

METHODOLOGICAL NOTES

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Classical mechanical analogs of relativistic effects

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Contents

Abstract. **The analogs of relativistic effects in classical mechanics, which are observed in the propagation of solitons in solids, are discussed. These effects are described by formulas similar to those of the special theory of relativity, with the speed of sound entering them in lieu of the speed of light. These parallels are shown to be a part of the correspondence between the soliton theory and field theories (in particular, electrodynamics). The effect of Lorentz-invariance breakdown in mechanical systems on dynamic soliton properties is considered. It is shown that supersonic solitons (in particular, dislocations) can propagate in such systems.**

1. Introduction

It is common knowledge that classical mechanics [1] embraces an analog of optics — linear acoustics. On the face of it, relativistic mechanics (the special theory of relativity) cannot have a classical analog. Meanwhile, the development of nonlinear dynamics has led to the discovery of classical particle-like objects — solitons [1] which are described by Lorentz-invariant equations and have, like relativistic particles, a continuous and bounded velocity spectrum $0 \leqslant v < v_s$, where v_s is the velocity of sound. The motion of solitons is accompanied by effects related to the finiteness of the speed of sound, which are similar to the relativistic effects in the special theory of relativity. Among these are the Lorentzian contraction of the traveling soliton width [1], the form variation of the mechanical stress field of a traveling topological defect [2], the Lorentzian velocity dependence of the soliton energy [1], etc. The formulas describing these effects are similar to the formulas of the special theory of relativity, with the speed of sound entering them in lieu of the speed of light in vacuum.

The effects stemming from the finiteness of the speed of sound in mechanical systems possessing Lorentzian symmetry will be referred to as 'relativistic'. In the scientific literature, this terminology has already been established, although it may well surprise the reader unfamiliar with the soliton theory. However, it comes as no surprise that different optical effects have analogs in acoustics. The names of these effects in optics and acoustics are the same: diffraction, interference, and dispersion. The use of common terms reflects the common character of the mathematical description of these effects despite the difference in magnitudes of the constants (the speeds of sound and light). The analogy between soliton dynamics and the special relativity may be considered as an extension of the analogy between acoustics and optics. Apart from the waves, the particles (in the special theory of relativity) and solitons (in classical mechanics) also present in this case. Once again, we can see similar effects: the Lorentzian dependence of the width of an object (a particle or a soliton) on its velocity, the equivalence relation between the mass and the rest energy of an object in the form $E = mc^2$ (c is the speed of light in the case of the special theory of relativity or the speed of sound in classical mechanics), the particle – antiparticle annihilation (the soliton – antisoliton annihilation in mechanics) with the emission of energy in the form of electromagnetic waves (sound waves in mechanics). Clearly this all is no a mere coincidence but is a reflection of the common character of the mathematical description of the processes occurring in mechanics and electrodynamics alike. This community stems from the finiteness of the rate of information transfer (the speed of sound or light) or, in

[1] Throughout this paper, by classical mechanics is meant the mechanics based on Newtonian laws, i.e., nonrelativistic mechanics.

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other words, from the retardation of signals and from the Lorentzian symmetry of dynamic equations. The community of the mathematical description also underlies the community of terminology. Employing the term 'relativistic' to describe the mechanical effects related to the finiteness of information transfer rate is therefore warranted to the same degree as employing the term 'diffraction' both for light and sound.

The analogy under discussion applies both to solitons in one-dimensional systems (the simplest case is mechanical models described by the sine-Gordon equation) and to, for instance, screw dislocations in a three-dimensional elastic continuum. However, the finiteness of information transfer rate in classical mechanics does not necessarily lead to the Lorentzian symmetry. In particular, there exist two rates of information transfer in a solid even in the isotropic case: the velocities of longitudinal and transverse sound waves. In the propagation description of any dislocations, except for the screw rectilinear ones, all physical quantities are therefore split into components which are Lorentz-transformable with different parameters (the velocities of longitudinal and transverse sound waves) [2]. In anisotropic bodies, the number of sound velocities in any direction increases up to three, making the description of the dynamics of topological effects all the more complicated. These cases still allow us to employ the Lorentz transformations. But in the investigation of real physical systems, quite often there is a need to include the terms which account for the gradient nonlinearity [for instance, terms of the form $\partial^2(u^3)/\partial x^2$] and the gradient dispersion (terms of the form $\partial^4 u/\partial x^4$) in the equations of motion of a continuous medium. This has the consequence that the equations of motion do not satisfy the Lorentz invariance condition. Nevertheless, effects related to the finiteness of information transfer rate still persist in the theory. The analogy with the special theory of relativity is retained at a qualitative level. It is shown below (see Section 2) that the soliton width, for instance, decreases with its velocity in these systems as well. However, the Lorentz relations do not apply to the description of the dynamic properties of these systems. The effects related to the finiteness of information transfer rate in the systems devoid of Lorentzian symmetry will be referred to as 'quasi-relativistic'. This term reflects the fact that the source of these effects, like of the corresponding effects in Lorentz-invariant models, is the finite rate of information transfer. The difference consists only in the symmetry of dynamic equations.

Some nonlinear equations not satisfying the Lorentz invariance condition admit supersonic soliton solutions. In Section 2 we will show that their existence violates neither the causality principle nor the other laws of physics. The supersonic solitons in different mechanical systems can possess a continuous velocity spectrum as well as a discrete one.

The situation is further complicated if the discrete structure of real physical systems is taken into account. Unfortunately, the equations which describe discrete mechanical systems do not, as a rule, possess analytical soliton solutions. That is why we can investigate their properties only with the aid of a physical experiment or numerical simulations.

The analogy between optics and acoustics can be extended even further. In the framework of classical mechanics, there exists an analog of classical electrodynamics, i.e., a theory which describes not only the propagation of electromagnetic waves but also their interaction with charged particles. As shown by Kosevich [3], an analog of this sort is the dynamic theory of dislocations, i.e., topological solitons in a crystal lattice. The dislocations correspond to electric charges in this theory, and the fields of elastic deformations and mechanical stresses to the electromagnetic field. More recently, Musienko and Koptsik [4] showed that the dynamic dislocation theory can be formulated as a four-dimensional gauge theory. The ambiguity of the potential of the dislocation elastic field (the so-called 'gauge freedom') is related to the ambiguity of the choice of the particle displacement field in the medium (in the continuous approximation) around a dislocation. The tensor rank of many physical quantities in the gauge dislocation theory is different from the dimensionality of their electrodynamic analogs. Furthermore, the distortion and mechanical stress tensors in the most general case are neither symmetric nor antisymmetric, while their corresponding electromagnetic field tensor is antisymmetric. Nevertheless, every formula and every effect in electrodynamics possess exact and unambiguous analogs in the framework of dislocation theory. The gauge theory of dislocations and disclinations constructed in the works of Kadić and Edelen [5] is also based on the analogy between the theory of defects and electrodynamics, but this analogy is radically different from that employed in Refs [3, 4]. This problem will be discussed in greater detail in Section 4.

To avoid misunderstanding, first of all we want to explain what we imply by the term 'soliton'. Mathematicians use the term soliton in reference to a localized particle-like solution of a totally integrable nonlinear system of equations describing a finite-energy excitation [6]. Physicists commonly give a broader definition and assume that a soliton is a localized stationary (or stationary on the average) perturbation of a homogeneous or spatially periodic nonlinear medium [7]. Topological solitons, i.e., ones possessing topological charges, make up a special class of solitons. This definition permits us to include in the list of topological solitons all topological defects in condensed matter: dislocations and disclinations in crystals, quantized vortices in superfluid liquid helium, vortex defects and domain walls in ferromagnets, disclinations in liquid crystals, Abrikosov vortices in superconductors, frustration lines in spin glasses, etc. The physical approach to the concept of a soliton is fundamentally different from the mathematical one: not only does it allow us to consider nonintegrable systems, but it also permits the assignment to solitons of those configurations for which the exact solutions of the corresponding nonlinear equations are unknown. In some cases, it is possible to find numerically such solutions and study their dynamic behavior by way of computer simulations. Moreover, experimental physicists can investigate the behavior of topological solitons in condensed matter even though the governing equations themselves, whose solutions are these solitons, may be unknown. The solitons that are void of a topological charge are termed dynamic.

We are reminded that topological charges are the elements of a homotopy group $\pi_i(V)$, where V is the order parameter space (it is sometimes also termed the degeneracy space) [8, 9]. V covers the range of all values which the order parameter (degeneracy parameter) can assume without a change in the energy of the system. For instance, the order parameter in a three-dimensional crystal is the three-dimensional vector of atomic displacements. Since the lattice displacement by lattice spacing results in the same structure which corresponds to the zero displacement, the crystal-lattice degeneracy space forms a cube whose opposite faces are equivalent, i.e., the three-dimensional torus surface T^3 in four-dimensional space [10]. Each element of the $\pi_i(V)$ group is given by the homotopy class of mappings $S^i \to V$ of an i-dimensional sphere. The choice of i is determined by the dimensionality v of the solitons under study (in the mathematical literature, such objects are also referred to as topological singularities): $i = d - v - 1$, where d is the dimensionality of the space M which harbors the solitons. For instance, measuring the charges of dislocations, which are point-like (i.e., zero-dimensional) topological solitons in the two-dimensional space and one-dimensional solitons in the three-dimensional space, requires choosing $i = 1$. The Burgers circuits γ used in the dislocation theory [10] are mapped onto closed circuits Γ in the order parameter space V. When there is no dislocation inside the Burgers circuit, its image Γ contracts into a point. This mapping corresponds to the unit element of the group $\pi_1(T^3)$, i.e., to the zero topological charge. If a dislocation is enclosed by the Burgers circuit, its image on the torus Γ is characterized by three topological invariants n_1, n_2, and n_3 — the winding numbers of the three circles that form the torus. The circuits characterized by different winding numbers correspond to different homotopic mapping classes, i.e., cannot be brought into coincidence through a continuous transformation. Each mapping class sets a specific topological charge value — the set of topological invariants (n_1, n_2, n_3). This set unambiguously determines the value of the Burgers vector $\mathbf{b} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$, where $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are the lattice translation vectors. A screw dislocation, i.e., a dislocation with the Burgers vector parallel to the defect line, is depicted in Fig. 1. In the continuous approximation, the Burgers vector constitutes the integral

$$b_i = \oint_L \mathrm{d}u_i = \oint_L \frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_k\,, \qquad (1)$$

where L is an arbitrary closed circuit enclosing the dislocation, and u_i is the vector of particle displacements in the medium.

This paper is concerned primarily with solitons in mechanical systems (except as otherwise noted). Particle displacement serves as the order parameter for these solitons. In solids, such solitons are the elementary plasticity carriers. The best-known example of solitons of this kind comprises dislocations in crystals. Presumably, less known is the fact that point (zero-dimensional) topological solitons can also be plasticity carriers in three-dimensional crystals. This applies, for instance, to three-dimensional polymer crystals consisting of parallel macromolecules. Such solitons strongly affect the mechanical properties (plasticity, strength) of solids. They also interact with the vibrational modes of a body, dissipate their energy, and thereby make a contribution to internal friction and thermal conductivity.

Of course, the function of order parameter can be fulfilled not only by displacement, but by other physical quantities as well. Other solitons correspond to a different choice of this parameter. At the present time, solitons are being investigated in virtually all branches of physics. Solitons have been discovered in liquids (solitons on the water surface) [11], in gaseous systems (Rossby waves) [11, 12], in plasmas (Langmuir, cyclotron, ion-sound solitons, etc.) [12], in different solids (in crystals, superconductors, spin glasses, etc.), and in optical fibers (optical solitons) [13]. The possible role of solitons in astrophysics [14] and elementary particle physics [15] is being discussed as well.

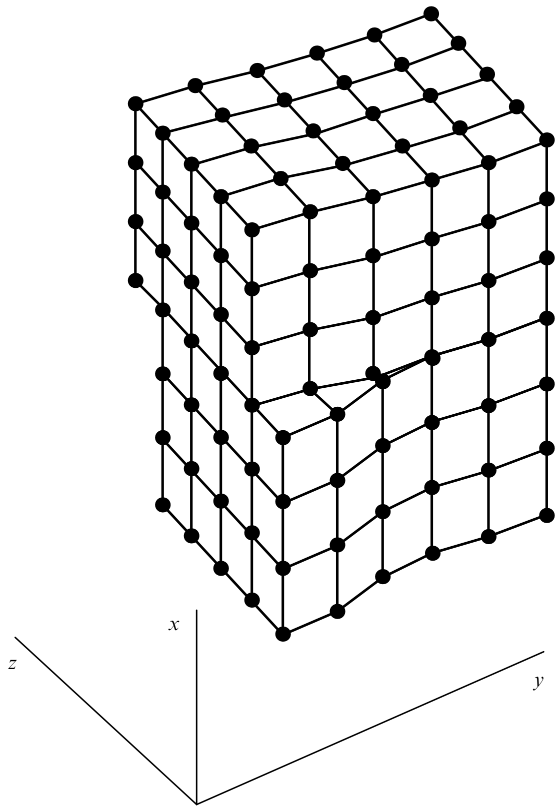

Figure 1. Screw dislocation in a three-dimensional crystal.

The concept of solitons is extensively exploited in biophysics. At present, Davydov solitons [13] are recognized to play a fundamental part in efficient dispersionless energy transfer in complex biological objects such as proteins, DNA, and other biomolecules. The peak corresponding to these solitons was discovered in the infrared spectra of crystalline acetanilide. Of considerable interest is the possible role of Davydov solitons in the mechanisms of nonthermal action of electromagnetic fields on living cells. A comprehensive review of theoretical and experimental papers dedicated to the study of Davydov solitons is given in Ref. [16]. Furthermore, other types of solitons are being investigated in biological entities (in particular, in DNA [17, 18], nerve fibers, and biological tissues [19]).

We shall not discuss the application of solitons in all these systems for the following reasons. First, the volume of our paper is limited. Second, the applications of soliton theory in different fields of physics are the concern of a series of monographs and review articles, which we do not want to repeat. The principal aim of our paper is to review those effects in soliton dynamics that are related to the finiteness of information transfer rate (both relativistic and quasi-relativistic effects). To greatly simplify for the reader the understanding of the physical nature of these effects, we decided to restrict our consideration to the simplest systems that allow the existence of solitons — mechanical systems. The intention in this paper is to compare the relativistic and quasi-relativistic effects in classical mechanics with similar effects in the special theory of relativity; to discuss the causes of the emergence of these effects in the framework of classical mechanics, as well as the analogies between soliton dynamics and gauge field theories (in particular, electrodynamics and gravitation theory), and to show that soliton dynamics exhibits quasi-relativistic effects which have no analogs in the special theory of relativity: in particular, formulas may appear which do not satisfy the Lorentz invariance condition, while solitons may propagate with supersonic velocities.

In Section 2 of our paper we consider relativistic and quasi-relativistic effects in the dynamics of solitons (both topological and dynamic) in one-dimensional systems. The analogy between the relativistic effects in the soliton theory and those in the special theory of relativity is drawn by the example of the best-known sine-Gordon equation. We review the results obtained by different authors, which are related to the supersonic propagation of solitons and soliton-like excitations.

Considered in Section 3 are relativistic and quasi-relativistic effects in the dynamics of dislocations (one-dimensional topological solitons). The nature of these effects is demonstrated to be more complicated than the nature of the effects of the special relativity, which is due to the existence of several velocities of sound in solids. We consider the origin of these effects in the mechanics of solids. Papers concerned with supersonic dislocations are also reviewed.

In Section 4 we discuss the analogies between the gauge theory of dislocations and disclinations and the fundamental field theories (electrodynamics and the gravitation theory).

In the concluding section we summarize the results of our work and discuss further prospects for studying the problems under discussion.

2. Relativistic and quasi-relativistic effects in soliton dynamics in one-dimensional systems

2.1 Dynamics of soliton solutions of the sine-Gordon equation

As the first example we consider the classical one-dimensional Frenkel – Kontorova dislocation model [2]. A chain of particles of mass m, coupled by linear springs with coefficients of elasticity k, resides in a periodic sinusoidal potential. This potential describes the interaction of the chain with some external object — a substrate. In a certain area, the chain may be stretched (or squeezed), so that the number of particles turns out to be smaller (or larger) by unity than the number of potential wells. This configuration is termed a kink (or, respectively, an antikink). Let a be the potential period, U_n the displacement of the nth particle relative to the nth well, and $u_n = U_n/a$. The equations of chain motion (i.e., Newton's second law) are written in the form

$$m\partial_t^2 u_n - k(u_{n+1} - 2u_n + u_{n-1}) + \frac{A}{2\pi a^2} \sin 2\pi u_n = 0 \,,$$

where $\partial_t \equiv \partial/\partial t$, A is a constant, and $-\infty < n < \infty$.

