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How were the Hilbert – Einstein equations discovered?

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Contents

Abstract. **The ways in which Albert Einstein and David Hilbert independently arrived at the gravitational field equations are traced. A critical analysis is presented of a number of papers in which the history of the derivation of the equations is viewed in a way that "radically differs from the standard point of view." The conclusions of these papers are shown to be totally unfounded.**

1. Introduction

Since the studies by J Earman and C Glymour [1], it has become clear that the equations of Albert Einstein's general relativity were discovered almost simultaneously, but with different methods by David Hilbert and Einstein.

In 1997, the article entitled "Belated Decision in the Hilbert – Einstein Priority Dispute" appeared in *Science* [2]; its authors claim that "...knowledge of Einstein's result may have been crucial to Hilbert's introduction of the trace term into his field equations." On this ground, they push forward their point of view that "radically differs from the standard point of view" and which is exposed at length in Ref. [3].

According to the standard point of view, Einstein and Hilbert discovered the gravitational field equations independently of each other and in different ways. The same question was the subject of paper [4]. What is the question? In Einstein's paper [5], the gravitational field equations

$$\sqrt{-g}R_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}T\right)$$

are given, where $g_{\mu\nu}$ is a metric tensor, $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor density for matter, and T is the trace of $T_{\mu\nu}$,

$$T = g^{\mu\nu}T_{\mu\nu}\,.$$

The authors of paper [2] assert that Hilbert, having gained knowledge of these equations and having seen the 'trace term' $(1/2)g_{\mu\nu}T$, also 'introduced' the trace term into his equations [6]

$$\sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right) = -\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} \qquad (1)$$

[in this case, the term $(1/2)g_{\mu\nu}R$, where the trace is $R = g^{\mu\nu}R_{\mu\nu}$].

We now consider in what field equation Hilbert needed, according to the authors of Ref. [2], to 'introduce the trace term.' The authors of Ref. [2] do not take into account that in Hilbert's approach, nothing can be 'introduced' in principle because everything is exactly determined by the world function (Lagrangian)

$$H = R + L\,,$$

introduced by Hilbert, which plays a key role in the derivation of the gravitational equations in the framework of the least action principle.

The authors of Ref. [2] produced their discovery when they became aware of the proofs of the Hilbert paper (in which, by the way, some parts are missing; see Ref. [7], where, in particular, the remaining parts of the proofs are reproduced) and saw that the gravitational field equations were there presented in the form of the variational derivative of $[\sqrt{g}R]$ in $g^{\mu\nu}$,

$$\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}} - \partial_k\,\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}_k} + \partial_k\partial_\ell\,\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}_{k\ell}} = -\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}\,, \qquad (2)$$

but there are no equations of form (1). Thereof they draw their conclusion that Hilbert did not derive the gravitation equations in form (1).

But even if this had been the case, then Hilbert would still have nothing to 'introduce' additionally because Eqn (2)

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turns exactly into Eqn (1) after some quite trivial calculations. *Things, however, do not go in the way that the authors of* [2] *write*. In order to show that the statement by the authors of Ref. [2] has no serious grounds, we have to give an account of the basics of Hilbert's work (see Section 2).

On the basis of his idea of the equivalence of acceleration and gravity, Einstein, in a joint article [8] with M Grossmann in 1913, identified the gravitational field with the metric tensor of a pseudo-Riemannian (below, just Riemannian) space. The tensor gravitational field was thus introduced. In this article, on the basis of a simple model, Einstein formulates the general energy-momentum conservation law

$$\partial_\nu\left(\sqrt{-g}\,\Theta_\sigma^\nu\right) + \frac{1}{2}\sqrt{-g}\,\Theta_{\mu\nu}\partial_\sigma g^{\mu\nu} = 0\,. \tag{3}$$

"The first three of these relations ($\sigma = 1,2,3$) express the momentum conservation law, the latter ($\sigma = 4$) that of energy conservation," Einstein wrote. Here, $\Theta_{\mu\nu}$ stands for the energy-momentum tensor of matter. It must be noted that such *a law of energy-momentum conservation for any matter system* was then introduced by Einstein just as a plausible physical assumption. In the same article, Grossmann showed that Eqn (3) is covariant under arbitrary transformations and can be written as

$$\nabla_\nu \Theta_\sigma^\nu = 0\,, \tag{4}$$

where ∇_ν is the covariant derivative with respect to the metric $g_{\mu\nu}$. Einstein posed the problem to construct gravitational equations of the form

$$\Gamma_{\mu\nu} = \varkappa\,\Theta_{\mu\nu}\,, \tag{5}$$

where $\Gamma_{\mu\nu}$ is a tensor constructed from the metric and its derivatives. We note that in the part of this article written by Grossmann, the possible use of the Ricci tensor $R_{\mu\nu}$ as $\Gamma_{\mu\nu}$ in Eqn (5) was discussed.

Grossmann writes: "But in the special case of an infinitesimally small static gravitational field, this tensor *does not reduce* to $\Delta\varphi$. Therefore, the extent to which the problem of gravitational field equations is related to the general theory of differential tensors associated with the gravitational field, remains an open problem."

Later, Einstein, following his own ideas, searched for $\Gamma_{\mu\nu}$ as a quantity that behaves as a tensor only under arbitrary *linear transformations*. He would follow this path till November 1915. At the end of June (beginning of July) 1915, Einstein spent a week in Göttingen and, as he recollected later, "gave there six two-hour lectures." It is evident that after having met him, Hilbert got interested in the problem.

Einstein's formulation of the problem and *his* identification of the gravitation potential with the metric tensor $g_{\mu\nu}$ of a Riemannian space appeared to be the key ones for Hilbert. That was sufficient for him in order to find the gravitational field equation proceeding from the least action principle (Hilbert's Axiom I) and from his profound knowledge of the theory of invariants. All this is clearly seen in paper [6] by Hilbert.

In Section 2, we describe Hilbert's approach to the derivation of the gravitational field equation and also critically review articles [2–4] devoted to the same question. In Section 3, we expound Einstein's approach to the derivation of the same field equations.

2. Hilbert's approach

We scrutinize Hilbert's approach [6]. He formulates Axiom I: "*The law of a physical event is determined by the world function H whose arguments are*

$$g_{\mu\nu}, g_{\mu\nu\ell} = \frac{\partial g_{\mu\nu}}{\partial x^\ell}\,, \qquad g_{\mu\nu\ell k} = \frac{\partial^2 g_{\mu\nu}}{\partial x^\ell \partial x^k}\,,$$
$$q_s, q_{s\ell} = \frac{\partial q_s}{\partial x^\ell} \qquad (\ell, k = 1,2,3,4)\,,$$

and the variation of the integral [1]

$$\int H\sqrt{g}\,\mathrm{d}\omega \tag{6}$$
$$\left(g = |g_{\mu\nu}|\,, \quad \mathrm{d}\omega = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\right)$$

vanishes for any of the 14 potentials $g_{\mu\nu}$, q_s". He writes further: "As to the world function H, additional axioms are needed for its unambiguous determination. If only second derivatives of the potentials $g^{\mu\nu}$ can enter the gravitational equations, then the function H must have the form

$$H = R + L\,, \tag{7}$$

where R is an invariant following from the Riemann tensor (scalar curvature of a four-dimensional manifold),

$$R = g^{\mu\nu}R_{\mu\nu}\,, \tag{8}$$
$$R_{\mu\nu} = \partial_\nu\Gamma^\alpha_{\mu\alpha} - \partial_\alpha\Gamma^\alpha_{\mu\nu} + \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\lambda\alpha}\,, \tag{9}$$

and L is a function of the variables $g^{\mu\nu}$, $g^{\mu\nu}_\ell$, q_s, and q_{sk} only. For simplicity, we additionally assume in what follows that L is independent of $g^{\mu\nu}_\ell$."

In the same paper, Hilbert writes: "Axiom I implies that variation in the 10 gravitational potentials yields the 10 Lagrange differential equations

$$\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}} - \partial_k\,\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}_k} + \partial_k\partial_\ell\,\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}_{k\ell}} = -\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}\,.\text{"} \tag{10}$$

It is easy to see from (8) and (9) that *both R and $R_{\mu\nu}$ involve second-order derivatives of the metric only linearly. Second-rank tensors with such properties are*

$$R_{\mu\nu} \quad \textit{and} \quad g_{\mu\nu}R\,. \tag{10a}$$

All other tensors with such properties are obtained as combinations of these tensors.

This conclusion, to some extent, was already known to Einstein, and he, mentioning second-rank tensors that could lead to the gravitational equations with derivatives of the order at most two, wrote in a letter to H A Lorentz on 19 January 1916 [9]: "...aside from tensors...

$$R_{\mu\nu} \quad \text{and} \quad g_{\mu\nu}R$$

there are no (arbitrary substitutions for covariant) tensors..." For mathematician Hilbert that was evident.

[1] Here and in what follows, unless stated otherwise, we change the equation numbers in quotations in accordance with our numbering. In Ref. [6], Hilbert used the notation $K_{\mu\nu}$ and K for the Ricci tensor and the scalar curvature. For these, and also for other quantities, we use modern notations.

For brevity, we follow Hilbert in introducing the notation

$$[\sqrt{g}R]_{\mu\nu} = \frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}} - \partial_k \frac{\partial\sqrt{g}R}{\partial g_k^{\mu\nu}} + \partial_k\partial_\ell \frac{\partial\sqrt{g}R}{\partial g_{k\ell}^{\mu\nu}} \tag{11}$$

for the left-hand side of the equation. Then Eqn (10) takes the form

$$[\sqrt{g}R]_{\mu\nu} = -\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}. \tag{12}$$

We note that Hilbert's method of derivation of the gravitational equations does not require concrete specification for the Lagrangian function of the matter system. *In paper* [6], *Theorem* II (*see the Appendix*), *Hilbert infers the identity*

$$\delta_{\mathrm{L}}(\sqrt{g}\,J) + \partial_\lambda(\delta x^\lambda \sqrt{g}\,J) = 0\,, \tag{12a}$$

where δ_{L} is the Lie variation and J is an arbitrary function invariant under coordinate transformations. He uses this identity in obtaining Eqn (48).

