Physics – Uspekhi **47** (7) 705 – 716 (2004)

METHODOLOGICAL NOTES

PACS numbers: 03.65.Ta, **03.67.–a**, 42.50.Tv

‘Interaction-free’ measurement: possibilities and limitations

S P Vyatchanin, F Ya Khalili

DOI: 10.1070/PU2004v047n07ABEH001821

Contents

1. **Introduction** **705**
2. **Minimal energy cost of the ‘interaction-free’ measurement** **706**
3. **Reduction of the state of the object due to the ‘interaction-free’ measurement** **707**
   3.1 ‘Yes – no’ measurements; 3.2 ‘Interaction-free’ measurement of the coordinate
4. **Measurement of the number of quanta in an electromagnetic cavity** **711**
   4.1 Measurement scheme; 4.2 ‘Usual’ measurement; 4.3 ‘Interaction-free’ measurement
5. **Possible applications** **713**
   5.1 A Fabry – Perot cavity instead of the opaque object; 5.2 ‘Interaction-free’ detection of single atoms in a Fabry – Perot cavity
6. **Conclusion** **713**
7. **Appendices** **714**
   7.1 A scheme of ‘interaction-free’ measurement with the quantum efficiency close to unity; 7.2 Proof of formulas (3.8) and (3.12); 7.3 Coordinate and momentum wave functions in a discretized state

References **716**

Abstract. The so-called ‘interaction-free’ measurement is a very interesting quantum effect that allows discovering the presence of an opaque object in a given spatial domain, with the probability that the object absorbs a photon being, in principle, as low as desired. This probability is bounded from below only by a value of the order of $1/\omega\tau$, where ω is the frequency of light and τ is the measurement time. This corresponds to the average absorbed energy of the order of $\hbar/\tau$. The ‘interaction-free’ technique can also be used to measure the coordinate of an object but only under the condition that the object is prepared in a special ‘discretized’ quantum state. Such is, for instance, the state of a ponderomotive meter of electromagnetic energy, which, in principle, enables the ‘interaction-free’ measurement of the energy contained in an electromagnetic cavity. Estimations show that with modern experimental equipment and with the help of ‘interaction-free’ measurement, single atoms can be registered inside optical cavities.

1. Introduction

The so-called ‘interaction-free’ measurement [1] [1] is one of the purely quantum phenomena where the wave and particle features of objects manifest themselves simultaneously. Such phenomena, which look paradoxical from the classical standpoint, are nowadays especially interesting due to their possible applications in quantum computing and quantum cryptography (see, e.g., monograph [2]).

In principle, an ‘interaction-free’ measurement can be realized by means of any setup where light (or electrons, atoms, etc.; to be specific, we speak of photons) can take any of two possible paths and hence gives rise to an interference pattern. This can be a two-slit interference experiment, a Mach – Zehnder interferometer (as in the original paper [1]), or a Michelson interferometer [3, 4]. In this work, we consider the last of these.

We consider a Michelson interferometer with the distances between the central beamsplitter and the mirrors N and E fixed such that all radiation from the light source is reflected back and no light hits the detector D (Fig. 1a). If an opaque object is then inserted in one of the interferometer arms (Fig. 1b), the interference is destroyed and the probability of detector firing becomes nonzero.

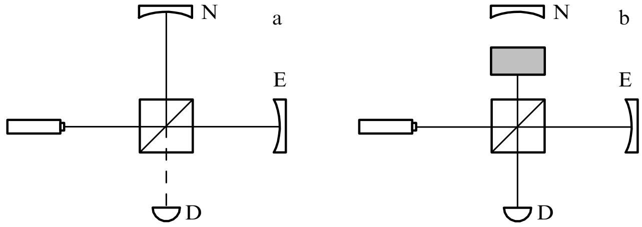

Figure 1. A simple version of the ‘interaction-free’ measurement scheme: (a) the lengths of the interferometer arms are chosen such that all light is reflected back to the laser; (b) if an opaque object is placed into one arm, the probability of photons hitting the detector becomes nonzero.

[1] This term is not quite appropriate, but we accept it because it is already used in the literature.

S P Vyatchanin, F Ya Khalili Physics Department,
M V Lomonosov Moscow State University,
Vorob’evy Gory, 119899 Moscow, Russian Federation
Tel. (7-095) 939 44 28, (7-095) 939 12 24. Fax (7-095) 932 88 20
E-mail: vyat@hbar.phys.msu.ru, farid@hbar.phys.msu.su

Received 16 February 2004, revised 4 March 2004
Uspekhi Fizicheskikh Nauk **174** (7) 765 – 777 (2004)
Translated by M V Chekhova; edited by A M Semikhatov

Let a single photon be fed into the interferometer. In the presence of the opaque object, there is no interference because the two evolution paths of the photon never meet again. Hence, the photon can be considered as a classical particle. There are three possible outcomes in this case. With the probability $p_- = R$, where R is the reflection coefficient of the central beamsplitter, the photon takes the path blocked by the object and is absorbed. We call this outcome the unsuccessful result. With the probability $p_+ = RT$, where $T = 1 - R$, it takes the second path and hits the detector (the successful result), and with the probability $p_0 = T^2$, it also takes the second path but returns to the source (goes left in Fig. 1b). In the last case, the photon can be registered by an auxiliary detector, and hence this outcome can be distinguished from the one in which the photon is absorbed.

The successful result (which is thus named because it is the goal of the experiment) allows us to unambiguously conclude that one of the interferometer paths is blocked. At the same time, somewhat conventionally, it allows us to assert that the photon did not take this path and did not interact with the object: otherwise, it would have been absorbed. This argument, based on the idea of a photon as a classical particle, has led to the term 'interaction-free measurement'.

The unsuccessful result also allows the object to be detected but in an 'uninteresting' way, via direct interaction with the photon.

The neutral result is similar to the situation of 'missing the target': the photon reaches the point that it could reach in the absence of the object. In this case, the experiment can be repeated — until either a successful or unsuccessful result is achieved, or until the total probability of 'target missing', $P_0 = p_0^N$ (where N is the number of experimental runs), becomes less than some given threshold. In this case, the probabilities of successful and unsuccessful results can be easily calculated as

$$P_+ = \frac{1 - p_0^N}{1 - p_0}\, p_+ \to \frac{p_+}{p_+ + p_-}\,, \tag{1.1a}$$

$$P_- = \frac{1 - p_0^N}{1 - p_0}\, p_- \to \frac{p_-}{p_+ + p_-}\,. \tag{1.1b}$$

This reasoning was recently confirmed in several experiments [5–7], one of them [7] using the interference of neutrons.

We stress at this point that the question of whether the interaction with the object occurs 'in reality' cannot be answered unambiguously, at least in the framework of modern physics. The answer can be different depending on the practiced interpretation of quantum mechanics. In addition to a simple 'yes' or 'no', the spectrum of answers given by various interpretations includes answers like 'yes, but in a parallel universe' or 'we do not know and there is no way to know'. In this paper, we try to approach the problem in a positivistic way, i.e., to consider some limitations for this class of measurements and some possibilities for its application from a 'consumer' standpoint.

An advanced version of the 'interaction-free' measurement has been proposed and experimentally realized in Ref. [4]. In that experiment, the probability P_+ (following Ref. [4], we call it the quantum efficiency of the setup) can be made as close to unity as desired. In Section 2, we consider this advanced version in detail and find the principal lower limit for the probability of absorbing a photon.

Among its other curious features, the 'interaction-free' measurement violates the simple logic of the Heisenberg microscope, which is 'a measurement of the coordinate entails a perturbation of the momentum'. In the 'interaction-free' measurement scheme, the object becomes definitely localized in space (its coordinate wave function is reduced), while there is no random force of the measurement device acting on the object and causing the perturbation of its momentum. It should be stressed at the very beginning that the Heisenberg uncertainty relation is by no means violated here: the Heisenberg microscope is just an illustration of the uncertainty relation but not its proof.

This problem is discussed in Section 3. In the same section, we also detail the analysis of the possibility of measuring the coordinate of an object in an 'interaction-free' way, as proposed in Ref. [3], by performing the above-described test for the object in a series of space points.

In Section 4, we consider the measurement of the number of quanta in an electromagnetic cavity. In this measurement, the number of quanta is transformed into the coordinate of a mechanical object and the coordinate is then measured in an 'interaction-free' way.

Finally, Section 5 is devoted to the experimental scheme where the opaque object is replaced by a Fabry–Perot cavity. We recall that at resonance, a Fabry–Perot cavity transmits light, i.e., acts as an 'absorber', and off-resonance, it reflects light, i.e., acts as a 'mirror'. It is known that the presence of an atom in the cavity shifts the resonance frequency; hence, an atom appearing inside the cavity can turn it from a 'mirror' into an 'absorber'. We show that modern experimental techniques allow performing 'interaction-free' detection of single atoms in optical cavities.

2. Minimal energy cost of the 'interaction-free' measurement

For the simple scheme of 'interaction-free' measurement shown in Fig. 1, the quantum efficiency is

$$P_+ = \frac{T}{1+T}\,. \tag{2.1}$$

As $T \to 1$, it becomes close to $1/2$. It was shown in Ref. [4] that the quantum efficiency can be made as close to unity as desired by introducing an additional mirror S into the 'South' arm of the Michelson interferometer. In this paper, we consider another scheme, containing an additional mirror in the 'West' arm (Fig. 2). The second mirror provides automatic recycling of light quanta in the case of neutral results. For this new scheme, the probabilities of both unsuccessful and neutral results can be made as small as desired even for a single measurement.

