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Electromagnetic wave propagation in a randomly inhomogeneous medium as a problem in mathematical statistical physics

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Contents

Abstract. **The major stages of how mathematical statistical physics has been used in the last fifty years to describe random medium electromagnetic wave (light) propagation are outlined. The statistical description is discussed either in terms of the scalar parabolic equation (quasioptical approximation) — when the governing parameters are needed, or by writing its functional integral solution — if the caustic structure of the wave field is to be analyzed.**

1. Introduction

The year 2003 marks the 50th anniversary of the publication of a widely known work by A M Obukhov [1] in which for the first time he considered diffraction effects associated with wave propagation in random media in the framework of perturbation theory. Earlier, similar studies had been carried out in the geometrical optics (acoustics) approximation. The method proposed in Obukhov's work has not lost its value so far. It provides a basic mathematical tool for a variety of technical applications. However, it was shown experimentally by later authors [2, 3] that wave field fluctuations grow rapidly with increasing distance as waves (light) propagate in a medium with random large-scale (compared with the wave length) inhomogeneities due to the effect of multiple forward scattering. Starting from a certain path length, calculations based on the perturbation theory become unacceptable regardless of its form (strong fluctuation regime). Monographs [4 – 13] and review papers [14 – 18] reflected the general state of the wave propagation theory in randomly inhomogeneous media at the time of their publication. These works contain a lengthy bibliography. In what follows, we focus on the main stages of the description of wave propagation in randomly inhomogeneous media from the standpoint of mathematical statistical physics.

Key issues in the description of deterministic problems appear to be

• transition from the Maxwell vector equations to the Leontovich scalar parabolic equation (quasioptical approximation [19]);

• the method of smooth perturbations (MSP) proposed by S M Rytov [20] for amplitude and phase fluctuations with diffraction effects taken into account (employed in the aforementioned work of A M Obukhov [1]);

• description of the solution of the parabolic equation in the operator form or in the form of a path integral [21, 22].

In our opinion, the following steps in the development of the theory are of primary importance for the statistical description of the problem:

• development of a statistical theory for the description of amplitude and phase fluctuations in the framework of the first MSP approximation for a random phase screen and in the case of a continuously distributed random medium;

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• development of a statistical theory for the wave field description in the framework of the approximation of dielectric permittivity fluctuations delta-correlated along the direction of wave propagation (equations for the characteristic wave field functional and coherence functions of different orders and for the derivation of the probability distribution law for wave field intensity fluctuations) [13, 23–25];

• analysis of the wave field caustic structure in a randomly inhomogeneous medium based on the statistical topography approach [26];

• analysis of statistical characteristics of reflected waves in random media [27, 28];

• going beyond the framework of the approximation of delta-correlated fluctuations of dielectric permittivity (diffusion approximation [13], equivalent to Chernov's local method [6] for the given problem).

These issues are briefly discussed in what follows.

2. Original stochastic equations and some corollaries thereof

2.1 Maxwell equations for a stationary problem

Propagation of a monochromatic electromagnetic wave of frequency ω in a stationary inhomogeneous medium is described by the Maxwell equations (see, e.g., Ref. [4])

$$\begin{aligned} &\operatorname{rot}\mathbf{E}(\mathbf{r}) = \mathrm{i}k\mathbf{H}(\mathbf{r})\,, \\ &\operatorname{rot}\mathbf{H}(\mathbf{r}) = -\mathrm{i}k\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\,, \\ &\operatorname{div}\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0\,, \end{aligned} \tag{1}$$

where $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are the electric and magnetic field strengths, respectively, $\varepsilon(\mathbf{r})$ is the dielectric permittivity of the medium, and $k = \omega/c = 2\pi/\lambda$ is the wave number (λ is the wave length and c is the wave propagation velocity). It is assumed that the magnetic permeability $\mu = 1$, the medium conductivity $\sigma = 0$, and time dependence of all the fields has the form $\exp(-\mathrm{i}\omega t)$.

Equations (1) can be rewritten as a closed-form equation for the electric field $\mathbf{E}(\mathbf{r})$,

$$\left[\Delta + k^2\varepsilon(\mathbf{r})\right]\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\big(\mathbf{E}(\mathbf{r})\boldsymbol{\nabla}\ln\varepsilon(\mathbf{r})\big)\,. \tag{2}$$

The magnetic field $\mathbf{H}(\mathbf{r})$ is then computed using equality

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mathrm{i}k}\operatorname{rot}\mathbf{E}(\mathbf{r})\,. \tag{3}$$

We are interested in the propagation of electromagnetic waves in a medium with weak dielectric permittivity fluctuations. We assume that

$$\varepsilon(\mathbf{r}) = 1 + \varepsilon_1(\mathbf{r})\,,$$

where $\varepsilon_1(\mathbf{r})$ is the fluctuating component of the dielectric permittivity ($\langle\varepsilon_1(\mathbf{r})\rangle = 0$). The smallness of $\varepsilon_1(\mathbf{r})$ means that $\langle|\varepsilon_1(\mathbf{r})|\rangle \ll 1$. Therefore, Eqn (2) can be written in a simplified form

$$[\Delta + k^2]\mathbf{E}(\mathbf{r}) = -k^2\varepsilon_1(\mathbf{r})\mathbf{E}(\mathbf{r}) - \boldsymbol{\nabla}\big(\mathbf{E}(\mathbf{r})\boldsymbol{\nabla}\varepsilon_1(\mathbf{r})\big)\,. \tag{4}$$

In Refs [29, 30], the perturbation theory was used to evaluate depolarization of a light wave in the real atmosphere at lengths of the order of 1 km; it was shown that depolarization is very small, and therefore the last term in the right-hand side of Eqn (4) can be ignored. This allows passing to the scalar equation

$$[\Delta + k^2]U(\mathbf{r}) = -k^2\varepsilon_1(\mathbf{r})U(\mathbf{r})\,. \tag{5}$$

It is necessary to formulate the boundary conditions for Eqn (5) and identify the source of radiation.

2.2 Helmholtz equation (boundary value problem)

We now assume that an inhomogeneous medium layer occupies a part of space $L_0 < x < L$ and that the point source has coordinates $(x_0, \mathbf{R}_0)$, where $\mathbf{R}$ denotes coordinates in the plane perpendicular to the x axis. The wave field $G(x, \mathbf{R}; x_0, \mathbf{R}_0)$ inside the layer is then described by the equation for the Green's function

$$\begin{aligned} &\left\{\frac{\partial^2}{\partial x^2} + \Delta_{\mathbf{R}} + k^2\big[1 + \varepsilon(x, \mathbf{R})\big]\right\} G(x, \mathbf{R}; x_0, \mathbf{R}_0) \\ &\quad = \delta(x - x_0)\,\delta(\mathbf{R} - \mathbf{R}_0)\,, \end{aligned} \tag{6}$$

where k is the wave number, $\Delta_{\mathbf{R}} = \partial^2/\partial\mathbf{R}^2$, and $\varepsilon(x, \mathbf{R}) = \varepsilon_1(\mathbf{r})$ is the deviation of the dielectric permittivity from unity. Equation (6) implies the condition for a derivative jump at x_0,

$$\begin{aligned} &\frac{\partial}{\partial x} G(x, \mathbf{R}; x_0, \mathbf{R}_0)\bigg|_{x=x_0+0} - \frac{\partial}{\partial x} G(x, \mathbf{R}; x_0, \mathbf{R}_0)\bigg|_{x=x_0-0} \\ &\quad = \delta(\mathbf{R} - \mathbf{R}_0)\,. \end{aligned} \tag{7}$$

We suppose that $\varepsilon(x, \mathbf{R}) = 0$ outside the medium layer; the wave field outside the layer is then described by the Helmholtz equation

$$\left\{\frac{\partial^2}{\partial x^2} + \Delta_{\mathbf{R}} + k^2\right\} G(x, \mathbf{R}; x_0, \mathbf{R}_0) = 0\,.$$

The continuity conditions for the functions G and $\partial G/\partial x$ must be satisfied at layer boundaries. In addition, radiation conditions must also be satisfied for Eqn (6) as $x \to \pm\infty$. These boundary conditions can be written as

$$\begin{aligned} &\left(\frac{\partial}{\partial x} + \mathrm{i}\sqrt{k^2 + \Delta_{\mathbf{R}}}\right) G(x, \mathbf{R}; x_0, \mathbf{R}_0)\bigg|_{x=L_0} = 0\,, \\ &\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2 + \Delta_{\mathbf{R}}}\right) G(x, \mathbf{R}; x_0, \mathbf{R}_0)\bigg|_{x=L} = 0\,. \end{aligned} \tag{8}$$

For a space that is boundless in $\mathbf{R}$, the operator $\sqrt{k^2 + \Delta_{\mathbf{R}}}$ in Eqns (8) is defined by the Fourier transformation. It can be regarded as an integral operator whose kernel is determined by the Green's function for a free space (see below).

Thus, the field of a point source in an inhomogeneous medium is described by the boundary-value problem in Eqns (6) and (8). This problem is equivalent to the integral equation

$$\begin{aligned} &G(x, \mathbf{R}; x_0, \mathbf{R}_0) = g_0(x - x_0, \mathbf{R} - \mathbf{R}_0) \\ &\quad + \int_{L_0}^{L} \mathrm{d}x' \int \mathrm{d}\mathbf{R}'\, g_0(x - x', \mathbf{R} - \mathbf{R}')\,\varepsilon(x', \mathbf{R}')\, G(x', \mathbf{R}'; x_0, \mathbf{R}_0)\,, \end{aligned} \tag{9}$$

where $g_0(x, \mathbf{R})$ is the Green's function in a free space.

In the three-dimensional case,

$$g_0(x,\mathbf{R}) = -\frac{1}{4\pi r}\exp(\mathrm{i}kr)\,, \qquad r=\sqrt{x^2+\mathbf{R}^2}\,, \tag{10}$$

and this function is described by the integral representation

$$g_0(x,\mathbf{R}) = \int g_0(\mathbf{q})\exp\left\{\mathrm{i}\sqrt{k^2-q^2}\,|x| + \mathrm{i}\mathbf{qR}\right\}\mathrm{d}\mathbf{q}\,,$$

$$g_0(\mathbf{q}) = \frac{1}{8\mathrm{i}\pi^2\sqrt{k^2-q^2}}\,.$$

The action of the operator $\sqrt{k^2+\Delta_{\mathbf{R}}}$, in this case on an arbitrary function $F(\mathbf{R})$, can be represented in the form of a linear integral operator,

$$\sqrt{k^2+\Delta_{\mathbf{R}}}\;F(\mathbf{R}) = \int K(\mathbf{R}-\mathbf{R}')\,F(\mathbf{R}')\,\mathrm{d}\mathbf{R}'\,,$$

whose kernel is defined by the equality (see, e.g., Refs [13, 31])

$$K(\mathbf{R}-\mathbf{R}') = \sqrt{k^2+\Delta_{\mathbf{R}}}\;\delta(\mathbf{R}-\mathbf{R}') = 2\mathrm{i}(k^2+\Delta_{\mathbf{R}})\,g_0(0,\mathbf{R}-\mathbf{R}')\,.$$

The corresponding kernel of the inverse operator is defined by

$$L(\mathbf{R}-\mathbf{R}') = (k^2+\Delta_{\mathbf{R}})^{-1/2}\delta(\mathbf{R}-\mathbf{R}') = 2\mathrm{i}g_0(0,\mathbf{R}-\mathbf{R}')\,.$$

If the point source is situated at the boundary of the layer $x_0 = L$, the wave field inside the layer at $L_0 < x < L$ is described by the equation

$$\left\{\frac{\partial^2}{\partial x^2} + \Delta_{\mathbf{R}} + k^2\big[1+\varepsilon(x,\mathbf{R})\big]\right\} G(x,\mathbf{R};L,\mathbf{R}_0) = 0 \tag{11}$$

with the boundary conditions following from conditions (7) and (8),

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) G(x,\mathbf{R};L,\mathbf{R}_0)\bigg|_{x=L_0} = 0\,,$$
$$\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) G(x,\mathbf{R};L,\mathbf{R}_0)\bigg|_{x=L} = -\delta(\mathbf{R}-\mathbf{R}_0)\,. \tag{12}$$

Equivalent to the boundary-value problem in Eqns (11) and (12) is the integral equation

$$G(x,\mathbf{R};L,\mathbf{R}_0) = g(x-L,\mathbf{R}-\mathbf{R}_0) + \int_{L_0}^{L}\mathrm{d}x'\int\mathrm{d}\mathbf{R}'\,g(x-x',\mathbf{R}-\mathbf{R}')\,\varepsilon(x',\mathbf{R}')\,G(x',\mathbf{R}';L,\mathbf{R}_0)\,, \tag{13}$$

corresponding to the point $x_0 = L$ in Eqn (9).

We note that all these boundary-value problems can be reduced to problems with the initial conditions for an auxiliary parameter L taking advantage of the so-called embedding method (see, e.g., Refs [13, 31]). The basic equation for a back-scattered field is then a nonlinear integro-differential equation. A marked simplification is feasible for waves in stratified media (one-dimensional problems) when the equations of the embedding method are transformed into ordinary differential equations with the given initial conditions; this case allows a sufficiently comprehensive statistical analysis [13, 31].

If a wave $u_0(x,\mathbf{R})$ falls onto a layer of the medium $L_0 < x < L$ from the region $x < L_0$ (in the positive direction of the x axis), the wave field $U(x,\mathbf{R})$ inside the layer satisfies the Helmholtz equation

$$\left\{\frac{\partial^2}{\partial x^2} + \Delta_{\mathbf{R}} + k^2\big[1+\varepsilon(x,\mathbf{R})\big]\right\} U(x,\mathbf{R}) = 0 \tag{14}$$

with the boundary conditions

$$\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) U(x,\mathbf{R})\bigg|_{x=L} = 0\,,$$
$$\left(\frac{\partial}{\partial x} + \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) U(x,\mathbf{R})\bigg|_{x=L_0} = 2\mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\;u_0(L,\mathbf{R})\,. \tag{15}$$

The field $U(x,\mathbf{R})$ can be represented as

$$U(x,\mathbf{R}) = u_1(x,\mathbf{R}) + u_2(x,\mathbf{R})\,,$$
$$\frac{\partial}{\partial x}\,U(x,\mathbf{R}) = \mathrm{i}k\sqrt{k^2+\Delta_{\mathbf{R}}}\,\{u_1(x,\mathbf{R}) + u_2(x,\mathbf{R})\}\,, \tag{16}$$

where two functions, $u_1(x,\mathbf{R})$ and $u_2(x,\mathbf{R})$, are considered instead of one, $U(x,\mathbf{R})$. These functions describe waves that propagate in positive and negative directions of the x axis, respectively, and are related to the field $U(x,\mathbf{R})$ by equalities ensuing from (16),

$$u_1(x,\mathbf{R}) = -\frac{\mathrm{i}}{2\sqrt{k^2+\Delta_{\mathbf{R}}}}\left(\frac{\partial}{\partial x} + \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) U(x,\mathbf{R})\,,$$
$$u_2(x,\mathbf{R}) = \frac{\mathrm{i}}{2\sqrt{k^2+\Delta_{\mathbf{R}}}}\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) U(x,\mathbf{R})\,. \tag{17}$$

Differentiation of (17) with respect to x and the use of Eqn (14) yield a system of equations for the functions $u_1(x,\mathbf{R})$ and $u_2(x,\mathbf{R})$ with the boundary conditions ensuing from (15) that have the form [32]

$$\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) u_1(x,\mathbf{R}) = \frac{\mathrm{i}k^2}{2\sqrt{k^2+\Delta_{\mathbf{R}}}}\,\varepsilon(x,\mathbf{R})\,U(x,\mathbf{R})\,,$$
$$\left(\frac{\partial}{\partial x} + \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) u_2(x,\mathbf{R}) = -\frac{\mathrm{i}k^2}{2\sqrt{k^2+\Delta_{\mathbf{R}}}}\,\varepsilon(x,\mathbf{R})\,U(x,\mathbf{R})\,,$$
$$u_1(L_0,\mathbf{R}) = u_0(L,\mathbf{R})\,, \qquad u_2(L,\mathbf{R}) = 0\,. \tag{18}$$

The function $u_2(x,\mathbf{R})$ describes a wave propagating in the direction opposite to that of the incident wave, i.e., a back-scattered field.