Then *Hilbert proves the very important theorem* III (see the Appendix): "Let J be an invariant depending only on the components $g^{\mu\nu}$ and their derivatives; the variational derivatives of $\sqrt{g}\,J$ in $g^{\mu\nu}$ are denoted, as earlier, as $[\sqrt{g}\,J]_{\mu\nu}$. If $h^{\mu\nu}$ is an arbitrary contravariant tensor, then the quantity

$$\frac{1}{\sqrt{g}}\,[\sqrt{g}\,J]_{\mu\nu}\,h^{\mu\nu} \tag{13}$$

is also invariant; if one substitutes the standard tensor $p^{\mu\nu}$ instead of $h^{\mu\nu}$ and writes

$$[\sqrt{g}\,J]_{\mu\nu}\,p^{\mu\nu} = (i_s\,p^s + i_s^\ell\,p_\ell^s)\,, \tag{14}$$

where the expressions

$$i_s = [\sqrt{g}\,J]_{\mu\nu}\partial_s\,g^{\mu\nu}\,, \tag{15}$$

$$i_s^\ell = -2[\sqrt{g}\,J]_{\mu s}\,g^{\mu\ell} \tag{16}$$

depend only on $g^{\mu\nu}$ and their derivatives, then

$$i_s = \frac{\partial i_s^\ell}{\partial x^\ell}\,, \tag{17}$$

in the sense that this equation holds identically for all arguments, i.e., $g^{\mu\nu}$ and their derivatives."

Hilbert applies this theorem to the case where $J = R$. Then identity (17) becomes

$$\partial_\ell\{[\sqrt{g}R]_s^\ell\} + \frac{1}{2}[\sqrt{g}R]_{\mu\nu}\,\frac{\partial g^{\mu\nu}}{\partial x^s} \equiv 0\,. \tag{18}$$

This identity is similar to (3), and it can therefore be also written in form (4),

$$\nabla_\ell\,[\sqrt{g}R]_s^\ell \equiv 0\,. \tag{19}$$

We see that the covariant derivative of the variational derivative $[\sqrt{g}R]_s^\ell$ is equal to zero. Thus, on the basis of (12), we obtain

$$\nabla^\ell\left\{\frac{\partial\sqrt{g}L}{\partial g^{s\ell}}\right\} = 0\,. \tag{20}$$

According to Hilbert, the energy-momentum tensor density $T_{\mu\nu}$ of the matter system is defined as

$$T_{\mu\nu} = \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}\,, \tag{21}$$

and equality (20) can be written as a covariant conservation law of the energy-momentum tensor of the matter system,

$$\nabla_\nu T_\mu^\nu = 0\,. \tag{22}$$

Hilbert was the first to give definition (21) of the energy-momentum tensor of the matter system and to show that this tensor satisfies Eqn (22); in this way, he substantiated Einstein's assumption in Ref. [8]. Hilbert thus *found the gravitational field equation*[2]

$$[\sqrt{g}R]_{\mu\nu} = -\varkappa\,T_{\mu\nu}\,, \tag{23}$$

from which *the law of covariant conservation of energy-momentum* (22) *follows exactly*.

Multiplying both parts of Eqn (23) by $g^{\mu\nu}$ and summing over the indices μ and ν, we obtain

$$g^{\mu\nu}[\sqrt{g}R]_{\mu\nu} = -\varkappa\,T\,. \tag{24}$$

The left-hand side of Eqn (24) involves an invariant that contains second derivatives linearly. But there exists only one such invariant, R. Hence, we obtain the equation

$$\sqrt{g}\,\beta R = -\varkappa\,T\,, \tag{25}$$

where β is an arbitrary constant.

Summarizing, we can say that the gravitational field equations were found *by Hilbert and thus the problem posed by Einstein in 1913 was solved*. Equations (23) are identical to Eqns (1). They differ only in form. Below, we see that according to Hilbert, Eqns (23) are easily transformed to form (1). Hilbert, both in the proofs and in paper [6], wrote: "In the following I want ... to establish ... a new system of fundamental equations of physics." And further: "my fundamental equations", "my theory".

Hilbert could not write so if he did not consider *himself* the author of the "fundamental equations of physics."

The tensor density $[\sqrt{g}R]_{\mu\nu}$ in Eqn (23) also involves, by construction (11), the second-order derivatives only linearly, and therefore from (10a), this energy density is given by

$$[\sqrt{g}R]_{\mu\nu} = \sqrt{g}(R_{\mu\nu} + \alpha g_{\mu\nu}R)\,. \tag{26}$$

Expression (26) was quite evident for Hilbert. The authors of Refs [2 – 4] may find this difficult to understand, but this is their personal affair. From (26), we express the left-hand side of Eqn (24) as

$$g^{\mu\nu}[\sqrt{g}R]_{\mu\nu} = \sqrt{g}(4\alpha + 1)R\,, \tag{27}$$

which completely agrees with (25). Just about these general reasonings Hilbert wrote: "...which is clear without calculations if we recall that R is the only invariant and $R_{\mu\nu}$ is the only (besides $g_{\mu\nu}$) second-order tensor that can be constructed from $g_{\mu\nu}$ and its first and second derivatives $g_k^{\mu\nu}$, $g_{k\ell}^{\mu\nu}$."

[2] Original paper [6] by Hilbert corresponds to the system of units where $\varkappa = 1$.

The authors of paper [2] (see also Ref. [3]) write in this connection: "This argument is, however, untenable, because there are many other tensors of second rank and many other invariants that can be constructed from the Riemann tensor."

This statement by the authors of [2] *has no relation to Hilbert's exact argument because the authors of* [2, 3] *have overlooked the main point*: *the construction of the gravitational equations containing derivatives of* $g^{\mu\nu}$ *of the order not higher than two*. Hilbert specially wrote about that in his paper [6]: "If only second-order derivatives of the potentials $g^{\mu\nu}$ can enter the gravitational equations, then the function H must have the form

$$H = R + L.\text{"}$$

Therefore, Hilbert was *absolutely right* that in this case, there is only one invariant R and *two tensors* $R_{\mu\nu}$ and $g_{\mu\nu}R$ that contain second derivatives of the gravitational potential $g^{\mu\nu}$ linearly. All other tensors with such properties are linear combinations of these tensors.

Likewise, the author of paper [4] is wrong when he writes (p. 1360): "But variational derivation of the equations is then absent, and the correct form of the equations (with the 'half' term) is justified (not quite correctly) by the uniqueness of the Ricci tensor and the scalar curvature as generally covariant quantities depending only on $g^{\mu\nu}$'s and their first and second derivatives."

It is quite surprising to see the author of paper [4] writing about Hilbert's paper that "...variational derivation... is then absent." He must have *forgotten* the well-known circumstance that the Lagrange equations, which were presented by Hilbert, are a consequence of the least action principle (Axiom I of Hilbert). Therefore, *the variational derivation of the gravitational field equation is contained in Hilbert's paper* [6].

How can the authors of Refs [2–4] make up their minds to analyze and to judge Hilbert's papers [6] if they do not understand the essence of his exact mathematical arguments? The authors of papers [2, 3] write further: "Even if one requires the tensors and invariants to be linear in the Riemann tensor, the crucial coefficient of the trace term remains undetermined by such an argument." This is again *wrong*. The coefficient is easily determined. Hilbert proved the identity (19)

$$\nabla_\sigma [\sqrt{g} R]_\mu^\sigma \equiv 0 \,. \tag{28}$$

With Eqn (26), in the local frame where the Christoffel symbols are zero, identity (28) takes the simple form

$$\partial_\sigma (R_\mu^\sigma + \alpha \delta_\mu^\sigma R) \equiv 0 \,. \tag{29}$$

From (8) and (9), we find

$$\partial_\mu R = K_\mu \,, \qquad \partial_\sigma R_\mu^\sigma = \frac{1}{2} K_\mu \,, \tag{30}$$

where

$$K_\mu = g^{\nu\sigma} g^{\lambda\rho} \partial_\sigma \partial_\nu \partial_\mu g_{\lambda\rho} - g^{\nu\sigma} g^{\alpha\lambda} \partial_\sigma \partial_\alpha \partial_\mu g_{\lambda\nu} \,. \tag{31}$$

Using these expressions, we obtain

$$\partial_\sigma (R_\mu^\sigma + \alpha \delta_\mu^\sigma R) = \left(\frac{1}{2} + \alpha\right) K_\mu \equiv 0 \,,$$

whence

$$\alpha = -\frac{1}{2} \,, \tag{32}$$

and therefore,

$$[\sqrt{g} R]_{\mu\nu} = \sqrt{g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) , \tag{33}$$

i.e.,

$$\sqrt{g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = -\varkappa T_{\mu\nu} \,. \tag{34}$$

Thus 'the critical coefficient' that is of concern to the authors of Refs [2, 3] is obtained in Hilbert's approach in a trivial way by using ordinary derivatives accessible to a first-year university student. It is also clear that the trace term $(1/2) g_{\mu\nu} R$ does not arise as a result of some arbitrary 'introduction' into the field equations formulated by Hilbert; it *is inherent* there.

Later, in 1921, in paper 60 in [10] (see Ref. [19]), in writing the gravitational equations, Einstein would construct the geometric part of the gravitational equations using the tensor

$$R_{\mu\nu} + a g_{\mu\nu} R \,,$$

i.e., in the same way as was done earlier by Hilbert in transforming gravitational equations (12) to form (34).

The creative endeavor of the authors of Refs [2, 3] is crowned with the following thoughtful conclusion: "Taken together, this sequence suggests that knowledge of Einstein's result may have been crucial to Hilbert's introduction of the trace term into his field equations."

It is difficult to imagine that reading Hilbert's paper can lead one to such an idea. But the authors of Ref. [2] prove that this is possible. We remind them that *in Hilbert's formalism, one does not need to introduce anything*. *As soon as one wrote the world function H in the form*

$$H = R + L \,,$$

and established Theorem III, *the rest was just a matter of calculational techniques and nothing more*.

Thus, the analysis that we have undertaken on the judgements of the authors of Ref. [2] shows that *all their reproofs to Hilbert are either wrong or do not concern him. Therefore, all their arguments supporting the point of view* '*that radically differs*' *from the standard one are untenable*.

Before publication of Einstein's paper with the trace term, Hilbert has already obtained equality (33). Using (19) and (33), we find

$$\nabla_\nu \left(R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R \right) \equiv 0 \,. \tag{35}$$

But this is the Bianchi identity.