It is shown in Appendix 7.1 that if the matching condition

$$T = \frac{T_S}{T_S + T_W} \Leftrightarrow R = \frac{T_W}{T_S + T_W} \tag{2.2}$$

is satisfied, where T_S and T_W are the transmission coefficients of the respective mirrors S and W and the light pulse duration τ is sufficiently long such that the pulse bandwidth is much narrower than that of the interferometer, $\tilde{\tau}^{-1}$, then the probabilities of the unsuccessful and neutral results are

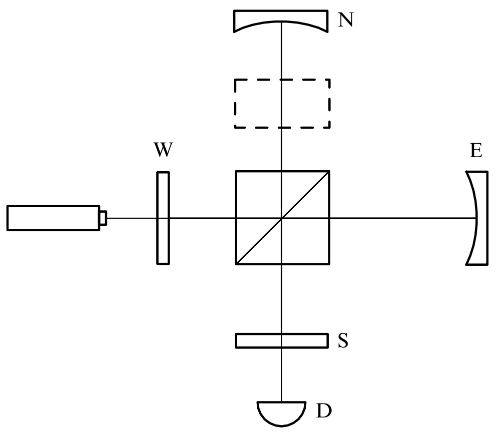

Figure 2. Using additional mirrors in the 'South' and 'West' arms, one can make the quantum efficiency of the 'interaction-free' measurement as close to unity as desired.

equal, respectively, to

$$p_- = \frac{T_S + T_W}{4}, \quad p_0 = \left(\frac{\tilde{\tau}}{\tau}\right)^2. \tag{2.3}$$

For $T_{S,W} \ll 1$, the probability p_- can be as close to zero as necessary.

The conclusion follows that for an *unlimited measurement time*, the setup under consideration produces one bit of information (the answer to the question of whether the object is present in a given space domain) at the expense of the energy that can be made as small as desired. Indeed, the photon detector can be replaced, in principle, by a quantum nondemolition energy meter [8]. Such a meter would register a quantum passing through the 'South' arm without absorbing it; further, this quantum can be used again, for instance, in similar measurements.

At the same time, it is shown in Appendix 7.1 that as $T_{S,W} \to 0$, the transmission bandwidth of the interferometer also tends to zero. Therefore, the smaller the probability p_- required, the longer the light pulses should be. As a result, for a given τ, the average energy absorbed by the object per trial is of the order of

$$\mathcal{E} = \hbar\omega p_- \gtrsim \frac{\hbar}{\tau}, \tag{2.4}$$

which is in full agreement with the standard concept of the minimal energy consumed for obtaining one bit of information.

We emphasize that this energy is consumed *on the average*: in most cases, there is no energy loss, but seldom, with the probability $\sim 1/\omega\tau$, the photon goes 'in a wrong direction' and is absorbed by the object.

3. Reduction of the state of the object due to the 'interaction-free' measurement

3.1 'Yes – no' measurements

It is commonly supposed that localizing an object in space (i.e., obtaining information about its coordinate) must be accompanied by a perturbation of its momentum. Indeed, as a rule, under a measurement of the coordinate x, the relatively broad a priori coordinate wave function $\psi_{\rm apr}(x)$ transforms into a more narrow a posteriori wave function $\psi_{\rm apost}(x)$ (Figs 3a, b). Accordingly, the relatively narrow a priori wave function $\phi_{\rm apr}(p)$ in the momentum representation (Fig. 3c) transforms into a broader a posteriori wave function $\phi_{\rm apost}(p)$ (Fig. 3d). The additional random momentum is in this case provided by the meter.

An 'interaction-free' meter seems to cause no effect of this kind. Indeed, an object placed outside the interferometer cannot receive any momentum from the field inside the interferometer (Fig. 4a). If the object is in one arm of the interferometer and blocks this arm completely, its momentum is also not perturbed, because photons do not reach this arm (Fig. 4b). But there is a third case, the intermediate one, where the object *partly* blocks the interferometer arm. With some finite probability, the photon then reaches the North arm, is scattered by the object, and passes random momentum to it (Fig. 4c).

This situation occurs not only for the 'interaction-free' measurement. We consider the simple example shown in Fig. 5, which is a version of the well-known method of 'knife and slit'. The problem is to measure the position of a reflecting object M on the axis x. For this, a light beam is sent to the probable position of the object. The beam is either not reflected by the object (and is then registered by the detector D1, see Fig. 5a) or reflected and registered by the detector D2 (Fig. 5b). Evidently, there is no momentum perturbation in case (a). In case (b), perturbation of the x-

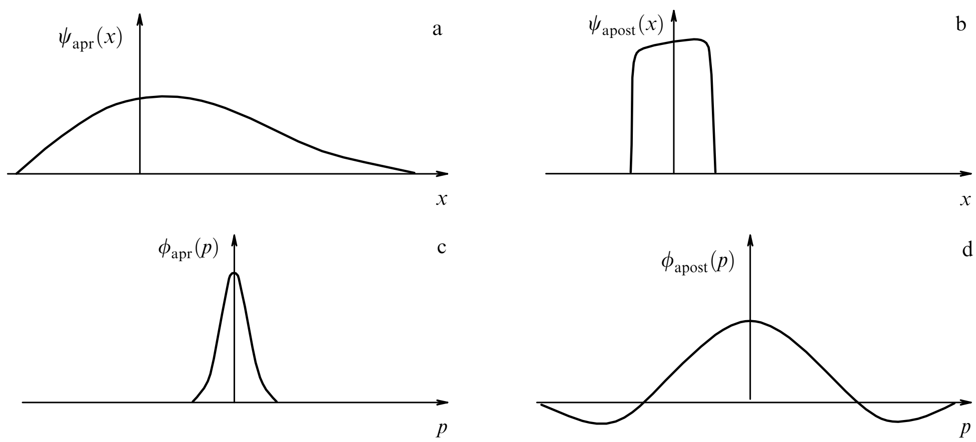

Figure 3. As a rule, localization of an object (i.e., narrowing of its coordinate wave function, top) is accompanied by a perturbation of its momentum (spreading of the momentum wave function, bottom).

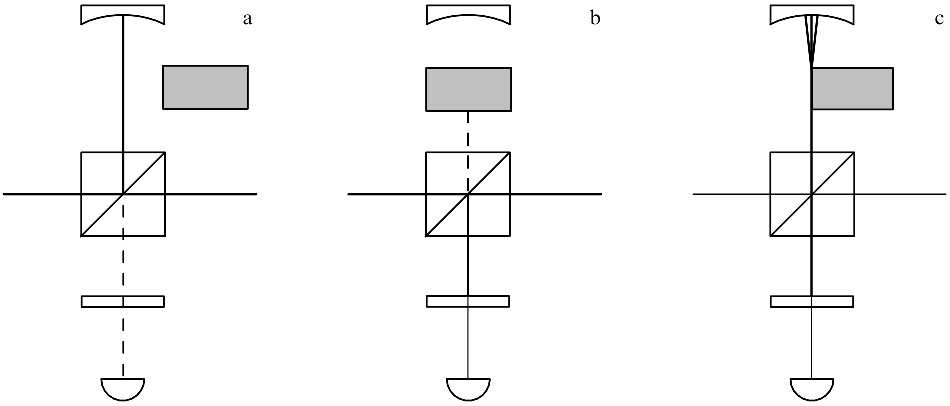

Figure 4. 'Interaction-free' measurement leads to a perturbation of the object momentum only if the object partly blocks the light.

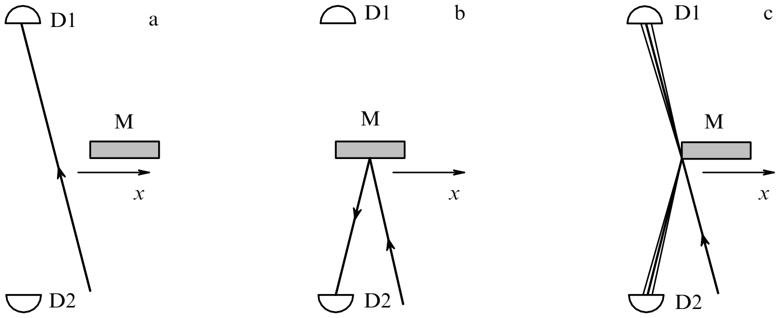

Figure 5. A simpler example of a 'yes – no' measurement. Here, the momentum of the object is also perturbed only in the case where the object partly blocks the light.

component of the momentum can be made as small as desired because reflection of light from a perfect mirror occurs with no momentum transferred to the mirror in the tangential (x) direction. The situation is the same as for 'interaction-free' measurements: the x-component of the momentum is perturbed only in the case where the beam hits the edge of the object and is diffracted by it (Fig. 5c).

In both examples, the meter provides information not about the coordinate of the object but about whether the coordinate lies within a given interval. Quantum measurements of this kind were first considered in monograph [9]. They consist of measuring binary observables (Eigenschaften), which can take only two values, 'yes' or 'no'. Hereafter, we call them 'yes – no' measurements.

Clearly, the initial state of the object may not allow realization of the case shown in Figs 4c and 5c. Let the initial coordinate wave function of the object have the form of a double-humped curve with the maximums at points $x = 0$ and $x = X > A$, the width of each maximum being $\delta x \ll X$ (Fig. 6a). In the momentum representation, this wave function has a cosine behavior with the period $2\pi\hbar/X$ and the envelope of the width $\Delta p \sim \hbar/2\delta x \gg \hbar/X$ (Fig. 6c).

According to our reasoning, a 'yes – no' measurement must result in zero perturbation of the momentum in this case. Indeed, after the measurement, only one hump of the coordinate wave function is left (Fig. 6b), and the wave function in the momentum representation loses its fine structure (Fig. 6d). But the width of the momentum wave function remains equal to its initial value Δp.

These considerations by no means contradict the Heisenberg uncertainty relation in its strict mathematical sense. Indeed, the rigorous Heisenberg inequality relates the minimal variances for two observables of an object measured in a *given quantum state*. This relation is not violated: the product of the coordinate and momentum uncertainties in the initial state (Fig. 6a) is much greater than $\hbar/2$, and therefore, although the coordinate variance is reduced due to the measurement, the product of the variances in the final state remains larger than or equal to $\hbar/2$ (Fig. 6b).