2.3 Quasioptical parabolic equation

Ignoring the effects associated with back scattering, that is, setting $u_2(x,\mathbf{R}) = 0$ in Eqns (18), we obtain the generalized parabolic equation

$$\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2+\Delta_{\mathbf{R}}}\right) U(x,\mathbf{R}) = \frac{\mathrm{i}k^2}{2\sqrt{k^2+\Delta_{\mathbf{R}}}}\,\varepsilon(x,\mathbf{R})U(x,\mathbf{R})\,,$$
$$U(L_0,\mathbf{R}) = u_0(L_0,\mathbf{R})\,, \tag{19}$$

which allows scattering at different angles (smaller than $\pi/2$, however) to be described. In the case of small-angle scattering

($\Delta_{\mathbf{R}} \ll k^2$), substitution of the field $U(x,\mathbf{R})$ in the form

$$U(x,\mathbf{R}) = \exp\left\{\mathrm{i}k(x-L_0)\right\} u(x,\mathbf{R})$$

in Eqn (19) yields an approximate quasioptical parabolic equation adequate for the description of waves in a medium with large-scale three-dimensional inhomogeneities,

$$\frac{\partial}{\partial x}\, u(x,\mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}} u(x,\mathbf{R}) + \frac{\mathrm{i}k}{2}\,\varepsilon(x,\mathbf{R}) u(x,\mathbf{R})\,, \qquad u(0,\mathbf{R}) = u_0(\mathbf{R})\,, \tag{20}$$

This equation has been successfully applied to the solution of many problems concerning wave propagation in the Earth atmosphere and oceans.

We note that both derivation and substantiation of the parabolic equation itself and the generalized parabolic equations have been considered in many publications.

Because Eqn (20) is a first-order equation in x with the initial condition at $x=0$, the causality condition in x (with the coordinate x playing the role of time) is fulfilled for (20); in other words, the following relation holds for the variational derivative:

$$\frac{\delta u(x,\mathbf{R})}{\delta\varepsilon(x',\mathbf{R}')} = 0 \quad \text{at} \quad x'<0\,, \quad x'>x\,. \tag{21}$$

For the variational derivative at $x = x'$, we have the equality

$$\frac{\delta u(x,\mathbf{R})}{\delta\varepsilon(x-0,\mathbf{R}')} = \frac{\mathrm{i}k}{2}\,\delta(\mathbf{R}-\mathbf{R}')\, u(x,\mathbf{R})\,. \tag{22}$$

In the general case, the quantity $\delta u(x,\mathbf{R})/\delta\varepsilon(x',\mathbf{R}')$ at $0 \leqslant x' < x$ can be expressed through the Green's function of Eqn (20) using the relation

$$\frac{\delta u(x,\mathbf{R})}{\delta\varepsilon(x',\mathbf{R}')} = \frac{\mathrm{i}k}{2}\, G(x,\mathbf{R};x',\mathbf{R}')\, u(x',\mathbf{R}')\,.$$

The Green's function $G(x,\mathbf{R};x',\mathbf{R}')$ satisfies the integral equation

$$G(x,\mathbf{R};x',\mathbf{R}') = g(x-x',\mathbf{R}-\mathbf{R}') + \frac{\mathrm{i}k}{2}\int_{x'}^{x} \mathrm{d}x'' \int \mathrm{d}\mathbf{R}''\, g(x-x'',\mathbf{R}-\mathbf{R}'') \times \varepsilon(x'',\mathbf{R}'')\, G(x'',\mathbf{R}'';x',\mathbf{R}')\,, \tag{23}$$

where the function

$$g(x-x',\mathbf{R}-\mathbf{R}') = \exp\left\{\frac{\mathrm{i}(x-x')}{2k}\,\Delta_{\mathbf{R}}\right\}\delta(\mathbf{R}-\mathbf{R}') = \frac{k}{2\pi\mathrm{i}(x-x')}\exp\left\{\frac{\mathrm{i}k(\mathbf{R}-\mathbf{R}')^2}{2(x-x')}\right\}$$

at $x > x'$ is the Green's function for Eqn (20) in the absence of inhomogeneities corresponding to the Fresnel expansion of the Green's function $g_0(x,\mathbf{R})$ in Eqn (10). As $x \to x'$, Eqn (23) becomes

$$G(x,\mathbf{R};x',\mathbf{R}')\Big|_{x\to x'} = g(x-x',\mathbf{R}-\mathbf{R}')\Big|_{x\to x'} = \delta(\mathbf{R}-\mathbf{R}')\,.$$

It should be noted that the Green's function $G(x,\mathbf{R};x',\mathbf{R}')$ describes the field of a spherical wave propagating from the point $(x',\mathbf{R}')$.

Integral equation (23) can be written in an equivalent form as a functional equation in variational derivatives,

$$\frac{\delta G(x,\mathbf{R};x',\mathbf{R}')}{\delta\varepsilon(\xi,\mathbf{R}_1)} = \frac{\mathrm{i}k}{2}\, G(x,\mathbf{R};\xi,\mathbf{R}_1)\, G(\xi,\mathbf{R}_1;x',\mathbf{R}')$$

with the 'initial' functional condition

$$G(x,\mathbf{R};x',\mathbf{R}')\Big|_{\varepsilon=0} = g(x-x',\mathbf{R}-\mathbf{R}')\,.$$

Introducing the wave field amplitude and phase in Eqn (20) in accordance with the formula

$$u(x,\mathbf{R}) = A(x,\mathbf{R})\exp\left\{\mathrm{i}S(x,\mathbf{R})\right\}$$

allows writing the equation for the wave field intensity $I(x,\mathbf{R}) = u(x,\mathbf{R})u^*(x,\mathbf{R})$ as

$$\frac{\partial}{\partial x}\, I(x,\mathbf{R}) + \frac{1}{k}\,\nabla_{\mathbf{R}}\left\{\nabla_{\mathbf{R}} S(x,\mathbf{R})\, I(x,\mathbf{R})\right\} = 0\,, \qquad I(0,\mathbf{R}) = I_0(\mathbf{R})\,. \tag{24}$$

It follows that in the general case of an arbitrary incident beam of waves, the wave power in the plane $x = \text{const}$ is conserved:

$$E_0 = \int I(x,\mathbf{R})\,\mathrm{d}\mathbf{R} = \int I_0(\mathbf{R})\,\mathrm{d}\mathbf{R}\,.$$

Equation (24) can be interpreted as a transfer equation for a conservative admixture in a potential velocity field, for which the cluster structure of the admixture field is known to arise (see, e.g., Refs [13, 33–35]). Hence, the realization of the wave intensity field must also have a cluster structure. In the case under consideration, this phenomenon is manifested in the form of caustic structures due to the effects of random focusing and defocusing of the wave field in a random medium. By way of example, photographs in Fig. 1 show cross sections of a transverse laser beam propagating in a turbulent medium under laboratory conditions [36] at different intensities of dielectric permittivity fluctuations. Similar photographs borrowed from Ref. [15] are presented in Fig. 2. These photographs have been obtained by numerical simulation described in Refs [37, 38]. The figures illustrate formation of wave field caustic structures. Figure 3 shows a pool with a clearly visible caustic structure of the wave field at the bottom. Such structures arise when there are refraction and reflection of light by a disturbed water surface; this phenomenon corresponds to scattering from the so-called phase screen.

2.3.1 Path-integral form of parabolic equation solution. The solution of parabolic equation (20) can be written in the form of a path integral. For this purpose, instead of (20), we consider the equation

$$\frac{\partial}{\partial x}\, u(x,\mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}} u(x,\mathbf{R}) + \frac{\mathrm{i}k}{2}\,\varepsilon(x,\mathbf{R})\, u(x,\mathbf{R}) + \big(\mathbf{v}(x)\nabla_{\mathbf{R}}\big)\, u(x,\mathbf{R})\,, \qquad u(0,\mathbf{R}) = u_0(\mathbf{R})\,, \tag{25}$$

where an arbitrary vector function $\mathbf{v}(x)$ is introduced. The solution of Eqn (25) is a functional of the function $\mathbf{v}(\xi)$, where

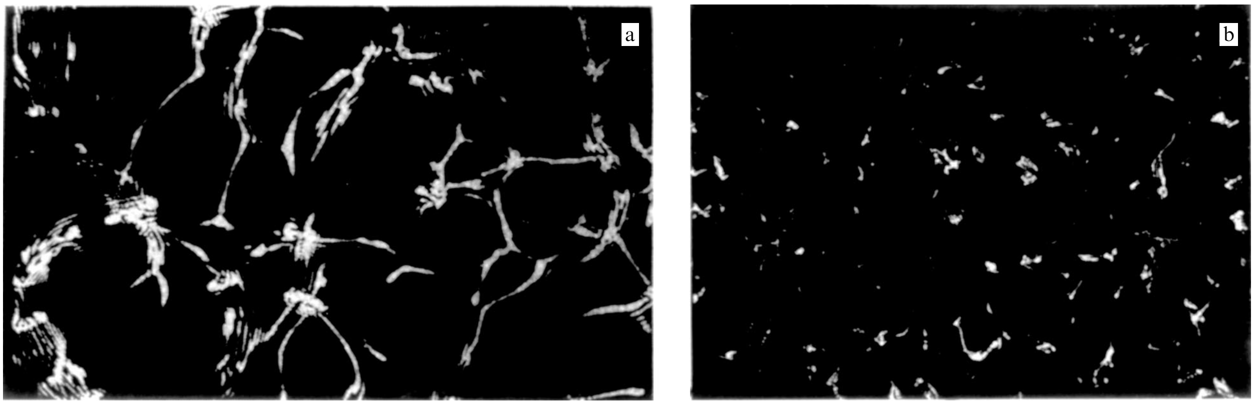

Figure 1. Cross section of a laser beam in a turbulent medium in the strong focusing (a) and strong fluctuation (b) regions (laboratory experiment).

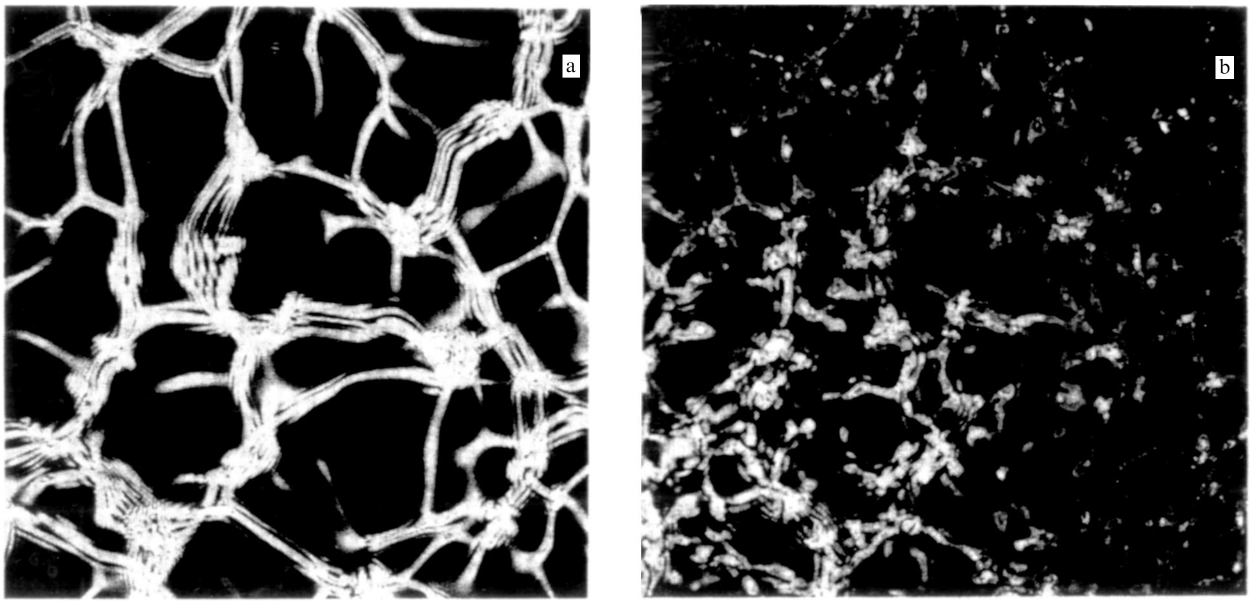

Figure 2. Cross section of a laser beam in a turbulent medium in the strong focusing (a) and strong fluctuation (b) regions (numerical simulation).

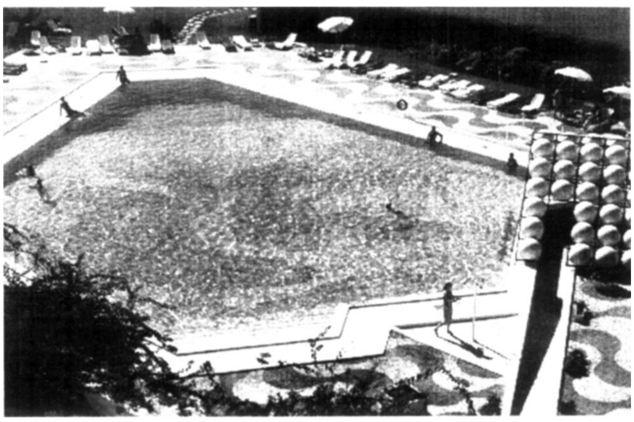

Figure 3. Caustic structure in a swimming pool.

$0 \leqslant \xi \leqslant x$, i.e., $u(x, \mathbf{R}) = u[x, \mathbf{R}, \mathbf{v}(\xi)]$. Then,

$$u(x, \mathbf{R}) \equiv u[x, \mathbf{R}]\big|_{\mathbf{v}=0} .$$

It follows from Eqn (25) that

$$\frac{\delta u(x, \mathbf{R})}{\delta \mathbf{v}(x-0)} = \nabla_{\mathbf{R}}\, u(x, \mathbf{R}) . \tag{26}$$

Hence, Eqn (25) can be rewritten as

$$\frac{\partial}{\partial x}\, u(x, \mathbf{R}) = \frac{\mathrm{i}}{2k}\, \frac{\delta^2}{\delta \mathbf{v}^2(x)}\, u(x, \mathbf{R}) + \frac{\mathrm{i}k}{2}\, \varepsilon(x, \mathbf{R})\, u(x, \mathbf{R}) + \big(\mathbf{v}(x)\nabla_{\mathbf{R}}\big)\, u(x, \mathbf{R}) .$$

The solution of this equation has the form

$$u(x, \mathbf{R}) = \exp\left\{\frac{\mathrm{i}}{2k}\int_0^x \frac{\delta^2}{\delta \mathbf{v}^2(\xi)}\, \mathrm{d}\xi\right\} w(x, \mathbf{R}) , \tag{27}$$

where $w(x, \mathbf{R})$ satisfies the equation

$$\frac{\partial}{\partial x}\, w(x, \mathbf{R}) = \frac{\mathrm{i}k}{2}\, \varepsilon(x, \mathbf{R}) w(x, \mathbf{R}) + \big(\mathbf{v}(x)\nabla_{\mathbf{R}}\big) w(x, \mathbf{R}) , \qquad w(0, \mathbf{R}) = u_0(\mathbf{R}) , \tag{28}$$

which is simpler than (25).