Poor knowledge of Hilbert's paper can be met not only in Ref. [2]. For instance, A Pais, in book [11], § 15.3, wrote: "Evidently Hilbert did not know the Bianchi identities either!" and further: "I repeat one last time that neither Hilbert nor Einstein was aware of the Bianchi identities in that crucial November 1915."

"Interesting enough, in 1917 the experts were not aware that Weyl's derivation of Eqn 15.4 (The identity in question. — *the authors*) by variational techniques was a brand new method for obtaining a long-known result," Pais continues.

Pais was right that Einstein did not know the Bianchi identity in that crucial November 1915. *All the rest in* [11] *that concerns Hilbert is wrong. It can be said that Hilbert did not know the Bianchi identity, indeed. He obtained it himself.* With the variational method, Hilbert proved a general identity (see Theorem III by Hilbert) from which, putting $J = R$, he also obtained the Bianchi identity. *Thus, it was not Weyl in 1917 but Hilbert in 1915 who obtained the Bianchi identity with the variational method.* Pais wrote in § 15.3 [11]: "In November 1915, neither Hilbert nor Einstein was aware of this royal road to the conservation laws. Hilbert had come close."

The authors of Ref. [3] write similarly: "...Hilbert did not discover royal road to the formulation of the field equations of general relativity. In fact, he did not formulate these equations at all..."

All this is wrong. It was Hilbert who found the shortest and most general way to construct the gravitational equations. He found the Lagrange function of the gravitational field, R, which yields the gravitational equations automatically via the least action principle. *This is just how these equations are obtained in modern presentations of Einstein's general theory of relativity.* It is a pity that Pais seems to have looked through Hilbert's paper superficially; the same is true for the authors of Refs [2, 3].

Later, in 1924, Hilbert wrote [12]: "In order to define the expression $[\sqrt{g}R]_{\mu\nu}$ one first chooses the frame in such a way that all $g_s^{\mu\nu}$, taken at the world point, vanish. We thus find

$$[\sqrt{g}R]_{\mu\nu} = \sqrt{g}\left[R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right]. \text{''} \tag{36}$$

The authors of Ref. [2] write, concerning this: "To summarize: Initially Hilbert did not give the explicit form of the field equations; then, after Einstein had published his field equations, Hilbert claimed that no calculation is necessary; finally, he conceded that one is."

This statement is a creation of the mind of the authors of Ref. [2]. There is no reason to believe that Hilbert did not himself obtain the explicit form of the field equations. They can be obtained in an elementary way from Eqns (23) and expression (26) with the use of identity (28). *Can one seriously assume that Hilbert was unable to obtain* (33) *from* (28)? Hilbert's addition made in 1924 does not mean a "recognition that calculation is necessary." He introduced it as a reminder of a simple method for finding a tensor. But this in no way invalidated his exact argument ("...clear without calculation").

The authors of Refs [2, 3] claim, referring to the proofs, that Hilbert had the gravitational equation only in form (23). Equation (23) contains the derivatives

$$\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}}\,,\quad \frac{\partial\sqrt{g}R}{\partial g_k^{\mu\nu}}\,,\quad \frac{\partial\sqrt{g}R}{\partial g_{k\ell}^{\mu\nu}}\,. \tag{37}$$

It is impossible to imagine a theoretical physicist or mathematician who would not calculate these derivatives and explicitly obtain the differential equations containing only the derivatives $g_k^{\mu\nu}$, $g_{k\ell}^{\mu\nu}$. As we have seen, it was not necessary for Hilbert to calculate them, because he determined the structure of the expression $[\sqrt{g}R]_{\mu\nu}$ from the general and rigorous mathematical statements, which made the calculation of the 'critical coefficient' trivial.

That is why the conclusion of the authors of papers [2–4], *that Hilbert did not obtain the "explicit form of the gravitational field equations" cannot be true.* It also contradicts, as we see in Section 4, the correspondence between Einstein and Hilbert, from which everything becomes absolutely clear, and no additional arguments are needed. There exists no more decisive argument than the evidence from Einstein himself. *But precisely this most important evidence from Einstein was left out by the authors of* [2, 3], who focused their analysis on unpublished (and incomplete) materials of Hilbert.

The evidence from Einstein in his letter of 18 November 1915 to Hilbert unambiguously disproves any false conjectures about Hilbert's paper [6]. Thus, the '*archive finding*' of the authors of Ref. [2], as a matter of principle, cannot shatter the evidence from Einstein himself. One could stop further discussion of the question here. But alongside their arguments, the authors of Refs [2–4] make erroneous conclusions about Hilbert's paper [6], and we therefore have to specially concentrate on this.

Even if one does not follow Hilbert's general statements, it is still possible, using definition (11), to perform simple differentiation and to express the tensor density $[\sqrt{g}R]_{\mu\nu}$ in terms of the Ricci tensor density and the scalar density $\sqrt{g}R$. The first term in (11) can be written in the form

$$\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}} = \sqrt{g}\,R_{\mu\nu} + \frac{\partial\sqrt{g}}{\partial g^{\mu\nu}}\,R + \sqrt{g}\,g^{\alpha\beta}\,\frac{\partial R_{\alpha\beta}}{\partial g^{\mu\nu}}\,. \tag{38}$$

Because

$$\frac{\partial\sqrt{g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\,\sqrt{g}\,g_{\mu\nu}\,, \tag{39}$$

we obtain

$$\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}} = \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right) + \sqrt{g}\,g^{\alpha\beta}\,\frac{\partial R_{\alpha\beta}}{\partial g^{\mu\nu}}\,. \tag{40}$$

From (11) and (40), we have

$$[\sqrt{g}R]_{\mu\nu} = \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right) + \left\{\sqrt{g}\,g^{\alpha\beta}\,\frac{\partial R_{\alpha\beta}}{\partial g^{\mu\nu}} - \partial_k\,\frac{\partial\sqrt{g}R}{\partial g_k^{\mu\nu}} + \partial_k\partial_\ell\,\frac{\partial\sqrt{g}R}{\partial g_{k\ell}^{\mu\nu}}\right\}.$$

It is easy to see that the sum of the terms in braces is identically equal to zero. The simplest way is to use the local Riemannian frame where the Christoffel symbols are zero. In such a simple, although not very elegant way, we arrive again at the expression

$$[\sqrt{g}R]_{\mu\nu} = \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right).$$

The authors of paper [3] wrote: "In both the Proofs and the published version of paper [6], Hilbert erroneously claimed that one can consider the last four equations (i.e., electromagnetic field equations. — *the authors*) as a consequence of the 4 identities that must hold, according to his Theorem I, between the 14 differential equations..."

Things, however, are not as the authors of Ref. [3] suppose. Theorems I and II are formulated for J, which is invariant under arbitrary transformations of the four world parameters. According to these theorems, there exist *four* *identities* for any invariant. In his paper, Hilbert considers two invariants, R and L. He composes the general invariant H of these two invariants:

$$H = R + L .$$

In Hilbert's notation, the gravitation equations have the form

$$[\sqrt{g}R]_{\mu\nu} = -\varkappa\, T_{\mu\nu} .$$

Hilbert chooses the invariant L as a function of the variables $g^{\mu\nu}$, q_σ, $\partial_\nu q_\sigma$, and he therefore obtains the generalized Maxwell equations

$$[\sqrt{g}L]^{\nu} = 0 , \qquad (41)$$

where

$$[\sqrt{g}L]^{\nu} = \frac{\partial\sqrt{g}L}{\partial q_\nu} - \partial_\mu\left(\frac{\partial\sqrt{g}L}{\partial(\partial_\mu q_\nu)}\right) . \qquad (42)$$

Then, on the basis of Theorem II, Hilbert establishes that the Lagrange function L depends on the derivatives of the potential q_ν only through the combination $F_{\mu\nu}$, i.e.,

$$L(F_{\mu\nu}) , \qquad (43)$$

where

$$F_{\mu\nu} = \partial_\mu q_\nu - \partial_\nu q_\mu . \qquad (44)$$

This does not forbid L to explicitly depend on q_ν, of course. Based on this, Hilbert chooses the Lagrangian in the form

$$L = \alpha Q + f(q) , \qquad (45)$$

where

$$Q = F_{\mu\nu}F_{\lambda\sigma}\, g^{\mu\sigma}g^{\nu\lambda} , \qquad q = q_\mu q_\nu\, g^{\mu\nu} , \qquad (46)$$

and α is a constant.

Hilbert then notes that the equations of electrodynamics "can be considered as a consequence of the equations of gravity."

According to Theorem II, the *four identities* hold for the invariant L:

$$\nabla_\mu T^{\mu}_{\nu} = F_{\mu\nu}[\sqrt{g}L]^{\mu} + q_\nu\partial_\mu[\sqrt{g}L]^{\mu} . \qquad (47)$$

It follows from identity (47) that if the equations of motion (41) for the matter system hold, then the covariant conservation law

$$\nabla_\mu T^{\mu}_{\nu} = 0$$

is satisfied for the matter system. But if we follow Hilbert in using gravitational equations (34) in identity (47), we obtain Hilbert's equations

$$F_{\mu\nu}[\sqrt{g}L]^{\mu} + q_\nu\partial_\mu[\sqrt{g}L]^{\mu} = 0 , \qquad (48)$$

which were assigned number 27 in his paper [6]. Equations (48) must be *compatible with the equations that follow from the least action principle with the same Lagrangian L*. This is only possible in the case where the *generalized Maxwell equations* hold:

$$[\sqrt{g}L]^{\nu} = 0 . \qquad (49)$$

Therefore, the author of paper [4] is completely wrong considering that "in the case of gauge-noninvariant Mie's theory with a Lagrangian of type (45), one has in general to use not the generalized Maxwell equations (49) but equations (48)."

This statement contradicts the least action principle, i.e., Hilbert's Axiom I. Thus, *four identities* (47) *due to Theorem II and equations of gravitation* (34) *lead to four equations* (48), *which are compatible with the generalized Maxwell equations obtained on the basis of Hilbert's Axiom* I. This is what Hilbert emphasized in paper [6]: "...from gravitation equations (10), there indeed follow 4 mutually independent linear combinations (48) *of the electrodynamics equations* (41) (authors' emphasis) together with their first derivatives."