To go over to a continuum description, we assume the spatial scale measurement unit to be equal to $\varepsilon^{-1}a$, where $\varepsilon \ll 1$. The small parameter ε characterizes the ratio between the interparticle distance and the typical spatial scale of the solution. With this choice of the unit of measurement, the interparticle distance is equal to ε. Then, we can replace the discrete quantities u_n with a continuously varying parameter $u(n)$ (the variable n is also now assumed to be continuous) and make use of the Taylor expansion

$$u_{n+1} - u_n = \varepsilon\partial_n u + \frac{1}{2}\,\varepsilon^2\partial_n^2 u + \ldots$$

making it possible to obtain one nonlinear partial differential equation in lieu of the initial infinite system of equations. In the principal approximation, it constitutes the well-known sine-Gordon equation [1, 15]

$$\frac{1}{c^2}\partial_t^2 u - \frac{1}{a^2}\,\partial_n^2 u + \frac{A}{2\pi k a^4} \sin 2\pi u = 0 \,, \qquad (2)$$

where $\partial_n \equiv \partial/\partial n$, and $c = a(k/m)^{1/2}$ is the speed of sound in the chain under consideration. We go over to the system of units whereby $a = 1, A = 1, k = 1$ and denote the continuous spatial variable as x. Then, the kink (topological soliton) is described by the solution of Eqn (2):

$$u(x, t) = \frac{2}{\pi} \arctan \exp \frac{x - vt}{\gamma} \,, \qquad (3)$$

where $\gamma = (1 - v^2/c^2)^{1/2}$, and v is the kink velocity. The dependence of u on x for a fixed t is plotted in Fig. 2. Therefore, the kink represents a localized extension region. Hereinafter in this section we will consider only such solitons which effect a particle displacement by some value. For topological solitons, this quantity is equal to the topological charge. If the kink travels all the way through the chain, from the left end to the right one, the chain will shift by one period to the left relative to the substrate. Conversely, an antikink would shift the chain to the right. Therefore, topological solitons are the elementary carriers of plasticity in this model.

From formula (3) it follows that a substantial part of the variation of the quantity u occurs in the region about the kink center of width

$$L = \left(1 - \frac{v^2}{c^2}\right)^{1/2} . \qquad (4)$$

This quantity is termed the kink width. Therefore, the kink width depends on its velocity according to the Lorentz law,

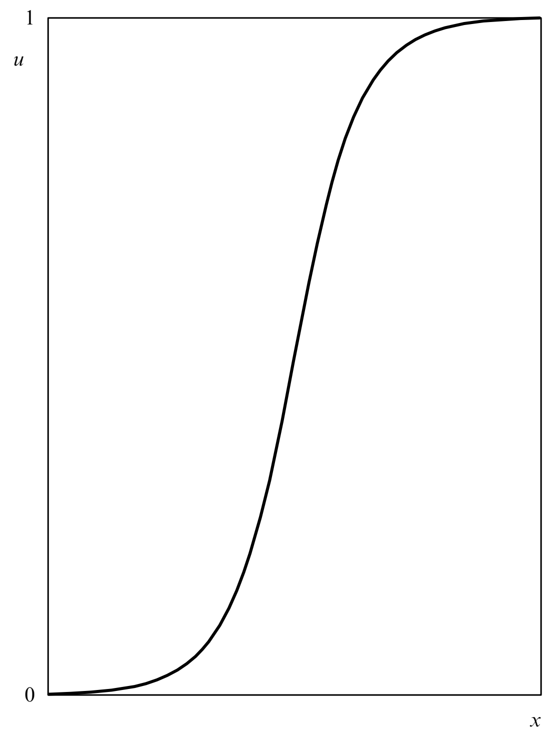

Figure 2. Soliton solution (kink) of the sine-Gordon equation.

much as the length of a moving object varies in the special theory of relativity. But in place of the speed of light in formula (4) there appears the speed of sound in the chain involved.

The kinetic energy of the chain is given by

$$T = \frac{m}{2} \sum_n (\partial_t u_n)^2 \,. \tag{5}$$

The potential energy of the chain is written as

$$U = \frac{k}{2} a^2 \sum_n (u_{n+1} - u_n)^2 + \frac{A}{4\pi^2} \sum_n (1 - \cos 2\pi u_n) \,. \tag{6}$$

The total kink energy equals $E = T + U$. On going over to the continuous limit, the sums in formulas (5) and (6) are replaced with integrals. We substitute the solution (3) into these integrals to obtain the final result [1]

$$E = E_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} , \tag{7}$$

where E_0 is the energy of an immobile kink (the rest energy). Therefore, the energy of a classical kink varies with its velocity by the well-known relativistic law. From formula (7) it follows that the kinetic energy of a kink with a velocity $v \ll c$ is described by the same formula as the kinetic energy of a classical particle, $E = Mv^2/2$, where M is the effective inertial mass of the kink. It is related to the rest energy of the kink by the well-known equivalence relationship $E_0 = Mc^2$. We emphasize that these relativistic results were obtained in the framework of classical Newtonian mechanics.

A kink and an antikink can annihilate at collision. As this takes place, their energy is emitted in the form of sound waves. This phenomenon is similar to a particle–antiparticle annihilation (electron–positron, for instance) with the emission of energy in the form of electromagnetic waves. In some cases, a bound state of a soliton and an antisoliton — a breather — may be produced.

Scott [20] developed a simple mechanical transmission line which enabled him to carry out experimental investigations of the soliton dynamics for the sine-Gordon equation. This line comprises a chain of pendulums attached to a horizontal string. The dynamics of this system is described by the sine-Gordon equation (2). Then, the variable $u(x)$ is the angle of deviation of the pendulum at a point x from the equilibrium position. This system enables one to observe relativistic effects in mechanics with the naked eye. In particular, Scott observed the Lorentzian contraction of width of a moving soliton, described in the foregoing.

The sine-Gordon equation is employed to describe the behavior of different physical systems. Zubova et al. [21] showed that the dynamics of a polymer chain in a polyethylene crystal is, under the assumption that the neighboring chains are immobile, described by the sine-Gordon equation. Then, the kinks (antikinks) constitute stretching (or, respectively, contraction) defects, i.e., chain portions being stretched (contracted) by the lattice constant relative to the neighboring chains. The same authors [22] showed that the system of equations of the sine-Gordon type describes other defects existing in crystalline polyethylene — twistons. Polyethylene crystals consist of parallel plane zigzag polymer chains. Twistons are localized regions of twisting a polymer chain through 180° with stretching (or contraction) by a half period of the chain several dozen periods long. Savin and Manevich [23, 24] investigated topological defects in crystalline polytetrafluoroethylene, which bore resemblance to twistons. These defects are localized regions of the rotation of the polymer chain by an angle of $14\pi/13$ with a shift along the molecular axis equal to 1/13 of the period (recall that the polytetrafluoroethylene molecule in the crystal has the shape of the spiral 13/6, i.e., contains 13 CF_2 groups per 6 turns).

Therefore, it is possible to arrive at different relativistic laws and effects in the framework of classical soliton theory in one-dimensional space. The relationships constructed are Lorentz-invariant. But in the framework of classical mechanics it is also possible to obtain more complex relativistic formulas which do not satisfy the Lorentz invariance condition.

2.2 Supersonic dynamic solitons

The sine-Gordon equation considered above takes into account the nonlinear interatomic interaction described by the sine function dependent on the field components. But nonlinear interactions in real solids are much more complicated and may involve gradient terms. The inclusion of these interactions has the result that the equation of atomic motion loses the Lorentz invariance. It would appear natural that these equations would possess soliton solutions to which the supersonic velocities correspond. It has been the investigation of supersonic nontopological solitons (in particular, the soliton solutions of the Korteweg–de Vries equation) that has marked the beginning of the modern stage of nonlinear dynamics, related to the advent of the method of the inverse scattering transform [25]. Since the late 1960s, such solitons have been vigorously studied both analytically and numerically.

Supersonic dynamic solitons in a discrete one-dimensional lattice were analytically investigated for the first time by Toda [26–28]. He considered an atomic chain with an exponential interaction potential (such chains have come to be known as Toda lattices). Let a be the lattice constant, and

$$\rho_n \equiv u_{n-1} - u_n \tag{8}$$

is the decrease of the distance between the neighboring atoms of mass M due to their displacements u_n. The interatomic interaction energy is given by

$$U = \sum_n \Phi(\rho_n) \,,$$

where

$$\Phi(\rho_n) = \frac{\varkappa}{b} \left\{ -\rho_n + \frac{1}{b} \left[\exp b\rho_n - 1\right] \right\} , \tag{9}$$

$\varkappa$ and b are constants. In the limit $b \to 0$, the Toda chain transforms into a harmonic chain, and in the opposite limit $b \to \infty$ into a chain consisting of solid spheres.

The equations of atomic motion are of the form

$$M\partial_t^2 \rho_n = \frac{\varkappa}{b} \left(\exp b\rho_{n+1} + \exp b\rho_{n-1} - 2\exp b\rho_n\right) . \tag{10}$$

There exists a soliton solution for these equations (Fig. 3):

$$\rho_n = \frac{1}{b} \ln \left\{ 1 + \frac{\sinh^2 qa}{\cosh^2 [q(na - vt)]} \right\} , \tag{11}$$

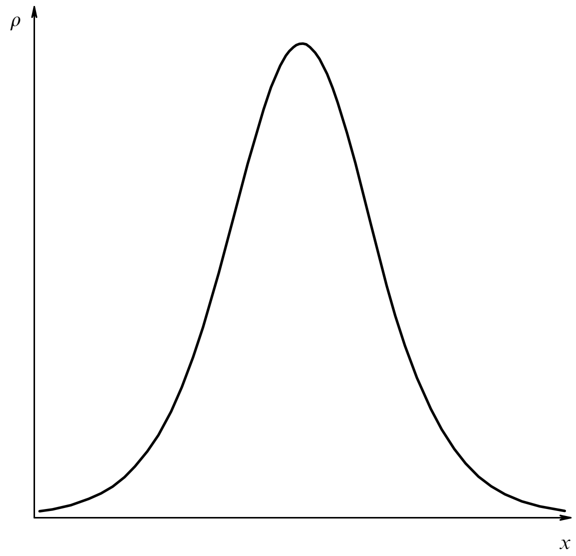

Figure 3. Supersonic dynamic soliton in the Toda lattice.

where the soliton velocity is defined as

$$v = \frac{c}{qa} \sinh qa \,, \tag{12}$$

and $c = a\sqrt{\varkappa/M}$ is the velocity of longitudinal sound waves in the harmonic chain, i.e., at $b = 0$. The parameter q characterizes the reciprocal of the soliton width: $q \approx 2\pi/L$, where L is the soliton width. This soliton is devoid of topological charge, i.e., is dynamic (use is sometimes made of the term 'acoustic soliton'). From formula (12) it follows that acoustic solitons (11) are always supersonic. For $v \to c$, the soliton amplitude tends to zero, and the width to infinity (Fig. 4). Therefore, the penetration of the sound barrier by a soliton is impossible in this system. Supersonic solitons are produced with velocities $v > c$.

When the soliton width L is large in comparison with the interatomic distance, namely, $qa \ll 2\pi$, it is possible to move to the continuous approximation. Then, the equations (10) of motion of the system assume the form of the Boussinesq equation

$$\left[\partial_t^2 - c^2\left(\partial_x^2 + \frac{a^2}{12}\,\partial_x^4\right)\right]\rho - bc^2\partial_x^2\rho^2 = 0 \,. \tag{13}$$

This equation possesses a soliton solution

$$\rho = \frac{a^2q^2}{b}\,\mathrm{sech}^2 q\zeta \,, \tag{14}$$

where $\zeta = x - x_0 - vt$, and

$$v = c\left(1 + \frac{a^2q^2}{3}\right)^{1/2} .$$

The velocity dependence of the soliton width is described by the expression

$$L = \frac{2\pi ac}{\sqrt{3(v^2 - c^2)}} \,. \tag{15}$$

The energy of a supersonic soliton equals $E = K + W$, where the kinetic energy is given by

$$K = \frac{M}{2a}\int\left[\partial_t u(\zeta)\right]^2 \mathrm{d}\zeta = \frac{2q^3}{3b^2}\,Mv^2a \,,$$

and the potential energy is written down as

$$W = \frac{1}{a}\int \Phi\left[\rho(\zeta)\right] \mathrm{d}\zeta = \frac{2q^3}{3b^2}\,\varkappa a^3\left(1 + \frac{4}{15}\,q^2a^2\right) .$$

When the velocity of a supersonic soliton approaches the speed of sound, its energy (both kinetic and potential) tends to zero (Fig. 5). Therefore, soliton behavior in the supersonic velocity region is the reverse of the behavior of soliton solutions of Lorentz-invariant equations in the subsonic region, where the soliton energy tends to infinity, and soliton width to zero, when its velocity approaches the speed of sound.

Toda surmised that supersonic dynamic solitons should make a significant contribution to the thermal conductivity of dielectric crystals. Recent numerical experiments showed [29] that the heat transfer by dynamic solitons which are close in properties to Toda solitons is indeed observed in argon crystals. In the view of the authors of Ref. [29], the energy transfer along the [110] direction in a face-centered cubic crystal can be regarded as a process occurring in the one-dimensional lattice embedded in an external potential produced by the neighboring atoms of the crystal. With increasing temperature, the ballistic contribution to the thermal conductivity, stemming from the solitons, decreased, while the diffusion (phonon) contribution increased.

The role of solitons in heat transfer in crystals was investigated not only in numerical, but in real experiments as well. Narayanamurti and Varma [30] undertook an experimental investigation of heat pulse propagation in sodium fluoride (NaF) crystals along the [100] direction at temperatures from 1.4 to 4.2 K. This temperature range corresponds to the ballistic regime of thermal conduction. When the pulse energy exceeded some threshold value, they observed the formation of a soliton from the thermal pulse. With increasing pulse energy, the soliton amplitude increased and the soliton width decreased.

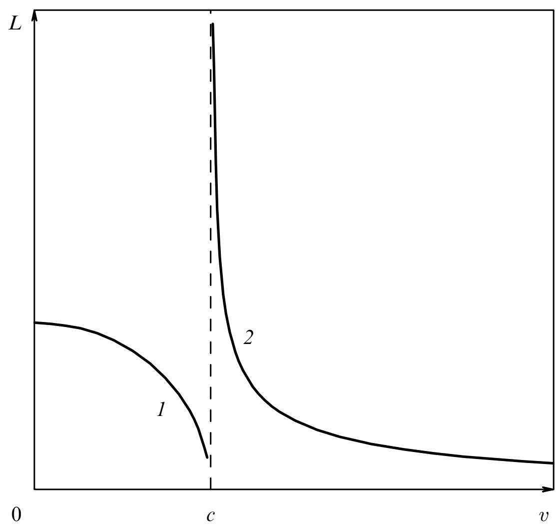

Figure 4. Width L of different solitons vs. their velocity v: *1* — for a kink of the sine-Gordon equation (Lorentzian dependence), and *2* — for supersonic dynamic solitons (in particular, the soliton in the Toda lattice).

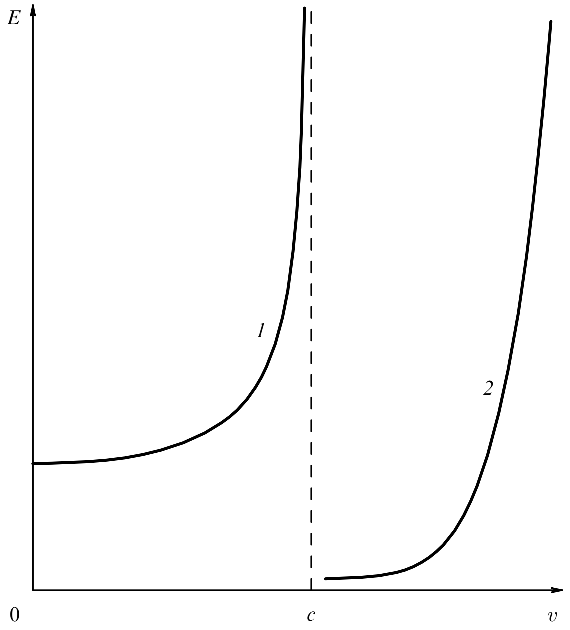

Figure 5. Energy E of different solitons as a function of their velocity v: *1* — for the kink of the sine-Gordon equation (Lorentzian dependence), and *2* — for supersonic dynamic solitons (in particular, the soliton in the Toda lattice).