Thus, the solution of Eqn (20) can be written in the operator form as

$$u(x,\mathbf{R}) = \exp\left\{\frac{\mathrm{i}}{2k}\int_0^x \frac{\delta^2}{\delta\mathbf{v}^2(\xi)}\,\mathrm{d}\xi\right\} u_0\left(\mathbf{R}+\int_0^x \mathbf{v}(\xi)\,\mathrm{d}\xi\right)$$
$$\times\exp\left\{\frac{\mathrm{i}k}{2}\int_0^x \varepsilon\left(\xi,\mathbf{R}+\int_\xi^x \mathbf{v}(\eta)\,\mathrm{d}\eta\right)\mathrm{d}\xi\right\}\bigg|_{\mathbf{v}(x)=0}. \quad (29)$$

For a plane incident wave, $u_0(\mathbf{R}) = u_0$ and Eqn (29) is simplified:

$$u(x,\mathbf{R}) = u_0\exp\left\{\frac{\mathrm{i}}{2k}\int_0^x \frac{\delta^2}{\delta\mathbf{v}^2(\xi)}\,\mathrm{d}\xi\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}k}{2}\int_0^x \varepsilon\left(\xi,\mathbf{R}+\int_\xi^x \mathbf{v}(\eta)\,\mathrm{d}\eta\right)\mathrm{d}\xi\right\}\bigg|_{\mathbf{v}(x)=0}.$$

Equation (29) can also be rewritten in the form of a path integral. For this purpose, the function $\mathbf{v}(x)$ in Eqn (28) should be formally considered a stochastic Gaussian process with the correlation function

$$\langle v_\alpha(x)v_\beta(x')\rangle = \frac{\mathrm{i}}{k}\,\delta_{\alpha\beta}\delta(x-x'). \quad (30)$$

The ensemble average of Eqn (28) over random function $\mathbf{v}(x)$ with the aid of expression (26) and the Furutsu–Novikov formula of type (48) (see Section 3.3) then yields the closed-form equation

$$\frac{\partial}{\partial x}\langle w(x,\mathbf{R})\rangle_{\mathbf{v}} = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}}\langle w(x,\mathbf{R})\rangle_{\mathbf{v}} + \frac{\mathrm{i}k}{2}\,\varepsilon(x,\mathbf{R})\langle w(x,\mathbf{R})\rangle_{\mathbf{v}},$$
$$\langle w(0,\mathbf{R})\rangle_{\mathbf{v}} = u_0(\mathbf{R}),$$

which coincides with Eqn (20). Hence,

$$u(x,\mathbf{R}) = \int D\mathbf{v}(x)\,u_0\left(\mathbf{R}+\int_0^x \mathbf{v}(\xi)\,\mathrm{d}\xi\right)$$
$$\times\exp\left\{\frac{\mathrm{i}k}{2}\int_0^x\left[\mathbf{v}^2(\xi)+\varepsilon\left(\xi,\mathbf{R}+\int_\xi^x \mathbf{v}(\eta)\,\mathrm{d}\eta\right)\right]\mathrm{d}\xi\right\} \quad (31)$$

with the integral measure

$$D\mathbf{v}(x) = \frac{\prod_{\xi=0}^x \mathrm{d}\mathbf{v}(\xi)}{\int\ldots\int\prod_{\xi=0}^x \mathrm{d}\mathbf{v}(\xi)\exp\left\{(\mathrm{i}k/2)\int_0^x \mathbf{v}^2(\xi)\,\mathrm{d}\xi\right\}}.$$

Evidently, the two forms of the solution in Eqns (29) and (31) are equivalent. Indeed, it is possible to rewrite Eqn (31) as

$$u(x,\mathbf{R}) = \langle w[x,\mathbf{R};\mathbf{v}(\xi)+\mathbf{y}(\xi)]\rangle_{\mathbf{v}}\big|_{\mathbf{y}=0}$$
$$= \left\langle\exp\left\{\int_0^x \mathbf{v}(\xi)\,\frac{\delta}{\delta\mathbf{y}(\xi)}\,\mathrm{d}\xi\right\}\right\rangle_{\mathbf{v}} w[x,\mathbf{R};\mathbf{y}(\xi)]\bigg|_{\mathbf{y}=0}$$
$$= \exp\left\{\frac{\mathrm{i}}{2k}\int_0^x\frac{\delta^2}{\delta\mathbf{y}^2(\xi)}\,\mathrm{d}\xi\right\} w[x,\mathbf{R};\mathbf{y}(\xi)]\bigg|_{\mathbf{y}=0},$$

which coincides with expression (27).

2.3.2 Hopf equation. We now consider the functional

$$\varphi[x;v(\mathbf{R}'),v^*(\mathbf{R}')] = \varphi[x;v,v^*]$$
$$= \exp\left\{\mathrm{i}\int\left[u(x,\mathbf{R}')v(\mathbf{R}')+u^*(x,\mathbf{R}')v^*(\mathbf{R}')\right]\mathrm{d}\mathbf{R}'\right\}, \quad (32)$$

where the wave field $u(x,\mathbf{R})$ is a solution of Eqn (20) and $u^*(x,\mathbf{R})$ is the complex conjugate function. Differentiating (32) with respect to x and using dynamic equation (20) along with its complex conjugate yields the equality

$$\frac{\partial}{\partial x}\,\varphi[x;v,v^*]$$
$$= -\frac{1}{2k}\int\left[v(\mathbf{R})\Delta_{\mathbf{R}}u(x,\mathbf{R}) - v^*(\mathbf{R})\Delta_{\mathbf{R}}u^*(x,\mathbf{R})\right]\varphi[x;v,v^*]\,\mathrm{d}\mathbf{R}$$
$$-\frac{k}{2}\int\varepsilon(x,\mathbf{R})\left[v(\mathbf{R})u(x,\mathbf{R}) - v^*(\mathbf{R})u^*(x,\mathbf{R})\right]\varphi[x;v,v^*]\,\mathrm{d}\mathbf{R},$$

which can be written in the form of an equation in variational derivatives,

$$\frac{\partial}{\partial x}\,\varphi[x;v,v^*] = \frac{\mathrm{i}k}{2}\int\varepsilon(x,\mathbf{R})\widehat{M}(\mathbf{R})\varphi[x;v,v^*]\,\mathrm{d}\mathbf{R}$$
$$+\frac{\mathrm{i}}{2k}\int\left[v(\mathbf{R})\Delta_{\mathbf{R}}\,\frac{\delta}{\delta v(\mathbf{R})} - v^*(\mathbf{R})\Delta_{\mathbf{R}}\,\frac{\delta}{\delta v^*(\mathbf{R})}\right]\varphi[x;v,v^*]\,\mathrm{d}\mathbf{R} \quad (33)$$

with the Hermitian operator

$$\widehat{M}(\mathbf{R}) = v(\mathbf{R})\,\frac{\delta}{\delta v(\mathbf{R})} - v^*(\mathbf{R})\,\frac{\delta}{\delta v^*(\mathbf{R})}.$$

Equation (33) is equivalent to the original equation (20). A corollary of Eqn (33) is the expression for the variational derivative

$$\frac{\delta}{\delta\varepsilon(x-0,\mathbf{R})}\,\varphi[x;v,v^*] = \frac{\mathrm{i}k}{2}\,\widehat{M}(\mathbf{R})\varphi[x;v,v^*]. \quad (34)$$

3. Statistical averaging

We now consider the statistical description of the wave field.

3.1 General case of arbitrary statistics of the field $\varepsilon(x,\mathbf{R})$

We assume that the random field $\varepsilon(x,\mathbf{R})$ is a homogeneous isotropic random field with the characteristic functional

$$\Phi_\varepsilon[x;\psi(\xi,\mathbf{R}')] = \left\langle\exp\left\{\mathrm{i}\int_0^x \mathrm{d}\xi\int \mathrm{d}\mathbf{R}'\,\varepsilon(\xi,\mathbf{R}')\psi(\xi,\mathbf{R}')\right\}\right\rangle.$$

Instead of (33), we consider a more general equation

$$\frac{\partial}{\partial x}\,\varphi[x;v,v^*;\eta]$$
$$= \frac{\mathrm{i}k}{2}\int\left[\varepsilon(x,\mathbf{R})+\eta(x,\mathbf{R})\right]\widehat{M}(\mathbf{R})\varphi[x;v,v^*;\eta]\,\mathrm{d}\mathbf{R}$$
$$+\frac{\mathrm{i}}{2k}\int\left[v(\mathbf{R})\Delta_{\mathbf{R}}\,\frac{\delta}{\delta v(\mathbf{R})} - v^*(\mathbf{R})\Delta_{\mathbf{R}}\,\frac{\delta}{\delta v^*(\mathbf{R})}\right]\varphi[x;v,v^*;\eta]\,\mathrm{d}\mathbf{R} \quad (35)$$

with an arbitrary function $\eta(x,\mathbf{R})$. We then take the ensemble average of Eqn (35) over the random field $\varepsilon(x,\mathbf{R})$. For the characteristic functional of the solution of Eqn (20), supplemented by an arbitrary function $\eta(x,\mathbf{R})$,

$$\Phi[x;v(\mathbf{R}'),v^*(\mathbf{R}');\eta] = \Phi[x;v,v^*;\eta] = \langle\varphi[x;v,v^*;\eta]\rangle,$$

this gives a closed-form functional equation in variational derivatives [13],

$$\frac{\partial}{\partial x}\,\Phi[x;v,v^*;\eta] = \dot{\Theta}_x\left[x;\,\frac{\delta}{\mathrm{i}\delta\eta(\xi,\mathbf{R}')}\right]\Phi[x;v,v^*;\eta] + \frac{\mathrm{i}k}{2}\int \eta(x,\mathbf{R})\widehat{M}(\mathbf{R})\Phi[x;v,v^*;\eta]\,\mathrm{d}\mathbf{R} + \frac{\mathrm{i}}{2k}\left\{\int\left[v(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v(\mathbf{R}')} - v^*(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v^*(\mathbf{R}')}\right]\mathrm{d}\mathbf{R}'\right\} \times \Phi[x;v,v^*;\eta]\,, \qquad (36)$$

where the functional

$$\dot{\Theta}_x\big[x;\psi(\xi,\mathbf{R}')\big] = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Phi_\varepsilon\big[x;\psi(\xi,\mathbf{R}')\big]$$

is a derivative of the logarithm of the characteristic functional of $\varepsilon(x,\mathbf{R})$. Although Eqn (36) is linear, it cannot be solved at present. Assuming that $\eta(x,\mathbf{R})=0$, we obtain a nonclosed equation for the characteristic functional of the solution of (20) in the form

$$\frac{\partial}{\partial x}\,\Phi[x;v,v^*] = \left\langle \dot{\Theta}_x\left[x;\,\frac{\delta}{\mathrm{i}\delta\varepsilon(\xi,\mathbf{R}')}\right]\varphi[x;v,v^*]\right\rangle + \frac{\mathrm{i}}{2k}\left\{\int\left[v(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v(\mathbf{R}')} - v^*(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v^*(\mathbf{R}')}\right]\mathrm{d}\mathbf{R}'\right\} \times \Phi[x;v,v^*]\,. \qquad (37)$$

3.2 Approximation of the delta-correlated random field $\varepsilon(x,\mathbf{R})$

The wave field $u(x,\mathbf{R})$ in the plane x is functionally dependent only on the previous values of $\varepsilon(x,\mathbf{R})$, by virtue of the dynamic causality principle. However, there can be a statistical relation between $u(x,\mathbf{R})$ and subsequent values of $\varepsilon(x_1,\mathbf{R})$ $(x_1>x)$ because $\varepsilon(x',\mathbf{R}')$ values at $x'<x$ are correlated with $\varepsilon(\xi,\mathbf{R})$ values at $\xi>x$. The correlation between the field $u(x,\mathbf{R})$ and subsequent values of $\varepsilon(x',\mathbf{R}')$ is clearly apparent at $x'-x\sim l_\parallel$, where $l_\parallel$ is the longitudinal correlation radius of the field $\varepsilon(x,\mathbf{R})$. At the same time, the characteristic correlation radius of the field $u(x,\mathbf{R})$ is of the order of x in the longitudinal direction. Therefore, the problem under consideration involves the small parameter $l_\parallel/x$, which can be used for the construction of an approximate solution.

In the first approximation, it may be assumed that $l_\parallel/x\to 0$, meaning a transition to the approximation that the random field $\varepsilon(x,\mathbf{R})$ is delta-correlated in x. In this case, the values of the fields $u(\xi_i,\mathbf{R})$ at $\xi_i<x$ are not only functionally but also statistically independent of $\varepsilon(\eta_j,\mathbf{R}')$ values at $\eta_j>x$; in other words, the following equality holds at $\xi_i<x$ and $\eta_j>x$:

$$\left\langle\prod_{i,j}u(\xi_i,\mathbf{R}_i)\varepsilon(\eta_j,\mathbf{R}_j)\right\rangle = \left\langle\prod_i u(\xi_i,\mathbf{R}_i)\right\rangle\left\langle\prod_j\varepsilon(\eta_j,\mathbf{R}_j)\right\rangle\,. \qquad (38)$$

In this approximation, all cumulant functions of the random field $\varepsilon(x,\mathbf{R})$ have the structure (see, e.g., Ref. [13])

$$K_n(\mathbf{R}_1,x_1;\ldots;\mathbf{R}_n,x_n) = K_n(\mathbf{R}_1,\ldots,\mathbf{R}_n;x_1)\delta(x_1-x_2)\ldots\delta(x_{n-1}-x_n)\,,$$

and the following equality holds for the characteristic functional of field $\varepsilon(x,\mathbf{R})$:

$$\dot{\Theta}_x\left[x;\,\frac{\delta}{\mathrm{i}\delta\varepsilon(\xi,\mathbf{R}')}\right] = \dot{\Theta}_x\left[x;\,\frac{\delta}{\mathrm{i}\delta\varepsilon(x,\mathbf{R}')}\right].$$

For example, if [for the linear parabolic equation (20)] the field $\varepsilon(x,\mathbf{R})$ is assumed to be a homogeneous random field delta-correlated in x, then equation (37) for the characteristic functional $\Phi[x;v,v^*]$ of the solution of the problem takes the form

$$\frac{\partial}{\partial x}\,\Phi[x;v,v^*] = \left\langle \dot{\Theta}_x\left[x;\,\frac{\delta}{\mathrm{i}\delta\varepsilon(x,\mathbf{R}')}\right]\varphi[x;v,v^*]\right\rangle + \frac{\mathrm{i}}{2k}\left\{\int\left[v(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v(\mathbf{R}')} - v^*(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v^*(\mathbf{R}')}\right]\mathrm{d}\mathbf{R}'\right\}\Phi[x;v,v^*]\,,$$

which in view of (34) can be written in the closed operator form

$$\frac{\partial}{\partial x}\,\Phi[x;v,v^*] = \dot{\Theta}_x\left[x,\,\frac{k}{2}\,\widehat{M}(\mathbf{R}')\right]\Phi[x;v,v^*] + \frac{\mathrm{i}}{2k}\left\{\int\left[v(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v(\mathbf{R}')} - v^*(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v^*(\mathbf{R}')}\right]\mathrm{d}\mathbf{R}'\right\}\Phi[x;v,v^*]\,. \qquad (39)$$

For moments of the field $u(x,\mathbf{R})$,

$$M_{mn}(x;\mathbf{R}_1,\ldots,\mathbf{R}_m;\mathbf{R}'_1,\ldots,\mathbf{R}'_n) = \left\langle\prod_{p=1}^{m}\prod_{q=1}^{n}u(x;\mathbf{R}_p)u^*(x;\mathbf{R}'_q)\right\rangle, \qquad (40)$$

which are usually called the *coherence functions* of the order $2n$ at $m=n$, it follows from Eqn (39) that

$$\frac{\partial}{\partial x}\,M_{mn} = \frac{\mathrm{i}}{2k}\left(\sum_{p=1}^{m}\Delta_{\mathbf{R}_p} - \sum_{q=1}^{n}\Delta_{\mathbf{R}'_q}\right)M_{mn} + \dot{\Theta}_x\left[x;\,\frac{1}{k}\left(\sum_{p=1}^{m}\delta(\mathbf{R}'-\mathbf{R}_p) - \sum_{q=1}^{n}\delta(\mathbf{R}'-\mathbf{R}'_q)\right)\right]M_{mn}\,, \qquad (41)$$

due to the linearity of the original dynamic equation (20).