It must be stressed that Hilbert writes about a "*linear combination of the electrodynamics equations* (41)," but not expressions (42). Precisely here the authors of Refs [3, 4] find themselves a muddle.

We note that in the particular case where

$$L = \alpha Q , \qquad (50)$$

the second term in Eqns (48) vanishes identically and we obtain the equations

$$F_{\mu\nu}[\sqrt{g}L]^{\mu} = 0 .$$

It follows, therefore, that if the determinant $|F_{\mu\nu}|$ is not zero, the Maxwell equations hold,

$$[\sqrt{g}L]^{\mu} = 0 ,$$

in full agreement with the least action principle (Hilbert's Axiom I). *Therefore, the Maxwell equations are a consequence of gravitational equations* (34) *and four identities* (47). All this follows from Hilbert's article if one reads it attentively. Afterwards, Einstein, together with Infeld and Hoffmann in paper 117 in Ref. [10] (see Ref. [20]) and also Fock in Ref. [13], would obtain the equations of motion of a matter system from the gravitational equations.

It is noticed quite often that Hilbert obtained the gravitational field equation "...not for an arbitrary material system, but specifically basing on Mie's theory" [14]. That is not quite right. The method that Hilbert used is general and imposes no restrictions on the form of the function L. But the fact that the gravitational equations imply *four equations* for the material system looked attractive for Hilbert and he applied his general equations to Mie's theory. Such a unification of gravity and Mie's theory was not fruitful, but Hilbert's general method for obtaining the gravitational equations proved to be very far-reaching.

Now a few words about auxiliary noncovariant equations.

To solve a problem, it is always necessary to have a complete system of equations. There are only ten equations of general relativity. One still needs to add four equations, which cannot be generally covariant. These auxiliary conditions are called coordinate conditions and can be of various kinds. Hilbert meant exactly this when he wrote (see Proofs in

Ref. [7]): "As our mathematical Theorem shows us, the previous Axioms I and II[3] can give only 10 mutually independent equations for 14 potentials. On the other hand, due to general invariance, more than 10 essentially independent equations for the 14 potentials $g_{\mu\nu}$, q_s, are impossible, and, because we wish to hold on Cauchy's theory for differential equations and to give the basic equations of physics a definite character, an addition of auxiliary non-invariant equations to (4) and (5) is inevitable."[4]

This is a mathematical requirement and it is necessary for the theory. Hilbert tried to obtain these additional equations in the framework of the theory itself but failed to do so and did not include that into the published article.

Thus, the basic system of the 10 equations of general relativity is generally covariant, but the complete system of equations that is necessary to solve problems is not generally covariant because four equations expressing coordinate conditions cannot be tensorial; they are not generally covariant. A solution to the complete system of the gravitational field equations can always be written in any admissible coordinate system. Precisely here the notion of the chart atlas arises. *That is why the statement by the authors of* [2–4] *that Hilbert's theory is not generally covariant, in contrast with Einstein's theory, is wrong. The complete system of equations of both Hilbert and Einstein is not generally covariant.*

The only difference was that Hilbert tried to uniquely construct these noncovariant equations in the framework of the theory itself. This proved impossible. The equations were made quite arbitrary but not tensorial. They then determine the choice of a frame.

In this regard, J Synge [15] writes: "A number of coordinate conditions occur in the literature of relativity, designed for special purposes. To put matter in general form, we shall denote the coordinate conditions by

$$C_i = 0\,;$$

these are equations (perhaps differential equations) satisfied by metric tensor g_{ij}. They are of course *not* tensor equations, since they are satisfied only when coordinates are specially chosen."

What is the material on which the authors of paper [2] base their conclusions? In the so-called proofs of Hilbert's paper from which they proceeded, the invariants H and K are used but their definition is lacking in the existing part of the proofs. Hilbert writes in the proofs: "I would like to construct below a new system of basic equations of physics, following the axiomatic method and proceeding, essentially, from the three axioms." Evidently, Hilbert had to define the invariants H and K in order to do that.

It is impossible to imagine that Hilbert, having posed such an aim, did not define these fundamental quantities. *But this means that the parts absent from the proofs are absolutely essential and contain important information. Valid conclusions cannot be made without taking this key information into account.*

But the authors of Ref. [2] neglected this important issue and were in a hurry to conclude that Hilbert did not derive the gravitation equations in the form

$$\sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right) = -\varkappa T_{\mu\nu}\,.$$

They presented this conclusion to the scientific community in the popular and well-known journal *Science* [2]. For all that, the authors of Ref. [2] did not inform the readers that the so-called proofs that they used are incomplete. Only later, in Ref. [3], did they mention that. The authors of Ref. [2] claim that the proofs allowed them to substantiate their point of view "that radically differs from the standard" one. How could this be done on the basis of a preliminary and incomplete material?

Here is one more method of 'analysis' used by the authors of Ref. [3]: "Remarkably, in characterizing his system of equations, Hilbert deleted the word '*neu*', a clear indication that he had meanwhile seen Einstein's paper and recognized that the equations implied by his own variational principle are formally equivalent to those which Einstein had explicitly written down (because of where the trace term occurs), if Hilbert's stress-energy tensor is substituted for the unspecified one on the right-hand side of Einstein's field equations."

But everything written by the authors of Ref. [3] makes no sense because their '*clear indication*' actually disappears since Hilbert wrote quite clearly in published article [6]: "I would like to construct below ... a new system of fundamental equations."

It is extremely tactless to reach conclusions on Hilbert's ideas on the basis of his marginal remarks in preliminary unpublished materials. The system of gravitational equations obtained by Hilbert is indeed a *new* one. He obtained it without the knowledge that Einstein came to the same gravitational equations. Einstein wrote to Hilbert about this in a letter of 18 November 1915 (see Section 3). The way the authors of [3] chose to substantiate their '*radically different*' point of view is strange. The multi-page composition [3] abounds in both similarly doubtful arguments and erroneous statements. Such an approach to the study of the most important physics papers can hardly be considered professional, based on a profound analysis of the material.

In concluding this section, we note that Hilbert's papers under the general title "Die Grundlagen der Physik" are very important and instructive. It would be very good if theoreticians who deal with similar problems knew them. Thus, for instance, article [16] was published in *Physics – Uspekhi*. If the authors of this paper had read Hilbert's paper [17], published in 1917, they would have seen that the critical coordinate velocity v_{c}, which they calculated approximately, is in fact equal to

$$v_{\mathrm{c}} = \frac{1}{\sqrt{3}}\left(\frac{r-\alpha}{r}\right), \qquad \alpha = r_g = 2GM\,.$$

The acceleration is equal to zero at just this velocity. The velocity v_{c} depends on the radius, while the corresponding proper velocity v is independent of r and is given by

$$v = \frac{1}{\sqrt{3}}\,.$$

To obtain the critical coordinate velocity v_c in the first order in G, one needs to keep terms of the second order in G in the acceleration. A gravitational field does not exert action on a

[3] In accordance with Axiom II, the world function H is invariant under any coordinate transformation. (*Authors' note*.)

[4] The respective equation numbers (4) and (5) are given to gravitational field equations (10) and generalized Maxwell equations (41). (*Authors' note*.)

body if the latter moves with the velocity v_c under the action of some external force.

In paper [17], Hilbert obtains the equation

$$\frac{d^2 r}{dt^2} - \frac{3\alpha}{2r(r-\alpha)}\left(\frac{dr}{dt}\right)^2 + \frac{\alpha(r-\alpha)}{2r^3} = 0$$

and gives its integral,

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r-\alpha}{r}\right)^2 + A\left(\frac{r-\alpha}{r}\right)^3,$$

where A is a constant; for light, $A = 0$.

In particular, this implies formula (20) in paper [16] for the velocity,

$$\left(\frac{dr}{dt}\right)^2 = \frac{1}{3}\left(1 - \frac{r_g}{r}\right)^2\left(1 + \frac{2r_g}{r}\right),$$

which differs from the critical velocity v_c. At this velocity, the acceleration is not exactly equal to zero.

Hilbert writes further: "According to this equation, the acceleration is negative or positive, i.e., gravitation attracts or repulses depending on whether the absolute value of the velocity satisfies the inequality

$$\left|\frac{dr}{dt}\right| < \frac{1}{\sqrt{3}}\left(\frac{r-\alpha}{r}\right)$$

or the inequality

$$\left|\frac{dr}{dt}\right| > \frac{1}{\sqrt{3}}\left(\frac{r-\alpha}{r}\right).\text{"}$$

For light, Hilbert finds that

$$\left|\frac{dr}{dt}\right| = \frac{r-\alpha}{r},$$

and further notes: "The light propagating rectilinearly towards the center always experiences a repulsion according to the latter inequalities; its speed increases from 0 at $r = \alpha$ to 1 at $r = \infty$."

We note that the local speed of light is equal to 1 (in units of c). It is also necessary to note that the velocity v_c is not a solution of the initial equation.

One more remark. The authors of Ref. [16] write: "Maybe this is the reason why the proper time is sometimes called 'genuine', or 'physical' in the literature, with the meaning of these terms unexplained." And further: "As a result, many experts in general relativity consider coordinate-dependent quantities nonphysical, so to say 'second-quality' quantities. However, the coordinate time is even more important for some problems than the proper time τ."

Thus, as the authors of Ref. [16] notice, "...to speak about the proper time as a 'genuine' or 'physical' in contrast with the coordinate velocity is not logical." In vain the authors of Ref. [16] think that experts in general relativity do not understand the significance of coordinate quantities. All the description in general relativity is in terms of coordinate quantities. One cannot avoid them in principle. This has been well known for a long time.

As an example of the physical quantity, we consider the proper time, which differs from the coordinate time in that it does not depend on the choice of coordinates. As one sees, there is a difference, one that is quite essential. Another example is the coordinate velocity of light,

$$v = c\,\frac{\sqrt{g_{00}}}{1 - g_{0i}e^i/\sqrt{g_{00}}};$$

here $i = 1, 2, 3$; e^i is a unit vector in the three-dimensional Riemannian space.