The Boussinesq equation (13) describes the behavior of different physical systems. In this paper we shall not consider its application to fluid dynamics, described in a plethora of works. However, the Boussinesq equation also describes the dynamics of other objects, for instance, carbon nanotubes [31]. These structures are close in properties to fullerenes. At present, carbon nanotubes hold great promise as a material for practical use in nanotechnology. In particular, the possibility of using them for the production of nanoscale electronic devices is being looked into. A single-wall nanotube comprises a cylindrical two-dimensional surface made up of carbon atoms. The hexagonal surface structure is close to the structure of the atomic (001) plane in the graphite crystal, but the lengths of interatomic bonds and the angles between them are somewhat different from the corresponding characteristics of graphite because of the deformation arising from rolling a plane into a cylinder. Astakhova et al. [31] showed that the longitudinal displacements of carbon atoms in a nanotube (i.e., displacements parallel to the cylinder axis) are described by the Boussinesq equation. Therefore, supersonic deformation solitons can propagate through the tube, which is confirmed by numerical simulations [31]. It is conceivable that the investigation of these solitons will allow an explanation of the anomalously high values of the heat capacity and thermal conductivity of the nanotubes. Also of considerable interest is the study of the soliton influence on optomechanical and mechanoelectric phenomena in nanotubes.

The investigation of solitons in the Toda lattice brings up the natural question: are supersonic soliton-like solutions possible for one-dimensional chains with other interatomic interaction potentials $V(r)$? This problem was attacked by Collins [32]. We write the equations of motion for an atomic chain (10) in a more general form

$$M\partial_t^2\rho_n = \partial_\rho\left[V(\rho_{n+1}) + V(\rho_{n-1}) - 2V(\rho_n)\right]. \tag{16}$$

Let us assume that a soliton-like wave

$$\rho_n(t) = \rho(na - vt) = \rho(\zeta)$$

propagates along this chain, the wave form changing slowly with n. By employing expansion in a Taylor series it is possible to obtain the following relation for an arbitrary function $f(n)$:

$$\begin{aligned} f(n+1) + f(n-1) - 2f(n) &= \left[\exp\left(\frac{\mathrm{d}}{\mathrm{d}n}\right) + \exp\left(-\frac{\mathrm{d}}{\mathrm{d}n}\right) - 2\right] f(n) \\ &= \left[2\sinh\left(\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}n}\right)\right]^2 f(n)\,. \end{aligned}$$

Then, Eqn (16) can be written down as

$$M\,\frac{v^2}{a^2}\,\frac{\mathrm{d}^2\rho(\zeta)}{\mathrm{d}n^2} = \left[2\sinh\left(\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}n}\right)\right]^2 \partial_\rho V(\rho)\,.$$

Hence follows

$$M\,\frac{v^2}{a^2}\,\frac{\mathrm{d}^2}{\mathrm{d}n^2}\left[\frac{1}{2}\,\operatorname{cosech}\left(\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}n}\right)\right]^2 \rho(\zeta) = \partial_\rho V(\rho)\,.$$

We replace the function in square brackets with its Taylor expansion and ignore the terms of the higher order of smallness than $\mathrm{d}^2\rho/\mathrm{d}n^2$ to arrive at

$$M\,\frac{v^2}{a^2}\left(\rho - \frac{1}{12}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}n^2}\right) = \partial_\rho V(\rho)\,.$$

Integrating this equation gives

$$\frac{Mv^2}{24a^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}n}\right)^2 = \frac{Mv^2}{2a^2}\,\rho^{\,2} - V(\rho) + V(0)\,. \tag{17}$$

Employing the expansion of the potential $V(\rho)$ into the Taylor series

$$V(\rho) - V(0) = \frac{1}{2}\partial_\rho^2 V(0)\rho^2 + \frac{1}{6}\partial_\rho^3 V(0)\rho^3 + \ldots\,,$$

we can write down Eqn (17) in the form

$$\frac{Mv^2}{24a^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}n}\right)^2 = \frac{M}{2a^2}(v^2 - c^2)\rho^2 - \frac{1}{6}\partial_\rho^3 V(0)\rho^3 - \ldots\,, \tag{18}$$

where $c = a[\partial_\rho^2 V(0)/M]^{1/2}$ is the speed of sound. This equation allows the following conclusion: when the interatomic interaction potential is characterized by a stronger repulsion than the harmonic one, soliton-like supersonic compression waves are possible in this system. This conclusion follows from the nonnegativity of the quantity $(\mathrm{d}\rho/\mathrm{d}n)^2$ and the requirement $\rho \to 0$ when $\zeta \to \pm\infty$. Tension waves are unstable in this case.

The simplest particular case of the class of systems under discussion is the well-known Fermi–Pasta–Ulam (FPU) chain whose numerical investigation lent impetus to modern progress for the soliton theory. More recently, different authors performed analytical investigations of soliton propagation through this chain for different interatomic interaction potentials. In particular, Pnevmatikos [33] studied supersonic dynamic solitons in a one-dimensional chain of particles coupled via a polynomial interaction potential. In the continuous approximation, the dynamics of this chain is described by the modified Boussinesq equation

$$\partial_t^2 \varphi - \partial_x^2 (c^2 \varphi + B\varphi^3 - h\partial_x^2 \varphi) = 0\,, \tag{19}$$

where $\varphi \equiv \partial_x u$. The soliton solution of this equation, corresponding to a velocity of travel v, is of the form

$$u = \pm 2\left(-\frac{2h}{B}\right)^{1/2} \arctan\left\{\exp\left[\frac{2}{L}(x - vt) + x_0\right]\right\}, \tag{20}$$

where the soliton width equals

$$L = 2\left(\frac{h}{c^2 - v^2}\right)^{1/2}. \tag{21}$$

Solitons (20) may be both subsonic ($v < c$) and supersonic ($v > c$), depending on the sign of the dispersion parameter h.

Balabaev et al. [34, 35] proposed another modification of the Boussinesq equation to describe the nonlinear dynamics of a polymer molecule in a planar trans-zigzag conformation (Fig. 6). The authors of Refs [34, 35] investigated the dynamics of a polyethylene molecule in the united atom approximation. On passing to the continuous limit, their resultant equation for the longitudinal displacement u of the polymer chain took the form

$$\partial_t^2 u - c^2\left(\partial_x^2 u + \frac{3s^2}{n^2}\partial_x u\, \partial_x^2 u + \frac{s^2 l^2}{3}\,\partial_x^4 u\right) - \frac{s^4 l^2}{4n^2}\,\partial_t^2 \partial_x^2 u = 0\,, \tag{22}$$

where c is the speed of sound, l is the valence bond length (these bonds are assumed to be nonstretchable), $s = \sin(\theta_0/2)$, $n = \cos(\theta_0/2)$, and θ_0 is the equilibrium value of the valence angle.

Balabaev et al. [34, 35] found the soliton solution of Eqn (22):

$$u = \frac{d}{2}\left\{1 + \tanh\left[k(x - vt)\right]\right\}, \tag{23}$$

where v is the soliton velocity, k defines the reciprocal of the soliton width, and d is the chain displacement after the passage of the soliton. The parameters v and k should satisfy the conditions

$$k^2 + \frac{2n^2}{s^2 d}\left(\frac{4n^2}{3s^2} + 1\right)k - \frac{n^2}{l^2 s^4} = 0\,,$$

$$v^2 = c^2\left(1 + \frac{s^2 d}{2n^2}\,k\right).$$

Therefore, the soliton (23) turns out to be supersonic. Such solitons transfer mechanical excitations along an isolated polyethylene molecule. In the investigation of polymer crystals, the natural question arises as to how the properties of the soliton (23) change when this molecule is incorporated into the three-dimensional polyethylene crystal. The answer to this question will be provided in Section 2.3, when considering supersonic topological solitons.

The problem of supersonic soliton propagation in polyethylene molecules was also considered by Manevich and Savin [36, 37]. They proposed the following Lagrangian to describe the dynamics of a zigzag polymer chain:

$$L = \sum_n \left[\frac{1}{2}\,M(\partial_t u_n)^2 + \frac{1}{2}\,M(\partial_t v_n)^2 - V(\rho_n) - U(\theta_n)\right], \tag{24}$$

where $M = 14m_p$ is the mass of a CH_2 group (i.e., united atom), m_p is the proton mass, u_n and v_n are the longitudinal and transverse displacements of the nth united atom from the equilibrium position, respectively, the potential of the nth valence bond (i.e., the bond between the neighboring carbon atoms) is

$$V(\rho_n) = D_0\left\{1 - \exp\left[-\alpha(\rho_n - \rho_0)\right]\right\}^2,$$

ρ_n is the length of the nth valence bond, $\rho_0 = 0.153$ nm is the equilibrium magnitude of this bond, the potential of the nth valence angle θ_n (i.e., the angle formed by two valence bonds) is

$$U(\theta_n) = \frac{\gamma}{2}(\cos\theta_n - \cos\theta_0)^2\,,$$

and $\theta_0 = 113°$ is the equilibrium magnitude of the valence angle. Manevich and Savin [36, 37] used the following values of potential parameters: $D_0 = 334.72$ kJ mol^{-1}, $\alpha = 19.1$ nm^{-1}, and $\gamma = 130.122$ kJ mol^{-1}.

The supersonic dynamic solitons present in this model were investigated numerically. It turned out that in this system there exist supersonic solitons with a finite velocity spectrum $c < v \leqslant c_1$, where $c = 1$ is the dimensionless velocity of longitudinal sound waves in the polyethylene molecule, v is the dimensionless soliton velocity, and $c_1 = 1.0203$. In the soliton localization region there occurs a longitudinal stretching and transverse contraction of the molecular chain. With an increase in soliton velocity, its energy and amplitude increase monotonically, while the width decreases. For $v = c_1$, the soliton energy is $E = 4.6$ eV, and the width is $R = 0.53$ nm. Numerical simulations showed that the soliton is dynamically stable for all velocities in the $c < v \leqslant c_1$ range. The existence of the upper velocity spectrum boundary for supersonic solitons in the polyethylene molecule (the velocity c_1) and the exact value of this velocity have been found through numerical experiments. The relation of the velocity c_1 to the parameters of the model has yet to be theoretically investigated.

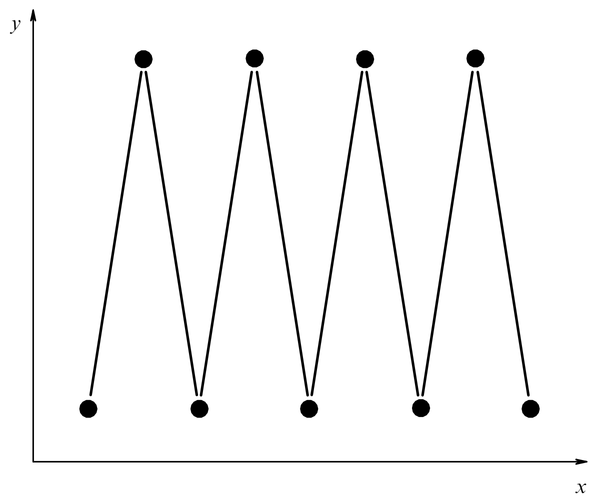

Figure 6. Schematic representation of the polymer chain in the plane trans-zigzag conformation.

The majority of authors who investigated the properties of solitons (including the supersonic ones) in one-dimensional systems restricted themselves to the consideration of the simplest case — the interaction only between the nearest neighbors in a one-dimensional chain. However, more complex kinds of interaction can also be realized in real physical systems. Remoissenet and Flytzanis [38] studied the soliton properties in a one-dimensional chain wherein the interaction was not confined to the nearest neighbors. They considered a chain of atoms of mass m with a lattice constant a. The Hamiltonian of this chain was of the form

$$H = \frac{1}{2}\sum_i \left[m(\partial_t u_i)^2 + W_i\right] + \frac{1}{2}\sum_{i\neq j} V_{ij}(u_i - u_j)^2 ,$$

where the potential

$$W_i = \frac{G}{2}\,(u_i - u_{i-1})^2 + \frac{A}{3}\,(u_i - u_{i-1})^3$$

characterizes the interaction between the nearest neighbors, and the potential

$$V_{ij} = \frac{J}{2}(1-r)\, r^{|i-j|-1}$$

the long-range interactions in the system. Here, J is a constant and the quantity r $(0 \leqslant r < 1)$ characterizes the interaction radius: for $r = 0$, the chain turns into a system wherein only the nearest neighbors interact, and for $r \to 1$ the atomic interaction radius becomes infinitely long. Using the designation $\varphi \equiv \partial_x u$ and disregarding higher-order derivatives, Remoissenet and Flytzanis [38] obtained the following equation of motion (the generalized Boussinesq equation) in the continuous approximation:

$$\partial_t^2\varphi - \partial_x^2\left[c^2(r)\varphi - p\,\varphi^2 - h(r)\,\partial_x^2\varphi - f(r)\,\partial_t^2\varphi\right] = 0\,, \quad (25)$$

where the speed of sound equals

$$c(r) = \left[\frac{1+r}{(1-r)^2}\,J' + G'\right]^{1/2} a\,,$$

$$J' = \frac{J}{m}\,, \quad G' = \frac{G}{m}\,, \quad p = 2A'a^3\,, \quad A' = \frac{A}{m}\,,$$

$$h(r) = \left[\frac{1+r}{(1-r)^2}\,J' + G'\right]\frac{a^4}{12} - \frac{ra^4}{(1-r)^2}\,G'\,,$$

$$f(r) = \frac{ra^2}{(1-r)^2}\,.$$

Equation (25) possesses the soliton solution

$$\varphi(x,t) = B(r)\,\mathrm{sech}^2\,\frac{x - vt}{L(r)}\,, \quad (26)$$

where the soliton amplitude is given by

$$B(r) = \frac{3}{2p}\left[v^2 - c^2(r)\right],$$

the soliton width is defined as

$$L(r) = 2\left[\frac{h'(r)}{v^2 - c^2(r)}\right]^{1/2},$$

and $h'(r) = h(r) + f(r)v^2$. At $r = 0$, Eqn (25) transforms into the Boussinesq equation (13), and soliton (26) to the soliton solution of the Boussinesq equation (14). In the limit $r \to 1$, when the atomic interaction radius tends to infinity, the width of the soliton becomes infinite, and its amplitude tends to zero.

A significant feature of the model under consideration is that it allows the existence of both subsonic and supersonic solitons. This is impossible in the system described by the Boussinesq equation. For $h'(r) > 0$, the solitons are supersonic $[v^2 > c^2(r)]$. When $p > 0$, they constitute extension regions; when $p < 0$, they form compression regions. For $h'(r) < 0$, the solitons propagate with subsonic velocities. In this case, the condition

$$v^2 < G'a^2 - \frac{(1-r)^2}{12r}\,c^2(r)$$

should be fulfilled. Therefore, in the subsonic area there exists a 'gap' (a forbidden velocity region) which separates the supersonic and subsonic parts of the soliton velocity spectrum. When the atomic interaction radius r shortens, the gap width increases until it occupies the entire subsonic area.

A simple generalization of the nonlinear chains considered in this section is a one-dimensional grain medium, i.e., a chain of macroscopic elastic granules. Nesterenko [39] was the first to theoretically show that there exist dynamic supersonic solitons in such a system, and he investigated their behavior in numerical experiments. More recently, Lazaridi and Nesterenko [40] observed these solitons in real experiments. For a one-dimensional grain medium, advantage was taken of a chain of steel balls 4.75 mm in diameter. One of the chain ends leaned against a rigid wall, while the other was subjected to a blow. The initial disturbance produced by the blow disintegrated rapidly into several solitons. Coste et al. [41] also observed such solitons in a chain of spherical granules. In these experiments, they used the balls manufactured of various materials: steel, bronze, tungsten carbide, etc. In particular, they observed solitons with a velocity of 1000 m s^{-1} and of width 3–4 cm (i.e., 4–5 granules) in the chain of small steel balls. The experimental data of Coste et al. [41] agree nicely with the theoretical predictions by Nesterenko [39].

The investigation of solitons in granular media is of importance not only from the fundamental standpoint, but from the practical one as well. In particular, the study of acoustic pulse propagation in soils is important for the understanding of the processes occurring during earthquakes and for building protection from earthquakes. Granular materials are employed in engineering, for instance, as dampers.

The study of solitons in granular media leads us to the conclusion that there is a close interrelation between supersonic dynamic solitons and shock waves. Indeed, the disintegration of a shock wave into several solitons was observed not only in the experiments by Lazaridi and Nesterenko [40]. Thus, Batteh and Powell [42] performed a series of numerical experiments on the propagation of shock waves in a one-dimensional chain of particles with the exponential Morse interaction potential. The Hamiltonian of the chain took the form

$$H = \frac{m}{2}\sum_{i=1}^{N}(\partial_t u_i)^2 + D\sum_{i=2}^{N}\left\{\exp\left[\alpha\,(u_{i-1} - u_i)\right] - 1\right\}^2,$$

where u_i is the displacement of the ith particle of mass m from the equilibrium position, while D and α are constants. This chain is close in properties to the Toda chain. Batteh and Powell [42] discovered that the propagation of a shock wave across a thermalized chain is accompanied by the formation of supersonic compression solitons with different velocities behind the wave front. The difference in soliton velocities is responsible for the broadening of the shock front and does not allow the stationary profile to set in. In the case of an initially nonthermalized chain, the amplitudes and velocities of all the solitons behind the front tended to the same fixed values. Peyrard et al. [43] discovered a similar effect in the simulation of detonation in the two-dimensional nitromethane crystal employing the molecular dynamics method[2]. They observed a detonation regime in which the shock front transformed into a chain of soliton-like excitations whose velocities decreased with distance to the front. This velocity difference resulted in a permanent broadening of the front of the shock wave in its propagation. The shock wave disintegration into separate solitons is attributed to the fact that the solitons are the stable elementary excitations in these nonlinear systems. That is why an unstable disturbance of arbitrary form (in particular, a shock wave) eventually disintegrates into solitons whose further breakup does not occur.