Equations (41) for moments of the wave field $u(x,\mathbf{R})$ for delta-correlated fluctuations of medium parameters can also be obtained by a different, physically more demonstrable method. It is illustrated by the derivation of the equation for the mean field $\langle u(x,\mathbf{R})\rangle$. For this purpose, the original stochastic equation (20) should be rewritten in the form of the integral equation

$$u(x,\mathbf{R}) = u_0(\mathbf{R})\exp\left\{\frac{\mathrm{i}k}{2}\int_0^x\varepsilon(\xi,\mathbf{R})\,\mathrm{d}\xi\right\} + \frac{\mathrm{i}}{2k}\int_0^x\mathrm{d}x\,\exp\left\{\frac{\mathrm{i}k}{2}\int_\xi^x\varepsilon(\eta,\mathbf{R})\,\mathrm{d}\eta\right\}\Delta_{\mathbf{R}}u(\xi,\mathbf{R}). \qquad (42)$$

In taking the ensemble average of Eqn (42) over the random field $\varepsilon(\xi,\mathbf{R})$, we take equality (38) into account. This leads to the closed integral equation

$$\langle u(x,\mathbf{R})\rangle = u_0(\mathbf{R})\left\langle \exp\left\{\frac{ik}{2}\int_0^x \varepsilon(\xi,\mathbf{R})\,d\xi\right\}\right\rangle + \frac{i}{2k}\int_0^x d\xi \left\langle \exp\left\{\frac{ik}{2}\int_\xi^x \varepsilon(\eta,\mathbf{R})\,d\eta\right\}\right\rangle \Delta_{\mathbf{R}}\langle u(\xi,\mathbf{R})\rangle\,. \quad (43)$$

For the transition from the integral equation to the differential one, we note that for delta-correlated fluctuations of medium parameters, the equality

$$\left\langle \exp\left\{\frac{ik}{2}\int_0^x \varepsilon(\eta,\mathbf{R})\,d\eta\right\}\right\rangle = \left\langle \exp\left\{\frac{ik}{2}\int_0^\xi \varepsilon(\eta,\mathbf{R})\,d\eta\right\}\right\rangle\left\langle \exp\left\{\frac{ik}{2}\int_\xi^x \varepsilon(\eta,\mathbf{R})\,d\eta\right\}\right\rangle$$

holds for any arbitrary point $0 \leqslant \xi \leqslant x$. Introducing the function

$$\Phi(x,\mathbf{R}) = \left\langle \exp\left\{\frac{ik}{2}\int_0^x \varepsilon(\eta,\mathbf{R})\,d\eta\right\}\right\rangle,$$

we can therefore rewrite Eqn (43) as

$$\langle u(x,\mathbf{R})\rangle = u_0(\mathbf{R})\Phi(x,\mathbf{R}) + \frac{i}{2k}\int_0^x \frac{\Phi(x,\mathbf{R})}{\Phi(\xi,\mathbf{R})}\,\Delta_{\mathbf{R}}\langle u(\xi,\mathbf{R})\rangle\,d\xi\,,$$

whence it is easy to derive the differential equation for $\langle u(x,\mathbf{R})\rangle$,

$$\frac{\partial}{\partial x}\langle u(x,\mathbf{R})\rangle = \frac{i}{2k}\,\Delta_{\mathbf{R}}\langle u(x,\mathbf{R})\rangle + \langle u(x,\mathbf{R})\rangle\,\frac{\partial}{\partial x}\ln\Phi(x,\mathbf{R})\,,$$
$$u(0,\mathbf{R}) = u_0(\mathbf{R})\,,$$

which coincides with Eqn (41) at $m = 1$, $n = 0$. Evidently, equations for arbitrary moments of the field $u(x,\mathbf{R})$ can be obtained in a similar way.

3.3 Approximation of the delta-correlated Gaussian random field $\varepsilon(x,\mathbf{R})$

In the general case, correlation splitting depends on the properties of the random field $\varepsilon(x,\mathbf{R})$. If $\varepsilon(x,\mathbf{R})$ is assumed to be a random homogeneous delta-correlated field with the correlation function

$$B_\varepsilon(x,\mathbf{R}) = A(\mathbf{R})\delta(x)\,, \qquad A(\mathbf{R}) = \int_{-\infty}^{\infty} B_\varepsilon(x,\mathbf{R})\,dx\,, \quad (44)$$

then

$$\Theta\left[x;\psi(\xi,\mathbf{R}')\right] = -\frac{1}{2}\int_0^x d\xi \iint d\mathbf{R}'\,d\mathbf{R}\;A(\mathbf{R}'-\mathbf{R})\psi(\xi,\mathbf{R}')\psi(\xi,\mathbf{R})$$

and Eqn (39) takes the closed operator form

$$\frac{\partial}{\partial x}\,\Phi[x;v,v^*] = -\frac{k^2}{8}\iint d\mathbf{R}'\,d\mathbf{R}\;A(\mathbf{R}'-\mathbf{R})\widehat{M}(\mathbf{R}')\widehat{M}(\mathbf{R})\Phi[x;v,v^*] + \frac{i}{2k}\left\{\int\left[v(\mathbf{R}')\Delta_{\mathbf{R}'}\,\frac{\delta}{\delta v(\mathbf{R}')} - v^*(\mathbf{R}')\Delta_{\mathbf{R}'}\frac{\delta}{\delta v^*(\mathbf{R}')}\right]d\mathbf{R}'\right\}\Phi[x;v,v^*]\,,$$

while equations (41) for moments of the wave field $u(x,\mathbf{R})$ take the form

$$\frac{\partial}{\partial x}\,M_{mn} = \frac{i}{2k}\left(\sum_{p=1}^m \Delta_{\mathbf{R}_p} - \sum_{q=1}^n \Delta_{\mathbf{R}'_q}\right)M_{mn} - \frac{k^2}{8}\,Q(\mathbf{R}_1,\ldots,\mathbf{R}_m;\mathbf{R}'_1,\ldots,\mathbf{R}'_n)M_{mn}\,, \quad (45)$$

where

$$Q(\mathbf{R}_1,\ldots,\mathbf{R}_m;\mathbf{R}'_1,\ldots,\mathbf{R}'_n) = \sum_{i=1}^m\sum_{j=1}^m A(\mathbf{R}_i-\mathbf{R}_j) - 2\sum_{i=1}^m\sum_{j=1}^n A(\mathbf{R}_i-\mathbf{R}'_j) + \sum_{i=1}^n\sum_{j=1}^n A(\mathbf{R}'_i-\mathbf{R}'_j)\,. \quad (46)$$

We note that correlation splitting for a Gaussian random field and its functional is feasible based on the so-called *Furutsu – Novikov formula* [41, 42] (see also Ref. [5])

$$\left\langle \varepsilon(x,\mathbf{R})\Phi\left[\varepsilon(x',\mathbf{R}')\right]\right\rangle = \int dx' \int d\mathbf{R}'\,B_\varepsilon(x-x',\mathbf{R}-\mathbf{R}') \times \left\langle \frac{\delta}{\delta\varepsilon(x',\mathbf{R}')}\,\Phi\left[\varepsilon(x',\mathbf{R}')\right]\right\rangle. \quad (47)$$

This formula holds for any functional $\Phi[\varepsilon(x',\mathbf{R}')]$ of the Gaussian random field $\varepsilon(x,\mathbf{R})$ and can be regarded as integration by parts in a functional space [43]. Assuming that the field $\varepsilon(x,\mathbf{R})$ is a Gaussian, homogeneous, and δ-correlated with correlation function (44), we rewrite formula (47) at $0 < x' \leqslant x$ as

$$\left\langle \varepsilon(x,\mathbf{R})\Phi\left[\varepsilon(x',\mathbf{R}')\right]\right\rangle = \frac{1}{2}\int A(\mathbf{R}-\mathbf{R}')\left\langle \frac{\delta}{\delta\varepsilon(x-0,\mathbf{R}')}\,\Phi\left[\varepsilon(x',\mathbf{R}')\right]\right\rangle d\mathbf{R}'\,. \quad (48)$$

We now write equations for the mean field $\langle u(x,\mathbf{R})\rangle$ and the second-order coherence function

$$\Gamma_2(x,\mathbf{R},\mathbf{R}') = \langle \gamma_2(x,\mathbf{R},\mathbf{R}')\rangle\,,$$
$$\gamma_2(x,\mathbf{R},\mathbf{R}') = u(x,\mathbf{R})u^*(x,\mathbf{R}')\,,$$

ensuing from (45) and (46) at $m = 1$, $n = 0$, and $m = n = 1$:

$$\frac{\partial}{\partial x}\langle u(x,\mathbf{R})\rangle = \frac{i}{2k}\,\Delta_{\mathbf{R}}\langle u(x,\mathbf{R})\rangle - \frac{k^2}{8}\,A(0)\langle u(x,\mathbf{R})\rangle\,,$$
$$\langle u(0,\mathbf{R})\rangle = u_0(\mathbf{R})\,, \quad (49)$$

$$\frac{\partial}{\partial x}\,\Gamma_2(x,\mathbf{R},\mathbf{R}') = \frac{i}{2k}(\Delta_{\mathbf{R}} - \Delta_{\mathbf{R}'})\Gamma_2(x,\mathbf{R},\mathbf{R}') - \frac{k^2}{4}\,D(\mathbf{R}-\mathbf{R}')\Gamma_2(x,\mathbf{R},\mathbf{R}')\,, \quad (50)$$
$$\Gamma_2(0,\mathbf{R},\mathbf{R}') = u_0(\mathbf{R})u_0^*(\mathbf{R}')\,.$$

We here introduce a new function

$$D(\mathbf{R}) = A(0) - A(\mathbf{R})\,,$$

related to the structure function of the random field $\varepsilon(x,\mathbf{R})$.

Equations (49) and (50) are readily solvable for an arbitrary function $D(\mathbf{R})$ and arbitrary initial conditions. For example, the mean wave field is expressed as

$$\langle u(x,\mathbf{R})\rangle = u_0(x,\mathbf{R})\exp\left(-\frac{\gamma}{2}\,x\right), \tag{51}$$

where $u_0(x,\mathbf{R})$ is the solution of the problem in the absence of fluctuations of medium parameters,

$$u_0(x,\mathbf{R}) = \int g(x,\mathbf{R}-\mathbf{R}')u_0(\mathbf{R}')\,\mathrm{d}\mathbf{R}',$$

the function $g(x,\mathbf{R})$ is the Green's function in a free space, and the quantity $\gamma = (k^2/4)A(0)$ is the coefficient of extinction.

Accordingly, using the variables

$$\mathbf{R}\to\mathbf{R}+\frac{1}{2}\,\boldsymbol{\rho}, \qquad \mathbf{R}'\to\mathbf{R}-\frac{1}{2}\,\boldsymbol{\rho}$$

we express the second-order coherence function as

$$\Gamma_2(x,\mathbf{R},\boldsymbol{\rho}) = \int \gamma_0\left(\mathbf{q},\boldsymbol{\rho}-\mathbf{q}\,\frac{x}{k}\right) \times\exp\left\{\mathrm{i}\mathbf{q}\mathbf{R}-\frac{k^2}{4}\int_0^x D\left(\boldsymbol{\rho}-\mathbf{q}\,\frac{\xi}{k}\right)\mathrm{d}\xi\right\}\mathrm{d}\mathbf{q}, \tag{52}$$

where

$$\gamma_0(\mathbf{q},\boldsymbol{\rho}) = \frac{1}{(2\pi)^2}\int\gamma_0(\mathbf{R},\boldsymbol{\rho})\exp(-\mathrm{i}\mathbf{q}\mathbf{R})\,\mathrm{d}\mathbf{R}.$$

Further analysis of the problem depends on the form of the initial conditions for Eqn (20) and on the fluctuation patterns of the field $\varepsilon(x,\mathbf{R})$.

For the incident plane wave case, where

$$u_0(\mathbf{R}) = u_0 = \mathrm{const}, \quad \gamma_0(\mathbf{R},\boldsymbol{\rho}) = |u_0|^2, \quad \gamma_0(\mathbf{q},\boldsymbol{\rho}) = |u_0|^2\delta(\mathbf{q}),$$

expressions (51) and (52) are markedly simplified and become

$$\langle u(x,\mathbf{R})\rangle = u_0\exp\left(-\frac{1}{2}\,\gamma x\right), \qquad \Gamma_2(x,\mathbf{R},\boldsymbol{\rho}) = |u_0|^2\exp\left(-\frac{1}{4}\,k^2xD(\boldsymbol{\rho})\right). \tag{53}$$

These expressions are independent of the effect of plane wave diffraction in a randomly inhomogeneous medium. A new statistical scale $\boldsymbol{\rho}_{\mathrm{cog}}$ then appears in the plane normal to the direction of wave propagation; it is found from the condition

$$\frac{1}{4}\,k^2xD(\boldsymbol{\rho}_{\mathrm{cog}}) = 1 \tag{54}$$

and is called the *coherence radius* of the field $u(x,\mathbf{R})$. The coherence radius depends on the wave length, distance covered by the wave, and statistical parameters of the medium.

Equations for higher-order coherence functions cannot be solved in the analytic form; their analysis requires either numerical or approximate methods.

3.4 Applicability conditions for the approximation of delta-correlated fluctuations of medium parameters and diffusion approximation for a wave field

We now consider applicability conditions for the approximation of delta-correlated fluctuations of the field $\varepsilon(x,\mathbf{R})$. It is possible to construct a theory of consecutive approximations to the functional dependence of statistical characteristics of the wave on the field $\varepsilon(x,\mathbf{R})$. The approximation of delta-correlated fluctuations discussed in the preceding paragraphs is the first step in this theory. Other approximations take the finiteness of the longitudinal radius of the field $\varepsilon(x,\mathbf{R})$ into account and lead to a system of closed integro-differential equations for wave field moments (see, e.g., Ref. [13]).

Thus, it is easy to demonstrate that the approximation of delta-correlated fluctuations of the field $\varepsilon(x,\mathbf{R})$ for the mean field $\langle u(x,\mathbf{R})\rangle$ is valid if the following three conditions are fulfilled:

$$l_\parallel \ll kl_\perp^2, \quad \sigma_\varepsilon^2k^2l_\parallel^2 \ll 1, \quad x \gg l_\parallel \quad \left(A(0)\sim\sigma_\varepsilon^2 l_\parallel\right), \tag{55}$$

where $l_\parallel$ and $l_\perp$ are the longitudinal and transverse correlation radii of the field $\varepsilon(x,\mathbf{R})$, respectively, and σ_ε^2 is its variance.