The coordinate velocity v is certainly measurable but depends on the choice of coordinates and can have an arbitrary value satisfying the condition

$$0 < v < \infty,$$

while the physical speed of light is exactly equal to c. As one can see, there is also a difference, and it is also very essential. Therefore, there is nothing 'illogical' in the use of notions of physical and coordinate velocities, contrary to the arguments made by the authors of Ref. [16].

3. Einstein's approach

Einstein wrote in 1913: "The theory stated in what follows arose from the conviction that the proportionality between the inertial and gravitational masses of bodies is an exact law of Nature, which must be manifested in the foundations of theoretical physics. I have tried to express this conviction in a number of previous works (*Ann. Phys.* 1911, **35**, 898; 1912, **38**, 355: papers 14 and 17 in "Collection of Scientific Works" I [10]), where an attempt was made to reduce the *gravitational* mass to the *inertial* one; this intention led me to the hypothesis that the gravity field (homogeneous in an infinitesimally small volume) can be physically substituted by an accelerated frame."

It is precisely this path that led Einstein to the conviction that in the general case, the gravitational field is characterized by the 10 space-time functions (metric coefficient of the Riemann space) $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu. \tag{51}$$

He further published a series of papers (articles 21, 22, 23, 25, 28, 29, 32) [10] about which he wrote later in paper 34 [10] (see Ref. [21]): "My efforts in recent years were directed toward basing a general theory of relativity, also for nonuniform motion, upon the supposition of relativity. I believed indeed to have found the only law of gravitation that complies with a reasonably formulated postulate of general relativity; and I tried to demonstrate the truth of precisely this solution in a paper* [8] that appeared last year in the 'Sitzungsberichte'.

Renewed criticism showed to me that this truth is absolutely impossible to show in the manner suggested. That this seemed to be the case was based upon a misjudgment. The postulate of relativity — as far as I demanded it there — is always satisfied if the Hamiltonian principle is chosen as a basis. But in reality, it provides no tool to establish the Hamiltonian function H of the gravitational field. Indeed, equation (77) l.c. which limits the choice of H says only that H has to be an invariant toward linear transformations, a

* Equations in that paper are quoted in what follows with the additional note 'l.c.' in order to keep them distinct from those in the present paper.

demand that has nothing to do with the relativity of accelerations. Furthermore, the choice determined by equations (78) l.c. does not determine equation (77) in any way.

For these reasons I lost trust in the field equations I had derived, and instead looked for a way to limit the possibilities in a natural manner. In this pursuit I arrived at the demand of general covariance, a demand from which I parted, though with a heavy heart, three years ago when I worked together with my friend Grossmann. As a matter of fact, we were then quite close to that solution of the problem, which will be given in the following.

Just as the special theory of relativity is based upon postulate that all equations have to be covariant relative to linear orthogonal transformations, so the theory developed here rests upon the *postulate of the covariance of all systems of equations relative to transformations with the substitution determinant 1*.

Nobody who really grasped it can escape from its charm, because it signifies a real triumph of the general differential calculus as founded by Gauss, Riemann, Christoffel, Ricci, and Levi-Civita."

Einstein chose the gravitational equation in the coordinate system where $\sqrt{-g}=1$ in the form[5]

$$\partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = -\varkappa T_{\mu\nu}\,, \tag{52}$$

where

$$\Gamma^\alpha_{\mu\nu} = -\frac{1}{2}\,g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,,$$

with being $T_{\mu\nu}$ the energy-momentum tensor for the material system. The left-hand side of Eqn (52) is obtained from the Ricci tensor under the condition $\sqrt{-g}=1$.

Einstein finds the Lagrange function for the gravitational field

$$L = g^{\sigma\tau}\Gamma^\alpha_{\sigma\beta}\Gamma^\beta_{\tau\alpha}\,. \tag{53}$$

With the relation

$$2\Gamma^\alpha_{\sigma\beta}\,\delta(g^{\sigma\tau}\Gamma^\beta_{\tau\alpha}) = \Gamma^\alpha_{\sigma\beta}\,\delta g^{\sigma\beta}_\alpha \tag{54}$$

taken into account, it is easy to obtain

$$\delta L = -\Gamma^\alpha_{\sigma\beta}\Gamma^\beta_{\tau\alpha}\,\delta g^{\sigma\tau} + \Gamma^\alpha_{\sigma\beta}\,\delta g^{\sigma\beta}_\alpha\,, \tag{55}$$

whence

$$\frac{\partial L}{\partial g^{\mu\nu}} = -\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}\,, \qquad \frac{\partial L}{\partial g^{\mu\nu}_\alpha} = \Gamma^\alpha_{\mu\nu}\,. \tag{56}$$

With the help of these formulas, gravitational equation (52) can be written as

$$\partial_\alpha\left(\frac{\partial L}{\partial g^{\mu\nu}_\alpha}\right) - \frac{\partial L}{\partial g^{\mu\nu}} = -\varkappa T_{\mu\nu}\,. \tag{57}$$

Multiplying (57) by $g^{\mu\nu}_\sigma$ and summing over the indices μ and ν, Einstein obtains

$$\partial_\lambda t^\lambda_\sigma = \frac{1}{2}\,T_{\mu\nu}\,\partial_\sigma g^{\mu\nu}\,, \tag{58}$$

where the quantity

$$t^\lambda_\sigma = \frac{1}{2\varkappa}\left(\delta^\lambda_\sigma L - g^{\mu\nu}_\sigma\,\frac{\partial L}{\partial g^{\mu\nu}_\lambda}\right) \tag{59}$$

characterizes the gravitational field. Taking the equality

$$\Gamma^\lambda_{\mu\nu}\,\partial_\sigma g^{\mu\nu} = 2g^{\alpha\mu}\Gamma^\nu_{\alpha\sigma}\Gamma^\lambda_{\mu\nu}$$

into account, one finds

$$t^\lambda_\sigma = \frac{1}{\varkappa}\left(\frac{1}{2}\,\delta^\lambda_\sigma g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - g^{\alpha\mu}\Gamma^\nu_{\alpha\sigma}\Gamma^\lambda_{\mu\nu}\right). \tag{60}$$

All further calculations are made in the reference frame where $\sqrt{-g}=1$. Einstein writes the basic equations of gravity (52) in the form

$$\partial_\alpha(g^{\nu\lambda}\Gamma^\alpha_{\sigma\nu}) - \frac{1}{2}\,\delta^\lambda_\sigma g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = -\varkappa(T^\lambda_\sigma + t^\lambda_\sigma)\,. \tag{61}$$

We show below how close to the true gravitational field equations Einstein was when writing his paper of 4 November 1915 (paper 34 in Ref. [10], see Ref. [21]).

From 1913, Einstein had mentioned, in one or another way, that the *quantity* t^λ_σ characterizing the gravitational field must enter the gravitational equation *in the same way* as the *quantity* t^λ_σ characterizing matter systems. For instance, he wrote in 1913 in paper [8]: "...the gravitational field tensor is a source of the field on equal footing with the tensor of matter systems $\Theta_{\mu\nu}$. Exceptional position of the gravitational field energy in comparison with all other kinds of energy would lead to inadmissible consequences." However, Einstein ignored this important intuitive argument when he wrote paper 34 in Ref. [10] (see Ref. [21]).

The argument regarding the symmetry between the quantities T^λ_σ and t^λ_σ is rather a product of Einstein's intuition, but not a general physical principle. The important point is that the transformation properties of these quantities are different. But intuition is a great asset if it leads to fulfillment of one's aims. This was precisely the case with Einstein. We note that as a rule, basic physical equations are not derived. Rather, they are guessed at on the basis of experimental data, general physical principles, and intuition. That is why it is sometimes difficult to logically explain in what way they are obtained by an author.

With the help of (60), it is easy to find the trace of t^λ_σ,

$$t = t^\lambda_\lambda = \frac{1}{\varkappa}\,g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}\,, \tag{62}$$

and to rewrite Einstein's equation (61) in the form

$$\partial_\alpha(g^{\nu\lambda}\Gamma^\alpha_{\sigma\nu}) = -\varkappa\left(T^\lambda_\sigma + t^\lambda_\sigma - \frac{1}{2}\,\delta^\lambda_\sigma t\right). \tag{63}$$

It is seen that there is no symmetry between the quantities T^λ_σ and t^λ_σ in Eqn (63). One can easily see that this symmetry can be restored in a simple way. We consider this in what follows. Based on (63), we derive the conservation laws. For this, we find the trace of the equations,

$$\partial_\alpha(g^{\nu\beta}\Gamma^\alpha_{\nu\beta}) = -\varkappa(T - t)\,. \tag{64}$$

[5] In this section, we use Einstein's notation.

We multiply both sides of Eqn (64) by $(1/2)\delta_\sigma^\lambda$ and subtract the result from (63):

$$\partial_\alpha\left(g^{\nu\lambda}\Gamma^\alpha_{\sigma\nu} - \frac{1}{2}\,\delta^\lambda_\sigma g^{\nu\beta}\Gamma^\alpha_{\nu\beta}\right) = -\varkappa\left(T^\lambda_\sigma + t^\lambda_\sigma - \frac{1}{2}\,\delta^\lambda_\sigma T\right). \quad (65)$$

It is easy to verify that the equalities

$$\partial_\lambda\partial_\alpha(g^{\nu\lambda}\Gamma^\alpha_{\sigma\nu}) = \frac{1}{2}\,\partial_\lambda\partial_\alpha\partial_\sigma g^{\alpha\lambda}, \quad (66)$$

$$\partial_\lambda\partial_\alpha\delta^\lambda_\sigma g^{\nu\beta}\Gamma^\alpha_{\nu\beta} = \partial_\lambda\partial_\alpha\partial_\sigma g^{\alpha\lambda} \quad (67)$$

hold. Using these equalities, we find from Eqn (65) that

$$\partial_\lambda(T^\lambda_\sigma + t^\lambda_\sigma) = \frac{1}{2}\,\delta^\lambda_\sigma\partial_\lambda T, \quad (68)$$

and similarly, using (58), we find the relation

$$\partial_\lambda T^\lambda_\sigma + \frac{1}{2}\,T_{\mu\nu}\partial_\sigma g^{\mu\nu} = \frac{1}{2}\,\delta^\lambda_\sigma\partial_\lambda T. \quad (69)$$

It is evident from this that Eqn (63) does not provide the conservation laws and there is no symmetry between T^λ_σ and t^λ_σ in Eqn (68). To restore the symmetry in (63) and (68) and to ensure that the conservation laws hold, we must make the substitution

$$T^\lambda_\sigma \to T^\lambda_\sigma - \frac{1}{2}\,\delta^\lambda_\sigma T. \quad (70)$$

Under (70), the trace of $T_{\mu\nu}$ changes as

$$T \to -T. \quad (71)$$

We note that symmetrization is not related to any assumptions on the structure of matter. Having performed this operation, we obtain the new gravitational equations

$$\partial_\alpha(g^{\nu\lambda}\Gamma^\alpha_{\sigma\nu}) = -\varkappa\left\{(T^\lambda_\sigma + t^\lambda_\sigma) - \frac{1}{2}\,\delta^\lambda_\sigma(T+t)\right\}. \quad (72)$$

The same replacement applied to (68) and (69) leads to the restoration of the conservation laws

$$\partial_\lambda(T^\lambda_\sigma + t^\lambda_\sigma) = 0 \quad (73)$$

and similarly,

$$\partial_\lambda T^\lambda_\sigma + \frac{1}{2}\,T_{\mu\nu}\,\partial_\sigma g^{\mu\nu} = 0. \quad (74)$$

Equations (73) and (74) arise only from new equations (72).