It is evident that elastic rods are close in properties to one-dimensional chains. The dynamics of rods would therefore be expected to obey similar nonlinear equations, and these equations also have soliton solutions. The corresponding equations were indeed obtained and their soliton solutions were found. Detailed reviews of the work concerned with solitons in elastic rods (as well as in plates and shells) appeared in the monographs of Ostrovsky and Potapov [44] and Erofeev et al. [45]. We will discuss the paper by Samsonov et al. [46], since they not only theoretically described the dynamic solitons in rods, but also observed them experimentally.

To describe the nonlinear dynamics of an elastic cylindrical rod (an acoustic waveguide), Dreĭden et al. [46] proposed the following modification of the Boussinesq equation:

$$\partial_t^2 u - c_1^2 \partial_x^2 u = \frac{1}{2}\,\partial_x^2\left[\frac{\beta}{\rho}\,u^2 + v^2 R^2(\partial_t^2 u - c_t^2\,\partial_x^2 u)\right], \quad (27)$$

where u are the longitudinal displacements of rod particles, $c_l = (E/\rho)^{1/2}$ is the velocity of longitudinal sound waves in the rod, ρ is the material density, $E = 2\mu(1+v)$ is the Young modulus, μ is the shear modulus, v is the Poisson coefficient, $c_t = (\mu/\rho)^{1/2}$ is the velocity of transverse sound waves, β is the material nonlinearity factor, and R is the cross sectional radius of the rod. In the derivation of Eqn (27) it was assumed that the deformations were small and the characteristic wavelength was far greater than the radius of the rod. Then, the cross sections of the rod remain flat under deformation and its side surface is left free from radial stress. Equation (27) possesses the soliton solutions

$$u_k(x,t) = D_k + A_k\,\mathrm{sech}^2\,\frac{x - v_k t}{L_k}\,, \quad (28)$$

where $k = 1, 2,\ D_1 = 0,\ D_2 = -2A_2/3,\ A_k$ is the soliton amplitude, the soliton velocity squared is

$$v_k^2 = c_1^2 \pm \frac{A_k\,\beta}{3\rho}\,, \quad (29)$$

and the soliton width squared is given by

$$L_k^2 = 2v^2 R^2\left[\frac{3(E-\mu)}{\beta A_k} \pm 1\right]. \quad (30)$$

The upper sign in relations (29) and (30) corresponds to the value of $k = 1$, and the lower to $k = 2$. When the nonlinearity factor is negative (this property is inherent, in particular, in the majority of metals, crystals of sodium chloride, and polystyrene), compression solitons can propagate through the rod; when β is positive (this is typical of glass and fused quartz), extension solitons are possible.

The condition $L_k^2 > 0$ imposes limitations on the velocity spectrum (28) of solitons. For the solutions u_1, this spectrum consists of two intervals: $0 < v_1 < c_t$ and $v_1 > c_1$. Therefore, the modification of the Boussinesq equation had the effect that subsonic nontopological solitons can propagate through the rod along with supersonic solitons. For the solutions u_2, only transonic velocities $c_t < v_2 < c_l$ are permissible.

To experimentally examine dynamic solitons, Dreĭden et al. [46] employed a 5.5-cm-long cylindrical polystyrene rod 1 cm in diameter. The rod was immersed in water. The solitons were excited in the sample when an unfocused pulse of a ruby laser irradiated an aluminum mirror located in water immediately in front of the input end of the sample. To measure the soliton characteristics, the authors of Ref. [46] used a holographic interferometer which recorded the rod images. The soliton amplitude was measured from the fringe displacements in the interference pattern. As a compression soliton passed through the rod, a Poisson widening of the side surface of the rod occurred. As a result, conic waves (the so-called Poisson waves) were excited in the fluid surrounding the rod. Observation of these waves enabled the measurement of the soliton velocity. The soliton width was measured directly from the photographs.

With the use of this facility, Dreĭden et al. [46] observed the propagation of compression solitons in the sample and measured their characteristics. In particular, they discovered a soliton of width $L = 8$ mm with a velocity $v = 2400$ m s^{-1}. The soliton parameters measured by the authors of Ref. [46] were found to be in satisfactory agreement with their theoretical predictions. More recently, Samsonov et al. [47] theoretically investigated the soliton propagation across a varied-diameter rod and experimentally examined solitons in such a sample.

Recently, Sharon et al. [48] experimentally discovered a new class of dynamic solitons in the investigation of crack propagation in glass. As a crack passed through a local inhomogeneity of the material, localized waves were excited at the crack front. They propagated a considerable distance without a significant change in amplitude and had a well-defined shape independent of their excitation conditions. Upon collision these waves retained their shape and amplitude, experiencing only a phase shift. This allowed Sharon et al. [48] to draw a conclusion about the soliton nature of such waves. Their velocity is close to that of Rayleigh waves.

2.3 Supersonic topological solitons

By now, different nontopological solitons in quasi-one-dimensional atomic chains have been thoroughly studied.

[2] We are reminded that the term molecular dynamics method is commonly used in reference to the method of numerical solution of physical problems by simulating the motion of particles (atoms, molecules) that make up the system under investigation.

When several such chains are placed some distance apart, a strongly anisotropic crystal forms. Polymer crystals containing parallel macromolecules give an example of such crystals. This brings up the natural question: how are nontopological solitons transformed in this case? In a three-dimensional system they acquire a topological charge which takes only discrete values. Indeed, each dynamic soliton displaces the chain over some distance. In a separate chain this displacement can be arbitrary. But in a crystal, the total displacement induced by a soliton (or a group of solitons) must be a multiple of the lattice constant. This is required for the soliton-carrying chain to be 'built into' the crystal. As we recede from a topological soliton (a defect), the crystal lattice structure should asymptotically approach the perfect (defect-free) structure. Therefore, the acquisition of a topological charge by a group of dynamic solitons imposes limitations on the displacements produced by these solitons. In some cases, these limitations lead to the formation of a discrete soliton-velocity spectrum. However, below in this section we will show that certain types of solitons possess a continuous velocity spectrum.

But after such a transformation, will the possibility of supersonic propagation persist for these, now topological, solitons? Balabaev et al. [34, 35] replied in the affirmative: they showed that in a three-dimensional system of loosely bound polymer chains (in other words, in a polymer crystal) there forms a bound state of several dynamic solitons, which has a topological charge. This group of solitons propagates with a supersonic velocity, like nontopological solitons in separate polymer chains. Similar results were somewhat earlier obtained by Savin [49] for a one-dimensional chain on a substrate. These cases are considered in greater detail below.

When the individual polymer chains forming a crystal are weakly coupled to each other, the possibility that the soliton localizes primarily in one chain persists after a group of dynamic solitons acquires a topological charge. In this case, the simplest model proves to be realistic — that is, the model wherein the weakly excited chains surrounding the strongly excited chain with the soliton are replaced by an external potential — the so-called 'substrate'. The overwhelming majority of investigations dedicated to the topological solitons were performed in this approximation. As a rule, only the chain–substrate interaction is taken into account in the analysis, but not the intrachain anharmonicity. Then, the problem reduces to the integration of the nonlinear Klein–Gordon equation, whose most studied special cases are the sine-Gordon equation discussed in the foregoing and an equation of the type φ^4. The localized solutions of these equations in the form of topological solitons possess, in conformity with the analogy discussed in our paper, a continuous velocity spectrum in the subsonic area. However, the simultaneous inclusion of the intrachain anharmonicity may radically change the situation.

The possibility for the supersonic propagation of a topological soliton was first demonstrated by Kosevich and Kovalev [50]. They considered the Frenkel–Kontorova model with the inclusion of the anharmonicity of interatomic interaction. In the continuous approximation, the equation of chain motion takes the form

$$m\partial_t^2 u = mc^2 \left[\partial_x^2 u + \frac{a^2}{12}\,\partial_x^4 u + \frac{\pi^2}{2}\,(\partial_x u)^2\,\partial_x^2 u\right] - \frac{2\pi}{a}\,U_0 \sin\frac{2\pi u}{a}\,, \tag{31}$$

where m, c, a, and U_0 are constants. The solution of this equation is a kink traveling with a velocity v:

$$u = \frac{2a}{\pi}\,\arctan\exp\left(-\frac{x - vt}{L_{\rm K}}\right), \tag{32}$$

where the kink width $L_{\rm K}$ is the solution of the following biquadratic equation

$$48\pi^2 U_0 L_{\rm K}^4 + 12ma^2(v^2 - c^2)\,L_{\rm K}^2 - mc^2a^4 = 0\,. \tag{33}$$

The dependence of the soliton width $L_{\rm K}$ on the soliton velocity v that follows from this equation is described by a rather cumbersome formula which we do not present here. The function $L_{\rm K}(v)$ is plotted in Fig. 7 (curve 3). Interestingly, Eqn (31) represents a 'unification' of two equations: the sine-Gordon equation [it can be obtained from Eqn (31) by discarding the second and third terms in the right-hand side] and the modified Boussinesq equation [it can be obtained from Eqn (31) by removing the substrate, i.e., putting $U_0 = 0$]. Accordingly, the Kosevich–Kovalev soliton (32) combines the properties of the kink of the sine-Gordon equation and the supersonic dynamic solitons considered in Section 2.2. Indeed, let us consider two limiting cases. When the interaction between the anharmonic chain and the substrate is strong, i.e., $U_0 \gg mc^2$, the last term on the left-hand side of Eqn (33) can be disregarded. Then, the velocity dependence of the soliton width is described by the well-known Lorentzian law:

$$L_{\rm L} = \frac{a}{2\pi}\sqrt{\frac{m}{U_0}}\,\sqrt{c^2 - v^2}\,.$$

This dependence is plotted in Fig. 7 (curve *1*).

The second limiting case corresponds to a free chain which does not interact with the substrate. Then, $U_0 = 0$, and the second and third terms remain on the left-hand side of Eqn (33). The velocity dependence of the soliton width

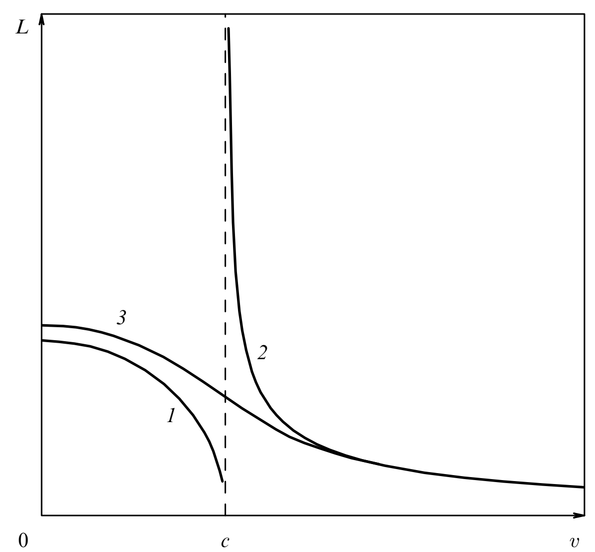

Figure 7. Widths L of different solitons as functions of their velocities v: *1* — for the kink of the sine-Gordon equation (the Lorentzian relationship); *2* — for supersonic dynamic solitons (in particular, for the soliton solution of the modified Boussinesq equation), and *3* — for the Kosevich–Kovalev soliton.

results in the form

$$L_{\mathrm{B}} = \frac{ac}{2\sqrt{3(v^2 - c^2)}} \,. \tag{34}$$

The function $L_{\mathrm{B}}(v)$ is depicted in Fig. 7 (curve *2*). Formula (34) for $L(v)$ differs from the similar expression for the width (15) of the soliton of the Boussinesq equation by a constant numerical factor.

Therefore, the function $L_{\mathrm{K}}(v)$ for the topological Kosevich – Kovalev soliton is the 'joining' of two solutions: the Lorentzian function $L_{\mathrm{L}}(v)$ for subsonic topological solitons and the function $L_{\mathrm{B}}(v)$ for supersonic dynamic solitons. Interestingly, the 'limiting solutions' $L_{\mathrm{L}}(v)$ and $L_{\mathrm{B}}(v)$ behave in the opposite way when the soliton velocity approaches the speed of sound: as this takes place, the width of the subsonic kink tends to zero, and the width of the supersonic dynamic soliton to infinity. The Kosevich – Kovalev soliton width $L_{\mathrm{K}}(v)$, which corresponds to the 'joining' of the solutions $L_{\mathrm{L}}(v)$ and $L_{\mathrm{B}}(v)$, remains nonzero and finite at $v = c$ and asymptotically tends to zero as $v \to \infty$. This soliton possesses a continuous velocity spectrum ranging from zero to infinity. The speed of sound for the Kosevich – Kovalev soliton (32) is not a singularity at all. Of course, when the velocity is high enough this soliton becomes so narrow that its width turns out to be of the same order of magnitude as the lattice constant. In this case, the continuous approximation is no longer valid and the description of chain dynamics calls for using a different equation instead of Eqn (31) to take into account the discrete nature of the system. However, provided that the interaction between the atoms in the chain is much stronger than their interaction with the substrate, i.e., $U_0 \ll mc^2$, the continuous approximation remains valid for rather high velocities from the supersonic area.

The Kosevich – Kovalev soliton possesses a continuous velocity spectrum in the supersonic area owing to the fact that both 'limiting cases' of Eqn (31) — the sine-Gordon equation and the modified Boussinesq equation — admit soliton solutions in the form of the same function (32), which differ only by the form of the dependence $L(v)$. As indicated below, the dynamic-to-topological soliton transformation upon embedding a one-dimensional atomic chain in the crystalline environment also occurs for other chains which may have other intrachain interactions. However, in those cases when the 'limiting equations' (i.e., the equations for the subsonic topological and supersonic dynamic solitons) possess solutions described by different functions, the velocity spectrum of the supersonic topological soliton becomes discrete. A similar 'joining' of subsonic and supersonic soliton solutions would be expected to occur for other equations as well, including those whose solutions cannot be represented in elementary functions.

Kosevich and Kovalev [50] considered yet another modification of the Frenkel – Kontorova model by including a cubic term in the atomic interaction potential. The potential accounting for the interaction with the substrate was taken to be of the polynomial form. The equation of chain motion assumed then the form

$$m\partial_t^2 u = mc^2\left(\partial_x^2 u + \frac{a^2}{12}\,\partial_x^4 u - 3\beta\partial_x u\,\partial_x^2 u\right) - \frac{2U_0}{a^4}\,u(a-u)(a-2u) \,. \tag{35}$$

This equation describes, in particular, the dynamics of a bistable molecular chain, i.e., a chain possessing two stable equilibrium states. Equation (35) admits the soliton solution

$$u = a\left[\exp\frac{3\beta(x - vt)}{a} + 1\right]^{-1} . \tag{36}$$

A characteristic feature of this soliton is that it can propagate only with a single velocity $v = v_0$:

$$v_0 = c\left(1 + \frac{3\beta^2}{4} - \frac{2U_0}{9\beta^2 mc^2}\right)^{1/2} .$$

When $\beta^4 > 8U_0/27mc^2$, this velocity is supersonic. For soliton (36), the speed of sound is not a singularity. It can propagate with this velocity and in so doing possess a finite energy. Below we provide other examples which indicate that the existence of a discrete velocity spectrum is the common property of many supersonic topological solitons.

