bjern had in mind when he stated that the gravitational potential abolishes the equal status of uniformly moving reference frames and leaves only one truly inertial reference frame.

3.5 Dynamical properties of space

The description of the dynamics of space in Bjern's theory by action (30) from the standpoint of the dynamical field theory reduces space to an ordinary continuous field.

Here is an example.

Melted pig iron is cast into a spherically shaped ball whose surface is rapidly cooled down. As a result, internal stresses develop in the iron ball. This means that after the ball is cut into small pieces, we cannot put them back together into a ball without restoring the stresses between the different neighboring parts, without deforming these parts. From the standpoint of differential geometry, this means that the ball's matter has an internal curvature.

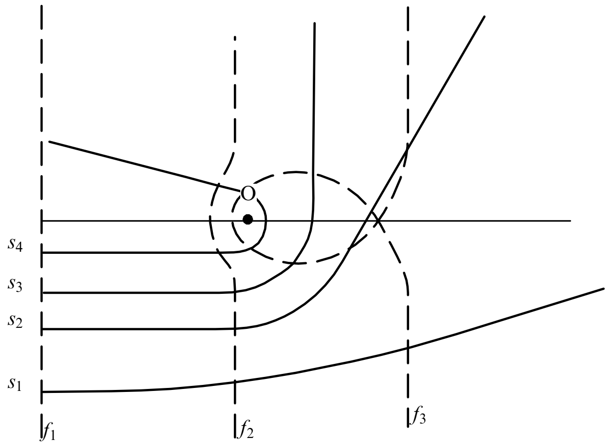

Figure 1. Distortion of the simultaneity front by a gravitating mass.

If we were able to place such a ball into a domain in space with exactly the same distribution of curvature as in the ball proper, the separate parts would merge without any stresses. The stresses occur because there is a mismatch between the internal curvature of the ball's matter and the curvature of the space into which this ball is placed.

If space were to exhibit no dynamical properties and acted only as a set of markers, it would tune its structure to match that of the curvature of the stressed ball. However, the existence of internal stresses not only in our virtual ball but also in thousands of real articles destroyed by internal stresses is proof that, from the energy standpoint, space favors minimum changes in its curvature generated by external bodies. Mathematically, this is expressed in Bjern–Lie formula (30) for gravitational action, where the term with the curvature in the potential energy enters with a very large factor, $c^4/(16\pi k)$. This means that even the smallest deviation from Euclidean space requires using a huge amount of energy.

Our space is (almost) perfectly Euclidean not because of the exquisiteness and beauty of Euclidean geometry but because such space has the minimal possible energy.

4. Conclusions

Niels Bjern presented a structure of space and time that was quite unexpected from the perspective of 20th-century physics. However, there is no hint of time transformations in his work. For him, time was unique and Newtonian, as it should have been in the 19th century.

His theory is the exact opposite of general relativity. While Einstein expected to establish, via gravity, the equal status of accelerating and uniformly moving observers, Bjern, on the contrary, concluded that the gravitational field selects only one inertial reference frame.

The physical base of both theories is the local equivalence principle. Bjern's inertial space is implemented in general relativity in a synchronous reference frame, in which time is the Newtonian absolute time. No other inertial reference frame exists because of the curvature of space.

In Bjern's theory, space proves to be a physical object with its own dynamics, involved in the dynamics of other fields on an almost equal status, while in general relativity, all the dynamics is hidden 'behind' the four-dimensional geometry.

Bjern's theory and general relativity in the synchronous reference frame have the same expression for the action, but in Bjern's theory there are nine quantities to be varied (six components of the spatial metric and three components of the absolute velocity), while in general relativity, all ten components of the four-dimensional metric are to be varied. If in general relativity we vary only the above-mentioned nine components, we arrive at a theory that slightly differs from general relativity, a theory of global time developed by the present author, whose main differences setting it apart from general relativity are global time and a nonzero Hamiltonian. This, however, is not a topic for the history of physics.

I am very grateful to Anna Florence, the great-granddaughter of Niels Bjern and a Russian – Norwegian translator by profession, who not only showed me the manuscript file of *Archiv for Naturvidenskab* but also helped me to find all the works of Bjern and translated them into Russian.

5. Appendix. List of Bjern's works [1]

1. On the plane flow of a viscous liquid (1888).
2. On vortices in a viscous liquid (1889)
3. An inertial reference frame inside a ball (1891).
4. Universal inertial reference frame (1894).
5. Motion of bodies in an inertial reference frame (1894).
6. A light ray in a gravitational field (1895).
7. Variation of light frequency in a gravitational field (1896).
8. The dynamics of the cosmos (1897).
9. The dynamics of space (together with Sophus Lie) (1989).
10. The equations of electrodynamics in a noninertial reference frame (1900).
11. The gravitational wave (1901).
12. The dynamics of a spherical world (1903).
13. Motion of liquid in a gravitational field (1904).
14. On the rotation of Mercury's perihelion (1909).

Note from the Editorial Board to the English translation. In publishing the very interesting paper about Niels Bjern's works, the Editorial Board must note that it does not have the manuscripts of Bjern's papers (or copies thereof). We hope that our colleagues outside Russia, especially those working in Norway, would give due attention to this material.

References

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2. Landau L D, Lifshitz E M *Teoriya Polya* (The Classical Theory of Fields) (Moscow: Nauka, 1988) [Translated into English (Oxford: Pergamon Press, 1983)]
3. Misner C W, Thorne K S, Wheeler J A *Gravitation* (San Francisco: W.H. Freeman, 1973) [Translated into Russian (Moscow: Mir, 1977)]
4. Arnowitt R, Deser S, Misner C W *Phys. Rev.* **116** 1322 (1959)
5. Painlevé P *C.R. Acad. Sci.* (Paris) **173** 677 (1921)

[1] All the works were published in the manuscript journal *Archiv for Naturvidenskab*.