In paper 35 in Ref. [10] (see Ref. [22]), which is an addendum to paper 34 in Ref. [10] (see Ref. [21]), Einstein takes a further step and chooses the gravitational equations in the form

$$R_{\mu\nu} = -\varkappa T_{\mu\nu}, \quad (75)$$

which is generally covariant under arbitrary coordinate transformations. He abandons the condition $\sqrt{-g} = 1$. In the frame where $\sqrt{-g} = 1$, these equations are equivalent to Eqns (52). But Eqn (52) provides neither the symmetry between T^λ_σ and t^λ_σ nor the conservation laws and the replacement operation in (70) and (71) was required for symmetrization; it would therefore be natural to make replacement in the initial equations (75) as well. In this way, we obtain the new gravitation equations

$$R_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}T\right). \quad (76)$$

Exactly these equations were obtained by Einstein several days later and then published in paper 37 in Ref. [10] (see Ref. [5]). We note that Einstein found the conservation law equations (73) still with the gravitational equations (63). This circumstance probably satisfied him, and he did not pay attention to the symmetry breaking between T^λ_σ and t^λ_σ in Eqns (63). But his method of deriving the conservation laws led to a situation where the choice of the frame $\sqrt{-g} = 1$ was possible only if the trace of the matter tensor was equal to zero. Instead of restoring the symmetry via (70) and (71), Einstein chose another, more radical, way. In paper 35 in Ref. [10] (see Ref. [22]), he proposed a new physical idea that "in reality, only the quantity $T^\mu_\mu + t^\mu_\mu$ is positive, while T^μ_μ vanishes." Such an approach restored the symmetry, but despite its radicalness, was not fruitful and this idea existed but a short time.

Later, Einstein returned to his old idea on symmetry and obtained the gravitational field equations (76) in paper 37 in Ref. [10] (see Ref. [5]). He mentioned there: "*As it is not difficult to see, our additional term leads to that energy tensors of the gravitational field and of matter enter Eqn* (9) *in the same way*." There is some inexactitude in this statement. *A gravitational field energy tensor, does not exist in general relativity*. But taken as a heuristic notion, it led Einstein directly to his goal.

We see that Einstein's path led him inevitably to the same equations that were also obtained by Hilbert. It is quite evident that Einstein obtained them independently. Moreover, he gained them through much suffering for several years.

For a better understanding of what is written above, of no small importance is the quite vivid correspondence between Hilbert and Einstein, which took place during the period of their work on the gravitational field equations. This correspondence bears witness that no '*radically different*' point of view, other than the standard one, can exist, as a matter of principle.

4. Einstein – Hilbert correspondence

From Einstein to Hilbert

Berlin, Sunday, 7 November 1915

"Highly esteemed Colleague,

With return post I am sending you the correction to a paper in which I changed the gravitational equations, after having myself noticed about 4 weeks ago that my method of proof was a fallacious one. My colleague Sommerfeld wrote that you also have found a hair in my soup that has spoiled it entirely for you. I am curious whether you will take kindly to this new solution.

With cordial greetings, yours

A. Einstein

When may I expect the mechanics and history week to take place in Göttingen? I am looking forward to it very much."

From Einstein to Hilbert

Berlin, Friday, 12 November 1915

"Highly esteemed Colleague,

I just thank you for the time being for your kind letter. The problem has meanwhile made new progress. Namely, it is possible to exact *general* covariance from the postulate $\sqrt{-g} = 1$; Riemann's tensor then delivers the gravitation equations directly. If my present modification (which does not change the equations) is legitimate, then gravitation must play a fundamental role in the composition of matter. My own curiosity is interfering with my work! I am sending you two copies of last year's paper. I have only two other intact copies myself. If someone else needs the paper, he can easily purchase one, of course, for 2M (as an Academy offprint).

Cordial greetings, yours

Einstein"

From Hilbert to Einstein

Göttingen, 13 November 1915

"Dear Colleague,

Actually, I first wanted to think of a very palpable application for physicists, namely reliable relations between the physical constants, before obliging with my axiomatic solution to your great problem. But since you are so interested, I would like to lay out my theory in very complete detail on the coming Tuesday, that is, the day after the day after tomorrow (the 16th of this mo.). I find it ideally handsome mathematically and absolutely compelling according to axiomatic method, even to the extent that not quite transparent calculations do not occur at all and therefore rely on its factuality. As a result of gen. math. law, the (generalized Maxwellian) electrody. eqs. as a math. consequence of the gravitation eqs., such that gravitation and electrodynamics are actually nothing different at all. Furthermore, my energy concept forms the basis: $E = \sum(e_s T^s + e_{ih} t^{ih})$, which is likewise a general invariant, and from this then also follow from a very simple axiom the 4 missing "space-time equations" $e_s = 0$. I derived most pleasure in the discovery already discussed with Sommerfeld that normal electrical energy results when a specific absolute invariant is differentiated from the gravitation potentials and then g is set $= 0.1$. My request is thus to come for Tuesday. You can arrive at 3 or 1/2 past 5. The Math. Soc. meets at 6 o'clock in the auditorium building. My wife and I would be very pleased if you stayed with us. It would be better still if you came already on Monday, since we have the phys. colloquium on Monday, 6 o'clock, at the phys. institute. With all good wishes and in the hope of soon meeting again, yours,

Hilbert

As far as I understand your new paper, the solution given by you is entirely different from mine, especially since my e_s's must also necessarily contain the electrical potential. "

From Einstein to Hilbert

Berlin, Monday, 15 November 1915

"Highly esteemed Colleague,

Your analysis interests me tremendously, especially since I often racked my brains to construct a bridge between gravitation and electromagnetics. The hints your give in your postcards awaken the greatest of expectations. Nevertheless, I must refrain from travelling to Göttingen for the moment and rather must wait patiently untill I can study your system from the printed article; for I am tired out and plagued with stomach pains besides. If possible, please send me a correction proof of your study to mitigate my impatience.

With best regards and cordial thanks, also to Mrs. Hilbert, yours,

A. Einstein"

16 November 1915, Hilbert gave a talk. The author of paper [18] writes about that:

"'Grundgleichungen der Physik' was the title of Hilbert's lecture to the Göttingen Mathematical Society of November 16. It was also the title under which his communication in the letter of invitation circulated among the Academy members between November 15 and the meeting of November 20..." He mentions also: "The invitation for the meeting of 20 November was issued on November 15 and was, as always, circulated among the members to confirm their participation and announce any communications they intended to present at the meeting. Into this invitation Hilbert wrote: 'Hilbert legt vor in die Nachrichten: Grundgleichungen der Physik'."

"*In response to Einstein's request*", as the author of Ref. [18] notes, "*Hilbert had to report his findings in correspondence to Einstein* (authors' emphasis), unfortunately lost. He probably sent Einstein the manuscript of his lecture to the Göttingen Mathematical Society, or a summary of its main points."

From Einstein to Hilbert

Berlin, 18 November, 1915

"Dear Colleague,

The system you furnish agrees — as far as I can see — exactly with what I found in the last few weeks and have presented to the Academy (authors' emphasis). The difficulty was not in finding generally covariant equations for the $g_{\mu\nu}$'s; for this is easily achieved with the aid of Riemann's tensor. Rather, it was hard to recognize that these equations are a generalization, that is, simple and natural generalization of Newton's law. It has just been in the last few weeks that I succeeded in this (I sent you my communications), whereas 3 years ago with my friend Grossmann I had already taken into consideration the only possible generally covariant equations, which have now been shown to be the correct ones. We had only heavy-heartedly distanced ourselves from it, because it seemed to me that the physical discussion yielded an incongruency with Newton's law. The important thing is that the difficulties have now been overcome. Today I am presenting to the Academy a paper in which I derive quantitatively out of general relativity, without any guiding hypothesis, the perihelion motion of Mercury discovered by Le Verrier. No gravitation theory had achieved this untill now.

Best regards, yours

Einstein"

Such is the content of Einstein's reply letter. There does not exist an argument more forcible than the words in the letter, written by Einstein himself: "*The system you furnish agrees — as far as I can see — exactly with what I found in the last few weeks and have presented to the Academy*." An evidence more exact cannot exist in principle, but just this evidence remained aside in Refs [2 – 4]. This evidence alone by Einstein is already sufficient to exclude completely and forever any attempts to push forward a point of view "*radically different*" from the standard. The authors of

Refs [2, 3] made a whole series of other wrong conclusions about Hilbert's paper. That is why we had to consider their compositions in some detail in Section 2.

We nonetheless suppose that Einstein received the gravitational equations in form (12) from Hilbert, i.e.,

$$[\sqrt{g}R]_{\mu\nu} = -\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} \,. \tag{77}$$

It is improbable that Einstein would have agreed that *these equations* are in accord with *his equations*,

$$R_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}T\right), \tag{78}$$

where the Ricci tensor enters explicitly. To agree with this, Einstein would have needed to calculate the derivatives

$$\frac{\partial\sqrt{g}R}{\partial g^{\mu\nu}}\,,\quad \frac{\partial\sqrt{g}R}{\partial g_k^{\mu\nu}}\,,\quad \frac{\partial\sqrt{g}R}{\partial g_{k\ell}^{\mu\nu}}\,.$$

But he did not calculate them at that time. He wrote about that later, in a letter to H A Lorentz of 19 January 1916 [9]: "I avoided the somewhat involved computation of the $\partial R/\partial g^{\mu\nu}$'s and $\partial R/\partial g_\sigma^{\mu\nu}$'s by setting up the tensor equations directly. But the other way is certainly also workable and even more elegant mathematically."