Equations of the second approximation for the coherence function $\Gamma_2(x,\mathbf{R},\boldsymbol{\rho})$ can be obtained and analyzed in a similar way. In the case of a plane incident way, the applicability conditions for the approximation of the delta-correlated fluctuations of $\varepsilon(x,\mathbf{R})$ for the function $\Gamma_2(x,\mathbf{R},\boldsymbol{\rho})$ are given by

$$\rho \ll x, \qquad kx\left|\nabla A(\boldsymbol{\rho})\right| \ll 1. \tag{56}$$

It must be emphasized that conditions (55) and (56) are virtually independent because they impose constraints on different parameters. Specifically, conditions (56) may prove to be satisfied when the condition $\sigma_\varepsilon^2k^2l_\parallel^2 \ll 1$ is violated. We also note that conditions (56) impose limitations only on local characteristics of fluctuations of the field $\varepsilon(x,\mathbf{R})$ and can therefore be also written for a turbulent medium, whereas the quantity $\gamma = k^2A(0)/4$ is determined by the largest-scale fluctuations of the field $\varepsilon(x,\mathbf{R})$.

We now turn to the application of the diffusion approximation to the description of statistical properties of the solution of parabolic equation (20). We note that this approximation for the problem in question is close in 'spirit' to Chernov's local method [6]; it is physically more relevant than the formal approximation of the delta-correlated field $\varepsilon(x,\mathbf{R})$, takes the finiteness of the longitudinal correlation radius of $\varepsilon(x,\mathbf{R})$ into consideration, and describes wave propagation in a medium with inhomogeneities elongated parallel to the direction of propagation [39, 40]. It is assumed in the diffusion approximation that effects of random inhomogeneities at scales of the order of $l_\parallel$ are insignificant. In this case, equations for average values of products of wave fields are written down exactly, and the functional dependence of all quantities at scales $l_\parallel$ is defined by dynamic equations in the absence of fluctuations of the medium parameters (see, e.g., Ref. [13]).

Thus, in the diffusion approximation, the variational derivative is described by the deterministic equation

$$\left(\frac{\partial}{\partial x}-\frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}}\right)\frac{\delta u(x,\mathbf{R})}{\delta\varepsilon(x',\mathbf{R}')} = 0$$

with the stochastic initial condition

$$\left.\frac{\delta u(x,\mathbf{R})}{\delta\varepsilon(x',\mathbf{R}')}\right|_{x=x'+0} = \frac{\mathrm{i}k}{2}\,\delta(\mathbf{R}-\mathbf{R}')u(x',\mathbf{R}).$$

Therefore,

$$\frac{\delta u(x,\mathbf{R})}{\delta\varepsilon(x',\mathbf{R}')} = \frac{\mathrm{i}k}{2}\exp\left\{\frac{\mathrm{i}(x-x')}{2k}\,\Delta_{\mathbf{R}}\right\}\left[\delta(\mathbf{R}-\mathbf{R}')u(x',\mathbf{R})\right].$$

In the framework of the diffusion approximation, the wave field $u(x', \mathbf{R})$ is related to the field $u(x, \mathbf{R})$ by the equality

$$u(x', \mathbf{R}) = \exp\left\{-\frac{\mathrm{i}(x - x')}{2k}\,\Delta_{\mathbf{R}}\right\} u(x, \mathbf{R})\,,$$

which results from the solution of problem (20) in the absence of fluctuations. Therefore,

$$\left\langle \frac{\delta u(x, \mathbf{R})}{\delta\varepsilon(x', \mathbf{R}')} \right\rangle = \frac{\mathrm{i}k}{2} \exp\left\{\frac{\mathrm{i}(x - x')}{2k}\,\Delta_{\mathbf{R}}\right\} \times \left[\delta(\mathbf{R} - \mathbf{R}') \exp\left\{-\frac{\mathrm{i}(x - x')}{2k}\,\Delta_{\mathbf{R}}\right\}\left\langle u(x, \mathbf{R})\right\rangle\right],$$

and it can be shown that

$$\left\langle u(x, \mathbf{R})\right\rangle = \frac{1}{(2\pi)^2}\int \mathrm{d}\mathbf{q}\int \mathrm{d}\mathbf{R}'\, u_0(\mathbf{R}') \exp\left\{\mathrm{i}\mathbf{q}(\mathbf{R} - \mathbf{R}') - \mathrm{i}\,\frac{\mathbf{q}^2 x}{2} - \frac{k^2}{2}\int_0^x D(x', \mathbf{q})\,\mathrm{d}x'\right\} \qquad (57)$$

in the diffusion approximation.

At path lengths $x \gg l_\parallel$, where $l_\parallel$ is the longitudinal correlation radius of the field $\varepsilon(x, \mathbf{R})$, expression (57) is simplified and takes the form

$$\left\langle u(x, \mathbf{R})\right\rangle = \frac{1}{(2\pi)^2}\int \mathrm{d}\mathbf{q}\int \mathrm{d}\mathbf{R}'\, u_0(\mathbf{R}') \exp\left\{\mathrm{i}\mathbf{q}(\mathbf{R} - \mathbf{R}') - \mathrm{i}\,\frac{\mathbf{q}^2 x}{2} - \frac{k^2}{2}\, xD(\mathbf{q})\right\}, \qquad (58)$$

where

$$D(\mathbf{q}) = \pi\int \Phi_\varepsilon\left(\frac{1}{2k}(\mathbf{q}'^2 - 2\mathbf{q}'\mathbf{q}), \mathbf{q}'\right)\mathrm{d}\mathbf{q}'\,,$$

and $\Phi_\varepsilon(q_1, \mathbf{q})$ is the three-dimensional spectral function of the field $\varepsilon(x, \mathbf{R})$:

$$B_\varepsilon(x, \mathbf{R}) = \int_{-\infty}^{\infty} \mathrm{d}q_1 \int \mathrm{d}\mathbf{q}\, \Phi_\varepsilon(q_1, \mathbf{q}) \exp(\mathrm{i}q_1 x + \mathrm{i}\mathbf{q}\mathbf{R})\,.$$

We note that the delta-correlated approximation corresponds to the coefficient $D(\mathbf{q})$ of the form

$$D(\mathbf{q}) = \pi\int \Phi_\varepsilon(0, \mathbf{q}')\,\mathrm{d}\mathbf{q}'\,.$$

For an incident plane wave $u_0(\mathbf{R}) = u_0$; therefore, equality (58) yields the $\mathbf{R}$-independent expression

$$\left\langle u(x, \mathbf{R})\right\rangle = u_0 \exp\left\{-\frac{1}{2}\,k^2 xD(0)\right\},$$

$$D(0) = \pi\int \Phi_\varepsilon\left(\frac{\mathbf{q}'^2}{2k}, \mathbf{q}'\right)\mathrm{d}\mathbf{q}'\,,$$

for which the applicability condition is

$$\frac{k^2}{2}\, D(0) l_\parallel \ll 1\,.$$

Similarly, it is possible to derive equations for higher-order moments of the field $u(x, \mathbf{R})$.

We note that neither the approximation of the field $\varepsilon(x, \mathbf{R})$ delta-correlated in x nor the diffusion approximation is applicable in the case of $\varepsilon(x, \mathbf{R}) = \varepsilon(\mathbf{R})$ or stratified media $\varepsilon(x, \mathbf{R}) = \varepsilon(z)$. In either case, the field $\varepsilon(x, \mathbf{R})$ has a formally infinite correlation radius along the x axis.

3.5 Amplitude and phase fluctuations of the wave field (method of smooth perturbations)

We now consider a statistical description of amplitude and phase fluctuations of a given wave.

We introduce the phase and the amplitude of the wave field and the complex wave phase using the formula

$$u(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\left\{\mathrm{i}S(x, \mathbf{R})\right\} = \exp\left\{\mathrm{i}\varphi(x, \mathbf{R})\right\},$$

where

$$\varphi(x, \mathbf{R}) = \chi(x, \mathbf{R}) + \mathrm{i}S(x, \mathbf{R})\,,$$

$\chi(x, \mathbf{R}) = \ln A(x, \mathbf{R})$ is the level of wave amplitude, and $S(x, \mathbf{R})$ is the wave phase fluctuations relative to the phase kx of the incident wave. Proceeding from parabolic equation (20), it is possible to obtain a nonlinear equation of the so-called method of smooth perturbations (MSP) of Rytov for the complex phase $\varphi(x, \mathbf{R})$,

$$\frac{\partial}{\partial x}\,\varphi(x, \mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}}\varphi(x, \mathbf{R}) + \frac{\mathrm{i}}{2k}\left[\nabla_{\mathbf{R}}\varphi(x, \mathbf{R})\right]^2 + \frac{\mathrm{i}k}{2}\,\varepsilon(x, \mathbf{R})\,. \qquad (59)$$

In the case of a plane incident wave considered below, it may be assumed that $u_0(\mathbf{R}) = 1$, and hence $\varphi(0, \mathbf{R}) = 0$.

The separation of Eqn (59) into real and imaginary parts leads to the system of equations

$$\frac{\partial}{\partial x}\,\chi(x, \mathbf{R}) + \frac{1}{2k}\,\Delta_{\mathbf{R}}S(x, \mathbf{R}) + \frac{1}{k}\left[\nabla_{\mathbf{R}}\,\chi(x, \mathbf{R})\right]\left[\nabla_{\mathbf{R}}\,S(x, \mathbf{R})\right] = 0\,, \qquad (60)$$

$$\frac{\partial}{\partial x}\,S(x, \mathbf{R}) - \frac{1}{2k}\,\Delta_{\mathbf{R}}\chi(x, \mathbf{R}) - \frac{1}{2k}\left[\nabla_{\mathbf{R}}\,\chi(x, \mathbf{R})\right]^2 + \frac{1}{2k}\left[\nabla_{\mathbf{R}}\,S(x, \mathbf{R})\right]^2 = \frac{k}{2}\,\varepsilon(x, \mathbf{R})\,. \qquad (61)$$

If the function $\varepsilon(x, \mathbf{R})$ is sufficiently small, the iteration series in the field $\varepsilon(x, \mathbf{R})$ can be constructed for the solution of Eqns (60) and (61). In this case, the so-called first MSP approximation corresponds to Gaussian fields $\chi(x, \mathbf{R})$ and $S(x, \mathbf{R})$, whose statistical characteristics are found by statistically averaging the respective iteration series. For example, the second moments of these fields (including variances of all quantities) are derived from a linearized system of equations (69) and (61), that is, from the system of equations

$$\begin{aligned} &\frac{\partial}{\partial x}\,\chi_0(x, \mathbf{R}) = -\frac{1}{2k}\,\Delta_{\mathbf{R}}S_0(x, \mathbf{R})\,, \\ &\frac{\partial}{\partial x}\,S_0(x, \mathbf{R}) = \frac{1}{2k}\,\Delta_{\mathbf{R}}\,\chi_0(x, \mathbf{R}) + \frac{k}{2}\,\varepsilon(x, \mathbf{R})\,. \end{aligned} \qquad (62)$$

The mean values are then obtained by directly averaging Eqns (60) and (61). A wave field in a randomly inhomogeneous system was first described in this way by A M Obukhov [1].

The linear system of equations (62) can be solved with the use of the Fourier transformation in the transverse coordinate. Introducing Fourier transforms of all the fields and the

Fourier transform of the random field $\varepsilon(x, \mathbf{R})$ as

$$\chi_0(x, \mathbf{R}) = \int \chi_{\mathbf{q}}^0(x) \exp(\mathrm{i}\mathbf{q}\mathbf{R})\, \mathrm{d}\mathbf{q},$$

$$S_0(x, \mathbf{R}) = \int S_{\mathbf{q}}^0(x) \exp(\mathrm{i}\mathbf{q}\mathbf{R})\, \mathrm{d}\mathbf{q},$$

$$\varepsilon(x, \mathbf{R}) = \int \varepsilon_{\mathbf{q}}(x) \exp(\mathrm{i}\mathbf{q}\mathbf{R})\, \mathrm{d}\mathbf{q},$$

we obtain the solution of the system of equations (62) in the form

$$\begin{aligned} \chi_{\mathbf{q}}^0(x) &= \frac{k}{2}\int_0^x \varepsilon_{\mathbf{q}}(\xi) \sin \frac{q^2}{2k}(x-\xi)\, \mathrm{d}\xi, \\ S_{\mathbf{q}}^0(x) &= \frac{k}{2}\int_0^x \varepsilon_{\mathbf{q}}(\xi) \cos \frac{q^2}{2k}(x-\xi)\, \mathrm{d}\xi. \end{aligned} \tag{63}$$

In computing concrete integrals involving the random field $\varepsilon(x, \mathbf{R})$ delta-correlated in x, it is easy to obtain the correlation function of a Gaussian random field $\varepsilon_{\mathbf{q}}(x)$:

$$\langle \varepsilon_{\mathbf{q}_1}(x_1)\varepsilon_{\mathbf{q}_2}(x_2)\rangle = 2\pi\delta(x_1 - x_2)\delta(\mathbf{q}_1 + \mathbf{q}_2)\Phi_\varepsilon(0, \mathbf{q}_1). \tag{64}$$

We note that if the field $\varepsilon(x, \mathbf{R})$ is nonvanishing only in a finite layer $(0, \Delta x)$ and $\varepsilon(x, \mathbf{R}) = 0$ at $x > \Delta x$, then formula (64) is replaced by the expression

$$\langle \varepsilon_{\mathbf{q}_1}(x_1)\varepsilon_{\mathbf{q}_2}(x_2)\rangle = 2\pi\delta(x_1 - x_2)\theta(\Delta x - x)\delta(\mathbf{q}_1 + \mathbf{q}_2)\Phi_\varepsilon(0, \mathbf{q}_1).$$

For the fluctuations of $\varepsilon(x, \mathbf{R})$ caused by *turbulent temperature pulsations* in the earth's atmosphere, we obtain that in a broad range of wave numbers, the three-dimensional spectral density is given by

$$\Phi_\varepsilon(\mathbf{q}) = AC_\varepsilon^2 q^{-11/3} \qquad (q_{\min} \ll q \ll q_{\max}), \tag{65}$$

where $A = 0.033$ is a numerical constant and C_ε^2 is the structural characteristic of dielectric permittivity fluctuations depending on external parameters of the medium. In certain cases, the integrals describing statistical characteristics of the wave field amplitude and phase fluctuations and those containing the spectral function of form (65) diverge. Then the phenomenological spectral function of the form

$$\Phi_\varepsilon(\mathbf{q}) = \Phi_\varepsilon(q) = AC_\varepsilon^2 q^{-11/3} \exp\left(-\frac{\mathbf{q}^2}{\varkappa_m^2}\right) \tag{66}$$

is used, where $\varkappa_m$ is the wave number corresponding to the turbulence microscale.