On the other hand, in the quantum measurement theory, the uncertainty relation is often understood as an inequality relating the error in the measurement of some observable to the perturbation caused by the measurement of some other observable. Just this kind of the uncertainty relation is illustrated by the Heisenberg microscope. A rigorous analysis shows that it does not always hold. We do not linger here on this rather nontrivial subject; a discussion of its various aspects can be found, e.g., in monographs [10, 11] and paper [12].

On the other hand, we give a rigorous definition of the 'momentum perturbation'. The momentum perturbation cannot be measured by the increase of the momentum uncertainty after the measurement; indeed, in some cases the meter acts on the object with a random force but the momentum uncertainty of the object not only has no increase but even reduces. (An example of such behavior is given in the Conclusion.) We assume the measure of the momentum perturbation to be

$$(\delta p)^2_{\text{pert}} = \left\langle (\hat{p}_{\text{final}} - \hat{p})^2 \right\rangle , \tag{3.1}$$

where

$$\hat{p}_{\text{final}} = \hat{\mathcal{U}}^+ \hat{p} \hat{\mathcal{U}} \tag{3.2}$$

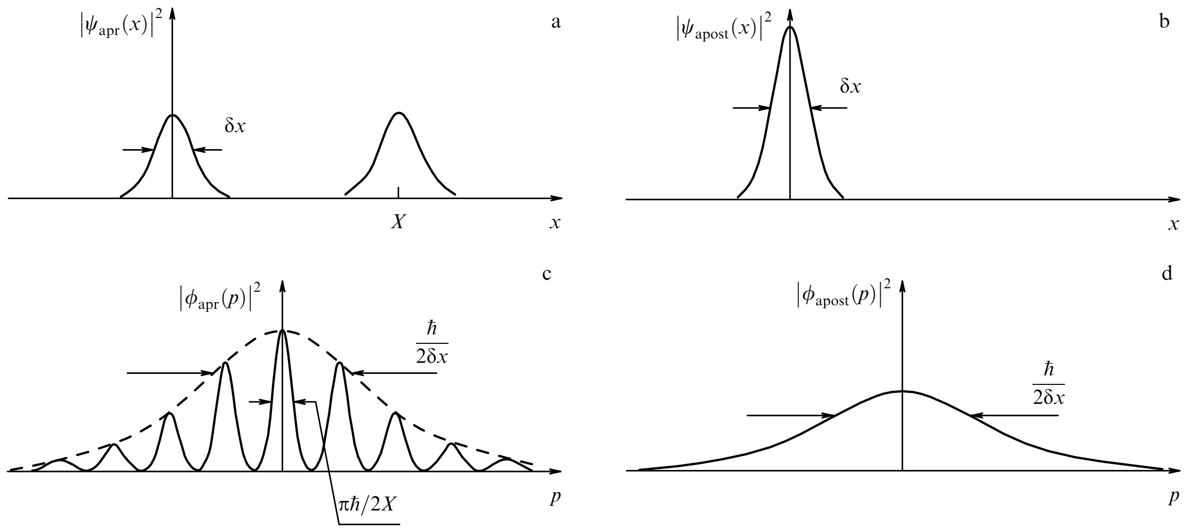

Figure 6. For a 'yes – no' measurement, perturbation of the momentum can be zero.

is the operator of the object momentum after the measurement (in the Heisenberg picture) and $\hat{\mathcal{U}}$ is the joint evolution operator for the object and the meter. Measure (3.1) agrees well with the intuitive view of the perturbation of an object by a meter. In particular, in simple 'linear' measurement schemes [10],

$$\hat{p}_{\rm final} - \hat{p} = \int_{t_1}^{t_2} \hat{F}(t)\,{\rm d}t\,, \tag{3.3}$$

where $\hat{F}(t)$ is the random force of the measurement device acting on the object and represented by an operator in the Hilbert space of the device and t_1 and t_2 are the time moments of the beginning and the end of the measurement.

Let the width of the light beam in these examples be δx and the transverse size of the object be $A \gg \delta x$. We assume that the coordinate x of the object is the coordinate of its middle point and that the light beam crosses the x axis at $x = 0$. Then for $|x| < (A - \delta x)/2$, the meter gives the answer 'yes' with certainty and for $|x| > (A + \delta x)/2$, it gives the answer 'no' with certainty. In the intermediate cases, any answer is possible.

If the initial wave function of the object is $\psi_{\rm apr}(x)$, then in the case of the answer 'yes', the object goes into the state with the wave function

$$\psi_+(x) = \frac{\Omega_+(x)\psi_{\rm apr}(x)}{\sqrt{w_+}}\,, \tag{3.4}$$

and in the case of the answer 'no', into the state with the wave function

$$\psi_-(x) = \frac{\Omega_-(x)\psi_{\rm apr}(x)}{\sqrt{w_-}}\,. \tag{3.5}$$

Here, Ω_+ is the reduction function, which is equal to unity for $|x| < (A - \delta x)/2$ and zero for $|x| > (A + \delta x)/2$, and falls from unity to zero as $|x|$ increases from $(A - \delta x)/2$ to $(A + \delta x)/2$, and

$$\Omega_-(x) = \sqrt{1 - \Omega_+^2(x)} \tag{3.6}$$

and

$$w_\pm = \int_{-\infty}^{\infty} \Omega_\pm^2(x)\,|\psi_{\rm apr}(x)|^2\,{\rm d}x \tag{3.7}$$

are the respective probabilities of the 'yes' and 'no' answers.

In Appendix 7.2, it is shown that the momentum perturbation in a 'yes – no' measurement is

$$(\delta p)^2_{\rm pert} = \int_{-\infty}^{\infty} \left(\left| \frac{{\rm d}\Omega_+(x)}{{\rm d}x} \right|^2 + \left| \frac{{\rm d}\Omega_-(x)}{{\rm d}x} \right|^2 \right) |\psi_{\rm apr}(x)|^2\,{\rm d}x\,, \tag{3.8}$$

and therefore, the momentum is perturbed in such a measurement only if the a priori wave function is nonzero in the 'gray' domain where the reduction function falls from unity to zero and its derivative differs from zero. But in the case shown in Fig. 6, ${\rm d}\Omega_\pm/{\rm d}x = 0$ wherever $\psi_{\rm apr} \neq 0$.

A well-known example of a 'yes – no' measurement is the 'which-way' procedure [13]. This procedure determines, in a nonperturbating way (as in quantum nondemolition measurements), which of the two possible paths was taken by a particle in a two-slit interferometer. The idea of the which-way procedure is shown in Fig. 7. Here, D1 and D2 are detectors that click when the particle passes them. The

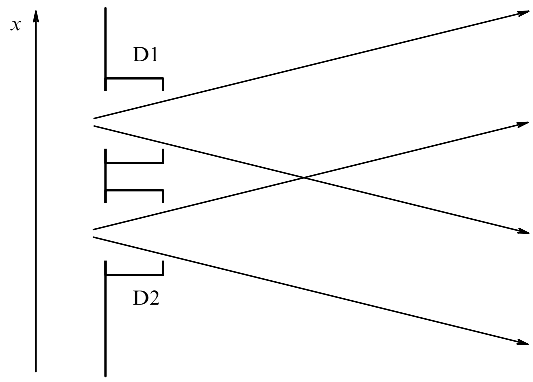

Figure 7. The which-way procedure: detectors D1 and D2 perform a nondemolition registration of a particle passing through their apertures.

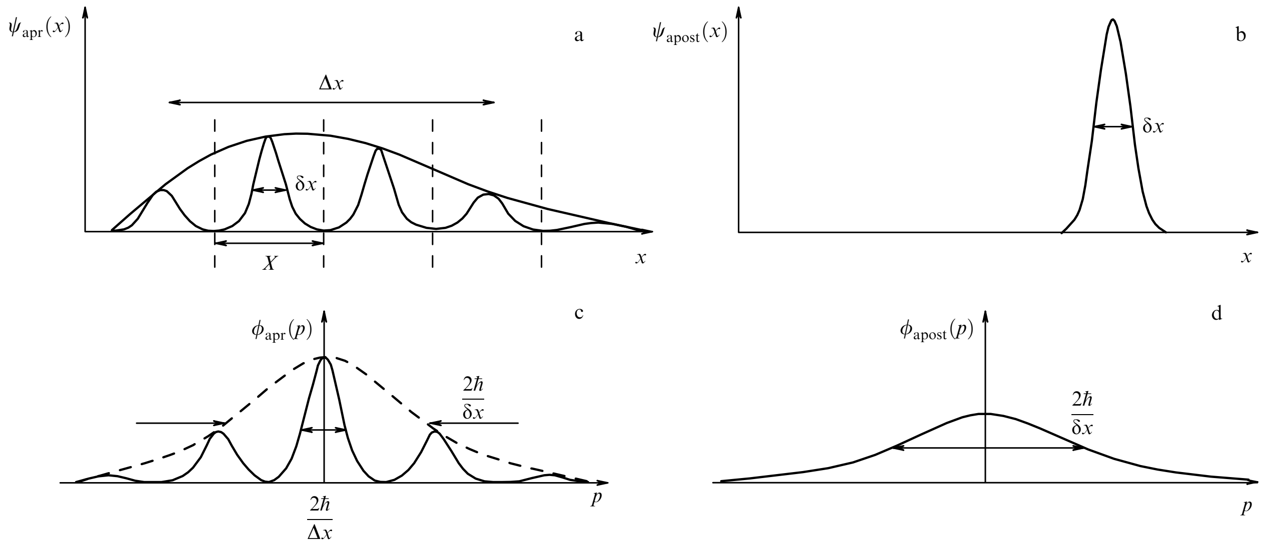

Figure 8. In an 'interaction-free' measurement, the mean-square perturbation of the object momentum can be equal to zero if the object was prepared beforehand in a 'discretized' state.

apertures of the detectors must be larger than the widths of both interfering beams. It is also important that in accordance with the basic principle of quantum nondemolition measurements, the detectors should not provide any additional information about the state of the passing particle, for instance, about the value of its coordinate. Then the state of the passing particles changes as shown in Fig. 6; in accordance with (3.8), their momentum perturbation is equal to zero.