It is also improbable that Hilbert, knowing that the Ricci tensor enters the Einstein equations (*he was informed of that in the letter from Einstein of 7 November 1915*), *could send him his equations in form* (77). *No doubt, Einstein received from Hilbert the equations in the form*

$$\sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right) = -\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}\,, \tag{79}$$

because it was not difficult for Hilbert to find, from general considerations and practically without computations, as we have seen above, the equality

$$[\sqrt{g}R]_{\mu\nu} = \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right).$$

In the letter to Hilbert of 18 November 1915, Einstein wrote: "*The system you furnish agrees — as far as I can see — exactly to what I found...*"

This is easily verified by comparing Eqns (78) and (79). Einstein's words "*as far as I can see*" were possibly caused by the energy-momentum tensor density being defined in Hilbert's paper as

$$\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}\,,$$

where L is a function of $g^{\mu\nu}$, q_σ, and $q_{\sigma\nu}$. Such a definition was new and unknown to Einstein. He needed time to understand its essence. But Einstein replied to Hilbert immediately. Later on, in paper 42 in Ref. [10] (see Ref. [23]), Einstein would use precisely this definition of the energy-momentum tensor. In this paper, he introduced, like Hilbert, a function $\mathfrak{M}$ of the variables $g^{\mu\nu}$, $q_{(\rho)}$, $q_{(\rho)\alpha}$ and wrote the energy-momentum tensor density as

$$\mathfrak{T}_{\mu\nu} = -\frac{\partial\mathfrak{M}}{\partial g^{\mu\nu}}\,.$$

Therefore, it is impossible to understand on what ground the authors of Ref. [3] try to conclude quite the opposite: "The new energy expression that Hilbert now took over from Einstein...". As we have just seen, it is absolutely wrong. Namely, Einstein adopted the definition of the energy-momentum tensor density from Hilbert and used it in paper 42 in Ref. [10] (see Ref. [23]).

Furthermore, the authors of Ref. [3] conclude: "...Einstein's generalization of Hilbert's derivation made it possible to regard the latter as merely representing a problematic special case."

All this is wrong. Hilbert's method is general; it allows obtaining the gravitational equations without assuming a specific form of the Lagrange function L of the matter system. Therefore, there was no (and could not be any) generalization of Hilbert's derivation. It is a different matter that afterwards Hilbert applied his method to the concrete case of Mie's theory.

As we have already mentioned in Section 2, the transformation of (77) to (79) was not a great labor for Hilbert with the help of Theorem III, proven by him.

Therefore, the proofs, moreover incomplete, cannot show that Hilbert did not write the gravitational field equations in form (79).

5. Conclusion

The analysis in Sections 2 and 3 shows that Einstein and Hilbert independently discovered the gravitational field equations. Their pathways were different but they led to exactly the same result. Nobody copied from the other. Therefore, no "*belated decision in the Einstein – Hilbert priority dispute*," about which the authors of Ref. [2] wrote, can be taken. Moreover, the very Einstein – Hilbert dispute never took place. Everything is absolutely clear: *both authors did everything to give both their names to the gravitational field equations*. But general relativity is Einstein's theory.

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6. Appendix

Below, for pedagogical purposes, we give a detailed proof of Hilbert's theorems II and III.

Theorem II. If H is an invariant that depends on $g_{\mu\nu}$, $\partial_\lambda g_{\mu\nu}$, $\partial_\sigma\partial_\lambda g_{\mu\nu}$, A_ν, and $\partial_\lambda A_\nu$, then an infinitesimal contravariant vector δx^s satisfies the identity

$$\delta_{\mathrm{L}}(\sqrt{g}H) = \partial_s(\sqrt{g}H\,\delta x^s)\,, \tag{A.1}$$

where δ_{L} is the Lie variation.

To prove this theorem, we consider the integral

$$S = \int_\Omega \mathrm{d}^4x\,\sqrt{g}H\,. \tag{A.2}$$

We make an infinitesimal coordinate transformation

$$x'^\nu = x^\nu + \delta x^\nu\,, \tag{A.3}$$

where δx^ν is an arbitrary infinitesimal four-vector. The integral then remains unchanged and therefore the variation

$\delta_c S$ vanishes:

$$\delta_c S = \int_{\Omega'} \mathrm{d}^4 x' \sqrt{g'} H' - \int_{\Omega} \mathrm{d}^4 x \sqrt{g} H = 0 \,. \tag{A.4}$$

The first integral can be written as

$$\int_{\Omega'} \mathrm{d}^4 x' \sqrt{g'} H' = \int_{\Omega} J \sqrt{g'} H' \, \mathrm{d}^4 x \,, \tag{A.5}$$

where J is the Jacobian of the transformation,

$$J = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} \,. \tag{A.6}$$

The Jacobian of transformations (A.3) is given by

$$J = 1 + \partial_\lambda \delta x^\lambda \,. \tag{A.7}$$

Expanding $\sqrt{g'} H'$ in the Taylor series, we find

$$\sqrt{g'(x')} H'(x') = \sqrt{g'(x)} H'(x) + \delta x^\lambda \, \partial_\lambda (\sqrt{g} H) \,. \tag{A.8}$$

Due to (A.5), (A.7), and (A.8), equality (A.4) assumes the form

$$\delta_c S = \int_{\Omega} \mathrm{d}^4 x \left[\delta_{\mathrm{L}} (\sqrt{g} H) + \partial_\lambda (\sqrt{g} H \, \delta x^\lambda) \right] = 0 \,, \tag{A.9}$$

where $\delta_{\mathrm{L}}(\sqrt{g} H)$ is the Lie variation,

$$\delta_{\mathrm{L}}(\sqrt{g} H) = \sqrt{g'(x)} H'(x) - \sqrt{g(x)} H(x) \,. \tag{A.10}$$

The Lie variation commutes with partial derivatives:

$$\delta_{\mathrm{L}} \partial_\lambda = \partial_\lambda \delta_{\mathrm{L}} \,. \tag{A.11}$$

The Lie variation of $\sqrt{g} H$ is

$$\delta_{\mathrm{L}}(\sqrt{g} H) = P_g(\sqrt{g} H) + P_q(\sqrt{g} H) \,, \tag{A.12}$$

where

$$P_g(\sqrt{g} H) = \frac{\partial \sqrt{g} H}{\partial g_{\mu\nu}} \delta_{\mathrm{L}} g_{\mu\nu} + \frac{\partial \sqrt{g} H}{\partial(\partial_\lambda g_{\mu\nu})} \partial_\lambda \delta_{\mathrm{L}} g_{\mu\nu} + \frac{\partial \sqrt{g} H}{\partial(\partial_\sigma \partial_\lambda g_{\mu\nu})} \partial_\sigma \partial_\lambda \delta_{\mathrm{L}} g_{\mu\nu} \,, \tag{A.13}$$

$$P_q(\sqrt{g} H) = \frac{\partial \sqrt{g} H}{\partial A_\lambda} \delta_{\mathrm{L}} A_\lambda + \frac{\partial \sqrt{g} H}{\partial(\partial_\sigma A_\lambda)} \partial_\sigma \delta_{\mathrm{L}} A_\lambda \,. \tag{A.14}$$

Because the volume Ω is arbitrary, we obtain the desired Hilbert identity from (A.9),

$$\delta_{\mathrm{L}}(\sqrt{g} H) + \partial_\lambda (\sqrt{g} H \, \delta x^\lambda) \equiv 0 \,, \tag{A.15}$$

where

$$\delta_{\mathrm{L}}(\sqrt{g} H) = P_g(\sqrt{g} H) + P_q(\sqrt{g} H) \,. \tag{A.16}$$

Theorem III. If an invariant H depends on $g_{\mu\nu}$, $\partial_\lambda g_{\mu\nu}$, and $\partial_\sigma \partial_\lambda g_{\mu\nu}$, then the variational derivative

$$\frac{\delta \sqrt{g} H}{\delta g_{\mu\nu}} = \sqrt{g} \, G^{\mu\nu} = \frac{\partial \sqrt{g} H}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial \sqrt{g} H}{\partial(\partial_\lambda g_{\mu\nu})} + \partial_\sigma \partial_\lambda \frac{\partial \sqrt{g} H}{\partial(\partial_\sigma \partial_\lambda g_{\mu\nu})} \tag{A.17}$$

satisfies the identity

$$\nabla_\lambda G^{\lambda\nu} \equiv 0 \,, \tag{A.18}$$

or, in another form,

$$\partial_\lambda (\sqrt{g} \, G^\lambda_\rho) + \frac{1}{2} \sqrt{g} \, G_{\lambda\sigma} \partial_\rho g^{\lambda\sigma} \equiv 0 \,, \tag{A.19}$$

where ∇_λ is the covariant derivative in the Riemannian space.