More recently, Savin [49] investigated this model employing a modified substrate potential. After this modification, Eqn (35) assumed the form

$$(1 - v^2)\,\partial_\zeta^2 u + \frac{1}{12}\,\partial_\zeta^4 u - 3\beta\partial_\zeta u\,\partial_\zeta^2 u - 4Gu(u^2 - 1) = 0 \,, \tag{37}$$

where $\zeta = x - vt$ and use is made of the system of units wherein $a = 1$, $c = 1$, and $m = 1$. In the limit $G \to 0$, equation (37) upon one integration passes into the Boussinesq equation

$$\frac{1}{12}\,\partial_\zeta^2\varphi + (1 - v^2)\varphi - \frac{3\beta}{2}\varphi^2 = 0 \,,$$

where $\varphi \equiv \partial_\zeta u$. We are reminded that the solution of this equation is a supersonic acoustic soliton traveling with a velocity v:

$$\varphi = \beta^{-1}(1 - v^2)\,\mathrm{sech}^2\, q\zeta \,, \tag{38}$$

where the reciprocal of the soliton width equals $q = \left[3(v^2 - 1)\right]^{1/2}$. With a substrate, this soliton acquires a topological charge

$$Q(v) = u(+\infty) - u(-\infty) = \int_{-\infty}^{+\infty}\varphi(\zeta)\,\mathrm{d}\zeta = -\left[\frac{4(v^2 - 1)}{3\beta^2}\right]^{1/2} = -1 \,. \tag{39}$$

This relation fixes the velocity of the supersonic soliton. Formula (39) is also valid when N identical acoustical solitons exist in the chain. It then assumes the form

$$NQ(v_N) = -1 \,.$$

From this equation it follows that the velocity of each acoustic soliton is given by

$$v_N = \left[1 + \frac{3}{4}\left(\frac{\beta}{N}\right)^2\right]^{1/2} .$$

Therefore, Savin [49] showed that in the limit $G \to 0$ the topological soliton conforming to the Frenkel – Kontorova model with a cubic anharmonicity of interatomic interactions possesses a finite discrete supersonic velocity spectrum for which the speed of sound is an accumulation point.

Savin [49] performed a numerical investigation of the dynamics of topological solitons in this chain for a small height $G = 10^{-3}$ of the potential barrier. Numerical simulations showed that the topological soliton always possesses a continuous subsonic velocity spectrum $0 \leqslant v \leqslant v_0 < 1$. With an increase in the anharmonicity parameter β, the upper boundary of the continuous spectrum tends toward the speed of sound: $v_0 \to 1$ as $\beta \to \infty$. The numerical simulations confirmed that the soliton displays a finite discrete supersonic velocity spectrum $v = v_n$, $n = 1, \ldots, N$, where $v_1 > \ldots > v_N > 1$. The number N of admissible supersonic velocity values and the velocity values v_n themselves are increasing monotonically with an increase in the anharmonicity parameter β.

Solution (38) found in the continuous approximation is not exact for the soliton in a discrete chain. That is why the real values of supersonic velocities ($\bar{v}_n$) are different from the calculated ones (v_n). Since the width of a supersonic soliton decreases with its velocity, the accuracy of results obtained in the continuous approximation is simultaneously lowered. The difference $\triangle v_n = v_n - \bar{v}_n$ therefore increases as the velocity v_n recedes from the speed of sound. When the supersonic kink velocity $v > \bar{v}_n$, its motion is accompanied by the emission of phonons, resulting in the deceleration of the soliton. For the velocity $v = \bar{v}_n$, the emission vanishes and the kink travels at this constant velocity. In a substrate-free chain ($G = 0$), the kink disintegrates into n uncoupled acoustic solitons and a subsonic phonon radiation.

More recently, Zolotaryuk et al. [51] performed a numerical investigation of the behavior of topological solitons in the 'Toda chain on a substrate' model. Like the Kosevich – Kovalev model, this model represents a modification of the Frenkel – Kontorova model with an anharmonic interatomic interaction. An exponential potential [the Toda potential (9)] was chosen as the potential of interatomic interaction, and a sinusoid was selected as the substrate potential. As in the case considered above, the velocity spectrum of topological solitons is discrete. A special feature of the 'Toda chain on a substrate' model is the discreteness of the soliton velocity spectrum not only in the supersonic area, but in the subsonic as well. This spectrum is bounded from below, i.e., there exists a minimal velocity of solitons in this chain. With an increase in anharmonicity parameter b entering potential (9) (for small b), the number of soliton velocities in the spectrum increases and the lower spectral boundary decreases. With $b \to \infty$, the properties of solitons in this chain approach the properties of ordinary Toda solitons. Braun [52] also investigated the Frenkel – Kontorova model with an exponential interatomic interaction to arrive at the conclusion that the solitons in this chain can propagate with both supersonic and subsonic velocities.

Zolotaryuk et al. [53] also investigated the supersonic motion of topological solitons in a bistable chain. They considered a one-dimensional chain of molecules coupled by hydrogen bonds: – X-H – X-H – X-H –. This structure was modeled with a chain consisting of alternating atoms of different mass: protons of mass m, and heavy ions X^- of mass M. Let $q_n = y_n/a$ be the dimensionless displacement of the proton in the nth hydrogen bond, $Q_n = Y_n/a$ be the dimensionless displacement of the ion, and a be the period of both sublattices. We introduce the following variables: $R_n = (Q_n + Q_{n+1})/2$ for the center-of-mass displacement of the cell formed by the nth and $(n+1)$th heavy ions, $\rho_n = Q_{n+1} - Q_n$ for the relative displacement of these ions, and $u_n = q_n - R_n$ for the nth proton displacement relative to the center of the segment connecting the nth and $(n+1)$th ions. The chain is free, for it does not experience any external forces.

In the continuous approximation, the soliton dynamics in this chain is described by the equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}u} = \frac{(1 - v^2)\,\partial\Phi/\partial u}{(v^2 - s^2)\,\partial\Phi/\partial\rho}\,, \tag{40}$$

where the potential is given by

$$\Phi = \frac{\rho^2}{2}\left[\left(1 + \frac{1}{\mu}\right)v^2 - \left(1 + \frac{s^2}{\mu}\right)\right] + \mu\Omega_0^2\left[V(u) - g_0\rho\right],$$

c and c_0 are the respective speeds of sound in the proton and ionic sublattices, the dimensionless soliton velocity v is expressed in units of the velocity c, $s = c_0/c$, $\mu = m/M$, $V(u)$ is the ion – proton interaction potential, while g_0 and Ω_0 are constants. An analytical solution of Eqn (40) was found in Ref. [53]:

$$\begin{aligned} \frac{\mathrm{d}u_{\mathrm{K}}}{\mathrm{d}\zeta} &= \pm\gamma_1\Omega_0\left[2V(u_{\mathrm{K}})\right]^{1/2}, \\ \rho_{\mathrm{K}} &= \mp\frac{\mu\Omega_0\left[2V(u_{\mathrm{K}})\right]^{1/2}}{\gamma_1(s^2 - v^2)}\,, \end{aligned} \tag{41}$$

where $u_{\mathrm{K}}(\zeta)$ is the proton component of the soliton solution, which is a kink (or an antikink), $\zeta = x - v\tau$, and $\tau = ct/a$ is the dimensionless time. The kink satisfies the boundary conditions $u_{\mathrm{K}}(\pm\infty) = \pm q_0$ and constitutes a negative ionic defect, while the antikink corresponds to the opposite boundary conditions $u_{\mathrm{K}}(\pm\infty) = \mp q_0$ and constitutes a positive ionic defect. The chain may reside in one of the two degenerate ground states: – X-H – X-H – X-H – or – H-X – H-X – H-X –. The solitons transfer the chain from one state to the other. The quantities $\pm q_0$ stand for the local minima of the potential $V(u)$, which correspond to the two equilibrium proton positions. The quantity

$$\gamma_1 = \left(\frac{1}{1 - v^2} + \frac{\mu}{s^2 - v^2}\right)^{1/2} \tag{42}$$

is the generalization of the Lorentzian factor to the case of a one-dimensional problem with two speeds of sound. On passing to the limit of 'frozen' heavy ions, $\mu \to 0$ and γ_1 transforms into a conventional Lorentzian factor.

The supersonic soliton solutions to Eqn (40) possess an electric charge. But all quasi-relativistic effects in this problem are related only to the speeds of sound. Therefore, the supersonic motion of the soliton (41) exhibits a purely mechanical effect. From formula (42) it follows that this soliton possesses two allowed velocity areas: the subsonic area $0 < v < s$, and the supersonic one (it can also be termed transonic, because it lies between the velocities of sound for the proton and ionic sublattices) $c_1 < v < 1$, where

$$c_1 = \left(\frac{s^2 + \mu}{1 + \mu}\right)^{1/2}.$$

When the soliton falls within the supersonic velocity range, the ionic sublattice deformation ρ_{K} changes sign: the kink is accompanied with a localized compression ($\rho_{\mathrm{K}} < 0$) in the subsonic area, and with extension ($\rho_{\mathrm{K}} > 0$) in the supersonic one, while the antikink, conversely, is attended with extension in the subsonic area and with compression in the supersonic one.

The following result was obtained for the soliton energy in the continuous approximation [53]:

$$E = \left(\frac{2^{1/2}}{\gamma_1}\right)^3 mc^2\Omega_0\left[\frac{1}{(1-v^2)^2} + \frac{\mu s^2}{(s^2-v^2)^2}\right] \times \int_0^{q_0} \left[V(u)\right]^{1/2}\,\mathrm{d}u\,.$$

It follows from this formula that the soliton energy E in the subsonic range $0 < v < s$ tends to infinity for $v \to s$. In the supersonic area, the soliton energy for velocities close to c_1 decreases sharply with velocity. However, on further increase in soliton velocity the soliton energy resumes its increase and tends to infinity as $v \to 1$ (Fig. 8).

Also investigated in Ref. [53] was the subsonic and supersonic soliton propagation in this system in the situation where the potential V depends not only on u, but on ρ as well. In this case, the effects related to the motion of the center of mass R_n were ignored. However, numerical simulations of the soliton motion in this system showed that this approximation (which the authors of Ref. [53] termed the second approximation) was in poor agreement with the results of simulations. The results outlined above agree well with the data of numerical experiments.

Zolotaryuk et al. [53] used the following values of the system parameters in their simulations: $a = 0.276$ nm, $\mu^{-1} = 17$, and $s = 0.1$. Numerical experiments showed that the supersonic soliton motion is unstable, with the exception of the $v = 0.67 - 0.69$ velocity range. The supersonic solitons which initially possess other velocities propagate with variable velocities until they either reach the stable domain or go over to the subsonic region. The collision of two supersonic solitons results in their reflection accompanied by a strong emission of sound waves. Upon collision, both solitons fall within the subsonic region. In the case where the chain interacted with the substrate, only subsonic solitons were observed in numerical simulations.

The model investigated in Ref. [53] describes proton transport in molecular systems with the chains of hydrogen bonds. Such chains are encountered in various substances, for instance, in ice and proteins. Solids with hydrogen bonds are void of free electrons and should therefore be insulators. In spite of this, charge transfer does take place in them owing to the proton transport. As is known from experiments, when the electric field and the chain of hydrogen bonds coincide in direction, the conductivity of such a material turns out to be three orders of magnitude higher than on application of the field in the perpendicular direction [54]. The proton transport plays a significant part in biosystems, in particular, in the energy transformation at the cellular level [55, 56]. At present, it is believed that chains of the polar groups of amino acid residuals of transmembrane protein molecules form proton channels in mitochondria. The bacteriorhodopsin molecule constitutes one proton pump of this kind. In this case, the proton path lies through the amino acid residuals containing the OH hydroxyl group.

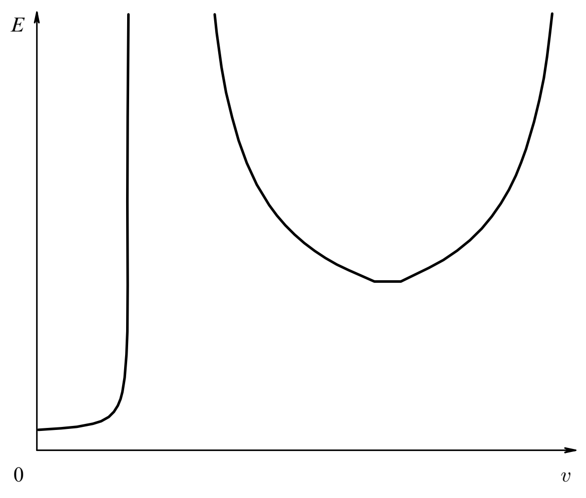

Figure 8. Energy E of the topological soliton in a bistable chain as a function of its velocity v.

The supersonic motion of topological solitons in bistable energy-nondegenerate systems was investigated by Manevich, Smirnov and co-workers [56–60]. In particular, Manevich and Smirnov [59] considered a quasi-one-dimensional model of a diatomic molecular crystal. Here, molecules form a flat zigzag-like chain. The Hamiltonian of the system has the form

$$H = \sum_j \left[\frac{M}{2}(\partial_t u_j)^2 + \frac{m}{2}(\partial_t w_j)^2 + U(w_j - u_j) + \frac{\delta}{2}(u_{j+1} - w_j)^2 + \frac{K}{2}(u_{j+1} - u_j)^2 + \frac{k}{2}(w_{j+1} - w_j)^2\right],$$

where M and m are the masses of 'heavy' and 'light' particles, u_j and w_j are their displacements from the equilibrium positions in the lattice, and δ, K, and k are the rigidities of intermolecular bonds. The intramolecular potential U has two energy-nondegenerate minima.

Let us introduce new variables: the center-of-mass displacements $\chi_j = (Mu_j + mw_j)/(M+m)$ of the molecules, the intramolecular coordinates $\varphi_j = w_j - u_j$, $M_t = M + m$, $\mu = Mm/M_t$, $c^2 = (\delta + K + k)/M_t$, $c_1^2 = (\delta - Km/M - kM/m)/M_t$, $\beta = \left[\delta(M-m)/2 + kM - Km\right]/M_t$, $\gamma = a\delta/6$, $\omega^2 = \delta/\mu a^2$, $h = a^2c^2/12$, and a is the lattice constant. The crystal dynamics in the continuous approximation is then described by the system of equations

$$\partial_t^2\chi - c^2\partial_x^2\chi - h\partial_x^4\chi + \frac{(\delta/a)\,\partial_x\varphi - \beta\partial_x^2\varphi + \gamma\partial_x^3\varphi}{M_t} = 0\,,$$

$$\partial_t^2\varphi + c_1^2\partial_x^2\varphi + \omega^2\varphi + \frac{(\partial_\varphi U/a^2) - (\delta/a)\,\partial_x\chi - \beta\partial_x^2\chi - \gamma\partial_x^3\chi}{\mu} = 0\,.$$

As shown in Ref. [59], the solution of this equation is a supersonic topological soliton traveling with a velocity v:

$$\varphi = \varphi_k\frac{1 - \tanh(\zeta/d)}{2}\,,$$

$$\partial_\zeta\chi = -\frac{a\varphi_k}{2M_t(v^2 - c^2)}\left\{\delta\left[1 - \tanh\left(\frac{\zeta}{d}\right)\right] - \frac{a\beta}{d}\,\mathrm{sech}^2\left(\frac{\zeta}{d}\right) - \left(a\gamma + \frac{\delta h}{v^2 - c^2}\right)\frac{\mathrm{sech}^2(\zeta/d)\tanh(\zeta/d)}{2d^2}\right\},$$

where $\zeta = x - vt$, and d is a constant. This soliton transfers the system from the initial state ($\varphi = 0$) to the intermediate one ($\varphi = \varphi_k$). The latter is nonequilibrium, because the center-of-mass velocities of the molecules are nonzero in this state. The subsequent crystal relaxation from the intermediate state to the final state (with zero center-of-mass velocities of the molecules) proceeds rather far from the soliton and does not significantly affect its dynamics. The initial and intermediate states correspond to the two minima of the effective potential Φ, i.e., are defined by the condition

$$\partial_\varphi \Phi(\varphi_k, v_k) = 0\,. \tag{43}$$

For the specific soliton velocity v_k, these minima are equal:

$$\Phi(0, v_k) = \Phi(\varphi_k, v_k)\,. \tag{44}$$

The topological soliton in this system can propagate only with a certain velocity defined by Eqns (43) and (44).

This model and its modifications are employed to describe the propagation of detonation waves in solid explosives, topochemical reactions (in particular, combustion) in molecular crystals, the proton transfer along the chains of hydrogen bonds, and the structural transitions in polymers [56]. In all these cases, the topological soliton effects the 'state transport', i.e., transfers the system from the initial state to the intermediate one. Subsequently, the system relaxes from the intermediate state to the final one. Among the topochemical reactions which proceed by the soliton mechanism, mention should be made of the solid-phase polymerization of diacetylene [56]. As a result of the reaction, the molecular monomer monocrystal transforms to the polymer monocrystal without participation of the liquid intermediate state. In this case, the displacement of the topological soliton by a distance equal to the lattice constant corresponds to an elementary act of growth of the polymer chain (i.e., to the displacement of one monomer molecule and its addition to the chain). The soliton stopping signifies the chain termination and the formation of a defect in the polymer crystal.