It is interesting to note that the which-way measurement is in fact a 'reversed version' of the 'interaction-free' measurement. Indeed, one of the detectors in Fig. 7, for instance, detector D2, can be removed. We then have the scheme of the 'interaction-free' measurement: particles taking the lower path and surely not having interacted with D1 still 'feel' its presence because they lose the fine structure of their momentum wave function. The only difference between the 'interaction-free' measurement and the which-way procedure is that in the latter, the interfering particles are the objects under study and the detector is the measurement device, while in the former, these roles are exchanged.

It is also worth mentioning Ref. [14], the authors of which claimed (in our opinion, erroneously) that destroying interference in the which-way procedure requires a direct local interaction of the particles with one of the detectors D1 and D2. There was a reply in Ref. [15]. The which-way procedure was also discussed in detail in monograph [11].

3.2 'Interaction-free' measurement of the coordinate

Evidently, the above procedure of 'interaction-free' binary detection of an object can be developed into the 'interaction-free' measurement of its coordinate x. For this, it suffices to scan the x axis by the device described above, with the scanning step $X \leqslant A$ determined by the required accuracy. The minimum value of this step is given by the width δx of the 'gray intervals' of the function $\Omega_+(x)$, i.e., the intervals where its x-derivative is nonzero. For the Michelson interferometer considered in Section 2, this value is slightly larger than the wavelength of the quanta used for the measurement.

We consider the scanning process in more detail. To be specific, we suppose that the initial wave function $\psi_{\rm apr}(x)$ differs from zero only at $x \geqslant 0$. The interferometer is first placed such that the optic axis of its 'north' arm has the coordinate $x = -A/2$ and is then stepwise moved to the right until the object is detected, i.e., until a photon hits the detector.

Let this happen at the nth step, i.e., for the coordinate x of the 'North' arm optic axis equal to $-A/2 + nX$. This means that the coordinate x of the middle of the object is somewhere between the points $(n-1)X$ and nX. In accordance with this, the a posteriori wave function of the object becomes

$$\psi_n(x) = \frac{\Omega_n(x)\psi_{\rm apr}(x)}{\sqrt{w_n}}\,, \tag{3.9}$$

where

$$w_n(x) = \int_{-\infty}^{\infty} \left|\Omega^2(x - nX)\right|^2 \left|\psi_{\rm apr}(x)\right|^2 \mathrm{d}x \tag{3.10}$$

is the probability of obtaining this particular result and the functions

$$\Omega_n(x) = \Omega(x - nX) \tag{3.11}$$

describe the reduction of the object wave function in this measurement. The function $\Omega(x)$ is equal to unity for $\delta x/2 < x < X - \delta x/2$ and zero for $x < -\delta x/2$ and $x > X + \delta x/2$.

It is shown in Appendix 7.2 that the perturbation of the object momentum in this procedure is

$$(\delta p)^2_{\rm pert} = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left|\frac{\mathrm{d}\Omega(x - nX)}{\mathrm{d}x}\right|^2 \left|\psi_{\rm apr}(x)\right|^2 \mathrm{d}x\,. \tag{3.12}$$

It is important that the factor $\left|\mathrm{d}\Omega(x - nX)/\mathrm{d}x\right|^2$ is nonzero only in the vicinities of the boundary points $x = nX$. Therefore, similarly to the previous case, the momentum perturbation occurs only if the a priori wave function $\psi_{\rm apr}(x)$ is nonzero in the neighborhoods of these points, i.e., if the cases shown Fig. 4c are possible.

On the other hand, let the initial wave function of the object, $\psi_{\rm apr}$, be given by a set of peaks $\psi_1(x)$ with widths $\delta x < X$, placed at a distance X from each other, such that the values of the wave function at the points nX are equal to zero:

$$\psi(x) = \sum_{n=0}^{\infty} \Psi_n \psi_1(x - nX) \tag{3.13}$$

(see Fig. 8a). Let the total width Δx of the wave function be sufficiently large, $\Delta x \gg X$, such that the wave function envelope is almost constant within the interval X. It follows from (3.11) that for this initial state of the object, there must be no momentum perturbation in the above procedure of 'interaction-free' measurement.

It is shown in Appendix 7.3 that the corresponding wave function $\phi_{\mathrm{apr}}(p)$ in the momentum representation also consists of a set of peaks but the distance between the peaks is $P = 2\pi\hbar/X$ (Fig. 8c). The width of the envelope of $\phi_{\mathrm{apr}}(p)$ is determined by the width of a single peak of $\psi_{\mathrm{apr}}(x)$,

$$\Delta p \sim \frac{\hbar}{\delta x}\,, \tag{3.14}$$

and vice versa: the width of a single peak of the wave function $\phi_{\mathrm{apr}}(p)$ is determined by the width of the envelope of $\psi_{\mathrm{apr}}(x)$:

$$\delta p \sim \frac{\hbar}{\Delta x}\,. \tag{3.15}$$

Of the entire set of the $\phi_{\mathrm{apr}}(p)$-peaks, only one peak remains after the measurement:

$$\psi_{\mathrm{apost}}(x) = \psi_1(x - nX) \tag{3.16}$$

(see Fig. 8b). After the measurement, by virtue of formula (7.24), the wave function in the momentum representation becomes proportional to the envelope of the initial momentum wave function (Fig. 8d).

It is important that the width of the momentum wave function, i.e., the momentum uncertainty, does not increase. Only the 'fine structure' of the wave function changes, its peaks being smeared. No random momentum transfer from the meter to the object (as mentioned at the beginning of this section) is then to occur, and hence, such a 'discretized' quantum state enables an 'interaction-free' measurement of the coordinate of the object.

4. Measurement of the number of quanta in an electromagnetic cavity

4.1 Measurement scheme

In principle, the measurement of the 'discretized' coordinate for a mechanical degree of freedom can be used to measure the number of quanta in an electromagnetic cavity. We note that the procedure described in this section is to be considered as a gedankenexperiment rather than a real measurement scheme. At the same time, it is of considerable interest from the methodological standpoint.

This procedure is based on the scheme of a ponderomotive meter of the number of quanta [8], which was initially proposed as a gedankenexperiment but later became a basis for practical schemes of nondemolition electromagnetic energy measurement.

We consider an electromagnetic cavity (for instance, a microwave cavity) with one wall, of mass m, being movable (Fig. 9); the frequency of the cavity then depends on the

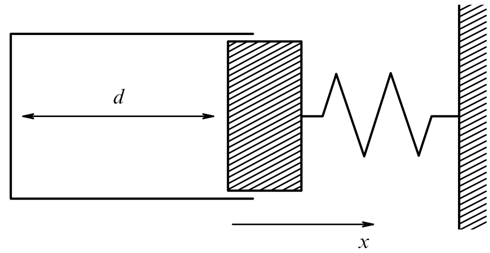

Fig. 9. The idea of a ponderomotive meter of electromagnetic energy: the force of the electromagnetic pressure acting on the piston is proportional to the number of quanta in the cavity.

coordinate of the wall:

$$\omega_{\mathrm{e}}(x) = \omega_{\mathrm{e}}\left(1 - \frac{x}{d}\right). \tag{4.1}$$

Let the wall, together with a rigidity $m\omega_{\mathrm{m}}^2$ attached to it, form a mechanical oscillator with the eigenfrequency ω_{m}. The coordinate of the oscillator is measured with respect to its equilibrium position in the absence of the cavity.

Due to the electromagnetic pressure, the oscillator coordinate operator evolves in the Heisenberg picture as

$$\hat{x}(t) = \hat{x}\cos\omega_{\mathrm{m}}t + \frac{\hat{p}}{m\omega_{\mathrm{m}}}\sin\omega_{\mathrm{m}}t + \frac{\hbar\omega_{\mathrm{e}}(1 - \cos\omega_{\mathrm{m}}t)}{m\omega_{\mathrm{m}}^2 d}\left(\hat{n} + \frac{1}{2}\right). \tag{4.2}$$

where $\hat{n}$ is the operator of the number of quanta in the cavity. At the time instant $t = \pi/\omega_{\mathrm{m}}$, this formula becomes

$$\hat{x}\left(\frac{\pi}{\omega_{\mathrm{m}}}\right) = -\hat{x} + \frac{2\hbar\omega_{\mathrm{e}}}{m\omega_{\mathrm{m}}^2 d}\left(\hat{n} + \frac{1}{2}\right). \tag{4.3}$$

Hence, a measurement of the coordinate x at time t allows determining the number of quanta n to an accuracy depending on the initial coordinate uncertainty Δx_{init} and the accuracy of the coordinate measurement Δx_{meas}:

$$(\Delta n_{\mathrm{meas}})^2 = \left(\frac{m\omega_{\mathrm{m}}^2 d}{2\hbar\omega_{\mathrm{e}}}\right)^2 \left[(\Delta x_{\mathrm{init}})^2 + (\Delta x_{\mathrm{meas}})^2\right]. \tag{4.4}$$

It is known that the measurement of the number of quanta in a cavity is accompanied by a perturbation of the cavity phase. We consider the mechanism of this perturbation in more detail. As a starting point, we consider the 'usual' measurement and then pass to the 'interaction-free' measurement.