To prove this theorem, we consider the integral

$$\int_{\Omega} \sqrt{g} H \, \mathrm{d}^4 x$$

over a finite region of the four-dimensional world. The translation vector δx^σ in (A.3) must vanish together with its derivatives on the 3-dimensional boundary of the region Ω. This implies the vanishing of the field variations and their derivatives on the boundary of this region. Using Hilbert identity (A.15), we find

$$\int_{\Omega} \delta_{\mathrm{L}}(\sqrt{g} H) \, \mathrm{d}^4 x = 0 \,. \tag{A.20}$$

In our case,

$$\delta_{\mathrm{L}}(\sqrt{g} H) = P_g(\sqrt{g} H) \,. \tag{A.21}$$

Expression (A.13) can be written as

$$P_g(\sqrt{g} H) = \frac{\delta \sqrt{g} H}{\delta g_{\mu\nu}} \, \delta_{\mathrm{L}} g_{\mu\nu} + \partial_\lambda S^\lambda \,, \tag{A.22}$$

where the vector S^λ is given by

$$S^\lambda = \left[\frac{\partial \sqrt{g} H}{\partial(\partial_\lambda g_{\mu\nu})} - \partial_\sigma \left(\frac{\partial \sqrt{g} H}{\partial(\partial_\sigma \partial_\lambda g_{\mu\nu})} \right) \right] \delta_{\mathrm{L}} g_{\mu\nu} + \frac{\partial \sqrt{g} H}{\partial(\partial_\sigma \partial_\lambda g_{\mu\nu})} \, \partial_\sigma \delta_{\mathrm{L}} g_{\mu\nu} \,. \tag{A.23}$$

Substituting (A.22) in (A.20), we find

$$\int_{\Omega} \frac{\delta \sqrt{g} H}{\delta g_{\mu\nu}} \, \delta_{\mathrm{L}} g_{\mu\nu} \, \mathrm{d}^4 x = 0 \,. \tag{A.24}$$

We now find the variation $\delta_{\mathrm{L}} g_{\mu\nu}$ under transformations (A.3). The metric tensor $g_{\mu\nu}$ is transformed as

$$g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \, g_{\lambda\sigma}(x) \,.$$

Hence, for transformation (A.3), we find

$$\delta_{\mathrm{L}} g_{\mu\nu}(x) = -\delta x^\sigma \partial_\sigma g_{\mu\nu} - g_{\mu\sigma} \partial_\nu \delta x^\sigma - g_{\nu\sigma} \partial_\mu \delta x^\sigma \,. \tag{A.25}$$

Accounting for the equality

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - g_{\lambda\mu} \Gamma^\lambda_{\sigma\nu} - g_{\lambda\nu} \Gamma^\lambda_{\sigma\mu} = 0 \,, \tag{A.26}$$

we can write the Lie derivative in the covariant form,

$$\delta_{\mathrm{L}} g_{\mu\nu} = -g_{\mu\sigma} \nabla_\nu \delta x^\sigma - g_{\nu\sigma} \nabla_\mu \delta x^\sigma \,. \tag{A.27}$$

Substituting this expression in integral (A.24), we obtain

$$\int_\Omega \mathrm{d}^4 x \, \frac{\delta\sqrt{g}H}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \nabla_\nu \delta x^\sigma = 0 \,. \tag{A.28}$$

Equation (A.28) can be written as

$$\int_\Omega \left[\nabla_\nu \left(\frac{\delta\sqrt{g}H}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \delta x^\sigma \right) - \delta x^\sigma \nabla_\nu \left(\frac{\delta\sqrt{g}H}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \right) \right] \mathrm{d}^4 x = 0 \,. \tag{A.29}$$

We note that

$$\nabla_\nu \left(\frac{\delta\sqrt{g}H}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \delta x^\sigma \right) = \partial_\nu \left(\frac{\delta\sqrt{g}H}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \delta x^\sigma \right). \tag{A.30}$$

Due to (A.30), the integral of the first term in the left-hand side of (A.29) vanishes and Eqn (A.29) becomes

$$\int_\Omega \delta x^\sigma \, \nabla_\nu G^\nu_\sigma \, \mathrm{d}^4 x = 0 \,. \tag{A.31}$$

Here, in accordance with definition (A.17), we introduce the mixed tensor

$$\sqrt{g} G^\nu_\sigma = \frac{\delta\sqrt{g}H}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \,.$$

Because the vector δx^σ is arbitrary, we find the desired Hilbert identity

$$\nabla_\nu G^\nu_\sigma \equiv 0 \tag{A.32}$$

or, in more detail,

$$\nabla_\nu G^\nu_\sigma = \partial_\nu G^\nu_\sigma - \Gamma^\lambda_{\sigma\nu} G^\nu_\lambda + \Gamma^\nu_{\nu\lambda} G^\lambda_\sigma \equiv 0 \,. \tag{A.33}$$

Taking the expressions

$$\Gamma^\lambda_{\sigma\nu} = \frac{1}{2} \, g^{\lambda\rho} (\partial_\sigma g_{\nu\rho} + \partial_\nu g_{\sigma\rho} - \partial_\rho g_{\sigma\nu}) \,, \qquad \partial_\lambda \sqrt{g} = \sqrt{g} \, \Gamma^\nu_{\nu\lambda} \,, \tag{A.34}$$

into account, we find

$$\nabla_\rho (\sqrt{g} \, G^\rho_\rho) = \partial_\rho (\sqrt{g} \, G^\rho_\rho) + \frac{1}{2} \sqrt{g} \, G_{\lambda\sigma} \partial_\rho \, g^{\lambda\sigma} \equiv 0 \,. \tag{A.35}$$

This identity was first obtained by Hilbert in 1915.

Applying this identity to the invariant $H = R$, where R is the scalar curvature, Hilbert obtained the Bianchi identity

$$\nabla_\nu \left(R^{\mu\nu} - \frac{1}{2} \, g^{\mu\nu} R \right) \equiv 0 \,. \tag{A.36}$$

A detailed account of this is given in the main text of this article.

We now apply Theorem II to the invariant L, which depends on A_ν, $\partial_\lambda A_\nu$, $g_{\mu\nu}$, and $\partial_\lambda g_{\mu\nu}$. From (A.22), we have that

$$P_g(\sqrt{g}L) = \frac{\delta\sqrt{g}L}{\delta g_{\mu\nu}} \, \delta_\mathrm{L} \, g_{\mu\nu} + \partial_\lambda S^\lambda_1 \,, \tag{A.37}$$

where

$$S^\lambda_1 = \frac{\partial\sqrt{g}L}{\partial(\partial_\lambda g_{\mu\nu})} \, \delta_\mathrm{L} g_{\mu\nu} \,. \tag{A.38}$$

Likewise,

$$P_q(\sqrt{g}L) = \frac{\delta\sqrt{g}L}{\delta A_\lambda} \, \delta_\mathrm{L} A_\lambda + \partial_\lambda S^\lambda_2 \,, \tag{A.39}$$

where

$$S^\lambda_2 = \frac{\partial\sqrt{g}L}{\partial(\partial_\lambda A_\sigma)} \, \delta_\mathrm{L} A_\sigma \,. \tag{A.40}$$

It follows from (A.15), (A.37), and (A.39) that

$$\int_\Omega \left[\frac{\delta\sqrt{g}L}{\delta g_{\mu\nu}} \, \delta_\mathrm{L} g_{\mu\nu} + \frac{\delta\sqrt{g}L}{\delta A_\lambda} \, \delta_\mathrm{L} A_\lambda \right] \mathrm{d}^4 x = 0 \,. \tag{A.41}$$

We now find the Lie variation of the field variable A_λ. According to the transformation law for the vector A_λ, we have

$$A'_\lambda(x') = \frac{\partial x^\nu}{\partial x'^\lambda} \, A_\nu(x) \,. \tag{A.42}$$

Hence, for transformation (A.3), we find

$$A'_\lambda(x + \delta x) = A_\lambda(x) - A_\nu(x) \partial_\lambda \delta x^\nu \,. \tag{A.43}$$

Expanding the left-hand side in the Taylor series, we obtain

$$\delta_\mathrm{L} A_\lambda = A'_\lambda(x) - A_\lambda(x) = -\delta x^\nu \, \partial_\nu A_\lambda - A_\nu(x) \, \partial_\lambda \delta x^\nu \,, \tag{A.44}$$

or, in the covariant form,

$$\delta_\mathrm{L} A_\lambda = -\delta x^\sigma \nabla_\sigma A_\lambda - A_\sigma \nabla_\lambda \delta x^\sigma \,. \tag{A.45}$$

Substituting (A.27) and (A.45) in (A.41), we find

$$\int_\Omega \mathrm{d}^4 x \left[2\nabla_\nu \left(\frac{\delta\sqrt{g}L}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \right) - \frac{\delta\sqrt{g}L}{\delta A_\lambda} \, \nabla_\sigma A_\lambda + \nabla_\lambda \left(\frac{\delta\sqrt{g}L}{\delta A_\lambda} \, A_\sigma \right) \right] \delta x^\sigma = 0 \,. \tag{A.46}$$

Because the transformation vector δx^σ is arbitrary, we obtain the identity

$$2\nabla_\nu \left(\frac{\delta\sqrt{g}L}{\delta g_{\mu\nu}} \, g_{\mu\sigma} \right) = (\nabla_\sigma A_\lambda - \nabla_\lambda A_\sigma) \, \frac{\delta\sqrt{g}L}{\delta A_\lambda} - A_\sigma \nabla_\lambda \left(\frac{\delta\sqrt{g}L}{\delta A_\lambda} \right). \tag{A.47}$$

According to Hilbert, the energy-momentum tensor density is defined by the expression

$$T^{\mu\nu} = -2 \, \frac{\delta\sqrt{g}L}{\delta g_{\mu\nu}} \,. \tag{A.48}$$

Identity (A.47) assumes the form

$$\nabla_\nu T^\nu_\sigma = A_\sigma \nabla_\lambda \left(\frac{\delta\sqrt{g}L}{\delta A_\lambda} \right) + (\nabla_\lambda A_\sigma - \nabla_\sigma A_\lambda) \, \frac{\delta\sqrt{g}L}{\delta A_\lambda} \,, \tag{A.49}$$

or

$$\nabla_\nu T^\nu_\sigma = A_\sigma \partial_\lambda \left(\frac{\delta\sqrt{g}L}{\delta A_\lambda} \right) + (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda) \, \frac{\delta\sqrt{g}L}{\delta A_\lambda} \,. \tag{A.50}$$

When the gravitational equations hold, Theorem III leads to the equality

$$\nabla_\nu T^\nu_\sigma = 0\,, \tag{A.51}$$

and, hence, identity (A.50) transforms into the equation assigned number (28) by Hilbert in Ref. [6]:

$$A_\sigma \partial_\lambda \left(\frac{\delta\sqrt{g}L}{\delta A_\lambda} \right) + (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)\, \frac{\delta\sqrt{g}L}{\delta A_\lambda} = 0\,. \tag{A.52}$$

But this equation always holds due to Hilbert's Axiom I, because

$$\frac{\delta\sqrt{g}L}{\delta A_\lambda} = 0\,. \tag{A.53}$$

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Authors' addition to the English translation

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