In the foregoing we considered the propagation of the supersonic dynamic soliton (23) in a zigzag polymer macromolecule. Balabaev et al. [34, 35] studied the following problem: how the properties of this soliton change when the molecule is incorporated in a polymer crystal? For the first time, they investigated the effects related to the acquisition of a topological charge by the dynamic soliton owing to a weak interchain coupling. The research was conducted not only in the 'anharmonic chain on a substrate' approximation, as in the works by Kosevich and Kovalev [50] and Savin [49], but with the inclusion of the mobility of all molecular chains in the crystal as well. They considered a polyethylene crystal consisting of parallel polymer chains in the plane trans-zigzag conformation. The authors of Ref. [35] discovered, both analytically and numerically, the solution of the equations describing the dynamics of this crystal in the approximation of immobile neighboring chains. It constitutes a supersonic topological extension soliton traveling with a velocity $v = 1.49 \times 10^4$ m s^{-1} $= 1.012c$, where c is the speed of sound in the chain. Balabaev et al. [34, 35] also reported the results of molecular-dynamics simulations of the behavior of extension solitons (vacancies) in a three-dimensional polymer crystal, when all the chains are immobile. The extension soliton was observed to propagate with a supersonic velocity $v = 1.5 \times 10^4$ m s^{-1} $= 1.019c$. When disregarding the interaction between the soliton-carrying chain and the neighboring chains, the supersonic topological soliton disintegrated into four dynamic solitons propagating with close velocities. Ignoring the interaction signifies the removal of the crystalline environment of the soliton-carrying chain. In other words, this chain was instantly extracted from the crystal in such a way that the relative velocities and displacements of none of the atoms changed at the instant of extraction. Of course, such an operation can be performed only in a numerical experiment. Naturally, the law of topological charge conservation is violated in this case, for it is precisely the crystalline environment that fixes the magnitude of the topological charge. The observed soliton disintegration proves that the supersonic topological solitons in polyethylene constitute the bound states of several dynamic solitons. We are reminded that similar behavior of supersonic solitons in the one-dimensional system was earlier discovered by Savin [49]. Balabaev et al. [34, 35] observed this phenomenon in a three-dimensional crystal for the first time.

To summarize the discussion of the properties of supersonic solitons, we emphasize their radical difference from the recently discovered supraluminal electromagnetic solitons [61, 62]. These supraluminal solitons can propagate only through nonequilibrium media — they do not carry information. By contrast, the supersonic topological solitons considered in our paper propagate in equilibrium systems and carry information.

3. Relativistic and quasi-relativistic effects in dislocation dynamics

We now move on to multidimensional mechanical systems. In this section and in Section 4 we restrict our consideration to the mechanics of continua that obey the relations of the linear theory of elasticity. Although our attention has heretofore been focused on different nonlinear models, in this case such a restriction is fully justified. The occurrence of a topological charge is a geometrical property, and it is therefore not necessary that the dynamic equations be nonlinear. Topological solitons, and dislocations in particular, can exist in linear systems as well. Naturally, real crystals are always nonlinear, but the nonlinear properties of a crystal lattice are most pronounced near the dislocation center, about its core, and they do not exert an appreciable influence on a variety of dislocation effects. That is the reason why the equations of the linear theory of elasticity are extensively used in the study of dislocations [2, 63]. When considering the analogs of relativistic effects in dislocation physics (of course, for not-too-high defect velocities) we can well restrict ourselves to the linear theory of elasticity.

3.1 Dynamics of screw dislocations

First, we consider a rectilinear screw dislocation in a three-dimensional crystal, namely, the dislocation parallel to the z-axis and traveling with a velocity v in the direction of the x-axis (see Fig. 1). We will pass to the continuous approximation and assume the continuous medium to be isotropic. Then, the displacements u_3 of the medium particles around the dislocation are the solutions of the equation

$$\mu\left(\partial_1^2 + \partial_2^2\right)u_3 = \rho\,\partial_t^2 u_3 \tag{45}$$

subject to an additional boundary condition: the solution to equation (45) satisfies the definition of the Burgers vector (1), which assumes, in this instance, the form

$$b = \oint_L \mathrm{d}u_3 \,.$$

From this point on, the subscript i on the u_i variable denotes the displacement direction: 1 — displacement in the x-axis direction, 2 — in the y-axis direction, and 3 — in the z-axis direction; μ is the shear modulus; ρ is the continuum density, and $\partial_1 \equiv \partial/\partial x$. Equation (45) is invariant under the Lorentz transformations with a parameter $c_{\mathrm{t}} = (\mu/\rho)^{1/2}$ (i.e., transformations in which the velocity c_{t} of transverse sound waves occupies the place of the speed of light). Its solution corresponding to a moving screw dislocation has the form [2]

$$u_3 = \frac{b}{2\pi} \arctan \frac{\gamma y}{x - vt} \,, \tag{46}$$

where $\gamma = (1 - v^2/c_{\mathrm{t}}^2)^{1/2}$. The dislocation is adopted as the origin of coordinates. The displacements u_1 and u_2 are zero in this case.

The mechanical stress about the dislocation takes on the form

$$\sigma_{13} = \mu \partial_1 u_3 = -\frac{\mu b \gamma y}{2\pi\left[(x - vt)^2 + \gamma^2 y^2\right]} \,,$$

$$\sigma_{23} = \mu \partial_2 u_3 = \frac{\mu b \gamma (x - vt)}{2\pi\left[(x - vt)^2 + \gamma^2 y^2\right]} \,. \tag{47}$$

We now move to cylindrical coordinates. The only nonzero component of the mechanical stress tensor is then described by the formula

$$\sigma_{\varphi z}(r, \varphi) = \frac{\mu b \gamma}{2\pi r(\cos^2\varphi + \gamma^2 \sin^2\varphi)} \,, \tag{48}$$

where $r^2 = (x - vt)^2 + y^2$, and the angle φ is measured from the x-axis. Figure 9 shows the sections through the surface defined by the condition $\sigma_{\varphi z} = \mathrm{const}$, which are produced by the plane $z = \mathrm{const}$ in the cases $v = 0$ and $0 < v < c_{\mathrm{t}}$. The same dependence on the coordinates and the velocity is inherent in the electric field intensity of an infinite rectilinear charged rod parallel to the z-axis and moving with a velocity v in the x-axis direction:

$$\mathbf{E}(r, \varphi) = \frac{2q\mathbf{r}\gamma}{r^2(\cos^2\varphi + \gamma^2 \sin^2\varphi)} \,, \tag{49}$$

where q is the linear charge density of the rod, and $\mathbf{r} = (x - vt, y)$ is the radius vector from the rod to the point of field observation. In formula (49), $\gamma = (1 - v^2/c^2)^{1/2}$, where c is the speed of light. The origin of the analogy between relations (48) and (49) will be elucidated in Section 4, where we will discuss the parallels between the dynamic theory of topological defects and the fundamental field theories, namely, electrodynamics and gravitation theory.

The kinetic dislocation energy constitutes the kinetic energy of the elastic continuum in the vicinity of the defect:

$$T = \frac{1}{2} \int \rho \partial_t u_i \partial_t u_i \, \mathrm{d}V \,. \tag{50}$$

We substitute the displacement field (46) of the screw dislocation in formula (50) to obtain [63, 64]

$$T = \frac{\rho}{2} \int (\partial_t u_3)^2 \, \mathrm{d}V = \frac{E_0 v^2}{2 c_{\mathrm{t}}^2 \gamma} \,,$$

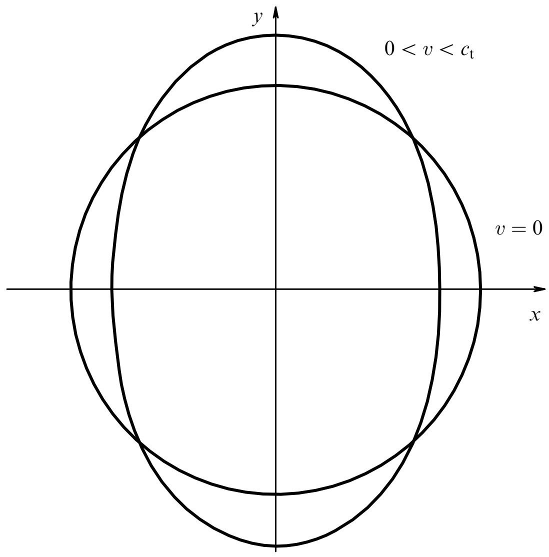

Figure 9. Sections through the surface defined by the condition $\sigma_{\varphi z} = \mathrm{const}$, which are produced by the plane $z = \mathrm{const}$ for an immobile screw dislocation and a dislocation traveling with a velocity ranging $0 < v < c_{\mathrm{t}}$.

with the rest energy of the screw dislocation being equal to

$$E_0 = \frac{\mu b^2}{4\pi} \ln \frac{R}{r_0} \,,$$

where R and r_0 are the limits of integration with respect to r in the cylindrical coordinates. R is generally taken to be equal to the distance between the dislocation and crystal boundary, and r_0 equal to the lattice constant. The potential energy U of the defect is the energy of its elastic field. In the continuous approximation it is defined as

$$U = \frac{1}{2} c_{idfh} \int \partial_d u_i \partial_h u_f \, \mathrm{d}V \,, \tag{51}$$

where c_{idfh} is the tensor of elastic modules. We substitute the displacement field (46) into formula (51) to obtain

$$U = \frac{E_0}{\gamma} \left(1 - \frac{v^2}{2 c_{\mathrm{t}}^2}\right) .$$

The total energy of the screw dislocation is given by

$$E = T + U = \frac{E_0}{(1 - v^2/c_{\mathrm{t}}^2)^{1/2}} \,.$$

Therefore, the energy of the screw dislocation depends on its velocity by the Lorentzian law.

3.2 Dynamics of edge dislocations

We consider an edge dislocation in a two-dimensional crystal, shown schematically in Fig. 10. The Frenkel – Kontorova model discussed above provides an approximate description of atomic behavior in the glide plane of the edge dislocation (i.e., in the plane passing through the dislocation, parallel to the Burgers vector). We move to the continuous approximation, assuming the medium to be isotropic and obedient to the relations of the linear theory of elasticity. Then, the displace-

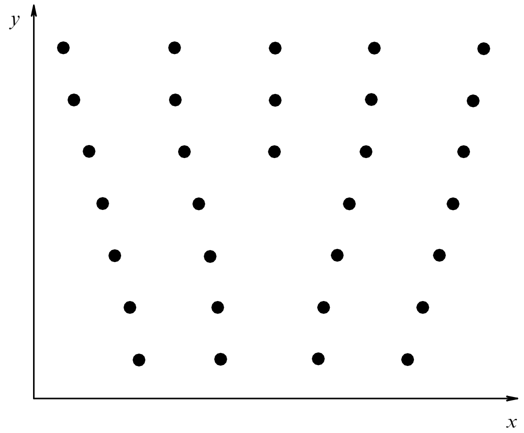

Figure 10. Edge dislocation in a two-dimensional crystal.

ments of the particles in the medium about the edge dislocation with the Burgers vector $\mathbf{b} = (b, 0)$, the dislocation traveling with a velocity $\mathbf{v} = (v, 0)$, are solutions of the system of equations [2]

$$\mu \partial_j \partial_j u_i + (\mu + \lambda) \partial_i \partial_j u_j = \rho \partial_t^2 u_i \tag{52}$$

subject to an additional boundary condition: the solution of Eqn (52) satisfies the definition of the Burgers vector (1). Here, λ is the Lamé constant. In the two-dimensional crystal there exist longitudinal and transverse sound waves. Their velocities are always different. It is therefore reasonable that the formulas which describe the motion of an edge dislocation [in particular, Eqn (52)] are not Lorentz-invariant. The solution of Eqn (52), which corresponds to the traveling edge dislocation, is of the form [64]

$$u_1(x, y, t) = \frac{bc_t^2}{\pi v^2} \left[\arctan \frac{y(1 - v^2/c_l^2)^{1/2}}{x - vt} + \left(\frac{v^2}{2c_t^2} - 1 \right) \arctan \frac{y(1 - v^2/c_t^2)^{1/2}}{x - vt} \right],$$

$$u_2(x, y, t) = \frac{bc_t^2}{2\pi v^2} \left\{ \frac{v^2/(2c_t^2) - 1}{(1 - v^2/c_t^2)^{1/2}} \times \ln \left[(x - vt)^2 + \left(1 - \frac{v^2}{c_t^2} \right) y^2 \right] + \left(1 - \frac{v^2}{c_l^2} \right)^{1/2} \ln \left[(x - vt)^2 + \left(1 - \frac{v^2}{c_l^2} \right) y^2 \right] \right\}, \tag{53}$$

where $c_l = \left[(\lambda + 2\mu)/\rho \right]^{1/2}$ is the velocity of longitudinal sound waves. The dislocation is adopted as the origin of the coordinates. The mass–energy equivalence relation also changes. For the edge dislocation it takes the following form [2, 64]

$$E_0 = \frac{Mc_t^2}{1 + c_t^4/c_l^4}, \tag{54}$$

where M is the dislocation mass.

The displacements (53) can be divided into two additive components u_t and u_l which are Lorentz-transformable with different parameters (the speeds of sound c_t and c_l) [2]. However, this separation would be impossible for the dislocation energy and momentum. Indeed, from formulas (50) and (51) it follows that in the expression for dislocation energy there appear Lorentz-transformable terms with the parameter c_t and the parameter c_l, as well as the cross terms arising from the multiplication of the derivatives of the displacements u_t and u_l. Consequently, if we know, for instance, the value of the rest energy of an edge dislocation, we cannot find its energy for some velocity v by the direct application of the Lorentz transformations or some more complex local transformations. First, it is necessary to calculate the displacements of the particles of the medium, which are produced by the given dislocation, resolve them into the components u_t and u_l, apply the Lorentz transformations of the two different types to these components, and then calculate the energy of the dislocation traveling with a velocity v. Therefore, in the soliton theory in a two-dimensional space, unlike the one-dimensional case considered above, the Lorentz transformations lose their significance as a universal instrument for the calculation of dislocation characteristics at different velocities. The equations of dynamic dislocation theory (see Section 4) do not satisfy the Lorentz invariance condition, either.

Fushchich and Nakonechnyi [65] showed that the wave equation describing the propagation of elastic waves in an isotropic continuous medium is invariant with respect to the nonlocal conformal group C(3,1) containing integro-differential transformations. This group plays the same part in the elasticity theory of isotropic continuum as the Lorentz group in electrodynamics. However, the situation is much more complicated in the description of dislocation motion in an anisotropic medium, where sound waves with three different velocities can generally propagate in any direction. To date, an analog of the Lorentz group has not been found for such an instance. Nevertheless, in the framework of the dislocation theory it is possible to obtain analogs of different formulas of the special theory of relativity. Therefore, the dislocation theory may well be referred to as quasi-relativistic. But its relationships become Lorentz-invariant [this can be verified by the example of formulas (53) and (54)] only in the limiting case, when the velocity c_l of longitudinal sound waves tends to infinity. Only in the case of a rectilinear screw dislocation in a three-dimensional isotropic continuum are all the formulas Lorentz-invariant irrespective of the value of c_l.

3.3 On the origin of relativistic and quasi-relativistic effects in dislocation theory

All the results of Section 3.2 remain in force for a rectilinear edge dislocation in a three-dimensional crystal as well. The Burgers vector of a rectilinear screw dislocation is parallel to the dislocation line, and the Burgers vector of an edge dislocation is perpendicular to this line. In the most general case, most often encountered in real crystals, the angle made by this vector with the dislocation is equal to neither 0 nor $\pi/2$, this angle assuming different values at various points of the dislocation line (because the direction of the Burgers vector always remains invariable, and the dislocation direction may arbitrarily change). Therefore, one and the same dislocation may be an edge dislocation in one portion of the line, and a screw dislocation in another portion. We have shown that the dynamic theory of screw dislocations satisfies the Lorentz invariance condition, and the theory of edge dislocations does not. Consequently, there is bound to exist a continuous passage from the Lorentz-invariant relations to the formulas which do not satisfy the Lorentz invariance condition. All these results may be obtained employing the same general relations of dislocation theory.

It is important to remember that all the relativistic and quasi-relativistic relationships given above were derived in the framework of the classical mechanics of an elastic continuum. They can therefore be obtained by purely classical methods which are ordinarily not invoked in relativistic physics. In particular, expressions (46) and (53) can be found by employing the Mura formulas for dislocation-induced distortions [66]. These relatively complex relations assume a simpler form in the four-dimensional notation [4]:

$$\beta_{jn}(x_f) \equiv \partial_n u_j(x_f) = c_{\mathrm{t}}^{-2} C^{iabd} e_{nhag} \int_{\Omega} J^h{}_i{}^g(x'_f) \times \partial_d G_{bj}(x_f - x'_f)\,\mathrm{d}\Omega', \qquad (55)$$

where $b, i, j = 1, 2, 3;$ $a, d, f, g, h, n = 0, 1, 2, 3;$ $c_{\mathrm{t}} = (c^{1212}/\rho)^{1/2}$, $\partial_0 \equiv c_{\mathrm{t}}^{-1}\partial_t$,

$$C^{iabd} = \begin{cases} c^{iabd} & \text{for} \quad a = 1, 2, 3, \\ -\delta^{bi}\delta^{ad}\rho c_{\mathrm{t}}^2 & \text{for} \quad a = 0. \end{cases}$$

Here, c^{iabd} is the three-dimensional tensor of elastic modules, δ^{bi} is the Kronecker symbol, e_{nhag} is the four-dimensional completely antisymmetric Levi–Civita tensor, $e^{0123} = 1$,

$$J^h{}_i{}^g(x'_f) = \tau^h b_i V^g \delta(x'_f - x_f{}^0) \qquad (56)$$

is the dislocation flux density tensor, τ^h is a unit vector tangential to the dislocation line, $V^g = = (c_{\mathrm{t}}, -\mathbf{V})$ is the four-dimensional dislocation velocity, $\mathbf{V}$ is the three-dimensional dislocation velocity vector, $\delta(x'_f - x_f{}^0)$ is the Dirac delta function, $x_f{}^0$ are the coordinates of the dislocation line, $\mathrm{d}\Omega' = \mathrm{d}V'\,\mathrm{d}(c_{\mathrm{t}}t')$ is a volume element in the four-dimensional spacetime, and G_{bj} is the tensor Green function of the equations of classical linear theory of elasticity of the continuum under consideration. Our attention is engaged by the fact that the above expression for the four-dimensional dislocation velocity is different from the usual formula of the special theory of relativity. This is due to the fact that the formulas of dynamic dislocation theory are in general not Lorentz-invariant.