4.2 'Usual' measurement

In the joint evolution of the mechanical oscillator and the electromagnetic cavity, the cavity frequency (4.1) does not have a precise value because the oscillator coordinate (4.2) has no precise value either. Therefore, the electromagnetic cavity acquires a random phase shift

$$\hat{\varphi} = \frac{\omega_{\mathrm{e}}}{d}\int_0^{\tau} \hat{x}(t)\,\mathrm{d}t\,, \tag{4.5}$$

where τ is the interaction time. If $\tau = \pi/\omega_{\mathrm{m}}$, then the component of φ depending on the initial state of the mechanical oscillator (i.e., the perturbation proper) is

$$\delta\hat{\varphi} = \frac{2\omega_{\mathrm{e}}}{m\omega_{\mathrm{m}}^2 d}\,\hat{p}\,, \tag{4.6}$$

the uncertainty of this value being

$$\Delta\varphi_{\mathrm{pert}} = \frac{2\omega_{\mathrm{e}}}{m\omega_{\mathrm{m}}^2 d}\,\Delta p_{\mathrm{init}}\,, \tag{4.7}$$

where Δp_{init} is the initial uncertainty of the oscillator momentum. It is easy to see that the accuracy of measuring the number of quanta in (4.4) and phase perturbation (4.7) satisfy the uncertainty relation

$$\Delta n_{\mathrm{meas}}\Delta\varphi_{\mathrm{pert}} = \frac{\sqrt{(\Delta x_{\mathrm{init}})^2 + (\Delta x_{\mathrm{meas}})^2}\,\Delta p_{\mathrm{init}}}{\hbar} \geqslant \frac{\Delta x_{\mathrm{init}}\Delta p_{\mathrm{init}}}{\hbar} \geqslant \frac{1}{2}\,. \tag{4.8}$$

In the procedure under consideration, perturbation of the phase of the electromagnetic cavity is determined by the initial momentum uncertainty Δp_{init} of the mechanical oscillator. But the procedure can easily be modified such that the perturbation of the phase is independent of the initial momentum uncertainty but is proportional to the momentum perturbation $\delta\hat{p}$ caused by the measurement of the oscillator coordinate. This can be done by extending the interaction between the oscillator and the cavity from the time instant π/ω_{m}, when the coordinate is measured, until the time instant $2\pi/\omega_{\mathrm{m}}$. Then, in the time interval between π/ω_{m} and $2\pi/\omega_{\mathrm{m}}$, the operator $\hat{x}(t)$ is given by

$$\hat{x}(t) = \hat{x}\cos\omega_{\mathrm{m}}t + \frac{\hat{p}+\delta\hat{p}}{m\omega_{\mathrm{m}}}\sin\omega_{\mathrm{m}}t + \frac{\hbar\omega_{\mathrm{e}}(1-\cos\omega_{\mathrm{m}}t)}{m\omega_{\mathrm{m}}^2 d}\left(\hat{n}+\frac{1}{2}\right), \tag{4.9}$$

where $\delta\hat{p}$ is the perturbation of the oscillator momentum caused by the coordinate measurement. From the mathematical standpoint, $\delta\hat{p}$ is an operator in the Hilbert space of the coordinate meter, which is typically interpreted as resulting from back-reaction of the random force of the meter fluctuation on the oscillator.

It follows from formulas (4.2) and (4.9) that at $\tau = 2\pi/\omega_{\mathrm{m}}$, the terms in (4.5) proportional to x and p both vanish, and the phase perturbation becomes

$$\delta\hat{\varphi} = \frac{2\omega_{\mathrm{e}}}{m\omega_{\mathrm{m}}^2 d}\,\delta\hat{p}\,. \tag{4.10}$$

Uncertainty relation (4.8) then takes the form

$$\Delta n_{\mathrm{meas}}\Delta\varphi_{\mathrm{pert}} = \frac{\sqrt{(\Delta x_{\mathrm{init}})^2 + (\Delta x_{\mathrm{meas}})^2}\,\Delta p_{\mathrm{meas}}}{\hbar} \geqslant \frac{\Delta x_{\mathrm{meas}}\Delta p_{\mathrm{meas}}}{\hbar} \geqslant \frac{1}{2}\,, \tag{4.11}$$

where Δp_{meas} is the uncertainty of $\delta\hat{p}$.

4.3 'Interaction-free' measurement

The coordinate of a mechanical oscillator can be measured, in principle, by means of the device described in Section 3, i.e., in the 'interaction-free' way.

Let the oscillator be initially prepared in a state with the wave function $\psi_0(x)$ for which the mean coordinate is zero and the coordinate uncertainty is small compared to the coordinate shift due to the attraction force caused by a single quantum:

$$\Delta x_{\mathrm{init}} \ll \frac{\hbar\omega_{\mathrm{e}}}{m\omega_{\mathrm{m}}^2 d}\,. \tag{4.12}$$

The initial wave function of the electromagnetic cavity can be written as

$$\sum_{n=0}^{\infty} \Psi_n |n\rangle\,. \tag{4.13}$$

It can then be easily shown that at time $t = \pi/\omega_{\mathrm{m}}$, the 'oscillator + cavity' system passes to the entangled state

$$\int_0^{\infty} \mathrm{d}x \sum_{n=0}^{\infty} \Psi_n \psi_1(x - nX)|n\rangle|x\rangle\,, \tag{4.14}$$

where $|x\rangle$ are oscillator states with a given coordinate,

$$\psi_1(x) = \psi_0(-x) \tag{4.15}$$

[the sign is reversed because the oscillator coordinate has the same absolute value at $t = \pi/\omega_{\mathrm{m}}$ as at $t = 0$, see (4.2), but the opposite sign], and

$$X = \frac{2\hbar\omega_{\mathrm{e}}}{m\omega_{\mathrm{m}}^2 d}\,. \tag{4.16}$$

The coordinate probability distribution for state (4.14), similarly to state (3.13), consists of nonoverlapping peaks separated by the distance X. Therefore, the 'interaction-free' measurement of the number of quanta can be applied. As a result, the system passes into one of the states

$$|n\rangle \int_0^{\infty} \psi_1(x - nX)|x\rangle\,\mathrm{d}x\,, \tag{4.17}$$

with the probability $|\Psi_n|^2$. It is important that this does not involve any random force acting on the oscillator from the coordinate meter [there is no term with $\delta\hat{p}$ in (4.9); see also Section 3]. The electromagnetic cavity is perturbed at this time instant because of the initial momentum uncertainty of the oscillator, see formulas (4.5) – (4.7). This perturbation can be made zero, as we have already mentioned, by extending the interaction until the time instant $2\pi/\omega_{\mathrm{m}}$. We then have

$$\delta\hat{\phi} = \frac{\omega_{\mathrm{e}}}{d}\int_0^{2\pi/\omega_{\mathrm{m}}} \hat{x}(t)\,\mathrm{d}t \equiv 0\,. \tag{4.18}$$

As a result, both possible 'physical' sources of the cavity phase perturbation, the initial uncertainty of the oscillator momentum and the perturbation of this momentum caused by the coordinate measurement, are eliminated. Nevertheless, from the arbitrary initial state (4.13), the cavity passes into a state with a definite number of quanta $|n\rangle$, where the phase is totally uncertain.

One can easily find some similarity between the procedure we have just described and the which-way measurement mentioned in Section 3.1. Both procedures involve selecting one of the components of the initial wave function (in the present example, it is one of the wave functions $|n\rangle$ and in the which-way measurement, it is one of the paths for the particles) without any direct physical contact between the object and the meter.[2] In both cases, this selection makes the wave function of the object lose its fine interference structure.

5. Possible applications

5.1 A Fabry–Perot cavity instead of the opaque object

The resolving power X of the scheme considered above cannot be less than the width of the light beam, which, in its turn, much exceeds the wavelength of light: $X \gg \lambda$. The value of X can be reduced by several orders of magnitude by using the scheme shown in Fig. 10. In this scheme, the opaque body in the 'North' arm of the Michelson interferometer is replaced by a Fabry–Perot cavity with a movable back mirror. The scheme allows the coordinate of this mirror to be measured in the 'interaction-free' way.

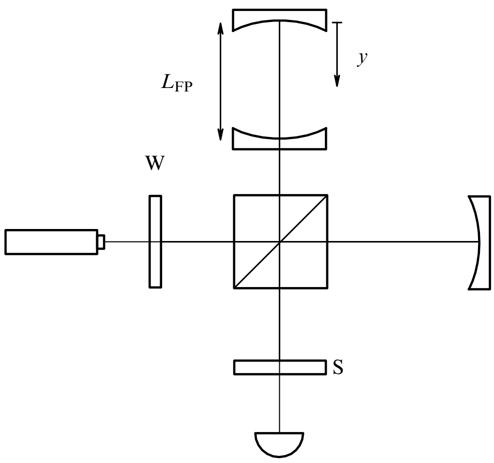

Figure 10. The scheme of 'interaction-free' measurement with the opaque object replaced by a Fabry–Perot cavity.

If both mirrors of the Fabry–Perot cavity have the same transmission coefficient $T_{FP} \ll 1$ and no losses, the coefficient of reflection from the cavity depends on the coordinate y of the back mirror,

$$K_{FP} = \frac{i(\omega - \omega_0)}{\gamma - i(\omega - \omega_0)} = \frac{iy}{\pi T_{FP}\lambda - iy}, \tag{5.1}$$

where

$$\omega_0 = \frac{\pi c n}{L_{FP}} \tag{5.2}$$

is the resonance frequency (n is an integer) and

$$\gamma = \frac{c T_{FP}}{2 L_{FP}} \tag{5.3}$$

is the half-width of the Fabry–Perot cavity transmission band. The origin is chosen such that $y = 0$ corresponds to the resonance.