For a medium that occupies a finite portion of space Δx, the statistical properties of amplitude fluctuations in the approximation being considered are described by the amplitude level variance, i.e., by the parameter

$$\sigma_0^2(x) = \langle \chi_0^2(x, \mathbf{R})\rangle,$$

which, in accordance with formulas (63) and (64), is given by

$$\begin{aligned} \sigma_0^2(x) &= \iint \mathrm{d}\mathbf{q}_1\, \mathrm{d}\mathbf{q}_2 \langle \chi_{\mathbf{q}_1}^0(x)\chi_{\mathbf{q}_2}^0(x)\rangle \exp\{\mathrm{i}(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{R}\} \\ &= \frac{\pi^2 k^2 \Delta x}{2}\int_0^\infty q\Phi_\varepsilon(q) \\ &\times \left\{1 - \frac{k}{q^2\Delta x}\left[\sin\frac{q^2 x}{k} - \sin\frac{q^2(x - \Delta x)}{k}\right]\right\} \mathrm{d}q. \end{aligned} \tag{67}$$

Equation (24) can be used to find the mean amplitude level. The ensemble average of this equation over the field $\varepsilon(x, \mathbf{R})$ for an incident plane wave yields the equality

$$\langle I(x, \mathbf{R})\rangle = 1,$$

which can be rewritten as

$$\begin{aligned} \langle I(x, \mathbf{R})\rangle &= \langle \exp\{2\chi_0(x, \mathbf{R})\}\rangle \\ &= \exp\{2\langle \chi_0(x, \mathbf{R})\rangle + 2\sigma_0^2(x)\} = 1. \end{aligned}$$

Thus, in the first approximation of MSP,

$$\langle \chi_0(x, \mathbf{R})\rangle = -\sigma_0^2(x).$$

The applicability condition for this approximation is evidently

$$\sigma_0^2(x) \ll 1.$$

In the first MSP approximation, the following equation holds for the wave intensity variance, called the *scintillation index*:

$$\beta_0^2(x) = \langle I^2(x, \mathbf{R})\rangle - 1 = \langle \exp\{4\chi_0(x, \mathbf{R})\}\rangle - 1 \approx 4\sigma_0^2(x).$$

This implies that in this approximation, the one-point probability distribution for the field $\chi(x, \mathbf{R})$ has the form

$$P(x; \chi) = \sqrt{\frac{2}{\pi\beta_0(x)}} \exp\left\{-\frac{2}{\beta_0(x)}\left(\chi + \frac{1}{4}\beta_0(x)\right)^2\right\}.$$

Therefore, the wave field intensity is a log-normal random field and its one-point probability density is given by

$$\begin{aligned} P(x; I) &= \frac{1}{I\sqrt{2\pi\beta_0(x)}} \\ &\times \exp\left\{-\frac{1}{2\beta_0(x)}\ln^2\left(I\exp\left\{\frac{1}{2}\beta_0(x)\right\}\right)\right\}. \end{aligned} \tag{68}$$

Moments of the wave field intensity are then described by the equalities

$$\langle I^n(x, \mathbf{R})\rangle = \langle \exp\{2n\chi(x, \mathbf{R})\}\rangle = \exp\{2n(n-1)\sigma_0^2(x)\}.$$

Two limiting asymptotic cases are typically considered in statistical analysis.

The first case corresponds to the assumption that $\Delta x \ll x$ and is called the *random phase screen*. In this case, a wave, having passed through a thin layer of the fluctuating medium, propagates further into an empty space. The thin layer hosts only phase fluctuations of the wave field that thereafter undergo transformation into amplitude fluctuations by virtue of the nonlinearity of Eqns (60) and (61).

The second case refers to a *continuous medium*, i.e., the condition $\Delta x = x$.

The two cases are considered in more detail below for weak wave field fluctuations.

Random phase screen ($\Delta x \ll x$). In this case, the amplitude level variance is described by the expression

$$\sigma_0^2(x) = \frac{\pi^2 k^2 \Delta x}{2}\int_0^\infty q\Phi_\varepsilon(q)\left\{1 - \cos\frac{q^2 x}{k}\right\} \mathrm{d}q \tag{69}$$

ensuing from Ref. (67).

If fluctuations of the field $\varepsilon(x, \mathbf{R})$ are caused by turbulent pulsation of the medium, the function $\Phi_\varepsilon(q)$ is described by formula (65) and integral (69) is easy to compute. Hence, we obtain the expression

$$\sigma_0^2(x) = 0.144\, C_\varepsilon^2 k^{7/6} x^{5/6} \Delta x\,, \tag{70}$$

and therefore the scintillation index is

$$\beta_0^2(x) = 0.563\, C_\varepsilon^2 k^{7/6} x^{5/6} \Delta x\,. \tag{71}$$

As regards phase fluctuations, the quantity describing the wave incidence angle at the point $(x, \mathbf{R})$ is of immediate physical interest:

$$\alpha(x, \mathbf{R}) = \frac{1}{k} \nabla_{\mathbf{R}} S(x, \mathbf{R})\,.$$

The formula for its variance, analogous to (69), is

$$\left\langle \alpha^2(x, \mathbf{R}) \right\rangle = \frac{\pi^2 \Delta x}{2} \int_0^\infty q \Phi_\varepsilon(q) \left\{ 1 + \cos \frac{q^2 x}{k} \right\} \mathrm{d}q\,.$$

Continuous medium ($\Delta x = x$). In this case, the amplitude level variance is described by the formula

$$\sigma_0^2(x) = \frac{\pi^2 k^2 x}{2} \int_0^\infty q \Phi_\varepsilon(q) \left\{ 1 - \frac{k}{q^2 x} \sin \frac{q^2 x}{k} \right\} \mathrm{d}q\,.$$

The parameters $\sigma_0^2(x)$ and $\beta_0^2(x)$ for turbulent medium fluctuations are given by the expressions

$$\begin{aligned} \sigma_0^2(x) &= 0.077\, C_\varepsilon^2 k^{7/6} x^{11/6}\,, \\ \beta_0^2(x) &= 0.307\, C_\varepsilon^2 k^{7/6} x^{11/6}\,. \end{aligned} \tag{72}$$

The variance of the wave incidence angle at the point $(x, \mathbf{R})$ is then given by

$$\left\langle \alpha^2(x, \mathbf{R}) \right\rangle = \frac{\pi^2 x}{2} \int_0^\infty q \Phi_\varepsilon(q) \left\{ 1 + \frac{k}{q^2 x} \sin \frac{q^2 x}{k} \right\} \mathrm{d}q\,.$$

In a similar way, it is possible to consider the variance of the amplitude level gradient; the spectral function $\Phi_\varepsilon(q)$ must then be described by formula (66). For the variance $\sigma_{\mathbf{q}}^2(x) = \langle [\nabla_{\mathbf{R}} \chi(x, \mathbf{R})]^2 \rangle$ in the case of a turbulent medium filling the entire space and under condition that the so-called wave parameter $D(x) = \varkappa_m^2 x/k \gg 1$, we obtain the equation (see, e.g., Ref. [8])

$$\begin{aligned} \sigma_{\mathbf{q}}^2(x) &= \frac{k^2 \pi^2 x}{2} \int_0^\infty q^3 \Phi_\varepsilon(q) \left\{ 1 - \frac{k}{q^2 x} \sin \frac{q^2 x}{k} \right\} \mathrm{d}q \\ &= \frac{1.476}{L_{\mathrm{f}}^2(x)}\, D^{1/6}(x) \beta_0(x)\,, \end{aligned} \tag{73}$$

where we introduce a natural and medium-independent length scale in the plane $x = \text{const}$, i.e., the size of the first Fresnel zone $L_{\mathrm{f}}(x) = \sqrt{x/k}$ that measures the transitional light – shadow diffraction region at the edge of a non-transparent screen (see, e.g., Refs [8]).

In the general case, the validity condition for the first MSP approximation for amplitude fluctuations is

$$\sigma_0^2(x) \ll 1\,.$$

The region of amplitude fluctuations in which this equality is fulfilled is called the *weak fluctuation region*. In the *strong fluctuation region* [where $\sigma_0^2(x) \geqslant 1$], it is necessary to consider the entire system of equations (60) and (61).

As regards fluctuations of the wave incidence angle at the observation point, which are related to fluctuations of the quantity

$$\alpha(x, \mathbf{R}) = \frac{1}{k} \nabla_{\mathbf{R}} S(x, \mathbf{R})\,,$$

they are fairly well described by the first MSP approximation even for larger values of the parameter $\sigma_0^2(x)$.

We note that the approximation of the delta-correlated random field $\varepsilon(x, \mathbf{R})$ for Eqn (20) imposes practically no constraints on amplitude fluctuations. This makes the equations for moments of $u(x, \mathbf{R})$ obtained in the previous paragraphs also valid in the region of strong amplitude fluctuations. Statistical characteristics of wave field intensities in this case are analyzed in what follows.

4. Path-integral description of the solution

We now consider a statistical description of wave field characteristics in a medium with random inhomogeneities based on the functional description of the solution of the problem Eqns (29) and (31).

4.1 Asymptotic analysis of plane wave intensity fluctuations

We consider the statistical moment of the field $u(x, \mathbf{R})$,

$$M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n}) = \left\langle \prod_{k=1}^{n} u(x, \mathbf{R}_{2k-1}) u^*(x, \mathbf{R}_{2k}) \right\rangle. \tag{74}$$

In the approximation of the delta-correlated random field $\varepsilon(x, \mathbf{R})$, the function $M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n})$ satisfies Eqn (45) at $n = m$; for an incident plane wave, it is written in the variables $\mathbf{R}_k$ as an equation with the initial condition

$$\begin{aligned} &\left(\frac{\partial}{\partial x} - \frac{\mathrm{i}}{2k} \sum_{l=1}^{2n} (-1)^{l+1} \Delta_{\mathbf{R}_l} \right) M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n}) \\ &\quad = \frac{k^2}{8} \sum_{l,j=1}^{2n} (-1)^{l+j} D(\mathbf{R}_l - \mathbf{R}_j) M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n})\,, \end{aligned}$$

where the function $D(\mathbf{R})$ is described by the formula

$$D(\mathbf{R}) = A(0) - A(\mathbf{R}) = 2\pi \int \Phi_\varepsilon(0, \mathbf{q}) \left[1 - \cos(\mathbf{qR}) \right] \mathrm{d}\mathbf{q}\,,$$

and $\Phi_\varepsilon(0, \mathbf{q})$ is the three-dimensional spectrum of the field $\varepsilon(x, \mathbf{R})$ of the two-dimensional vector $\mathbf{q}$.

Using the path-integral description of $u(x, \mathbf{R})$ in (31) and averaging over $\varepsilon(x, \mathbf{R})$, we express $M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n})$ as

$$\begin{aligned} M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n}) &= \int \ldots \int \mathrm{D}\mathbf{v}_1(\xi) \ldots \mathrm{D}\mathbf{v}_{2n}(\xi) \\ &\times \exp\left\{ \frac{\mathrm{i}k}{2x} \sum_{j=1}^{2n} (-1)^{j+1} \int_0^x \mathbf{v}_j^2(\xi)\, \mathrm{d}\xi - \frac{k^2}{8} \sum_{j,l=1}^{2n} (-1)^{j+l+1} \right. \\ &\times \left. \int_0^x D\left(\mathbf{R}_j - \mathbf{R}_l + \int_\xi^x \left[\mathbf{v}_j(x') - \mathbf{v}_l(x') \right] \mathrm{d}x' \right) \mathrm{d}\xi \right\}. \end{aligned} \tag{75}$$

Formula (75) can be represented in the operator form

$$M_{nn}(x,\mathbf{R}_1,\ldots,\mathbf{R}_{2n}) = \prod_{l=1}^{2n} \exp\left\{\frac{\mathrm{i}}{2k}(-1)^{l+1}\int_0^x \frac{\delta^2}{\delta\mathbf{v}^2(\xi)}\,\mathrm{d}\xi\right\}$$
$$\times \exp\left\{-\frac{k^2}{8}\sum_{j,l=1}^{2n}(-1)^{j+l+1}\right.$$
$$\left.\times\int_0^x D\left(\mathbf{R}_j-\mathbf{R}_l+\int_{x'}^x\left[\mathbf{v}_j(\xi)-\mathbf{v}_l(\xi)\right]\mathrm{d}\xi\right)\mathrm{d}x'\right\}_{\mathbf{v}=0}. \quad (76)$$

If the points $\mathbf{R}_{2k-1}$ and $\mathbf{R}_{2k}$ coincide, the function $M_{nn}(x,\mathbf{R}_1,\ldots,\mathbf{R}_{2n})$ becomes

$$\left\langle\prod_{k=1}^{n} I(x,\mathbf{R}_{2k-1})\right\rangle,$$

which describes correlation characteristics of wave intensity. If we now set all $\mathbf{R}_l=\mathbf{R}$, the function

$$M_{nn}(x,\mathbf{R},\ldots,\mathbf{R}) = \Gamma_{2n}(x,\mathbf{R}) = \left\langle I^n(x,\mathbf{R})\right\rangle$$

describes the nth moment of the wave field intensity.

Before proceeding to the discussion of the asymptotic forms of the functions $\Gamma_{2n}(x,\mathbf{R})$ for the continuous random medium case, we consider a simpler problem of wave field fluctuations behind a random phase screen.

4.1.1 Random phase screen. We consider an inhomogeneous medium layer that is thin enough to allow a passing wave to acquire only a random phase increment

$$S(\mathbf{R}) = \frac{k}{2}\int_0^{\Delta x}\varepsilon(\xi,\mathbf{R})\,\mathrm{d}\xi\,, \quad (77)$$

with the wave amplitude unchanged. We assume, as before, that the random field $\varepsilon(x,\mathbf{R})$ is Gaussian and is delta-correlated in x. After passing through the inhomogeneous layer, the wave propagates further in the homogeneous environment, as described by the equation derived from (20) with $\varepsilon(x,\mathbf{R})=0$. The solution of this problem is described by the formulas

$$u(x,\mathbf{R}) = \exp\left\{\mathrm{i}\,\frac{x}{2k}\,\Delta_{\mathbf{R}}\right\}\exp\left\{\mathrm{i}S(\mathbf{R})\right\}$$
$$= \frac{k}{2\pi\mathrm{i}x}\int\exp\left\{\frac{\mathrm{i}k}{2x}\,\mathbf{v}^2+\mathrm{i}S(\mathbf{R}+\mathbf{v})\right\}\mathrm{d}\mathbf{v}\,, \quad (78)$$

which are finite-dimensional analogs of formulas (29) and (31).

We consider the function $M_{nn}(x,\mathbf{R}_1,\ldots,\mathbf{R}_{2n})$. Substituting (78) in (74) and averaging, we easily obtain the formula

$$M_{nn}(x,\mathbf{R}_1,\ldots,\mathbf{R}_{2n}) = \left(\frac{k}{2\pi x}\right)^{2n}\int\ldots\int\mathrm{d}\mathbf{v}_1\ldots\mathrm{d}\mathbf{v}_{2n}$$
$$\times\exp\left\{\frac{\mathrm{i}k}{2x}\sum_{j=1}^{2n}(-1)^{j+1}\mathbf{v}_j^2\right.$$
$$\left.-\frac{k^2\Delta x}{8}\sum_{j,l=1}^{2n}(-1)^{j+l+1}D(\mathbf{R}_j-\mathbf{R}_l+\mathbf{v}_j-\mathbf{v}_l)\right\}, \quad (79)$$

which is an analog of (75).