An analysis of the above examples of the relativistic and quasi-relativistic effects leads to the following conclusion: all these effects ensue from the retardation of signals in the propagation of solitons. The function of signals in these instances is fulfilled by longitudinal and transverse sound waves rather than electromagnetic ones, as in the special theory of relativity. Indeed, the contraction of width of a traveling kink by the Lorentzian law, which was considered in Section 2.1, is attributed to the fact that the chain atoms situated in front of the kink have no time to shift to the positions corresponding to the maximal kink width. In the time during which they shift, the kink approaches, and therefore the same difference of displacements u is observed in a narrower region L. The same retardation accounts for the shape variation of the elastic field of a traveling dislocation, which is demonstrated in Fig. 9.

The effect of signal retardation on the shape of displacement field is described by the Mura formula (55) which expresses the dislocation distortions in terms of the dynamic Green function. We would remind the reader of the definition of this function in the continuous theory of elasticity. The dynamic Green function is defined as the solution of the classical equations of motion of an elastic continuum [67]:

$$c_{ijnl}\partial_l \partial_j G_{nr}(\mathbf{x}, t) + \delta_{ir}\delta(\mathbf{x})\delta(t) = \rho\partial_t^2 G_{ir}(\mathbf{x}, t).$$

It is well known that the Green function $G_{ij}(\mathbf{x}, t)$ represents the component of elastic displacement in the x_i direction at a point $\mathbf{x}$ at the instant of time t, which is caused by a unit pulsed force applied in the x_j direction at the point $\mathbf{x} = 0$ at the instant of time $t = 0$. It is precisely the dynamic Green function that describes the retardation of signals in the system under different actions. The motion of dislocations is one such action. All the above-discussed relativistic and quasi-relativistic effects emerging in the soliton theory are eventually governed by the properties of the Green function and are consequences of the Mura formula (55). This formula represents the most general relation of dynamic dislocation theory: consequences of this formula are both the Lorentz-invariant relativistic formulas (Section 3.1) and the quasi-relativistic relations which do not satisfy the Lorentz invariance condition (Section 3.2). The Mura formula (55) was derived in the framework of the classical mechanics of an elastic continuum, and therefore all the above-discussed relativistic and quasi-relativistic formulas of soliton theory ensue from Newton's postulates.

From the consideration conducted it follows that only those quantities which are expressed in terms of dislocation distortions depend on the dislocation velocity according to the Lorentzian law. In other words, the relativistic and quasi-relativistic velocity dependences are inherent only in the physical quantities related to the retardation of signals in the soliton motion: the fields of elastic deformations and mechanical stresses generated by dislocations, the soliton energy, the momentum, the force exerted on the dislocation by the elastic field, etc. Those quantities that are not expressed in terms of the Green function, i.e., are not related to signal retardation (the dislocation velocity, the lattice parameters), do not depend on the soliton velocity in the relativistic manner. However, this does not hinder from using four-dimensional designations for these quantities [4].

3.4 Transonic and supersonic dislocations

In Section 2 we considered different examples of supersonic topological solitons propagating in one-dimensional systems. It would therefore appear natural that topological solitons in two- and three-dimensional systems (in particular, dislocations in crystals) would possess the capacity to travel faster than sound. Unfortunately, the problem of supersonic dislocation motion has been poorly studied in comparison with the problem of supersonic solitons in one-dimensional systems. Although the first papers on this subject date back to the 1940s, they are currently of no special interest because their authors endeavored to solve this problem in the framework of the linear theory of elasticity. From the above-considered examples of supersonic solitons it follows that solving the supersonic dislocation problem calls for the inclusion of nonlinear and dispersion terms in the equations of the theory of elasticity.

We introduce several new terms, because both transverse and longitudinal sound waves can propagate through two- and three-dimensional crystals. The dislocation velocity v, which is below the velocity c_{t} of transverse sound waves, will be referred to as subsonic velocity. The velocity v of a transonic dislocation satisfies the inequality $c_t < v < c_l$, where c_l is the velocity of longitudinal sound waves. The dislocation which travels faster than the longitudinal sound waves will be termed supersonic.

Hoover et al. [68] published the first paper on the problem of transonic dislocations that took into account the crystal lattice anharmonicity. They carried out numerical simulations of the propagation of edge dislocations in a closely packed hexagonal lattice. The lattice atoms obeyed a central-force interaction law. Hoover and co-workers observed the dislocation motion with a transonic velocity $v = 0.89c_l$. However, the authors provided no details on this dislocation propagation mode. More recently, Gumbsch and Gao [69] performed a closer investigation of this problem. They undertook numerical simulations of the motion of edge dislocations in a body-centered cubic tungsten lattice using the molecular dynamics method. The sample was subjected to an applied shear stress. For a light load, the dislocations traveled with subsonic velocities ($0.65c_t - 0.7c_t$). As the applied shear stress increased, they started moving with transonic velocities. The dislocation would then move slower, come to a halt, and stay at one point for 0.5 ps. During this pause, the broad core of the transonic dislocation would contract to the size of the subsonic defect core, following which the dislocation would move with a velocity of about $0.7c_t$. Subsonic dislocations were not observed to overcome the sound barrier: the dislocations produced possessed transonic velocities. As noted in the foregoing, this property is inherent in the solitons in one-dimensional systems — they can also be produced with supersonic velocities.

When the load was gradually increased, the velocity of transonic dislocations was observed to rise from $1.38c_t$ to $1.5c_t$. Since the dislocation is a source of the elastic field, its motion with a velocity higher than the velocity of the field itself (i.e., faster than the speed of sound) is bound to be accompanied by the emission of sound waves (this effect is the acoustic analog of the well-known Vavilov–Cherenkov effect). Gumbsch and Gao [69] observed this emission, though only in that part of the elastic field of the dislocation where the crystal was stretched. No emission was observed in the area of the compression field.

Under still heavier applied loads, the dislocation moved faster than the longitudinal sound waves. In this case, the configuration of the dislocation core differed little from the configuration of the transonic dislocation core, but the anisotropy of dislocation radiation was still stronger. One wave front of longitudinal oscillations was observed in the area of the extension field, while from four to five localized fronts were observed in the area of the compression field. The linear theory of elasticity predicts in this case the emergence of two symmetric pairs of wave fronts (one pair corresponds to the emission of transverse sound waves, and the other to the emission of longitudinal ones). It is conceivable that the additional fronts correspond to the nonlinear modes of the sound spectrum.

The results of Gumbsch and Gao [69] were later borne out by the works of other researchers. Shi et al. [70] also carried out a series of numerical experiments on the tungsten lattice and observed the transonic motion of edge dislocations in that crystal. Koizumi et al. [71] discovered the transonic motion of screw dislocations in their numerical experiments. They modeled a three-dimensional cubic lattice. The atomic rows parallel to the z-axis moved in that lattice as perfectly rigid rods. When the dislocation velocity exceeded $0.7c_t$, the dislocation field of mechanical stress gave birth to pairs of dislocations with the Burgers vectors of the opposite sign (this process is the reverse of the annihilation of a topological soliton and an antisoliton).

Rosakis [72] came up with a modification of the Peierls model describing transonic and supersonic dislocations. We are reminded that the dislocation-bearing crystal in the Peierls model [2] was considered as a linear elastic continuum everywhere except in the glide plane of the dislocation, where account was taken of the discreteness of a real crystal and the nonlinearity of its elastic properties characterized by the potential $\Phi(u)$, where u is the atom displacement. Rosakis showed that adding the term describing the gradient nonlinearity of the form $c(\partial_x u)^2$, where the x-axis is the direction of dislocation motion, to the derivative $\Phi'(u)$ of the potential, makes possible the transonic and supersonic propagation of edge dislocations. It would be instructive to investigate the relation between this finding and the above-outlined results on the supersonic propagation of topological solitons in one-dimensional models with a gradient nonlinearity.

4. Gauge theory of line defects and fundamental field theories

In Section 3.1 we noted the equivalence (correct to a change of constants) between the formula describing the dependence of the mechanical stress field induced by a rectilinear screw dislocation on its velocity and coordinates and the formula describing the dependence of the electric field intensity of an infinite rectilinear charged rod on its coordinates and velocity. Clearly, this coincidence is not accidental. It is part of the analogy between the dynamic dislocation theory and electrodynamics [3, 4]. An understanding of this analogy is required for elucidating the linkage between the relativistic effects in electrodynamics and classical mechanics.

4.1 Gauge theory of line defects and electrodynamics

Indeed, dislocations are sources of elastic fields (the fields of mechanical stress and deformation), while electric charges are sources of electromagnetic fields. However, any defects in crystals (for instance, cracks) are sources of elastic fields. A distinctive property of linear defects (dislocations and disclinations) is their topological nature. This is precisely the reason for the gauge nature of the dynamic theory of these defects. The topological charges of defects (the Burgers vectors in the case of dislocations) are analogous to electric charges. These charges are conserved quantities. The gauge transformations in the theory of defects constitute an analog to the well-known gradient potential transformations in electrodynamics. By way of example we consider a rectilinear screw dislocation parallel to the z-axis (see Fig. 1). We introduce a cylindrical coordinate system and adopt the dislocation as the origin of the coordinates. In the continuous approximation (the continuum is assumed to be isotropic), the particle displacement field in the medium about an immobile dislocation is of the form $u_z = b\phi/2\pi$. The angle ϕ can be measured from any axis perpendicular to the dislocation. Moving from one such axis to another (i.e., the passage to another coordinate system) has the result that some constant is added to the displacements u_z. In doing this, the distortions $\beta_{jn} = \partial_n u_j$ and the mechanical stresses

$$\sigma_{bh} = C_{bhjn}\beta^{jn} \qquad (57)$$

remain unchanged. Once again, we draw an analogy with electrodynamics: in the gauge transformations of the potential A_j, the electromagnetic field tensor $F_{jn} = \partial_j A_n - \partial_n A_j$ (its components are the components of the electric **E** and magnetic **H** field vectors) remains unchanged.

The fact that the distortion (β_{jn}) and mechanical stress (σ_{jn}) tensors are precisely the analogs to the electromagnetic field tensors F_{jn} and H_{jn}, is easily verified by comparing the expression for the electromagnetic field energy density

$$W_{\mathrm{em}} = \frac{1}{16\pi}\left(-4F^{0i}H^0{}_i + F_{in}H^{in}\right) = \frac{1}{8\pi}\left(\mathbf{ED} + \mathbf{HB}\right),$$

where $i, n = 1, 2, 3$, **D** is the electric induction, and **B** is the magnetic induction, with the formula for the elastic field energy density (in the three-dimensional notation)

$$W_{\mathrm{el}} = \frac{1}{2}\left[\rho(\partial_t u_i)^2 + \beta_{in}\sigma_{in}\right].$$

Using the four-dimensional notation introduced above, the elastic field energy density can be written down in the form

$$W_{\mathrm{el}} = \frac{1}{2}\left(-\beta^{i0}\sigma_i{}^0 + \beta^{in}\sigma_{in}\right),$$

where $i, n = 1, 2, 3$.

It is pertinent to note a significant distinction between the gauge theory of defects and electrodynamics. In electrodynamics and all gauge theories of the Yang – Mills type, the gauge-field tensor F^a_{jn} ($a = 1$ in electrodynamics) is antisymmetric with respect to permutation of the indices j and n. In the theory of defects, the analogous tensors β_{jn} and σ_{jn} are, generally speaking, neither symmetric nor antisymmetric. In the framework of the linear theory of elasticity, in the majority of cases it is possible to ignore both the antisymmetric part of the distortion tensor describing small rotations of the continuum and the antisymmetric part of the mechanical stress tensor (ignore couple stresses). Then, instead of the distortion tensor, use should be made of the symmetric deformation tensor $\varepsilon_{jn} = (\partial_n u_j + \partial_j u_n)/2$. In this case, the analogy with the electromagnetic field tensor F_{jn} becomes all the more evident.

The difference in symmetry and tensor dimensionality of the quantities describing gauge fields leads us to the conclusion that there is no way to directly use the relations of electrodynamics or gauge theories of the Yang – Mills type in the gauge dislocation theory. In particular, the Lagrangian of the interaction of dislocations with elastic fields cannot have the form of the convolution of the in-medium particle displacement vector u_i and the dislocation flux density tensor $J^h{}_i{}^g$, because this is a tensor of rank 3. Next in this section we briefly outline the method for constructing the Lagrangian of the interaction of linear defects (dislocations and disclinations) with elastic fields, which was proposed by Musienko and Koptsik [4]. However, since such defects as disclinations are relatively little known, we first give several definitions.

The dislocations considered above are local breakdowns of the translation symmetry of a crystal. By contrast, disclinations represent local breakdowns of orientation crystal symmetry (i.e., the symmetry with respect to a rotation group). Disclinations possess topological charges — the Frank vectors (strictly speaking, pseudovectors). In the continuous approximation, the Frank vector is represented by the integral [73, 74]

$$\omega_i = \frac{1}{2}\oint_L e_{imn}\partial_s\partial_m u_n \,\mathrm{d}x_s,$$

where L is an arbitrary closed circuit enclosing the disclination.

Taking advantage of the four-dimensional notation introduced in Section 3.3, we represent the Lagrangian of elastic fields in an anisotropic continuum as

$$L_0 = -\frac{1}{2}C^{irjn}\,\partial_r u_i\,\partial_n u_j. \tag{58}$$

Here, $n, r = 0, 1, 2, 3$, and $i, j = 1, 2, 3$. In Section 3.3 we gave the Mura formula which expresses the dislocation distortions in terms of the dislocation density. The corresponding expressions for disclinations were derived by Kossecka and de Wit [75]. In the four-dimensional notation, their results take on the form

$$\begin{aligned}\partial_f u_n(x_a) = &-\frac{1}{c_{\mathrm{t}}^2}\int_\Omega e_{tghf}\,C^{jilh}\partial_i\,G_{jn}(x_a - x'_a)\,I^t{}_{lr}{}^g(x'_a)\\ &\times(x'^r - \tilde{x}^r)\,\mathrm{d}\Omega' - \frac{1}{2}\int_\Sigma C^{jilh}\,\partial_i\,G_{jn}\,(x_a - x'_a)\,e_{ghbq}\,\Omega_{lf}\\ &\times V^g\,\mathrm{d}f'^{bq}\delta(x'_a - x_a{}^0),\end{aligned} \tag{59}$$

where $a, f, g, h, i, t = 0, 1, 2, 3$, and $b, j, l, n, q, r = 1, 2, 3$; the disclination flux density tensor is given by

$$I^t{}_{lr}{}^g(x_i) = \tau^t\,\Omega_{lr}\,V^g\,\delta(x_i - x_i{}^0), \tag{60}$$

$\Omega_{nj} = e_{nij}\omega^i$ (at $n = 0$ or $j = 0$, the components of the tensor Ω_{nj} are assumed to be zero), $x_i{}^0$ are the coordinates of the disclination line, $\tilde{x}^r$ are the coordinates of the point of Frank-vector application to the disclination line, V^g is the disclination velocity, $\mathrm{d}f'^{bq}$ is the area element lying in the plane formed by the unit vectors e^b and e^q, and Σ is the disclination formation surface, i.e., an arbitrary surface bordering the disclination.