[2] Of course, this statement is conventional, as any verbal description of a quantum phenomenon: there certainly is a 'contact' for the a priori wave function of the object but there is no such contact for its a posteriori wave function.

We see that at resonance, the reflection coefficient is equal to zero, i.e., the Fabry–Perot cavity behaves as an absorbing object. On the other hand, for the detuning greater than γ, i.e., for $y \gg T_{FP}\lambda$, the reflection coefficient of the cavity is close to unity and the cavity is similar to a highly reflecting mirror.

Hence, the entire 'interaction-free' argument described in the previous section is applicable to this scheme. However, the scanning step Y is here bounded from below by $T_{FP}\lambda$. With modern high-quality mirrors, this value can be made 4–5 orders of magnitude smaller than the wavelength of light.

5.2 'Interaction-free' detection of single atoms in a Fabry–Perot cavity

In conclusion, we mention the possibility of using the scheme with the Fabry–Perot cavity for 'interaction-free' detection of a single atom inside a cavity. It is known that an atom placed inside a cavity can split its resonance frequency if this resonance frequency is close to the frequency of one of the atomic transitions,

$$\omega_0 \Rightarrow \omega_0 \pm g, \tag{5.4}$$

where g is the frequency of the Rabi oscillations. If the offset g exceeds the Fabry–Perot bandwidth γ, then the cavity with the atom inside behaves as an absorbing object and the cavity without the atom behaves as a highly reflecting mirror. For quanta with the frequencies $\omega_0 \pm g$, the situation is reversed.

A Fabry–Perot cavity satisfying the condition $g \gg \gamma$, together with the second condition $g \gg \gamma_\perp$ (where $\gamma_\perp$ is the decay constant of the atom in an excited state) necessary in such experiments, was demonstrated in Ref. [16]. The authors used a Fabry–Perot cavity with the finesse $\mathcal{F}_{FP} = \pi/T_{FP} = 4.2 \times 10^5$. For the wavelength of the transition $\lambda_{atom} \simeq \lambda_{FP} = 852.4$ nm, the cavity length $L_{FP} = 44.6$ µm, and the beam width $w_0 = 29$ µm, the Rabi frequency of a Rydberg cesium atom was

$$g = 2\pi \times 32.2 \times 10^6 \ \mathrm{s}^{-1}, \tag{5.5}$$

and the values of γ and $\gamma_\perp$ were

$$\gamma_\perp = 2.6 \times 10^6 \ \mathrm{s}^{-1}, \tag{5.6}$$

$$\gamma = \frac{\pi c}{2\mathcal{F}_{FP} L_{FP}} \simeq 25.3 \times 10^6 \ \mathrm{s}^{-1}. \tag{5.7}$$

For the cavity frequency detuned from the resonance by g, the reflection coefficient was very close to unity:

$$|K_{FP}| = \frac{g}{\sqrt{\gamma^2 + g^2}} \simeq 0.992. \tag{5.8}$$

This example shows that modern technology allows the 'interaction-free' detection of single atoms.

6. Conclusion

'Interaction-free' measurement is in no way the only example of a quantum measurement procedure where obtaining information about the coordinate of an object is not accompanied by an increase in its momentum uncertainty. For instance, it was shown in Refs [17, 18] that for a certain choice of the initial wave function of the object, the momentum uncertainty can *decrease* even after a 'usual' measurement of the coordinate. As regards the momentum perturbation calculated in accordance with (3.1) for a 'usual' measurement, one can easily see that it always satisfies the uncertainty relation $\langle(\delta\hat{p})^2\rangle(\Delta x)^2 \geqslant \hbar^2/4$, where $(\Delta x)^2$ is the accuracy of the coordinate measurement.

We also touch upon the accuracy limit for the 'interaction-free' measurement. The above consideration shows that if a single quantum participates in the procedure, the accuracy is the same as for a 'usual' measurement. It is of the order of the light wavelength λ if no additional optical cavity is used and of the order of $\lambda/\mathcal{F}_{\text{FP}}$ if a cavity of finesse $\mathcal{F}_{\text{FP}}$ is used. In a 'usual' measurement, the accuracy can be improved by increasing the number N of optical quanta: it becomes $\Delta x \propto 1/\sqrt{N}$ if the optical field is in a coherent state and $\Delta x \propto 1/N$ if the field is in an optimal nonclassical state. This method of improving the accuracy is also applicable to the 'interaction-free' measurement but at the expense of complicating the measurement protocol. However, we note once again that 'interaction-free' measurements are important not because of some additional advantages they provide for experiments but because they bring a deeper understanding of quantum measurement theory and quantum physics in general.

This work was supported in part by the Institute of Quantum Information of the California Institute of Technology, by the Russian Foundation for Basic Research, grant number 03-02-16975-a, and by state contracts of the Ministry of Industry and Science of the Russian Federation, numbers 40.02.1.1.1.1137 and 40.700.12.0086.

The authors express their deep gratitude to V B Braginskiĭ and M B Menskiĭ for fruitful discussions that have led them to writing this paper. One of the authors (F Ya Khalili) is also grateful to B G Englert and H Walther for extremely interesting discussions on the foundations of quantum theory.

7. Appendices

7.1 A scheme of 'interaction-free' measurement with the quantum efficiency close to unity

7.1.1 Field amplitudes. Let the amplitudes of the fields incident on the central beamsplitter and reflected from it be denoted as shown in Fig. 11 (according to the geographic coordinates). The *amplitude* reflection and transmission coefficients of the beamsplitter are denoted by $-r$ and $\mathrm{i}t$ and the corresponding coefficients of the 'West' and 'South' mirrors by $-r_{\text{W}}$, $\mathrm{i}t_{\text{W}}$ and $-r_{\text{S}}$, $\mathrm{i}t_{\text{S}}$, respectively. The 'North' and the 'East' mirrors are assumed to be perfectly reflecting. We suppose that in the 'East' arm, the light acquires the phase shift $\pi/2$ (multiplication by i), and in the 'South' arm, the phase shift $-\pi/2$ (multiplication by $-$i). This is necessary for the suppression of light on the detector. We also introduce the notation

$$\mathcal{R}_{\text{W}} = -r_{\text{W}} \exp(2\mathrm{i}\omega\tau_1)\,, \tag{7.1a}$$

$$\mathcal{R}_{\text{S}} = -r_{\text{S}} \exp(2\mathrm{i}\omega\tau_1)\,, \tag{7.1b}$$

$$\mathcal{R}_{\text{N}} = -r_{\text{N}} \exp(2\mathrm{i}\omega\tau_2)\,, \tag{7.1c}$$

$$\mathcal{R}_{\text{E}} = -\exp(2\mathrm{i}\omega\tau_2)\,, \tag{7.1d}$$

where $r_{\text{N}} = 0$ in the presence of the absorbing object and $r_{\text{N}} = 1$ in its absence, ω is the light frequency, $\tau_{1,2} = L_{1,2}/c$, L_1 is the length of the 'West' arm and the 'South' arm, and L_2 is the length of the 'North' arm and the 'East' arm.

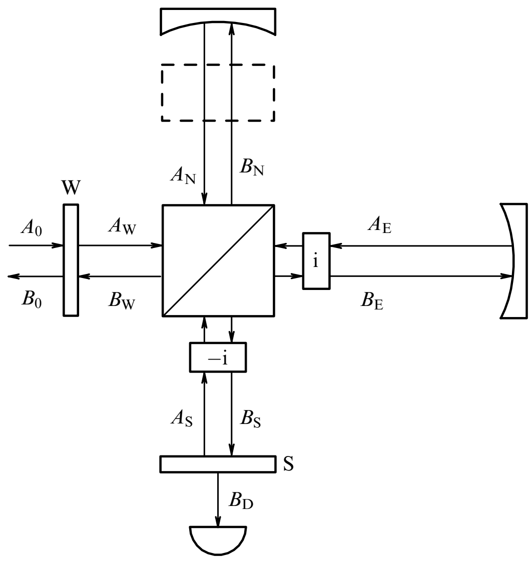

Figure 11. Notation used in the calculation of the minimal absorbed energy in the 'interaction-free' measurement.

The equations for the field amplitudes can then be written as

$$A_{\text{W}} = \mathcal{R}_{\text{W}} B_{\text{W}} + a_0\,, \qquad B_{\text{W}} = -rA_{\text{N}} - tA_{\text{E}}\,, \tag{7.2a}$$

$$A_{\text{S}} = \mathcal{R}_{\text{S}} B_{\text{S}}\,, \qquad B_{\text{S}} = -rA_{\text{E}} + tA_{\text{N}}\,, \tag{7.2b}$$

$$A_{\text{N}} = \mathcal{R}_{\text{N}} B_{\text{N}}\,, \qquad B_{\text{N}} = -rA_{\text{W}} + tA_{\text{S}}\,, \tag{7.2c}$$

$$A_{\text{E}} = \mathcal{R}_{\text{E}}\,, \qquad B_{\text{E}} = -rA_{\text{S}} - \mathrm{i}tA_{\text{W}}\,, \tag{7.2d}$$

$$B_0 = -r_{\text{W}} A_0 + \mathrm{i}t_{\text{W}} B_{\text{W}} \exp(\mathrm{i}\omega\tau_1)\,, \tag{7.2e}$$

$$B_{\text{D}} = \mathrm{i}t_{\text{S}} B_{\text{S}} \exp(\mathrm{i}\omega\tau_1)\,, \tag{7.2f}$$

where $a_0 = \mathrm{i}t_{\text{W}} \exp(\mathrm{i}\omega L_1) A_0$. The solution of this system of equations is