We first consider the case where $n=2$ and the observation points pairwise coincide,

$$\mathbf{R}_1=\mathbf{R}_2=\mathbf{R}'\,,\qquad \mathbf{R}_3=\mathbf{R}_4=\mathbf{R}''\,,\qquad \mathbf{R}'-\mathbf{R}''=\boldsymbol{\rho}\,.$$

Then the function

$$\Gamma_4(x;\mathbf{R}',\mathbf{R}',\mathbf{R}'',\mathbf{R}'') = \left\langle I(x,\mathbf{R}')I(x,\mathbf{R}'')\right\rangle$$

is the intensity covariance $I(x,\mathbf{R})=|u(x,\mathbf{R})|^2$. Introducing new integration variables

$$\mathbf{v}_1-\mathbf{v}_2=\mathbf{R}_1\,,\qquad \mathbf{v}_1-\mathbf{v}_4=\mathbf{R}_2\,,\qquad \mathbf{v}_1-\mathbf{v}_3=\mathbf{R}_3\,,$$
$$\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)=\mathbf{R}$$

in (79) (with $n=2$), we can integrate over $\mathbf{R}$ and $\mathbf{R}_3$ and thus obtain a simpler formula

$$\left\langle I(x,\mathbf{R}')I(x,\mathbf{R}'')\right\rangle$$
$$= \left(\frac{k}{2\pi x}\right)^2\iint\mathrm{d}\mathbf{R}_1\,\mathrm{d}\mathbf{R}_2\,\exp\left\{\frac{\mathrm{i}k}{x}\,\mathbf{R}_1(\mathbf{R}_2-\boldsymbol{\rho})\right.$$
$$\left.-\frac{k^2\Delta x}{4}\,F(\mathbf{R}_1,\mathbf{R}_2)\right\}, \quad (80)$$

where $\boldsymbol{\rho}=\mathbf{R}'-\mathbf{R}''$ and the function $F(\mathbf{R}_1,\mathbf{R}_2)$ is found from the equality

$$F(\mathbf{R}_1,\mathbf{R}_2) = 2D(\mathbf{R}_1)+2D(\mathbf{R}_2)-D(\mathbf{R}_1+\mathbf{R}_2)-D(\mathbf{R}_1-\mathbf{R}_2)\,,$$
$$D(\mathbf{R}) = A(0)-A(\mathbf{R})\,.$$

As $x\to\infty$, integral (80) has the asymptotic form

$$\left\langle I(x,\mathbf{R}')I(x,\mathbf{R}'')\right\rangle = 1+\exp\left\{-\frac{k^2\Delta x}{2}\,D(\boldsymbol{\rho})\right\}$$
$$+\pi k^2\Delta x\int\Phi_\varepsilon(\mathbf{q})\left[1-\cos\frac{\mathbf{q}^2x}{k}\right]$$
$$\times\exp\left\{\mathrm{i}\mathbf{q}\boldsymbol{\rho}-\frac{k^2\Delta x}{2}\,D\left(\frac{\mathbf{q}x}{k}\right)\right\}\mathrm{d}\mathbf{q}$$
$$+\pi k^2\Delta x\int\Phi_\varepsilon(\mathbf{q})\left[1-\cos\left(\mathbf{q}\boldsymbol{\rho}-\frac{\mathbf{q}^2x}{k}\right)\right]$$
$$\times\exp\left\{-\frac{k^2\Delta x}{2}\,D\left(\boldsymbol{\rho}-\frac{\mathbf{q}x}{k}\right)\right\}\mathrm{d}\mathbf{q}+\ldots \quad (81)$$

We note that in addition to ρ_{cog}, the problem has acquired another characteristic space dimension

$$r_0=\frac{x}{k\rho_{\mathrm{cog}}}\,. \quad (82)$$

With $\boldsymbol{\rho}=0$ set in Eqn (81), the following expression can be obtained for the squared intensity variance:

$$\beta^2(x) = \left\langle I^2(x,\mathbf{R})\right\rangle-1$$
$$= 1+\pi\Delta x\int q^4\Phi_\varepsilon(\mathbf{q})\exp\left\{-\frac{k^2\Delta x}{2}\,D\left(\frac{\mathbf{q}x}{k}\right)\right\}\mathrm{d}\mathbf{q}+\ldots \quad (83)$$

If fluctuations of the field $\varepsilon(x,\mathbf{R})$ in an inhomogeneous layer are caused by turbulence, such that $\Phi_\varepsilon(\mathbf{q})$ is described by formula (65), equality (83) leads to

$$\beta^2(x) = 1+0.429\,\beta_0^{-4/5}(x)\,, \quad (84)$$

where $\beta_0^2(x)$ is the variance calculated using the first MSP approximation for a random phase screen, Eqn (71).

The above considerations are readily generalized to higher moments of the field $u(x, \mathbf{R})$, specifically to the functions $\Gamma_{2n}(x, \mathbf{R}) = \langle I^n(x, \mathbf{R})\rangle$. In this case, formula (79) acquires the form

$$\langle I^n(x, \mathbf{R})\rangle = \left(\frac{k}{2\pi x}\right)^{2n} \int \ldots \int \mathrm{d}\mathbf{v}_1 \ldots \mathrm{d}\mathbf{v}_{2n} \times \exp\left\{\frac{\mathrm{i}k}{2x}\sum_{j=1}^{2n}(-1)^{j+1}\mathbf{v}_j^2 - F(\mathbf{v}_1, \ldots, \mathbf{v}_{2n})\right\},$$

where

$$F(\mathbf{v}_1, \ldots, \mathbf{v}_{2n}) = \frac{k^2\Delta x}{8}\sum_{j,l=1}^{2n}(-1)^{j+l+1}D(\mathbf{v}_j - \mathbf{v}_l),$$

whence it is easy to derive an asymptotic formula for $\langle I^n(x, \mathbf{R})\rangle$ as $x \to \infty$,

$$\langle I^n(x, \mathbf{R})\rangle = n!\left[1 + n(n-1)\,\frac{\beta^2(x) - 1}{4} + \ldots\right], \tag{85}$$

in which $\beta^2(x)$ is given by expression (84). This formula is discussed below after the wave propagation in a continuous randomly inhomogeneous medium is considered, because the results in both cases are very similar.

4.1.2 Continuous random medium. We consider the asymptotic form of higher moments of the wave field $M_{nn}(x, \mathbf{R}_1, \ldots, \mathbf{R}_{2n})$ propagating in a randomly inhomogeneous medium. The formal solution of this problem is provided by expressions (75) and (76). They differ from the formulas for a phase screen in the preceding section in that ordinary integration is replaced by a functional one.

In this case, as $x \to \infty$, the intensity variance

$$\beta^2(x) = \langle I^2(x, \mathbf{R})\rangle - 1$$

may be described by an asymptotic formula similar to (83),

$$\beta^2(x) = 1 + \pi\int_0^x \mathrm{d}x'(x - x')\int \mathrm{d}\mathbf{q}\, q^4\Phi_\varepsilon(\mathbf{q}) \times \exp\left\{-\frac{k^2x'}{2}\,D\left(\frac{\mathbf{q}}{k}(x - x')\right) - \frac{k^2}{2}\int_{x'}^x D\left(\frac{\mathbf{q}}{k}(x - x'')\right)\mathrm{d}x''\right\} + \ldots \tag{86}$$

For a turbulent medium, Eqn (86) implies that

$$\beta^2(x) = 1 + 0.861\left(\beta_0^2(x)\right)^{-2/5}, \tag{87}$$

where $\beta_0^2(x)$ is the wave field intensity variance computed using the first MSP approximation, Eqn (72).

We next consider the highest moments $\langle I^n(x, \mathbf{R})\rangle = \Gamma_{2n}(x, 0)$. Similarly to the case of a random phase screen, it is easy to deduce that for waves propagating in a randomly inhomogeneous medium, the expansion

$$\langle I^n(x, \mathbf{R})\rangle = n!\left[1 + n(n-1)\,\frac{\beta^2(x) - 1}{4} + \ldots\right] \tag{88}$$

holds for the variance of wave field intensity; this expansion coincides with expression (85) for the phase screen. Certainly, $\beta^2(x)$ is found using different formulas in either case.

Formula (89) gives the first two terms of the asymptotic expansion of the function $\langle I^n(x, \mathbf{R})\rangle$ as $\beta_0^2(x) \to \infty$. Because $\beta^2(x) \to 1$ as $\beta_0^2(x) \to \infty$, the second term in (88) is smaller than the first one at sufficiently large $\beta_0^2(x)$. Expression (88) makes sense only if

$$n(n-1)\,\frac{\beta^2(x) - 1}{4} \ll 1\,. \tag{89}$$

However, at fixed $\beta_0^2(x)$, there always exist numbers n such that condition (89) fails to be fulfilled. Therefore, formula (88) holds only for not very large n. Also, it should be noted that the asymptotic regime (88) as $\beta_0^2(x) \to \infty$ may be reached rather slowly.

Formula (88) leads to the probability density for the intensity with singularities. To avoid them, this formula may be approximated by the expression (see, e.g., Ref. [22])

$$\langle I^n(x, \mathbf{R})\rangle = n!\exp\left\{n(n-1)\,\frac{\beta^2(x) - 1}{4}\right\}, \tag{90}$$

with the corresponding probability density given by

$$P(x, I) = \frac{1}{\sqrt{\pi\left(\beta(x) - 1\right)}} \times \int_0^\infty \exp\left\{-zI - \frac{\left[\ln z - \left(\beta(x) - 1\right)/4\right]^2}{\beta(x) - 1}\right\}\mathrm{d}z\,. \tag{91}$$

We note that generally speaking, probability distribution (91) is inapplicable in the narrow vicinity of $I \sim 0$ [the greater the parameter $\beta_0^2(x)$, the narrower the vicinity]. This is so because formula (91) implies infinitely large values for moments of the quantity $1/I(x, \mathbf{R})$. However, for a finite value of $\beta_0^2(x)$ (no matter how large), the quantities $\langle 1/I^n(x, \mathbf{R})\rangle$ are also finite; hence, the equality $P(x, 0) = 0$ must be obeyed. Certainly, such a narrow vicinity of the point $I \sim 0$ has no effect on the behavior of moments (90) for large values of $\beta_0^2(x)$.

Asymptotic formulas (90) and (91) describe the transition to a saturated intensity fluctuation regime in which $\beta(x) \to 1$ as $\beta_0^2(x) \to \infty$. Accordingly, in this regime,

$$\langle I^n(x, \mathbf{R})\rangle = n!\,, \qquad P(x, I) = \exp(-I)\,. \tag{92}$$

Exponential probability distribution (92) implies that the complex field $u(x, \mathbf{R})$ is a Gaussian random field, with

$$u(x, \mathbf{R}) = A(x, \mathbf{R})\exp\{\mathrm{i}S(x, \mathbf{R})\} = u_1(x, \mathbf{R}) + \mathrm{i}u_2(x, \mathbf{R}),$$

where $u_1(x, \mathbf{R})$ and $u_2(x, \mathbf{R})$ are real and imaginary parts, respectively. The wave field intensity is then given by

$$I(x, \mathbf{R}) = A^2(x, \mathbf{R}) = u_1^2(x, \mathbf{R}) + u_2^2(x, \mathbf{R})\,.$$

Because the field $u(x, \mathbf{R})$ is Gaussian, the random fields $u_1(x, \mathbf{R})$ and $u_2(x, \mathbf{R})$ are also Gaussian and statistically independent, with the variances

$$\langle u_1^2(x, \mathbf{R})\rangle = \langle u_2^2(x, \mathbf{R})\rangle = \frac{1}{2}\,.$$

It is then natural to assume that their gradients

$$\mathbf{p}_1(x, \mathbf{R}) = \nabla_{\mathbf{R}} u_1(x, \mathbf{R}), \qquad \mathbf{p}_2(x, \mathbf{R}) = \nabla_{\mathbf{R}} u_2(x, \mathbf{R})$$

are statistically independent of the fields $u_1(x, \mathbf{R})$ and $u_2(x, \mathbf{R})$ and are Gaussian homogeneous isotropic fields in the plane $\mathbf{R}$ with the variances

$$\sigma_{\mathbf{p}}^2(x) = \langle \mathbf{p}_1^2(x, \mathbf{R}) \rangle = \langle \mathbf{p}_2^2(x, \mathbf{R}) \rangle . \tag{93}$$

As a result, the joint probability density of the fields $u_1(x, \mathbf{R})$ and $u_2(x, \mathbf{R})$ and their gradients $\mathbf{p}_1(x, \mathbf{R})$ and $\mathbf{p}_2(x, \mathbf{R})$ has the form

$$P(x; u_1, u_2, \mathbf{p}_1, \mathbf{p}_2) = \frac{1}{\pi^3 \sigma_{\mathbf{p}}^4(x)} \exp\left\{ -u_1^2 - u_2^2 - \frac{\mathbf{p}_1^2 + \mathbf{p}_2^2}{\sigma_{\mathbf{p}}^2(x)} \right\} . \tag{94}$$

We now consider the joint probability density of the wave field intensity $I(x, \mathbf{R})$ and the amplitude gradient

$$\boldsymbol{\varkappa}(x, \mathbf{R}) = \nabla_{\mathbf{R}} A(x, \mathbf{R}) = \frac{u_1(x, \mathbf{R})\, \mathbf{p}_1(x, \mathbf{R}) + u_2(x, \mathbf{R})\, \mathbf{p}_2(x, \mathbf{R})}{\sqrt{u_1^2(x, \mathbf{R}) + u_2^2(x, \mathbf{R})}} .$$

We then have

$$\begin{aligned} P(x; I, \boldsymbol{\varkappa}) &= \left\langle \delta\big(I(x, \mathbf{R}) - I\big) \delta\big(\boldsymbol{\varkappa}(x, \mathbf{R}) - \boldsymbol{\varkappa}\big) \right\rangle_{u_i, \mathbf{p}_i} \\ &= \frac{1}{\pi^3 \sigma_{\mathbf{p}}^4(x)} \int_{-\infty}^{\infty} \mathrm{d}u_1 \int_{-\infty}^{\infty} \mathrm{d}u_2 \int \mathrm{d}\mathbf{p}_1 \int \mathrm{d}\mathbf{p}_2 \exp\left\{ -u_1^2 - u_2^2 \right. \\ &\quad \left. - \frac{\mathbf{p}_1^2 + \mathbf{p}_2^2}{\sigma_{\mathbf{p}}^2(x)} \right\} \delta\left(u_1^2 + u_2^2 - I\right) \delta\left(\frac{u_1 \mathbf{p}_1 + u_2 \mathbf{p}_2}{\sqrt{u_1^2 + u_2^2}} - \boldsymbol{\varkappa} \right) \\ &= \frac{1}{2\pi \sigma_{\mathbf{p}}^2(x)} \exp\left\{ -I - \frac{\boldsymbol{\varkappa}^2}{2\sigma_{\mathbf{p}}^2(x)} \right\} . \end{aligned}$$

Therefore, the transverse amplitude gradient is independent of the wave field intensities and is a Gaussian homogeneous field with the variance

$$\langle \boldsymbol{\varkappa}^2(x, \mathbf{R}) \rangle = 2\sigma_{\mathbf{p}}^2(x) . \tag{95}$$

We note that the transverse amplitude gradient is also statistically independent of second derivatives of the wave field intensity with respect to the transverse coordinates.