We replace the distortions in the Lagrangian (58) with the sums of distortions produced by the applied elastic field and the distortions (55) and (59) arising from defects. We rearrange the resultant expression to find the Lagrangian of the interaction of elastic fields with the defects:

$$L_{\mathrm{int}} = -\frac{1}{c_{\mathrm{t}}}\,B_{gij}\,K^{gij},$$

where the four-dimensional tensor of the defect current is expressed as

$$\begin{aligned}K^{gij}(x_d) = e^{gajb}&\left[J_a{}^i{}_b(x_d) + I_a{}^{ir}{}_b\,(x_d)(\tilde{x}_r - x_r)\right] +\\ &+\frac{1}{2}\int_\Sigma \Omega^{gi}\,V_f e^{jfqt}\,\delta(x_d - x'_d)\,\mathrm{d}f'_{qt},\end{aligned}$$

$a, i, q, r, t = 1, 2, 3$, and $b, d, f, g, j = 0, 1, 2, 3$; B_{gij} is the tensor potential presenting an analog of the vector potential A_j in electrodynamics. The four-dimensional mechanical stress tensor and the tensor potential are related as

$$\sigma_{ij} = \partial^d\,B_{dij}.$$

The total Lagrangian of the medium with defects is written in the following way:

$$L = L_0 + L_{\mathrm{int}} + L_{\mathrm{m}}. \tag{61}$$

Here, L_m is the material Lagrangian which characterizes the energy of defects without the inclusion of their elastic interaction. Its analog in electrodynamics is the Dirac Lagrangian of the electron–positron field. The exact form of the Lagrangian L_m for dislocations and disclinations remains to be found. This is the reason why attempts to derive the Lagrangian of the interaction between defects and elastic fields employing the canonical gauge method — by substituting the covariant derivatives in the material Lagrangian for the partial ones — up to the present have not met with success. Several authors (for instance, Kadić and Edelen [5]) employed this method, but they relied on a radically different analogy between the gauge theory of linear defects and the field theories (in particular, electrodynamics). In the approach taken by Kadić and Edelen, the dislocation flux density turns out to be an analog to the intensity and induction of electric and magnetic fields. The material Lagrangian in this theory is the Lagrangian of the elastic field in the defect-free continuum. Kadić and Edelen described the interaction between the defects and the elastic fields by changing from partial derivatives in the elastic-field Lagrangian to covariant ones. We believe this approach to be incorrect. From the above consideration it is evident that the defects possessing a topological charge should be treated as analogs to charged particles rather than massless quantum-carriers of the interactions, as is done by Kadić and Edelen [5]. In our approach, as in Kosevich's work [3], the analog for the electromagnetic field is the elastic field.

We emphasize that it was precisely the analogy proposed by Kosevich [3] which was later employed in the construction of the theory of dislocation melting of two-dimensional crystals [76]. Such melting constitutes a topological phase transition, which is also known as the Berezinski–Kosterlitz–Thouless phase transition. The dislocation theory of melting is based on the analogy between the gas of edge dislocations and a two-dimensional Coulomb gas (a gas of charged particles).

By varying the field potentials, from expression (61) we obtain the equations for elastic fields:

$$\partial^h \sigma_{nh} = \frac{1}{c_t} C_{ngij} K^{gij} , \tag{62}$$

where $i, n = 1, 2, 3$, and $g, h, j = 0, 1, 2, 3$. For immobile dislocations, these equations (in three-dimensional notation) were found by Kosevich [3]. Equations (62) are similar to the second pair of the Maxwell equations

$$\partial_h H^{nh} = -\frac{4\pi}{c} j^n . \tag{63}$$

In the dislocation theory there also exists the analog to the first pair of the Maxwell equations. Since these equations are rather cumbersome, we write them for a disclination-free continuum:

$$\partial_g \partial_a u_n(x_b) - \partial_a \partial_g u_n(x_b) = e_{giah} J^i{}_n{}^h (x_b) . \tag{64}$$

Here, $n = 1, 2, 3$, and $a, b, g, h, i = 0, 1, 2, 3$. These equations constitute the definition of a dislocation and the statement that this continuum is void of disclinations. They are similar to the first pair of Maxwell equations

$$e^{ijlb} \partial_j F_{lb} = 0 . \tag{65}$$

As is well known, these equations are equivalent to the statement that there exist no magnetic charges (monopoles) in nature. If Eqns (64) are summed in threes, they acquire a form which is more similar to that of Eqns (65). Equations (62), like analogous Eqns (63), were obtained by varying the Lagrangian of the system. Equations (64) and (65) are not variational.

Invoking the principle of least action, from the Lagrangian (61) we find the force exerted by the elastic field on a unit length of the linear defect:

$$f_i = \frac{1}{c_t} \int_V K_{idj} \sigma^{dj} \, dV . \tag{66}$$

For a static dislocation, expression (66) in the three-dimensional notation assumes the form

$$f_i = e_{igt} \tau_g b_d \sigma_{dt} . \tag{67}$$

This is the well-known Peach–Kaeler force, being an analog to the Coulomb force in electrodynamics. We consider, for example, two parallel screw dislocations in an isotropic medium with the Burgers vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ parallel to the z-axis. We substitute the mechanical stress (47) produced by one of the dislocations in the right-hand side of formula (67). This gives the expression for the force of dislocation interaction in cylindrical coordinates:

$$\mathbf{F} = \frac{\mu(\mathbf{b_1} \cdot \mathbf{b_2}) \, \mathbf{r}}{2\pi r^2} ,$$

where $\mathbf{r}$ is the vector connecting the two dislocations. Consequently, the dislocations of like signs ($\mathbf{b_1} \cdot \mathbf{b_2} > 0$) repel each other, and the dislocations of opposite signs ($\mathbf{b_1} \cdot \mathbf{b_2} < 0$) attract each other.

In the dynamic case, there emerges a correction to force (67), which depends on the defect velocity. In the three-dimensional notation it takes the form

$$f_i = \rho v_n b_n e_{irl} \tau_r V_l .$$

Here, v_n is the velocity of the particles in the medium. This is the so-called dislocation Lorentz force found by Kosevich [3] in a different way.

Menskiĭ [77] showed that the gauge group of a theory is the representation of the fundamental group of the space of order parameter variation. The order parameters in the continuous theory of dislocations and disclinations are the particle displacements in the medium. The fundamental group in this case is $\mathrm{SO}(3) \triangleright \mathrm{T}(3)$, where $\triangleright$ symbolizes the semidirect product of the groups of rotations SO(3) and translations T(3) in a three-dimensional space. Consequently, the gauge group of this theory in the most general case is the group $\mathrm{SO}(3) \triangleright \mathrm{T}(3)$.

At present it is a wide-spread opinion (see, for instance, Ref. [15]) that the conservation of topological charges is not related to the symmetries of the Lagrangian of the system, i.e., is not a corollary of the Noether theorem. In the framework of the gauge theory of dislocations and disclinations, the laws of conservation of the topological charges of these defects can be obtained with the aid of the second Noether theorem, as shown by Musienko and Koptsik [4]. Indeed, let us consider the gauge transformations of the potential, which leave the values of the observed fields (mechanical stress and deformation tensors) unchanged:

$$B_{\gamma i\beta} \to B'_{\gamma i\beta} = B_{\gamma i\beta} + e_{\gamma\beta\delta\alpha} \partial^\alpha \varepsilon_i{}^\delta + \partial^\nu \partial^\alpha \xi_{\gamma i\beta\nu\alpha} . \tag{68}$$

Here, $i = 1, 2, 3$, the Greek indices assume the values from 0 to 3, $\varepsilon_i{}^{\delta}$ is an arbitrary tensor function of the coordinates and the time, and $\xi_{\gamma i\beta\nu\alpha}$ is also an arbitrary function of the coordinates and the time which is antisymmetric in the last two indices and satisfies the following conditions

$$\partial^{\nu}\partial^{\alpha}\xi_{\gamma i\beta\nu\alpha} \neq 0\,,$$

$$\partial^{\gamma}\partial^{\nu}\partial^{\alpha}\xi_{\gamma i\beta\nu\alpha} = 0\,.$$

According to the second Noether theorem, some conservation law corresponds to every gauge transformation. Transformations (68) with the parameters $\varepsilon_i{}^{\delta}$ lead to the conservation laws

$$\partial_{\mu}(J^{\mu}{}_i{}^{\nu} - J^{\nu}{}_i{}^{\mu}) - I^{\mu}{}_{i\mu}{}^{\nu} + I^{\nu}{}_{i\mu}{}^{\mu} = 0\,. \tag{69}$$

This is an analog to the law of electric charge conservation in electrodynamics, which also follows from the second Noether theorem. In the absence of disclinations in this continuum, expression (69) at $\nu = 0$ signifies that dislocations cannot terminate inside a crystal: they should either form closed loops or crop out at the surface, and for $\nu \neq 0$ it transforms into the continuity equation for the dislocation flux density. In the presence of disclinations, expression (69) at $\nu = 0$ signifies that dislocations may terminate at disclinations; and for $\nu \neq 0$ it signifies that the disclination motion [under the conditions defined by expression (69)] is accompanied by the production or cancellation of dislocations.

Transformations (68) with the parameters $\xi_{\gamma i\beta\nu\alpha}$ lead to similar disclination conservation laws

$$\partial_{\mu}(I^{\delta}{}_{i\gamma}{}^{\mu} - I^{\mu}{}_{i\gamma}{}^{\delta}) = 0\,. \tag{70}$$

The topological charge conservation laws (69) and (70) were previously found [78] in the three-dimensional form as a result of topological relationships. Therefore, the second Noether theorem makes it possible to deduce the continuity equations for the dislocation and disclination fluxes, which describe the conservation of the topological charges of linear defects in their motion, along with the static relationships characterizing the spatial continuity of the extended defects themselves and the equality of topological characteristics at different points of one and the same linear defect.

Musienko and Koptsik [79] generalized the above-outlined gauge theory of dislocations and disclinations to the case of crystals with complex lattices, i.e., crystals containing more than one atom in the elementary cell.

4.2 Gauge theory of line defects and gravitation theory

Many authors [80–82] have called attention to yet another analogy — that is, between the dynamic theory of linear defects and the gravitation theory. This analogy is closer than the analogy to electrodynamics considered above. Indeed, a crystal containing topological defects constitutes a manifold with non-Euclidean geometry. When studying the dynamic behavior of this crystal in the continuous approximation, we can introduce the concepts of metric and connectivity. The metric tensor g_{ij}, as is generally known, defines the magnitude of the square of the interval:

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,.$$

The connectivity Γ^i_{jn} characterizes the variation of the components of a vector under an infinitesimal parallel translation

$$\delta A^i = -\Gamma^i{}_{jn}A^j\,\mathrm{d}x^n\,.$$

In the case of a defect-free crystal, the metric is Euclidean and the connectivity is identically equal to zero. When our concern is only with the processes occurring on greater scales than the interatomic distance (i.e., when the continuous approximation is valid), the mechanical properties of the crystal are determined exclusively by its geometry. Therefore, the situation is the same as in the gravitation theory, where the interaction of particles depends on the geometric characteristics of spacetime.

The deformation of continuum (possibly including the formation of topological defects) is described by the mapping of the initial manifold: $x_i \to y_i(x)$. This manifold transforms the initial Euclidean metric to the metric

$$g_{ij} = \frac{\partial x^n}{\partial y^i}\,\frac{\partial x^l}{\partial y^j}\,\delta_{nl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\varepsilon_{ij}\,.$$

The curvature tensor

$$R_{ijn}{}^l = \partial_i\Gamma_{jn}{}^l - \partial_j\Gamma_{in}{}^l + \Gamma_{ir}{}^l\Gamma_{jn}{}^r - \Gamma_{jr}{}^l\Gamma_{in}{}^r$$

and the disclination flux density tensor (60) are related as

$$R^{ijnl} = e^{iajb}I_a{}^{nl}{}_b\,.$$

In the gravitation theory, as is well known, the source of torsion is particles and, in general, any objects possessing energy (e.g., an electromagnetic field).

The torsion tensor

$$T^g{}_{ij} = \Gamma^g{}_{ij} - \Gamma^g{}_{ji}$$

and the dislocation flux density tensor (56) are related as

$$T^{gij} = e^{iajb}J_a{}^g{}_b\,. \tag{71}$$

The source of torsion in the gravitation theory is the four-dimensional spin density tensor $S^{\lambda}{}_{\mu\nu} = v^{\lambda}S_{\mu\nu}$, where v^{λ} is the velocity of a particle possessing spin, and $S_{\mu\nu}$ is an antisymmetric tensor. Its spatial components make up the three-dimensional vector $\mathbf{s} = (S^{23}, S^{31}, S^{12})$, which is equal to the three-dimensional spin density in the rest frame of the particle. In particular, the spin density tensor of the Dirac field is [83]

$$S_{\lambda\mu\nu} = \Psi^{+}\gamma_{[\lambda}\gamma_{\mu}\gamma_{\nu]}\Psi\,,$$

where Ψ is the Dirac spinor, Ψ^{+} is the Dirac adjoint spinor, and γ_{μ} are the Dirac matrices; antisymmetrization is performed over the indices in the square brackets. The spin density and the torsion are related by the formula [83, 84]

$$T^{\lambda}{}_{\mu\nu} = \frac{16\pi G}{c^3}\left(S^{\lambda}{}_{\mu\nu} + \frac{1}{2}\,\delta^{\lambda}_{\mu}S_{\nu} - \frac{1}{2}\,\delta^{\lambda}_{\nu}S_{\mu}\right). \tag{72}$$

Here, G is the Newtonian constant of gravitation, c is the speed of light, and $S_{\mu} = S^{\lambda}{}_{\mu\lambda}$.

5. Conclusions

Thus, we have shown that the propagation of solitons in the framework of classical mechanics is accompanied by effects related to the finiteness of the velocity of information transmission. When the equations describing the dynamics of the system possess Lorentzian symmetry, these effects coincide in form with the effects of the special theory of relativity. In this case, the speed of sound appears in the corresponding formulas instead of the speed of light. Like many other authors, we call these effects relativistic. The equations describing some mechanical systems need not be Lorentz-symmetric. However, the effects arising from the finiteness of the velocity of information transmission exist in these systems, too. The analogy to the special theory of relativity is nevertheless retained at a qualitative level. In particular, in the Kosevich – Kovalev model (the modified Frenkel – Kontorova model with an intrachain anharmonicity), the soliton width decreases with its velocity, while its energy therewith increases. The same qualitative analogy can be traced for other relativistic effects as well. However, the mathematical description of soliton dynamics in the Kosevich – Kovalev model is different from the usual Lorentzian one. That is why we applied the term 'quasi-relativistic' to these effects, which is indicative of their qualitative similarity to the effects in the special theory of relativity, as well as of some distinctions. The most significant of these distinctions is the possibility of supersonic motion of topological solitons in systems devoid of the Lorentzian symmetry. We emphasize that the soliton's passage through the sound barrier does not lead to any paradoxes like the violation of the causality principle or the emergence of imaginary soliton energy. Unlike the supraluminal solitons in active media, which do not carry information [61, 62], the supersonic solitons considered in our paper propagate in passive media and convey information.

Owing to the absence of Lorentzian symmetry, classical mechanics possesses a greater diversity of different effects related to the finiteness of the velocity of information transmission than the special theory of relativity. Investigations in this field have been undertaken relatively recently, and new results would therefore be expected to emerge here. Of special interest is, in our view, the study of systems whose behavior is close to the Lorentzian one for low soliton velocities but which depart from Lorentzian behavior when the soliton velocity approaches the speed of sound. An example of such a system is provided by the Kosevich – Kovalev model.

The most part of the results, especially the theoretical ones, pertaining to the dynamics of solitons in general and topological supersonic solitons in particular were obtained in the study of one-dimensional systems. An important line of future investigation is the generalization of these results to the case of multidimensional (two- and three-dimensional) systems. The theory of solitons in such systems is being developed intensively [85]. In particular, it would be instructive to theoretically describe the supersonic motion of dislocations in crystals with the use of the available data on the supersonic motion of topological solitons in one-dimensional systems. Undeniably, such works would be beneficial from the fundamental, as well as applied, viewpoints.

The investigation of supersonic solitons, both dynamic and topological, is of considerable applied interest. Such solitons (for instance, dislocations) play a significant part in the processes occurring in solids under heavy loads. Their investigation is important for understanding the phenomena like the propagation of shock waves, detonation, tectonic processes (and, in particular, earthquakes), the plasticity and disruption of solids, and the ballistic regime of thermal conduction. The progress of nanotechnologies makes particularly topical the study of the dynamics of solitons, because they play a large role in the processes occurring at the nanoscale level.

In this work we restricted ourselves to the consideration of solitons in mechanical systems. However, as noted in the Introduction, solitons are the subject of research in quite different branches of physics — from biophysics to elementary particle physics. It is hoped that many results outlined in our review will find use not only in mechanics, but in other areas of physics as well.

The history of science testifies that the study of analogies between different phenomena has repeatedly fostered the production of new results. The most striking example in this area is the optical – mechanical analogy whose investigation led to the discovery of the Schrödinger equation. We hope that the knowledge and the employment of the analogies between the theory of classical solitons, on the one hand, and the relativity theory and gauge field theories, on the other hand, will prove to be beneficial for experts in different realms of physics.

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