$$\begin{aligned} A_{\text{W}} &= \frac{a_0}{\mathcal{D}}\left(1 - r^2\mathcal{R}_{\text{S}}\mathcal{R}_{\text{E}} - t^2\mathcal{R}_{\text{S}}\mathcal{R}_{\text{N}}\right), \\ B_{\text{W}} &= \frac{a_0}{\mathcal{D}}\left(r^2\mathcal{R}_{\text{N}} + t^2\mathcal{R}_{\text{E}} - \mathcal{R}_{\text{S}}\mathcal{R}_{\text{N}}\mathcal{R}_{\text{E}}\right), \end{aligned} \tag{7.3a}$$

$$\frac{A_{\text{S}}}{R_{\text{S}}} = B_{\text{S}} = \frac{rta_0(\mathcal{R}_{\text{E}} - \mathcal{R}_{\text{N}})}{\mathcal{D}}\,, \tag{7.3b}$$

$$\frac{A_{\text{N}}}{R_{\text{N}}} = B_{\text{N}} = \frac{ra_0(-1 + \mathcal{R}_{\text{S}}\mathcal{R}_{\text{E}})}{\mathcal{D}}\,, \tag{7.3c}$$

$$\frac{A_{\text{E}}}{R_{\text{E}}} = B_{\text{E}} = \frac{ta_0(-1 + \mathcal{R}_{\text{S}}\mathcal{R}_{\text{N}})}{\mathcal{D}}\,, \tag{7.3d}$$

$$\begin{aligned} B_0 = \frac{A_0}{\mathcal{D}}\big[&-r_{\text{W}} + (-r^2\mathcal{R}_{\text{N}} - t^2\mathcal{R}_{\text{E}} + \mathcal{R}_{\text{S}}\mathcal{R}_{\text{N}}\mathcal{R}_{\text{E}})\exp(2\mathrm{i}\omega\tau_1) \\ &+ r_{\text{W}}\mathcal{R}_{\text{S}}(r^2\mathcal{R}_{\text{E}} + t^2\mathcal{R}_{\text{N}})\big]\,, \end{aligned} \tag{7.3e}$$

where

$$\begin{aligned} \mathcal{D} = 1 &- r^2(\mathcal{R}_{\text{W}}\mathcal{R}_{\text{N}} + \mathcal{R}_{\text{S}}\mathcal{R}_{\text{E}}) - t^2(\mathcal{R}_{\text{W}}\mathcal{R}_{\text{E}} + \mathcal{R}_{\text{S}}\mathcal{R}_{\text{N}}) \\ &+ \mathcal{R}_{\text{W}}\mathcal{R}_{\text{S}}\mathcal{R}_{\text{N}}\mathcal{R}_{\text{E}}\,. \end{aligned} \tag{7.4}$$

7.1.2 The absorbing object is absent. In the absence of the absorbing object,

$$\mathcal{R}_{\mathrm{N}} = \mathcal{R}_{\mathrm{E}} = -\exp(2\mathrm{i}\omega\tau_2)\,, \tag{7.5}$$

and, accordingly,

$$B_{\mathrm{S}} = 0 \Rightarrow B_{\mathrm{D}} = 0\,, \tag{7.5a}$$

$$B_0 = \frac{-r_{\mathrm{W}} + \exp(2\mathrm{i}\omega\tau_{12})}{1 - r_{\mathrm{W}}\exp(2\mathrm{i}\omega\tau_{12})}\,A_0\,, \tag{7.5b}$$

where $\tau_{12} = \tau_1 + \tau_2$. In this case, therefore, all quanta are with certainty reflected back into the laser ($|B_0|^2 = |A_0|^2$).

7.1.3 The absorbing object is present. In the presence of the absorbing object ($\mathcal{R}_{\mathrm{N}} = 0$), the amplitudes B_{N}, B_0, and B_{D} satisfy the relations

$$K_{\mathrm{N}} \equiv \frac{B_{\mathrm{N}}}{A_0} = \frac{\mathrm{i}rt_{\mathrm{W}}\left[-1 + r_{\mathrm{S}}\exp(2\mathrm{i}\omega\tau_{12})\right]}{1 - (r^2 r_{\mathrm{S}} + t^2 r_{\mathrm{W}})\exp(2\mathrm{i}\omega\tau_{12})}\,, \tag{7.6a}$$

$$K_0 \equiv \frac{B_0}{A_0} = \frac{-r_{\mathrm{W}} + (t^2 + r^2 r_{\mathrm{S}} r_{\mathrm{W}})\exp(2\mathrm{i}\omega\tau_{12})}{1 - (r^2 r_{\mathrm{S}} + t^2 r_{\mathrm{W}})\exp(2\mathrm{i}\omega\tau_{12})}\,, \tag{7.6b}$$

$$K_{\mathrm{D}} \equiv \frac{B_{\mathrm{D}}}{A_0} = \frac{rtt_{\mathrm{S}}t_{\mathrm{W}}\exp(2\mathrm{i}\omega\tau_{12})}{1 - (r^2 r_{\mathrm{S}} + t^2 r_{\mathrm{W}})\exp(2\mathrm{i}\omega\tau_{12})}\,. \tag{7.6c}$$

The value of K_{D} is maximal under the matching condition

$$r^2 = \frac{1 - r_{\mathrm{W}}}{2 - r_{\mathrm{W}} - r_{\mathrm{S}}} \Leftrightarrow t^2 = \frac{1 - r_{\mathrm{S}}}{2 - r_{\mathrm{W}} - r_{\mathrm{S}}}\,. \tag{7.7}$$

Then

$$|K_{\mathrm{N}}|^2 = \frac{(1 + r_{\mathrm{W}})(2 - r_{\mathrm{S}} - r_{\mathrm{W}})}{4\mathcal{D}_0}\left(1 + \frac{4r_{\mathrm{S}}}{(1 - r_{\mathrm{S}})^2}\sin^2\omega\tau_{12}\right), \tag{7.8a}$$

$$|K_0|^2 = \frac{1}{4\mathcal{D}_0}\left[(1 - r_{\mathrm{W}})^2 + 4r_{\mathrm{W}}\left(\frac{1}{1 - r_{\mathrm{S}}} + \frac{1}{1 - r_{\mathrm{W}}}\right)\times\left(\frac{1}{1 - r_{\mathrm{S}}} + \frac{r_{\mathrm{S}}r_{\mathrm{W}}}{1 - r_{\mathrm{W}}}\right)\sin^2\omega\tau_{12}\right], \tag{7.8b}$$

$$|K_{\mathrm{D}}|^2 = \frac{(1 + r_{\mathrm{S}})(1 + r_{\mathrm{W}})}{4\mathcal{D}_0}\,, \tag{7.8c}$$

where

$$\mathcal{D}_0 = 1 + \left(\frac{1}{1 - r_{\mathrm{S}}} + \frac{1}{1 - r_{\mathrm{W}}}\right)\left(\frac{r_{\mathrm{S}}}{1 - r_{\mathrm{S}}} + \frac{r_{\mathrm{S}}}{1 - r_{\mathrm{W}}}\right)\sin^2\omega\tau_{12}\,. \tag{7.8d}$$

We now suppose that the reflection coefficients of the 'West' and 'South' mirrors are close to unity,

$$r_{\mathrm{S,W}} \approx 1 - \frac{T_{\mathrm{S,W}}}{2}\,, \qquad T_{\mathrm{S,W}} = t^2_{\mathrm{S,W}} \ll 1\,, \tag{7.9a}$$

the central frequency of the light pulse ω_0 corresponds to the resonance,

$$\sin\omega_0\tau_{12} = 0\,, \tag{7.9b}$$

and the length of the light pulse well exceeds the geometric size of the interferometer,

$$|\Omega| \equiv |\omega - \omega_0| \ll \frac{1}{\tau_{12}}\,. \tag{7.9c}$$

Formulas (7.8) then become

$$|K_{\mathrm{N}}|^2 \approx \frac{T_{\mathrm{S}} + T_{\mathrm{W}}}{4\mathcal{D}_0}\left(1 + \frac{16\Omega^2\tau_{12}^2}{T_{\mathrm{S}}^2}\right), \tag{7.10a}$$

$$|K_0|^2 \approx \frac{\Omega^2\tilde{\tau}^2}{\mathcal{D}_0}\,, \tag{7.10b}$$

$$|K_{\mathrm{D}}|^2 \approx \frac{1}{\mathcal{D}_0}\,, \tag{7.10c}$$

$$\mathcal{D}_0 = 1 + \Omega^2\tilde{\tau}^2\,, \tag{7.10d}$$

where

$$\tilde{\tau} = 2\left(\frac{1}{T_{\mathrm{S}}} + \frac{1}{T_{\mathrm{W}}}\right)\tau_{12} \tag{7.11}$$

is the inverse frequency band of the interferometer.