The second-order coherence function in the regime of strong intensity fluctuations is independent of diffraction phenomena and is described by expression (53),

$$\begin{aligned} \Gamma_2(x, \mathbf{R} - \mathbf{R}') &= \langle u(x, \mathbf{R}) u^*(x, \mathbf{R}') \rangle \\ &= \langle u_1(x, \mathbf{R}) u_1^*(x, \mathbf{R}') \rangle + \langle u_2(x, \mathbf{R}) u_2^*(x, \mathbf{R}') \rangle \\ &= \exp\left\{ -\frac{1}{4} k^2 x D\Big(\mathbf{R} - \mathbf{R}'\Big) \right\} , \end{aligned} \tag{96}$$

where $D(\mathbf{R}) = A(0) - A(\mathbf{R})$. Therefore, the quantity $\sigma_{\mathbf{p}}^2(x)$ in (93) is given by the expression

$$\sigma_{\mathbf{p}}^2(x) = \frac{k^2 x}{8} \Delta_{\mathbf{R}} D(\mathbf{R}) \bigg|_{\mathbf{R}=0} = -\frac{k^2 x}{8} \Delta_{\mathbf{R}} A(\mathbf{R}) \bigg|_{\mathbf{R}=0} ,$$

which of course coincides with formula (73) for turbulent fluctuations of the field $\varepsilon(x, \mathbf{R})$,

$$\sigma_{\mathbf{p}}^2(x) = \frac{1.476}{L_{\mathrm{f}}^2(x)} D^{1/6}(x) \beta_0(x) . \tag{97}$$

We note that the path-integral representation of the field $u(x, \mathbf{R})$ permits us to study the applicability limits for the approximation of a delta-correlated random field $\varepsilon(x, \mathbf{R})$ for wave intensity fluctuations. It turns out that all applicability conditions for a delta-correlated random field $\varepsilon(x, \mathbf{R})$ in the calculation of $\langle I^n(x, \mathbf{R}) \rangle$ coincide with the applicability conditions of the delta-correlated approximation for the quantity $\langle I^2(x, \mathbf{R}) \rangle$. In other words, the approximation of a delta-correlated random field $\varepsilon(x, \mathbf{R})$ does not affect the probability distribution for the wave field intensity.

For turbulent temperature pulsations in the weak fluctuation regime, the approximation of a delta-correlated random field $\varepsilon(x, \mathbf{R})$ is valid if the inequalities

$$\lambda \ll \sqrt{\lambda x} \ll x$$

are satisfied, with $\lambda = 2\pi/k$ being the wavelength.

On the other hand, in the strong fluctuation regime, the applicability condition for the approximation of a delta-correlated random field $\varepsilon(x, \mathbf{R})$ is

$$\lambda \ll \rho_{\mathrm{cog}} \ll r_0 \ll x ,$$

where ρ_{cog} and r_0 are given by formulas (54) and (82). All these inequalities have simple physical meaning. The delta-correlated approximation holds as long as the correlation radius of $\varepsilon(x, \mathbf{R})$ (its role for turbulent temperature pulsation is played by the size of the first Fresnel zone) is the least of all longitudinal scales in the problem of wave propagation in a randomly inhomogeneous medium. When the wave propagates in the strong fluctuation region, a longitudinal scale $\sim \rho_{\mathrm{cog}} \sqrt{kx}$ appears that gradually decreases and can eventually become smaller than the correlation radius of $\varepsilon(x, \mathbf{R})$ at a sufficiently large parameter $\beta_0^2(x)$. The delta-correlated approximation is no longer applicable in such a situation.

Inequalities in the preceding paragraphs may be regarded as the lower and upper bounds for the scale of the intensity correlation function. The delta-correlated approximation then holds only if any scale arising in the problem remains small compared with the track length.

4.2 Caustic structure of the wave field in a randomly inhomogeneous medium

The statistical characteristics of the wave field $u(x, \mathbf{R})$ considered in the preceding paragraphs, e.g., the mean field and second-order coherence function, by no means reflect the actual behavior of the wave field in individual realizations of medium parameters (see Figs 1–3). For a detailed analysis of the random wave field structure, it is possible to apply methods of statistical topography that help to understand how the caustic structure of a wave field is formed and which statistical parameters describe it. We note that the theory of large deviations of a random intensity field was first applied to the analysis of the problem of wave propagation in a turbulent medium in Ref. [44] (see also Ref. [10]).

4.2.1 Elements of statistical topography of a random wave intensity field. By virtue of spatial homogeneity, all one-point statistical characteristics of an incident plane wave (including probability density) are independent of the variable $\mathbf{R}$. It is therefore possible to introduce specific (calculated per unit area) values of selected physical quantities that rather comprehensively characterize the caustic structure of the wave field intensity. As mentioned above, the size $L_{\rm f}(x) = \sqrt{x/k}$ of the first Fresnel zone then serves as the natural scale length in the plane $x = \text{const}$ independent of medium parameters.

These quantities include:

• The deterministic curve, called the *typical realization* $I^*(x)$ of the wave field intensity $I(x, \mathbf{R})$, which is the median of its probability distribution $P(x; I)$ and is determined as the solution of the algebraic equation $\int_0^{I^*} P(x; I')\,\mathrm{d}I' = 1/2$. A property of the median is that for any segment of distances (X_1, X_2), the mean value of all segments of distances with $I^*(x) < I(x, \mathbf{R})$ is equal to the mean value of all segments of distances with $I^*(x) > I(x, \mathbf{R})$, i.e.,

$$\langle X\rangle_{I^*(x)<I(x,\mathbf{R})} = \langle X\rangle_{I^*(x)>I(x,\mathbf{R})} = \frac{1}{2}(X_2 - X_1)\,.$$

• The specific mean total area of regions in the plane $\{\mathbf{R}\}$ bounded by level lines with $I(x, \mathbf{R}) > I$,

$$\big\langle s(x, I)\big\rangle = \int_I^\infty P(x; I')\,\mathrm{d}I'\,.$$

Here, $P(x; I)$ is the probability density of the wave field intensity $I(x, \mathbf{R})$.

• The specific mean field power contained in these regions,

$$\big\langle e(x, I)\big\rangle = \int_I^\infty I' P(x; I')\,\mathrm{d}I'\,.$$

• The specific mean length of these contours,

$$\big\langle l(x, I)\big\rangle = L_{\rm f}(x)\big\langle\big|\mathbf{p}(x, \mathbf{R})\big|\delta\big(I(x, \mathbf{R}) - I\big)\big\rangle\,,$$

where $\mathbf{p}(x, \mathbf{R}) = \nabla_{\mathbf{R}} I(x, \mathbf{R})$ is the transverse gradient of the wave field intensity.

• The estimated mean excess of contours with the opposite orientation of normal vectors within the first Fresnel zone,

$$\big\langle n(x, I)\big\rangle = \frac{1}{2\pi}\,L_{\rm f}^2(x)\big\langle\varkappa(x, \mathbf{R}; I)\big|\mathbf{p}(x, \mathbf{R})\big|\delta\big(I(x, \mathbf{R}) - I\big)\big\rangle\,,$$

where $\varkappa(x, \mathbf{R}; I)$ is the level line curvature,

$$\varkappa(x, \mathbf{R}; I) = \frac{1}{p^3(x, \mathbf{R})}\left[-p_y^2(x, \mathbf{R})\,\frac{\partial^2 I(x, \mathbf{R})}{\partial z^2} - p_z^2(x, \mathbf{R})\,\frac{\partial^2 I(x, \mathbf{R})}{\partial y^2} + 2p_y(x, \mathbf{R})\,p_z(x, \mathbf{R})\,\frac{\partial^2 I(x, \mathbf{R})}{\partial y\,\partial z}\right].$$

We now consider the dynamics of all these quantities as functions of the distance x traveled by a wave [of the parameter $\beta_0(x)$].

4.2.2 Region of weak intensity fluctuations. The weak intensity fluctuation region is bounded by parameter values $\beta_0(x) \leqslant 1$; in this case, the wave field intensity has a log-normal character and is described by expression (68).

The typical realization of random intensity for this log-normal process falls off exponentially with path length,

$$I^*(x) = \exp\left\{-\frac{1}{2}\,\beta_0(x)\right\}.$$

On the other hand, $\big\langle I(x)\big\rangle = 1$, and statistics (e.g., moments $\big\langle I^n(x, \mathbf{R})\big\rangle$) are formed by large deviations of $I(x, \mathbf{R})$ with respect to this curve.

Moreover, various majorant estimates exist for the realizations of a log-normal process. For example, the inequality

$$I(x) < 4\exp\left\{-\frac{1}{4}\,\beta_0(x)\right\}$$

is fulfilled with probability $p = 1/2$ for individual realizations of the wave field intensity over the entire range of distances $x \in (0, \infty)$. Taken together, these facts suggest the onset of the formation of the caustic structure of wave field intensity.

As mentioned before, the description thus obtained holds at values of $\beta_0(x) \leqslant 1$. The method of smooth perturbations becomes invalid as the parameter $\beta_0(x)$ continues to grow and nonlinearity of the equation for the complex phase of the wave field must be taken into consideration. This fluctuation region, referred to as the *strong focusing region*, is very difficult for analytic studies. A further rise in parameter $\beta_0(x)$ $\big(\beta_0^2(x) \geqslant 10\big)$ leads to the saturation of statistical characteristics of intensity; this region of $\beta_0(x)$ variations is called the *region of strong intensity fluctuations*.

4.2.3 Region of strong intensity fluctuations. It follows from the expression for probability density (91) that the mean specific area of the regions inside which $I(x, \mathbf{R}) > I$ is given by

$$\big\langle s(x, I)\big\rangle = \frac{1}{\sqrt{\pi\big(\beta(x) - 1\big)}} \times \int_0^\infty \exp\left\{-zI - \frac{\big[\ln z - \big(\beta(x) - 1\big)/4\big]^2}{\beta(x) - 1}\right\}\frac{\mathrm{d}z}{z}\,, \qquad (98)$$

while the specific mean power concentrated in these regions is described by the expression

$$\big\langle e(x, I)\big\rangle = \frac{1}{\sqrt{\pi\big(\beta(x) - 1\big)}} \times \int_0^\infty \left(I + \frac{1}{z}\right)\exp\left\{-zI - \frac{\big[\ln z - \big(\beta(x) - 1\big)/4\big]^2}{\beta(x) - 1}\right\}\frac{\mathrm{d}z}{z}\,. \qquad (99)$$

We note the very slow dependence of the parameter $\beta(x)$ on $\beta_0(x)$. Specifically, $\beta(x) = 1$ corresponds to the limiting transition $\beta_0(x) \to \infty$, while the value $\beta(x) = 1.861$ corresponds to $\beta_0(x) = 1$.

Asymptotic formulas (98) and (99) describe the transition to a saturated intensity regime $\big(\beta(x) \to 1\big)$. Accordingly, in this regime,

$$P(I) = \exp(-I)\,, \qquad \big\langle s(I)\big\rangle = \exp(-I)\,, \qquad \big\langle e(I)\big\rangle = (I + 1)\exp(-I)\,, \qquad (100)$$

and hence specific values of the mean area and mean power above the level line depend only on I.

For the specific mean contour length, we obtain the expression

$$\begin{aligned}\langle l(x,I)\rangle &= L_{\mathrm{f}}(x)\langle|\mathbf{p}(x,\mathbf{R})|\delta(I(x,\mathbf{R})-I)\rangle\\ &= 2L_{\mathrm{f}}(x)\sqrt{I}\left\langle|\boldsymbol{\varkappa}(x,\mathbf{R})|\delta\Bigl(I(x,\mathbf{R})-I\Bigr)\right\rangle\\ &= 2L_{\mathrm{f}}(x)\sqrt{I}\langle|\mathbf{q}(x,\mathbf{R})|\rangle\exp(-I)\\ &= L_{\mathrm{f}}(x)\sqrt{2\pi\sigma_{\mathbf{q}}^2(x)I}\exp(-I)\,,\end{aligned} \tag{101}$$

where the variance of the amplitude level gradient in the saturated fluctuation regime coincides with the variance computed using the first MSP approximation. The maximum value in (101) is reached at $I = 1/\sqrt{2}$.

The mean specific number of contours in this region is estimated as

$$\begin{aligned}\langle n(x,I)\rangle &= \frac{L_{\mathrm{f}}^2(x)}{2\pi}\langle\varkappa(x,\mathbf{R},I)|\mathbf{p}(x,\mathbf{R})|\delta(I(x,\mathbf{R})-I)\rangle\\ &= -\frac{L_{\mathrm{f}}^2(x)}{2\pi}\sqrt{I}\langle\Delta_{\mathbf{R}}A(x,\mathbf{R})\delta(I(x,\mathbf{R})-I)\rangle\\ &= \frac{2L_{\mathrm{f}}^2(x)\sigma_{\mathbf{q}}^2(x)}{\pi}\left(I-\frac{1}{2}\right)\exp(-I)\,.\end{aligned} \tag{102}$$

The maximum value in (102) is reached at $I = 3/2$; the value at which the mean specific number of contours delineating the region $I(x,\mathbf{R}) > I_0$ coincides with the mean specific number of contours for which $I(x,\mathbf{R}) < I_0$ is $I_0 = 1/2$.

We note that formula (102) is inapplicable in the narrow vicinity of $I \sim 0$. For $I = 0$, it must be that $\langle n(x,0)\rangle = 0$.

It follows from expressions (101) and (102) that the mean length of level lines and the mean number of contours in the saturated fluctuation region continue to increase as the parameter $\beta_0(x)$ increases, even though the mean contour areas and the powers enclosed therein remain constant. The explanation lies in the dominant role played by the interference of partial waves coming from different directions.

The dynamics of level lines strongly depends on the balance between the focusing and defocusing of radiation by different portions of a turbulent medium. Focusing by large-scale inhomogeneities results in high-intensity peaks on the random relief. In the maximum focusing regime ($\beta_0(x) \sim 1$), about half of the total wave power is concentrated in such high narrow peaks. As the parameter $\beta_0(x)$ increases, defocusing of radiation prevails, which tends to smooth out high peaks and create a highly fragmented (interferential) landscape with a large number of smaller peaks near $I \sim 1$. Such a dynamic picture has been obtained both in laboratory experiments (Fig. 1b) and by numerical simulation (Fig. 2b).

The mean length and the mean number of contours depend on the wave parameter $D(x)$, in addition to $\beta_0(x)$; in other words, both grow as microscale inhomogeneities decrease. This dependence is due to small ripples generated by scattering from small inhomogeneities superimposed on the large-scale relief.

To summarize, we have attempted to provide a qualitative explanation of the field caustic structure of a plane light wave as it transversely propagates in a turbulent medium, as well as to quantify and estimate parameters characterizing the development of such a structure. Generally speaking, this problem may have many parameters. However, by confining its analysis to a fixed cross section, the solution for a plane wave at a constant parameter value can be described by a single parameter $\beta_0(x)$ — the intensity variance in the weak fluctuation regime. We have analyzed two extreme asymptotic cases corresponding to weak and strong intensity fluctuations. It should be borne in mind that the above asymptotic formulas are most likely applicable within a certain range depending on the intensity level I. Naturally, the applicability limits should extend as this level decreases.

The analysis of an intermediate case corresponding to the developed caustic structure region (most interesting in applications) would require the knowledge of the probability density of intensity and its transverse gradient for an arbitrary distance traveled by the wave. Such analysis is feasible using approximating expressions for the probability density of all parameter values or by means of numerical simulation.

5. Conclusion

We have considered the principal propositions of the theory of wave propagation in randomly inhomogeneous media developed during the past 50 years and have identified the main parameters characterizing wave propagation in a turbulent atmosphere. It is only natural that many questions either were discussed only in brief or were beyond the scope of this review. In our opinion, statistical analysis of the spatial caustic structure of wave fields appears to be one of the most promising lines of further research. Moreover, it is necessary to reconsider the problem of magnetic field depolarization in random media. As mentioned in Section 2.1, variances of parameters characterizing these effects are rather small (e.g., the mean value of the Umov–Pointing vector). But the problems concerning the spatial structure of correlational dependences of vector fields described by Maxwell equations (3) and (4) in the region of strong wave field fluctuations remain to be investigated. These problems can be examined by passing from Eqn (3) to a vector parabolic equation and thereafter using the approximation of delta-correlation of the dielectric permittivity field along the x axis.

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