For the probabilities of the neutral and negative results to be small, the frequency bandwidth of the light pulse τ^{-1} must be much smaller than the interferometer transmission bandwidth,

$$\tau \gg \tilde{\tau} \Leftrightarrow \Omega\tilde{\tau} \ll 1\,. \tag{7.12}$$

This leads to

$$|K_{\mathrm{N}}|^2 \approx p_- \approx \frac{T_{\mathrm{S}} + T_{\mathrm{W}}}{4}\,, \tag{7.13a}$$

$$|K_0|^2 \approx \Omega^2\tilde{\tau}^2 \Leftrightarrow p_0 \approx \frac{\tilde{\tau}^2}{\tau^2}\,. \tag{7.13b}$$

The average energy absorbed by the object per single photon is then

$$\langle\mathcal{E}\rangle = \hbar\omega_0 p_- = \hbar\omega_0\,\frac{T_{\mathrm{S}} + T_{\mathrm{W}}}{4}\,. \tag{7.14}$$

By virtue of formula (7.12) and the evident condition $\omega_0\tau_{12} \gg 1$, it follows from (7.14) that

$$\langle\mathcal{E}\rangle > \frac{\hbar}{\tau_{12}}\,\frac{(T_{\mathrm{S}} + T_{\mathrm{W}})}{4} = \frac{\hbar}{\tilde{\tau}}\left(\frac{1}{T_{\mathrm{S}}} + \frac{1}{T_{\mathrm{W}}}\right)\frac{T_{\mathrm{S}} + T_{\mathrm{W}}}{4} > \frac{\hbar}{\tilde{\tau}} > \frac{\hbar}{\tau}\,. \tag{7.15}$$

7.2 Proof of formulas (3.8) and (3.12)

If $|\psi_{\mathrm{apr}}\rangle$ is the initial wave function of the object and $|\psi_{\mathrm{meter}}\rangle$ the initial wave function of the meter, formula (3.1) can be written as

$$\left\langle(\delta\hat{p})^2\right\rangle = \langle\psi_{\mathrm{apr}}|\langle\psi_{\mathrm{meter}}|(\hat{\mathcal{U}}^+\hat{p}\hat{\mathcal{U}} - \hat{p})^2|\psi_{\mathrm{meter}}\rangle|\psi_{\mathrm{apr}}\rangle\,. \tag{7.16}$$

Let $\hat{k}$ be the operator corresponding to the observable of the meter that is directly registered in the schemes considered in Section 3. This operator has a discrete spectrum of eigenvalues ($k = 0, 1$ for the 'yes – no' measurement scheme in Section 3.1 and $k = 0, 1, \ldots$ for the measurement of the discretized coordinate in Section 3.2). The corresponding discrete set of eigenstates $\{|k\rangle\}$ satisfies the completeness condition

$$\sum_k |k\rangle\langle k| = 1\,. \qquad (7.17)$$

With this condition, we can rewrite formula (7.16) as

$$\begin{aligned}\big\langle(\delta\hat{p})^2\big\rangle &= \sum_k \langle\psi_{\rm apr}|\Big(\langle\psi_{\rm meter}|\hat{\mathcal{U}}^+|k\rangle\hat{p}^2\langle k|\hat{\mathcal{U}}|\psi_{\rm meter}\rangle \\ &\quad - \langle\psi_{\rm meter}|\hat{\mathcal{U}}^+|k\rangle\hat{p}\langle k|\hat{\mathcal{U}}|\psi_{\rm meter}\rangle\hat{p} \\ &\quad - \hat{p}\langle\psi_{\rm meter}|\hat{\mathcal{U}}^+|k\rangle\hat{p}\langle k|\hat{\mathcal{U}}|\psi_{\rm meter}\rangle \\ &\quad + \hat{p}\langle\psi_{\rm meter}|\hat{\mathcal{U}}^+|k\rangle\langle k|\hat{\mathcal{U}}|\psi_{\rm meter}\rangle\hat{p}\Big)|\psi_{\rm apr}\rangle \\ &= \sum_k \langle\psi_{\rm apr}|\big(\hat{\Omega}_k^+\hat{p}^2\hat{\Omega}_k - \hat{p}\hat{\Omega}_k^+\hat{p}\hat{\Omega}_k \\ &\quad - \hat{\Omega}_k^+\hat{p}\hat{\Omega}_k\hat{p} + \hat{p}\hat{\Omega}_k^+\hat{\Omega}_k\hat{p}\big)|\psi_{\rm apr}\rangle\,, \qquad (7.18)\end{aligned}$$

where

$$\hat{\Omega}_k \equiv \int_{-\infty}^{\infty} |x\rangle\Omega_k(x)\langle x|\,\mathrm{d}x = \langle k|\hat{\mathcal{U}}|\psi_{\rm meter}\rangle \qquad (7.19)$$

are the reduction operators for the measurement under consideration.

In the coordinate representation, formula (7.18) takes the form

$$\begin{aligned}\big\langle(\delta\hat{p})^2\big\rangle &= \hbar^2\sum_k \infty\int_{-\infty}^{\infty}\left(\left|\frac{\mathrm{d}\Omega_k(x)\psi_{\rm apr}(x)}{\mathrm{d}x}\right|^2\right. \\ &\quad - \psi^*_{\rm apr}(x)\Omega^*_k(x)\,\frac{\mathrm{d}\Omega_k(x)\psi_{\rm apr}(x)}{\mathrm{d}x} \\ &\quad - \frac{\mathrm{d}\Omega^*_k(x)\psi^*_{\rm apr}(x)}{\mathrm{d}x}\,\Omega_k(x)\psi_{\rm apr}(x) \\ &\quad \left. + \big|\Omega_k(x)\big|^2\left|\frac{\mathrm{d}\psi_{\rm apr}(x)}{\mathrm{d}x}\right|^2\right)\mathrm{d}x = \\ &= \hbar^2\sum_k\int_{-\infty}^{\infty}\left|\frac{\mathrm{d}\Omega_k(x)}{\mathrm{d}x}\right|^2\big|\psi_{\rm apr}(x)\big|^2\,\mathrm{d}x\,. \qquad (7.20)\end{aligned}$$

7.3 Coordinate and momentum wave functions in a discretized state

We consider a state with the coordinate-representation wave function

$$\psi(x) = \sum_{n=-\infty}^{\infty}\Psi_n\psi_1(x-nX) \approx \sqrt{X}\,\Psi(x)\sum_{n=-\infty}^{\infty}\psi_1(x-nX)\,, \qquad (7.21)$$

where the function $\psi_1(x-nX)$ is nonzero only for $0 < x < X$,

$$\Psi_k = \sqrt{X}\,\Psi\left(kX + \frac{X}{2}\right), \qquad (7.22)$$

and the envelope $\Psi(x)$ is almost constant within the interval X (Fig. 8a). In the momentum representation, this function takes the form

$$\begin{aligned}\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}}\sum_{n=-\infty}^{\infty}\Psi_n\int_{-\infty}^{\infty}\psi_1(x-nX)\exp\left(\frac{\mathrm{i}px}{\hbar}\right)\mathrm{d}x \\ &= \phi_1(p)\sum_{n=-\infty}^{\infty}\Psi_n\exp\left(\frac{\mathrm{i}pnX}{\hbar}\right), \qquad (7.23)\end{aligned}$$

where

$$\phi_1(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi_1(x)\exp\left(\frac{\mathrm{i}px}{\hbar}\right)\mathrm{d}x\,. \qquad (7.24)$$

The second factor in (7.23) is a periodic function with the period $P = 2\pi\hbar/X$,

$$\sum_{n=-\infty}^{\infty}\Psi_n\exp\left(\frac{\mathrm{i}pnX}{\hbar}\right) = \sqrt{P}\sum_{k=-\infty}^{\infty}\Phi(p-kP)\,, \qquad (7.25)$$

where the function $\Phi(p)$ is nonzero only for $|p| < P/2$. Moreover, because the coefficients Ψ_n vary insignificantly as the index changes by unity, summation in (7.23) can be replaced by integration:

$$\Phi(p) \approx \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Psi(x)\exp\left(\frac{\mathrm{i}px}{\hbar}\right)\mathrm{d}x\,. \qquad (7.26)$$

As a result, we obtain

$$\phi(p) = \sqrt{P}\,\phi_1(p)\sum_{k=-\infty}^{\infty}\Phi(p-kP)\,. \qquad (7.27)$$

In other words, the wave function in the momentum representation has the same form as in the coordinate representation, but the role of the slowly varying envelope is now played by the spectrum $\phi_1(p)$ of the narrow function $\psi_1(p)$ and the role of a narrow peak that determines the periodic pattern of the wave function is played by the spectrum $\Phi(p)$ of the slowly varying envelope $\Psi(y)$.

References

1. Elitzur A, Vaidman L *Found. Phys.* **23** 987 (1993)
2. Bouwmeester D, Ekert A K, Zeilinger A (Eds) *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation* (Berlin: Springer, 2000) [Translated into Russian (Moscow: Postmarket, 2002)]
3. White A G et al. *Phys. Rev. A* **58** 605 (1998)
4. Kwiat P G et al. *Phys. Rev. Lett.* **83** 4725 (1999)
5. Kwiat P G et al. *Phys. Rev. Lett.* **74** 4763 (1995)
6. du Marchie van Voorthuysen E H *Am. J. Phys.* **64** 1504 (1996)
7. Hafner M, Summhammer J *Phys. Lett. A* **235** 563 (1997)
8. Braginskii V B, Vorontsov Yu I, Khalili F Ya *Zh. Eksp. Teor. Fiz.* **73** 1340 (1977) [*Sov. Phys. JETP* **46** 705 (1977)]
9. von Neumann J *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton Univ. Press, 1955) [Translated into Russian (Moscow: Nauka, 1964)]
10. Braginsky V B, Khalili F Ya *Quantum Measurement* (Cambridge: Cambridge Univ. Press, 1992)
11. Mensky M B *Quantum Measurements and Decoherence. Models and Phenomenology* (Dordrecht: Kluwer Acad. Publ., 2000) [Translated into Russian (Moscow: Fizmatlit, 2001)]
12. Braginsky V B et al. *Phys. Rev. D* **67** 082001 (2003)
13. Scully M O, Englert B-G, Walther H *Nature* **351** 111 (1991)
14. Luis A, Sánchez-Soto L L *Phys. Rev. Lett.* **81** 4031 (1998)
15. Englert B-G, Scully M O, Walther H *J. Mod. Opt.* **47** 2213 (2000)
16. Ye J, Vernooy D W, Kimble H J, quant-ph/9908007
17. Vorontsov Yu I *Vestn. Mosk. Univ. Ser. 3. Fiz. Astron.* (6) 7 (1998)
18. Vorontsov Yu I *Vestn. Mosk. Univ. Ser. 3. Fiz. Astron.* (1) 23 (1999)
19. Vorontsov Yu I *Teoriya i Metody Makroskopicheskikh Izmereniĭ* (Theory and Methods of Macroscopic Measurements) (Moscow: Nauka, 1989)